An analytic perspective of Weil reciprocity

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Im memory of Emma Previato.

Abstract

In Cogdell et al., *LMS Lecture Notes Series* **459,** 393–427 (2020), the authors proved a type of Kronecker's limit formula associated to any divisor D on any smooth Kähler manifold X, assuming that D is smooth in codimension one. In the present article, it is shown how the aforementioned analogue of Kronecker's limit formula applies to reprove and generalize Weil reciprocity. More precisely, we extend Weil reciprocity to (suitably normalized) meromorphic modular forms of even weight on a smooth, compact Riemann surface, and present a variant of Weil reciprocity for a class of harmonic functions with special types of singularities on a finite volume quotient of a symmetric space or a compact, smooth projective Kähler variety. We also prove an integral version of Weil reciprocity for a compact, smooth projective Kähler variety.

1 Introduction

In its nascent form, Weil reciprocity is the following statement. Let Y be a smooth, compact Riemann surface, meaning a non-singular algebraic curve over $\mathbb{C}$. Let f and g be two meromorphic functions on Y, so then f and g can be viewed as elements of the function field $\mathbb{C}(Y)$. Let $D_f := \sum m(P)P$ and $D_g := \sum n(Q)Q$ be the divisors of f and g, respectively. If D_f and D_g are disjoint, then

$$f(D_g) = g(D_f) \tag{1}$$

where

$$f(D_g) = \prod_{Q\in D_g} f(Q)^{n(Q)} \quad \text{and} \quad g(D_f) = \prod_{P\in D_f} g(P)^{m(P)}.$$

Weil reciprocity (1) is attributed to [We40] over a finite field and is employed in the study of the Weil pairing, which itself is vital in the Weil's proof of the Riemann hypothesis for zeta functions attached to function fields over finite fields; see [We41]. At this time, (1) can be stated and proved in introductory textbooks; see page 242 [GH78] or Exercise

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2.11 of [Si86]. We note that the proof of (1) in [GH78] amounts to an application of the residue theorem.

If D_f and D_g are not distinct, one can state a generalization of (1) which we now describe. For every point $P \in Y$, define the local symbol

$$(f,g)_P = f^{\mathrm{ord}_P(g)}(P)/g^{\mathrm{ord}_P(f)}(P) \cdot (-1)^{\mathrm{ord}_P(f)\cdot \mathrm{ord}_P(g)}, \tag{2}$$

where (2) is interpreted as the constant term in a Laurent expansion of the quotient f/g at a common point $P \in D_f \cap D_g$. Note that for all but a finite number of P, one has that $(f,g)_P = 1$. In this notation, Weil reciprocity is the statement that

$$\prod_{P \in Y} (f,g)_P = 1. \tag{3}$$

In the case $Y = \mathbb{P}^1(\mathbb{C})$ with $f(x) = x - a$ and $g(x) = x - b$ for constants a and b, then Weil reciprocity states that $a - b = -(b - a)$, which is an enjoyable computation from (3).

In one of her first articles [Pr91], Emma Previato gave a proof of (3), though not stated in this notation, using a correspondence between certain differential operators on $\mathbb{C}(Y)$ and meromorphic functions on Y. One can find the formulation of (3) in Deligne's article [De91], who attributes the point of view to Tate and cites [Se79]. In fact, Tate's article [Ta68] develops an abstract theory of residues which permits the generalization of Weil reciprocity to function fields $K(Y)$ when K is not necessary equal to $\mathbb{C}$.

In a latter paper which Emma Previato wrote with J.-L. Brylinski [BP00], the authors refer to (3) as the "Weil-Tate reciprocity law". Futhermore, it is stated in [BP00] that the identity (3) is "is an essential tool in the abelian class field theory of its function field"; see page 89 of [BP00]. Indeed, in [HM16] the authors specifically state on page 88 that Weil reciprocity is one of the main tools at their dispose, and in fact use Weil reciprocity to prove the version of Artin reciprocity needed in their work; see section 4, in particular page 89, of [HM16]. Finally, let us call attention to the very interesting articles [AP02], [MP08], and [MNPP20], and references therein, which place Weil reciprocity as part of an algebraic theory which emanates from [Ta68] and is further developed in [ACK89] and complements the ideas from [Pr91].

In this note we will use the main results from [CJS20] and give a new proof of (1), from which (3) follows by a limiting argument as asserted in [Pr91]. In essence, our approach to Weil reciprocity begins with the spectral theory of certain integral kernels on X, as developed in [CJS20], from which we prove a general Kronecker's limit formula. From the Kronecker limit formula, we show that Weil reciprocity and its generalizations follow by studying properties of the Kronecker limit function. All of the properties of the Kronecker limit function needed to prove Weil reciprocity follow from properties of the integral kernels in question. We note that the setting of [CJS20] is to consider a general Kähler variety Y and divisor D which is smooth in codimension one. As such, we are able to obtain generalizations of Weil reciprocity using properties of the Green's function (the resolvent kernel) on Y.

More specifically, in Theorem 1 below we extend Weil reciprocity to (suitably normalized) meromorphic modular forms of even weight on a smooth, compact Riemann surface Y. Theorem 2 is a variant of Weil reciprocity for harmonic functions with Green's function type singularities on a finite volume quotient of a symmetric space or a compact, smooth projective Kähler variety, while in Theorem 3 we state an integral version of Weil reciprocity for a compact, smooth projective Kähler variety.

The paper is organized as follows. In section 2 below we introduce the notation, and discuss properties of the resolvent kernel and meromorphic forms in the setting of algebraic curves. If Y is an algebraic curve and the divisors under consideration have degree zero, then, in section 3 we obtain (1), hence (3). If the divisors do not have degree zero, in section 3.2 we show that our method yields a generalization of Weil reciprocity for holomorphic forms of non-zero weight, meaning sections of powers of the canonical bundle possibly twisted by a flat line bundle. In section 4 we show that many of our arguments in the case of algebraic curves will extend *mutatis mutandis* to the general setting of [CJS20]. In absence of an underlying complex structure, our generalization of Weil reciprocity involves the space of harmonic functions which have a finite number of specific singularities, one example of which is the logarithmic absolute value of meromorphic functions on a smooth algebraic curve. We will present various examples which illustrate the point of view presented here.

2 Preliminaries

We begin by recalling briefly the necessary notation and assertions. For further details, we refer to [CJS20] and references therein.

2.1 Notational set-up

Let Y be a smooth, compact Riemann surface of genus g. For simplicity, let us assume that $g > 1$. Let μ be any positive $(1, 1)$ form on Y, which provides a metric on Y as well as any power of the associated canonical bundle $\mathcal{K}$. For any local holomorphic coordinate z on Y, we can write

$$\mu(z) = \frac{i}{2}\partial_z\partial_{\bar{z}}\rho,$$

where ρ is the Kähler potential. The form μ is scaled so that

$$\int_Y \mu = \deg(\mathcal{K}) = 2g - 2$$

As an example, one can take $\rho(z) = \log(\mathrm{Im}(z))$ where z is a local holomorphic coordinate, so then μ is the hyperbolic metric on Y. Let f be a weight $2k$ modular form, meaning a holomorphic section of $\mathcal{K}^{\otimes k}$, with divisor D. Then the (pointwise) norm $\|f\|_\mu$ of f with respect to μ is

$$\|f\|^2_\mu(z) := e^{-2\pi k\rho(z)}|f(z)|^2, \tag{4}$$

so then

$$d_z d^c_z \log \|f\|_\mu(z) = \delta_D(z) - k\mu(z); \tag{5}$$

we refer to [La88] for the notation and scaling associated to the operators d_z and d^c_z.

2.2 Resolvent kernel and the prime form

For distinct points $z, w \in Y$, let $G_{Y;1/4}(z, w; s)$ denote the resolvent kernel of the Laplacian Δ_Y associated to the metric induced by μ on the space of continuous functions. As is common, in the notation of [CJS20], one takes $\rho_0 = 1/2$. For notational convenience, we will write $G(z, w; s) = G_{Y;1/4}(z, w; s)$. From the discussion and results within [CJS20], we have the following statements. The function $G(z, w; s)$ is defined for $\mathrm{Re}(s)$ sufficiently large, and it is symmetric in z and w, meaning that $G(z, w; s) = G(w, z; s)$. It is admits a meromorphic continuation to all $s \in \mathbb{C}$. In particular, its Laurent expansion near $s = 0$ is of the form

$$G(z, w; s) = \frac{1}{\mathrm{vol}_\mu(Y)} \frac{1}{s(s-1)} + G(z, w) + o(1) \quad \text{as } s \to 0; \tag{6}$$

see Corollary 6.1 and Remark 6.3 of [CJS20]. The function $G(z, w)$ is the Green's function which inverts the actions of the Laplacian on the space of smooth functions and is orthogonal to the constant functions. Specifically, this means that

$$\int\limits_Y G(z, w)\mu(z) = 0 \tag{7}$$

and

$$\mathrm{d}_z \mathrm{d}_z^c G(z, w) = \delta_w(z) - \mu(z); \tag{8}$$

see page 431 of [CJS20] as well as section 2.5 and references therein. In words, (8) implies that $G(z, w)$ is locally a harmonic function away from $z = w$, and it has a logarithmic singularity as z approaches w. More specifically, we can write

$$G(z, w) = \log \|H(z, w)\|_\mu^2 \tag{9}$$

where

$$\|H(z, w)\|_\mu^2 = e^{-(2\pi\rho(z))/c} e^{-(2\pi\rho(w))/c} |H(z, w)|^2 \tag{10}$$

and $c = 1/(2g - 2)$. The function $H(z, w)$ is equal to a constant multiple of Fay's prime form times a degenerate theta function as in [La82]; see [Fa73] and the discussion in section 7.5 of [JvPS18]. As such, we have that $H(z, w)$ is locally holomorphic in z and w, non-vanishing if $z \neq w$, has a first order zero as z approaches w, and is anti-symmetric meaning that $H(z, w) = -H(w, z)$.

In effect, the differential equation (8) and symmetry in z and w determines the function $H(z, w)$ in (10) up to a multiplicative constant. That constant is determined further, up to a multiplicative constant of modulus one, by (7). Without further considerations, the remaining constant of modulus one cannot be determined.

2.3 Product formula and normalization of modular forms

Any meromorphic modular form on a smooth, compact Riemann surface Y can be represented in terms of prime forms $H(z, w)$. Namely, we have the following lemma.

Lemma 1 *Let f be a meromorphic modular form on a smooth, compact Riemann surface Y and let $\mathrm{ord}_w(f)$ denote the order f at w. Define*

$$H_f(z) := \prod_{w \in D_f} H(z,w)^{\mathrm{ord}_w(f)}.$$

Then, there exists a complex constant c_f such that

$$f(z) = c_f H_f(z). \tag{11}$$

Proof: The argument is similar to that given in section 4.4 of [CJS22]. However, the proof is somewhat evident. Indeed, the ratio $f(z)/H_f(z)$ is holomorphic away from $z \in D_f$, and near D_f the numerator and denominator have singularities of the same order. Thus, one can apply the Riemann removable singularity theorem to conclude that $f(z)/H_f(z)$ extends to a bounded, non-zero, holomorphic function on Y, which is necessarily constant. □

Let f be an even weight $2k$ meromorphic modular form on Y, with multiplier system which may involve a one-dimensional unitary representation of the fundamental group of Y. In other terms, f is a meromorphic section of $\mathcal{K}^{\otimes k} \otimes \mathcal{L}_\chi$ where $\mathcal{L}_\chi$ is a flat line bundle. Let D_f denote the divisor of f, and let $\|f\|_\mu(z)$ be the pointwise norm of f as defined in (4). Then, from (5) we conclude that there is a constant c such that

$$\log \|f\|_\mu(z) = \sum_{P \in D_f} m(P) G(z,P) + c. \tag{12}$$

If we assume that

$$\int_Y \log \|f\|_\mu(z) \mu(z) = 0,$$

then, in view of (7) we normalize the constant c to be zero. In words, we can describe such a normalizaton as saying that f *has L^1-log norm equal to zero.*

3 A proof of Weil Reciprocity on algebraic curves

In this section we prove (1) and (3) and extend the Weil Reciprocity to the setting of meromorphic modular forms on Y of non-zero weight.

3.1 Weil Reciprocity for meromorphic modular forms

Let f, g be two meromorphic modular forms on Y. Then, f can be represented as (11), while, according to Lemma 1 we can write

$$g(z) = c_g H_g(z) \quad \text{where} \quad H_g(z) := \prod_{w \in D_g} H(z,w)^{\mathrm{ord}_w(g)}.$$

Recall that $D_f := \sum m(P)P$ and $D_g := \sum n(Q)Q$. For notational convenience, let us write

$$f(z) = c_f \prod_{P \in D_f} H(z,P)^{m(P)} \quad \text{and} \quad g(z) = c_g \prod_{Q \in D_g} H(z,Q)^{n(Q)}$$

Then

$$\begin{aligned}
f(D_g) = \prod_{Q \in D_g} f(Q)^{n(Q)} &= \prod_{Q \in D_g} \left(c_f^{n(Q)} \prod_{P \in D_f} H(Q,P)^{m(P)n(Q)} \right) \\
&= c_f^{\deg(D_g)} \prod_{P \in D_f} \left(\prod_{Q \in D_g} H(Q,P)^{m(P)n(Q)} \right) \\
&= c_f^{\deg(D_g)} c_g^{-\deg(D_f)} \prod_{P \in D_f} \left(c_g^{m(P)} \prod_{Q \in D_g} H(Q,P)^{m(P)n(Q)} \right) \\
&= c_f^{\deg(D_g)} c_g^{-\deg(D_f)} (-1)^{\deg(D_f)\deg(D_g)} \prod_{P \in D_f} \left(c_g^{m(P)} \prod_{Q \in D_g} H(P,Q)^{m(P)n(Q)} \right) \\
&= C \cdot g(D_f)
\end{aligned}$$

where

$$C = c_f^{\deg(D_g)} c_g^{-\deg(D_f)} (-1)^{\deg(D_f)\deg(D_g)}.$$

However, Y is a smooth, compact Riemann surface, hence $\deg(D_f) = \deg(D_g) = 0$, so then $C = 1$, which completes the proof of (1).

Let us now consider the case when D_f and D_g are not disjoint. As suggested in [Pr91], we will view (3) as a limiting case of (1) by arguing as follows. For simplicity, let us assume that D_f and D_g have a single point, R, in common with multiplicity $m(R)$ in D_f and $n(R)$ in D_g. Let $D'_f = D_f \setminus \{R\}$ and $D'_g = D_g \setminus \{R\}$. Let $\mathcal{D}$ be a holomorphic disc on Y containing R with holomorphic coordinate t. Let P_t and Q_t be two holomorphic functions of t with image in $\mathcal{D}$ such that $P_0 = Q_0 = R$ and $P_t \neq Q_t$ for $t \neq 0$. Define the functions

$$f_t(z) = c_f \prod_{P \in D'_f} H(z,P)^{m(P)} \cdot H(z,P_t)^{m(R)}$$

for any non-zero constant c_f, and similarly

$$g_t(z) = c_g \prod_{Q \in D'_g} H(z,Q)^{n(Q)} \cdot H(z,Q_t)^{n(R)}$$

By arguing as above, and setting $m(P_t) = m(R)$ and $n(Q_t) = n(R)$ we get that

$$\prod_{Q \in D'_g \cup \{Q_t\}} f_t(Q)^{n(Q)} = \prod_{P \in D'_f \cup \{P_t\}} g_t(P)^{m(P)}. \tag{13}$$

One now factors out the single term $H(P_t, Q_t)$ from both sides of (13). Using that $H(z,w)$ is anti-symmetric, (13) immediately implies (3) upon letting $t \to 0$.

3.2 Weil reciprocity for modular forms of non-zero weight

The following statement can be viewed as a Weil reciprocity for modular forms of even, nonzero weight $2k$.

Theorem 1 *Let Y be a smooth, compact Riemann surface and let f and g be two meromorphic modular forms of even, nonzero weight $2k_1$ and $2k_2$, respectively, and with divisors D_f and D_g, respectively. Assume that D_f and D_g are disjoint, and assume that both f and g are normalized to have L^1-log norm equal to zero. Then*

$$\prod_{Q\in D_g} \|f\|_\mu(Q) = \prod_{P\in D_f} \|g\|_\mu(P). \tag{14}$$

Proof: The proof is an immediate consequence of (12) with $c = 0$ and the symmetry of the Green's function $G(z, w)$ in the variables z and w. □

One could seek to extract from (14) a more detailed identity by observing that the ratio

$$\left(\prod_{Q\in D_g} \|f\|_\mu(Q)\right) / \left(\prod_{P\in D_f} \|g\|_\mu(P)\right)$$

is not only equal to 1 but, as one can see from (4) its dependence on μ is explicit. We will not pursue that line of inquiry here. Rather, we will use the above approach to Theorem 1 and obtain analogues of Weil reciprocity in other geometric settings.

4 Extensions of the main result

Our proof of Weil reciprocity (3) is based on the following points, which we list in increasing order of specificity. First, there exists a Green's function which satisfies (7) and (8), and the Green's function is symmetric in its two variables. With that information, we then have the trivial identity

$$\sum_{P\in A}\sum_{Q\in B} G(P,Q) = \sum_{Q\in B}\sum_{P\in A} G(Q,P) \tag{15}$$

where A and B are finite sets of points on the underlying Riemann surface Y. Second, one can express the Green's function using the holomorphic function theory on Y as in (9). In doing so, one is able to "drop the absolute values" and study Weil reciprocity for holomorphic functions since the aforementioned singularities are logarithmic. With these two points, our proof of (1) and (3) can be derived. Furthermore, the generalization (14) follow as above.

Let us now describe two other settings where our approach to Weil reciprocity applies.

For now, let X be either a finite volume quotient of a symmetric space or a compact, smooth projective Kähler variety. In either case, there is a well-define Laplacian which acts on the space of smooth functions, as well as a Green's function $G(z, w)$ which inverts the action of the Laplacian on the space orthogonal to the constant functions. The Green's function $G(z,w)$ has a singularity near $z=w$ whose order depends on the dimension of X. If X is a Riemann surface, then the singularity is, in local coordinates, equal to $\log|z-w|^2$. In general, the singularity is comparable to $|z-w|^{-(n-2)}$; see pages 94 and 109 of [Fo76]. For any function f on X, we will say that f *has a Green's function type singularity at* P if for some constant c_P, one has that

$$f(z) - c_P G(z,P) \text{ is bounded for } z \text{ near } P.$$

In the case when X is a hyperbolic 3-manifold, functions with Green's function type singularities are discussed beginning on page 6424 of [HIvPT19].

Theorem 2 *Let f and g be bounded functions on X which are harmonic except at a finite set of points which we denote by $D_f=\{P_j\}$ for f and $D_g=\{Q_i\}$ for g. Assume that D_f and D_g are disjoint sets and that at each point in D_f, respectively D_g, the function f, resp. g, has a Green's function type singularity. Further, assume that f and g are in $L^1\cap L^2(X)$. If*

$$\sum_{P_j\in D_f} c_{P_j}(f) = \sum_{Q_i\in D_g} c_{Q_i}(g) = 0, \tag{16}$$

then

$$\sum_{Q_i\in D_g} c_{Q_i}(g) f(Q_i) = \sum_{P_j\in D_f} c_{P_j}(f) g(P_j).$$

Proof: With the assumptions as above, any bounded harmonic function is constant. Hence, there are constants A_f and A_g such that

$$f(z) = \sum_{P_j\in D_f} c_{P_j}(f) G(z,P_j) + A_f \quad \text{and} \quad g(z) = \sum_{Q_i\in D_g} c_{Q_i}(g) G(z,Q_i) + A_g.$$

Actually, since the L^1-norm of G equals zero, it is straightforward to conclude that

$$A_f = \frac{1}{\mathrm{vol}_\mu(X)} \int\limits_X f(z)\mu(z)$$

and similarly for A_g. Therefore, in view of (16), and the symmetry of the Green's functions in the two variables we get

$$\sum_{Q_i\in D_g} c_{Q_i}(g) f(Q_i) = \sum_{Q_i\in D_g} c_{Q_i}(g) \sum_{P_j\in D_f} c_{P_j}(f) G(Q_i,P_j) = \sum_{P_j\in D_f} c_{P_j}(f) g(P_j).$$

□

Theorem 2 applies to the class $\mathcal{A}$ of functions considered in [HIvPT19] in the case when X is a certain finite volume quotient of hyperbolic three space.

Let us now state a generalization of Weil reciprocity which, in effect, is an integrated form of (15).

Theorem 3 *Let X be a compact, smooth projective Kähler variety. In a slight abuse of notation, let μ denote the volume form on whatever subvariety of X is under consideration. Let F and G be modular forms on X whose divisors are D_F and D_G, respectively. Assume that the forms are scaled to have L^1-log norm equal to zero, meaning*

$$\int_X \log \|F\|_\mu(z)\mu(z) = \int_X \log \|G\|_\mu(z)\mu(z) = 0.$$

Then

$$\int_{D_G} \log \|F\|_\mu(z)\mu(z) = \int_{D_F} \log \|G\|_\mu(w)\mu(w). \tag{17}$$

Proof: In the notation of [CJS20], Theorem 4 there is an absolute nonzero constant c_0 and a constant c_F such that

$$\int_{D_F} G(z,w)\mu(w) = c_0 \log \|F\|_\mu(z) + c_F.$$

The scaling of F is such that

$$c_F \mathrm{vol}_\mu(X) = \int_{D_F}\int_X G(z,w)\mu(z)\mu(w) - c_0 \int_X \log \|F\|_\mu(z)\mu(z) = -c_0 \int_X \log \|F\|_\mu(z)\mu(z) = 0.$$

Similarly,

$$\int_{D_G} G(z,w)\mu(w) = c_0 \log \|G\|_\mu(z),$$

from which the assertion follows. □

The following theorem is a generalization of Theorem 3

Theorem 4 *Let X be a compact, smooth projective Kähler variety. In a slight abuse of notation, let μ denote the volume form on whatever subvariety of X is under consideration. Let F and G be modular forms on X whose divisors are D_F and D_G, respectively. Let $\|\log \|F\|_\mu\|_1$ and $\|\log \|G\|_\mu\|_1$ denote the L^1-log norms of F and G respectively. Then* (17) *is equivalent to the statement that*

$$\mathrm{vol}_\mu(D_G)\|\log \|F\|_\mu\|_1 = \mathrm{vol}_\mu(D_F)\|\log \|G\|_\mu\|_1.$$

Proof: From the proof of Theorem 3 we have the following identity

$$c_0 \log \|F\|_\mu(z) = \int_{D_F} G(z,w)\mu(w) - c_F = \int_{D_F} G(z,w)\mu(w) + \frac{c_0}{\mathrm{vol}_\mu(X)}\|\log \|F\|_\mu\|_1, \tag{18}$$

and, similarly for G, the identity

$$c_0 \log \|G\|_\mu(z) = \int_{D_G} G(z,w)\mu(w) - c_G = \int_{D_G} G(z,w)\mu(w) + \frac{c_0}{\mathrm{vol}_\mu(X)}\|\log \|G\|_\mu\|_1. \tag{19}$$

The statement follows after integrating (18) over D_G and (19) over D_F (with the volume form μ). □

5 Concluding remarks

As an example, let X be an abelian variety with principal polarization Θ. Let f be the meromorphic function on X associated to the divisor $\sum n_j(\Theta + P_j)$ for a finite set of distinct points $\{P_j\}$ on X and integers $\{n_j\}$ such that $\sum n_j P_j$ is zero in the group law on X. Similarly, let g be the meromorphic function associated to the divisor $\sum m_i(\Theta + Q_i)$. Then Theorem 3 is the beginning of the reciprocity law due to Lang; see [La58] and, more specifically, page 80 of [LR15] (see also [Mi04] for further applicatons). As in the proofs of (1) and (3) on needs to "drop the absolute values" and use that, in the appropriate manner, the degrees of f and g are zero.

As another example, let X be a compact quotient of the complex 2-ball. In [KM81], the authors consider subspaces which are smooth algebraic curves and are totally geodesic. Let's assume such a subspace is also the divisor of an automorphic form on X (which is quite possibly true). Then the results from [CJS20] provide another means to undertake the analysis from [KM81]. Furthermore, the reciprocity law in Theorem 3 amounts to comparing the integrals of log-norm of modular forms on distinct Riemann surfaces which are the divisors in question. As a result, the evaluation can be viewed as a type of Rohrlich-Jensen formula; see [CJS22]. In doing so, one can further reduce the integration in Theorem 3 to comparing point evaluations of modular forms, as in (1). The details of this investigation will be developed elsewhere.

Finally, let us note that the role of the analysis in [CJS20] is to prove that the Green's function $G(z, w)$, and its integral transformations, can be viewed as a type of Kronecker limit formula. This interpretation is not needed in the generalizations of Weil reciprocity *per se,* but it may be useful in applications.

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