Math 6112 – Spring 2020
Problem Set 3
Due: Friday 31 January 2020

10. As an application of Yoneda's Lemma, show that there is a bijection between

    - the class of natural transformations between the functors $Hom_{\mathcal{C}}(A, -)$ and $Hom_{\mathcal{C}}(A', -)$ for two objects A, A' of $\mathcal{C}$
    - the set $Hom_{\mathcal{C}}(A, A')$.

11. Let G be a group and $\underline{G}$ the associated category as in Problem 1, so the category with a single object, call it $*$, and such that $Hom_{\underline{G}}(*, *) = G$.

    (a) Show that a covariant functor $F : \underline{G} \to \underline{Set}$ is determined by a set $X = F(*)$ and a left action of G on X. Call this functor F_X.

    (b) Show that a natural transformation $\eta : F_X \to F_Y$ determines a G-equivariant map $\eta : X \to Y$, i.e., $\eta(g \cdot x) = g \cdot \eta(x)$ for all $g \in G$ and $x \in X$.

    (c) Show that Yoneda's Lemma for the functor $F = Hom_{\underline{G}}(*, -)$ from $\underline{G}$ to $\underline{Set}$ gives Cayley's Theorem: G is isomorphc to a subgroup of $Sym(G)$, the permutations on G as a set.

12. Dualize Yoneda's Lemma to show that if F is a contravariant functor from $\mathcal{C}$ to $\underline{Set}$ and $A \in Ob(\mathcal{C})$, then any natural transformation of $Hom_{\mathcal{C}}(-, A)$ to F has the form $B \mapsto a_B$, where a_B is a map from $Hom_{\mathcal{C}}(B, A)$ to $F(B)$ determined by an element $a \in F(A)$ as $a_B : g \mapsto F(g)(a)$. Show that this gives a bijection of the set $F(A)$ with the class of natural transformations of $Hom(-, A)$ to F.

The next two exercises investigates the definition of kernels and cokernels in a categorical context. Let $\mathcal{C}$ be a category with a zero object, denoted $0_{\mathcal{C}}$. Let $f \in Hom(A, B)$.

We call a morphism $k \in Hom(K, A)$ a *kernel of f* if

(1) k is monic

(2) $fk = 0$ where $0 \in Hom(K, B)$ is defined by the composition $K \to 0_{\mathcal{C}} \to B$.

(3) for any $g \in Hom(G, A)$ such that $fg = 0$ there exists a $g' \in Hom(G, K)$ such that $g = kg'$. [Since k is monic, such a g' is unique.]

Dually, we call a morphism $c \in Hom(B, C)$ a *cokernel of f* if

(1) c is epic

(2) $cf = 0$ where $0 \in Hom(A, C)$ is defined by the composition $A \to 0_{\mathcal{C}} \to C$.

(3) for any $h \in Hom(B, H)$ such that $hf = 0$ there exists a $h' \in Hom(C, H)$ such that $h = h'c$. [Since c is monic, such a h' is unique.]

Note: Categorical kernels and cokernels are unique up to isomorphism.

13. In $R - \underline{mod}$ show that this recovers the usual notion of kernel and cokernal, that is
    (i) if $f \in Hom(A, B)$ and we let $K = ker(f) \subset A$ and $k : K \hookrightarrow A$ is the embedding of K into A, then $k \in Hom(K, A)$ is a kernel of f in the categorical sense for $R - \underline{mod}$.
    (ii) If $f \in Hom(A, B)$ and we let $C = B/f(A)$ and $c : B \to C$ the canonical quotient map, then $c \in Hom(B, C)$ is a cokernel of f in the categorical sense in $R - \underline{mod}$.

14. In $R - \underline{mod}$ show that
    (i) If $f \in Hom(A, B)$ is monic, then it is a kernel of its cokernel.
    (ii) If $f \in Hom(A, B)$ is epic, then it is a cokernel of its kernel.

Definition: A category $\mathcal{C}$ is called *abelian* if it is an additive category having the following additional properties:

(AC4) every morphism in $\mathcal{C}$ has a kernel and a cokernel.

(AC5) Every monic is a kernel of its cokernel and eveny epic is a cokernel of its kernel.

(AC6) Every morphism can be factored as $f = me$ where is e is epic and m is monic.