ANALYTIC THEORY OF L-FUNCTIONS FOR GL_n

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The purpose of this note is to describe the analytic theory of L-functions for cuspidal automorphic representations of GL_n over a global field. There are two approaches to L-functions of GL_n: via integral representations or through analysis of Fourier coefficients of Eisenstein series. In this note we will discuss the theory via integral representations.

The theory of L-functions of automorphic forms (or modular forms) via integral representations has its origin in the paper of Riemann on the ζ-function. However the theory was really developed in the classical context of L-functions of modular forms for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ by Hecke and his school. Much of our current theory is a direct outgrowth of Hecke's. L-functions of automorphic representations were first developed by Jacquet and Langlands for GL_2. Their approach followed Hecke combined with the local-global techniques of Tate's thesis. The theory for GL_n was then developed along the same lines in a long series of papers by various combinations of Jacquet, Piatetski-Shapiro, and Shalika. In addition to associating an L-function to an automorphic form, Hecke also gave a criterion for a Dirichlet series to come from a modular form, the so called converse theorem of Hecke. In the context of automorphic representations, the converse theorem for GL_2 was developed by Jacquet and Langlands, extended and significantly strengthened to GL_3 by Jacquet, Piatetski-Shapiro, and Shalika, and then extended to GL_n with Piatetski-Shapiro. What we have attempted to present here is a synopsis of this work. An expanded version of this note can be found in [1].

There is another body of work on integral representations of L-functions for GL_n which developed out of the classical work on zeta functions of algebras. This is the theory of principal L-functions for GL_n as developed by Godement and Jacquet [15, 20]. This approach is related to the one pursued here, but we have not attempted to present it.

The other approach to these L-functions is via the Fourier coefficients of Eisenstein series. This approach also has a classical history. In the context of automorphic representations, and in a broader context than GL_n, this approach was originally laid out by Langlands [29] but then most fruitfully pursued by Shahidi. Some of the major papers of Shahidi on this subject are [35, 36, 37, 38, 39, 40, 41]. In particular, in [38] he shows that the two approaches give the same L-functions for GL_n. We will not pursue this approach in these notes.

1. Fourier expansions

In this section we let k denote a global field, $\mathbb{A}$, its ring of adeles, and ψ will denote a continuous additive character of $\mathbb{A}$ which is trivial on k.

We begin with a cuspidal automorphic representation (π, V_π) of $\mathrm{GL}_n(\mathbb{A})$. For us, automorphic forms are assumed to be smooth (of uniform moderate growth) but not necessarily K_∞–finite at the archimedean places. This is most suitable for the analytic theory. For simplicity, we assume the central character ω_π of π is unitary. Then V_π is the space of smooth vectors in an irreducible unitary representation of $\mathrm{GL}_n(\mathbb{A})$. We will always use cuspidal in this sense: the smooth vectors in an irreducible unitary cuspidal automorphic representation. (Any other smooth cuspidal representation π of $\mathrm{GL}_n(\mathbb{A})$ is necessarily of the form $\pi = \pi^\circ \otimes |\det|^t$ with π° unitary and t real, so there is really no loss of generality in the unitarity assumption. It merely provides us with a convenient normalization.) By a cusp form on $\mathrm{GL}_n(\mathbb{A})$ we will mean a function lying in a cuspidal representation. By a cuspidal function we will simply mean a smooth function φ on $\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})$ satisfying $\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \varphi(ug)\, du \equiv 0$ for every unipotent radical U of standard parabolic subgroups of GL_n.

The basic references for this section are the papers of Piatetski-Shapiro [31, 32] and Shalika [42].

1.1. **The Fourier expansions.** If $f(\tau)$ is a holomorphic cusp form on the upper half plane $\mathfrak{H}$, say with respect to $\mathrm{SL}_2(\mathbb{Z})$, then f is invariant under integral translations, $f(\tau+1) = f(\tau)$ and thus has a Fourier expansion of the form

$$f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}.$$

If $\varphi(g)$ is a smooth cusp form on $\mathrm{GL}_2(\mathbb{A})$ then the translations correspond to the maximal unipotent subgroup $\mathrm{N}_2 = \left\{ n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}$ and $\varphi(ng) = \varphi(g)$ for $n \in \mathrm{N}_2(k)$. So, if ψ is any continuous character of $k\backslash\mathbb{A}$ we can define the ψ-Fourier coefficient or ψ-Whittaker function by

$$W_{\varphi,\psi}(g) = \int_{k\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi^{-1}(x)\, dx.$$

We have the corresponding Fourier expansion

$$\varphi(g) = \sum_{\psi} W_{\varphi,\psi}(g).$$

(Actually from abelian Fourier theory, one has

$$\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \sum_{\psi} W_{\varphi,\psi}(g)\psi(x)$$

as a periodic function of $x \in \mathbb{A}$. Now set $x = 0$.)

If we fix a single non-trivial character ψ of $k\backslash\mathbb{A}$, then the additive characters of the compact group $k\backslash\mathbb{A}$ are isomorphic to k via the map $\gamma \in k \mapsto \psi_\gamma$ where ψ_γ is the character $\psi_\gamma(x) = \psi(\gamma x)$. An elementary calculation shows that $W_{\varphi,\psi_\gamma}(g) = W_{\varphi.\psi}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right)$ if

$\gamma \neq 0$. If we set $W_\varphi = W_{\varphi,\psi}$ for our fixed ψ, then the Fourier expansion of φ becomes

$$\varphi(g) = W_{\varphi,\psi_0}(g) + \sum_{\gamma \in k^\times} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right).$$

Since φ is cuspidal

$$W_{\varphi,\psi_0}(g) = \int_{k\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\, dx \equiv 0$$

and the Fourier expansion for a cusp form φ becomes simply

$$\varphi(g) = \sum_{\gamma \in k^\times} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right).$$

We will need a similar expansion for cusp forms φ on $\mathrm{GL}_n(\mathbb{A})$. The translations still correspond to the maximal unipotent subgroup

$$\mathrm{N}_n = \left\{ n = \begin{pmatrix} 1 & x_{1,2} & & & * \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & x_{n-1,n} \\ 0 & & & & 1 \end{pmatrix} \right\},$$

but now this is non-abelian. This difficulty was solved independently by Piatetski-Shapiro [31] and Shalika [42]. We fix our non-trivial continuous character ψ of $k\backslash\mathbb{A}$ as above. Extend it to a character of N_n by setting $\psi(n) = \psi(x_{1,2} + \cdots + x_{n-1,n})$ and define the associated Fourier coefficient or Whittaker function by

$$W_\varphi(g) = W_{\varphi,\psi}(g) = \int_{\mathrm{N}_n(k)\backslash \mathrm{N}_n(\mathbb{A})} \varphi(ng)\psi^{-1}(n)\, dn.$$

Since φ is continuous and the integration is over a compact set this integral is absolutely convergent, uniformly on compact sets. The Fourier expansion takes the following form.

Theorem 1.1. *Let $\varphi \in V_\pi$ be a cusp form on $\mathrm{GL}_n(\mathbb{A})$ and W_φ its associated ψ-Whittaker function. Then*

$$\varphi(g) = \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash GL_{n-1}(k)} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right)$$

with convergence absolute and uniform on compact subsets.

The proof of this fact is an induction. It utilizes the *mirabolic subgroup* P_n of GL_n which seems to be ubiquitous in the study of automorphic forms on GL_n. Abstractly, a mirabolic subgroup of GL_n is simply the stabilizer of a non-zero vector in (either) standard representation of GL_n on k^n. We denote by P_n the stabilizer of the row vector $e_n = (0, \dots, 0, 1) \in k^n$. So

$$\mathrm{P}_n = \left\{ p = \begin{pmatrix} h & y \\ & 1 \end{pmatrix} \middle| h \in \mathrm{GL}_{n-1}, y \in k^{n-1} \right\} \simeq \mathrm{GL}_{n-1} \ltimes \mathrm{Y}_n$$

where

$$\mathrm{Y}_n = \left\{ y = \begin{pmatrix} I_{n-1} & y \\ & 1 \end{pmatrix} \middle| y \in k^{n-1} \right\} \simeq k^{n-1}.$$

Simply by restriction of functions, a cusp form on $\mathrm{GL}_n(\mathbb{A})$ restricts to a smooth cuspidal function on $\mathrm{P}_n(\mathbb{A})$ which remains left invariant under $\mathrm{P}_n(k)$. (A smooth function φ on $\mathrm{P}_n(\mathbb{A})$ which is left invariant under $\mathrm{P}_n(k)$ is called cuspidal if $\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \varphi(up)\, du \equiv 0$ for every standard unipotent subgroup $\mathrm{U} \subset \mathrm{P}_n$.) Since $\mathrm{P}_n \supset \mathrm{N}_n$ we may define a Whittaker function attached to a cuspidal function φ on $\mathrm{P}_n(\mathbb{A})$ by the same integral as on $\mathrm{GL}_n(\mathbb{A})$, namely

$$W_\varphi(p) = \int_{\mathrm{N}_n(k)\backslash \mathrm{N}_n(\mathbb{A})} \varphi(np)\psi^{-1}(n)\, dn.$$

One proves by induction on n that for a cuspidal function φ on $\mathrm{P}_n(\mathbb{A})$ we have

$$\varphi(p) = \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} p\right)$$

with convergence absolute and uniform on compact subsets.

To obtain the Fourier expansion on GL_n from this, if φ is a cusp form on $\mathrm{GL}_n(\mathbb{A})$, then for $g \in \Omega$ a compact subset the functions $\varphi_g(p) = \varphi(pg)$ form a compact family of cuspidal functions on $\mathrm{P}_n(\mathbb{A})$. So we have

$$\varphi_g(1) = \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_{\varphi_g}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}\right)$$

with convergence absolute and uniform. Hence

$$\varphi(g) = \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

again with absolute convergence, uniform for $g \in \Omega$.

1.2. **Whittaker models and multiplicity one.** Consider now the functions W_φ appearing in the Fourier expansion of a cusp form φ. These are all smooth functions $W(g)$ on $\mathrm{GL}_n(\mathbb{A})$ which satisfy $W(ng) = \psi(n)W(g)$ for $n \in N_n(\mathbb{A})$. If we let $\mathcal{W}(\pi,\psi) = \{W_\varphi \mid \varphi \in V_\pi\}$ then $\mathrm{GL}_n(\mathbb{A})$ acts on this space by right translation and the map $\varphi \mapsto W_\varphi$ intertwines V_π with $\mathcal{W}(\pi,\psi)$. $\mathcal{W}(\pi,\psi)$ is called the *Whittaker model* of π.

The notion of a Whittaker model of a representation makes perfect sense over a local field. Let k_v be a local field (a completion of k for example) and let (π_v, V_{π_v}) be an irreducible admissible smooth representation of $\mathrm{GL}_n(k_v)$. Fix a non-trivial continuous additive character ψ_v of k_v. Let $\mathcal{W}(\psi_v)$ be the space of all smooth functions $W(g)$ on $\mathrm{GL}_n(k_v)$ satisfying $W(ng) = \psi_v(n)W(g)$ for all $n \in \mathrm{N}_k(k_v)$, that is, the space of all smooth Whittaker functions on $\mathrm{GL}_n(k_v)$ with respect to ψ_v. This is also the space of the smooth induced representation $\mathrm{Ind}_{\mathrm{N}_v}^{\mathrm{GL}_n}(\psi_v)$. $\mathrm{GL}_n(k_v)$ acts on this by right translation. If we have a non-trivial continuous intertwining $V_{\pi_v} \to \mathcal{W}(\psi_v)$ we will denote its image by $\mathcal{W}(\pi_v,\psi_v)$ and call it a Whittaker model of π_v.

Whittaker models for a representation (π_v, V_{π_v}) are equivalent to continuous Whittaker functionals on V_{π_v}, that is, continuous functionals Λ_v satisfying $\Lambda_v(\pi_v(n)\xi_v) = \psi_v(n)\Lambda_v(\xi_v)$

for all $n \in \mathrm{N}_n(k_v)$. To obtain a Whittaker functional from a model, set $\Lambda_v(\xi_v) = W_{\xi_v}(e)$, and to obtain a model from a functional, set $W_{\xi_v}(g) = \Lambda_v(\pi_v(g)\xi_v)$. This is a form of Frobenius reciprocity, which in this context is the isomorphism between $\mathrm{Hom}_{\mathrm{N}_n}(V_{\pi_v}, \mathbb{C}_{\psi_v})$ and $\mathrm{Hom}_{\mathrm{GL}_n}(V_{\pi_v}, \mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{GL}_n}(\psi_v))$ constructed above.

The fundamental theorem on the existence and uniqueness of Whittaker functionals and models is the following.

Theorem 1.2. *Let (π_v, V_{π_v}) be a smooth irreducible admissible representation of $\mathrm{GL}_n(k_v)$. Let ψ_v be a non-trivial continuous additive character of k_v. Then the space of continuous ψ_v–Whittaker functionals on V_{π_v} is at most one dimensional. That is, Whittaker models, if they exist, are unique.*

This was first proven for non-archimedean fields by Gelfand and Kazhdan [14] and their results were later extended to archimedean local fields by Shalika [42].

A smooth irreducible admissible representation (π_v, V_{π_v}) of $\mathrm{GL}_n(k_v)$ which possesses a Whittaker model is called *generic* or *non-degenerate*. Gelfand and Kazhdan in addition show that π_v is generic iff its contragredient $\widetilde{\pi}_v$ is generic, in fact that $\widetilde{\pi} \simeq \pi^\iota$ where ι is the outer automorphism $g^\iota = {}^t g^{-1}$, and in this case the Whittaker model for $\widetilde{\pi}_v$ can be obtained as $\mathcal{W}(\widetilde{\pi}_v, \psi_v^{-1}) = \{\widetilde{W}(g) = W(w_n\, {}^t g^{-1}) \mid W \in \mathcal{W}(\pi, \psi_v)\}$.

As a consequence of the local uniqueness of the Whittaker model we can conclude a global uniqueness. If (π, V_π) is an irreducible smooth admissible representation of $\mathrm{GL}_n(\mathbb{A})$ then π factors as a restricted tensor product of local representations $\pi \simeq \otimes' \pi_v$ taken over all places v of k [10, 13]. Consequently we have a continuous embedding $V_{\pi_v} \hookrightarrow V_\pi$ for each local component. Hence any Whittaker functional Λ on V_π determines a family of local Whittaker functionals Λ_v on each V_{π_v} and conversely such that $\Lambda = \otimes' \Lambda_v$. Hence global uniqueness follows from the local uniqueness. Moreover, once we fix the isomorphism of V_π with $\otimes' V_{\pi_v}$ and define global and local Whittaker functions via Λ and the corresponding family Λ_v we have a factorization of global Whittaker functions

$$W_\xi(g) = \prod_v W_{\xi_v}(g_v)$$

for $\xi \in V_\pi$ which are factorizable in the sense that $\xi = \otimes' \xi_v$ corresponds to a pure tensor. As we will see, this factorization, which is a direct consequence of the uniqueness of the Whittaker model, plays a most important role in the development of Eulerian integrals for GL_n.

Now let us see what this means for our cuspidal representations (π, V_π) of $\mathrm{GL}_n(\mathbb{A})$. We have seen that for any smooth cusp form $\varphi \in V_\pi$ we have the Fourier expansion

$$\varphi(g) = \sum_{\gamma \in \mathrm{N}_{n-1}(k) \backslash GL_{n-1}(k)} W_\varphi \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right).$$

We can thus conclude that $\mathcal{W}(\pi,\psi) \neq 0$ and that π is (globally) generic with Whittaker functional

$$\Lambda(\varphi) = W_\varphi(e) = \int \varphi(ng)\psi^{-1}(n)\ dn.$$

Thus φ is completely determined by its associated Whittaker function W_φ. From the uniqueness of the global Whittaker model we can derive the Multiplicity One Theorem of Piatetski-Shapiro [32] and Shalika [42].

Multiplicity One: *Let (π, V_π) be an irreducible smooth admissible representation of $\mathrm{GL}_n(\mathbb{A})$. Then the multiplicity of π in the space of cusp forms on $\mathrm{GL}_n(\mathbb{A})$ is at most one.*

2. Eulerian integral representations

Let $f(\tau)$ again be a holomorphic cusp form of weight k on $\mathfrak{H}$ for the full modular group with Fourier expansion

$$f(\tau) = \sum a_n e^{2\pi i n\tau}.$$

Then Hecke [18] associated to f an L-function

$$L(s,f) = \sum a_n n^{-s}$$

and analyzed its analytic properties, namely continuation, order of growth, and functional equation, by writing it as the Mellin transform of f

$$\Lambda(s,f) = (2\pi)^{-s}\Gamma(s)L(s,f) = \int_0^\infty f(iy)y^s d^\times y.$$

An application of the modular transformation law for $f(\tau)$ under the transformation $\tau \mapsto -1/\tau$ gives the functional equation

$$\Lambda(s,f) = (-1)^{k/2}\Lambda(k-s,f).$$

Moreover, if f is an eigenfunction of all Hecke operators then $L(s,f)$ has an Euler product expansion

$$L(s,f) = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

There is a similar theory for cuspidal automorphic representations (π, V_π) of $\mathrm{GL}_n(\mathbb{A})$. For applications to the Langlands conjectures and to functoriality via the Converse Theorem we will need not only the standard L-functions $L(s,\pi)$ but the twisted L-functions $L(s,\pi\times\pi')$ for $(\pi', V_{\pi'})$ a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$ for $m < n$ as well.

The basic references for this section are Jacquet-Langlands [21], Jacquet, Piatetski-Shapiro, and Shalika [22], and Jacquet and Shalika [25].

2.1. **Integral representations for** GL_2**.** Let us first consider the L-functions for cuspidal automorphic representations (π, V_π) of $\mathrm{GL}_2(\mathbb{A})$ with twists by an idele class character χ, or what is the same, a (cuspidal) automorphic representation of $\mathrm{GL}_1(\mathbb{A})$, as in Jacquet-Langlands [21].

Following Jacquet and Langlands, who were following Hecke, for each $\varphi \in V_\pi$ we consider the integral
$$I(s;\varphi,\chi) = \int_{k^\times\backslash\mathbb{A}^\times} \varphi\begin{pmatrix} a & \\ & 1\end{pmatrix}\chi(a)|a|^{s-1/2}\ d^\times a.$$
Since a cusp form on $\mathrm{GL}_2(\mathbb{A})$ is rapidly decreasing upon restriction to $\mathbb{A}^\times$ as in the integral, it follows that the integral is absolutely convergent for all s, uniformly for $\mathrm{Re}(s)$ in an interval. Thus $I(s;\varphi,\chi)$ is an entire function of s, bounded in any vertical strip $a \le \mathrm{Re}(s) \le b$. Moreover, if we let $\widetilde{\varphi}(g) = \varphi({}^tg^{-1}) = \varphi(w_n\ {}^tg^{-1})$ then $\widetilde{\varphi} \in V_{\widetilde{\pi}}$ and the simple change of variables $a \mapsto a^{-1}$ in the integral shows that each integral satisfies a functional equation of the form
$$I(s;\varphi,\chi) = I(1-s;\widetilde{\varphi},\chi^{-1}).$$
So these integrals individually enjoy rather nice analytic properties.

If we replace φ by its Fourier expansion from Section 1 and unfold, we find
$$\begin{aligned} I(s;\varphi,\chi) &= \int_{k^\times\backslash\mathbb{A}^\times} \sum_{\gamma\in k^\times} W_\varphi\begin{pmatrix} \gamma a & \\ & 1\end{pmatrix}\chi(a)|a|^{s-1/2}\ d^\times a \\ &= \int_{\mathbb{A}^\times} W_\varphi\begin{pmatrix} a & \\ & 1\end{pmatrix}\chi(a)|a|^{s-1/2}\ d^\times a \end{aligned}$$
where we have used the fact that the function $\chi(a)|a|^{s-1/2}$ is invariant under $k^\times$. By standard gauge estimates on Whittaker functions [22] this converges for $Re(s) >> 0$ after the unfolding. As we have seen in Section 1, if $W_\varphi \in \mathcal{W}(\pi,\psi)$ corresponds to a decomposable vector $\varphi \in V_\pi \simeq \otimes' V_{\pi_v}$ then the Whittaker function factors into a product of local Whittaker functions
$$W_\varphi(g) = \prod_v W_{\varphi_v}(g_v).$$
Since the character χ and the adelic absolute value factor into local components and the domain of integration $\mathbb{A}^\times$ also factors we find that our global integral naturally factors into a product of local integrals
$$\int_{\mathbb{A}^\times} W_\varphi\begin{pmatrix} a & \\ & 1\end{pmatrix}\chi(a)|a|^{s-1/2}\ d^\times a = \prod_v \int_{k_v^\times} W_{\varphi_v}\begin{pmatrix} a_v & \\ & 1\end{pmatrix}\chi_v(a_v)|a_v|^{s-1/2}\ d^\times a_v,$$
with the infinite product still convergent for $\mathrm{Re}(s) >> 0$, or
$$I(s;\varphi,\chi) = \prod_v \Psi_v(s;W_{\varphi_v},\chi_v)$$
with the obvious definition of the local integrals
$$\Psi_v(s;W_{\varphi_v},\chi_v) = \int_{k_v^\times} W_{\varphi_v}\begin{pmatrix} a_v & \\ & 1\end{pmatrix}\chi_v(a_v)|a_v|^{s-1/2}\ d^\times a_v.$$
Thus each of our global integrals is Eulerian.

In this way, to π and χ we have associated a family of global Eulerian integrals with nice analytic properties as well as for each place v a family of local integrals convergent for $\mathrm{Re}(s) >> 0$.

2.2. **Integral representations for** $\mathrm{GL}_n \times \mathrm{GL}_m$ **with** $m < n$**.** Now let (π, V_π) be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ and $(\pi', V_{\pi'})$ a cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$ with $m < n$. Take $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$. At first blush, a natural analogue of the integrals we considered for GL_2 with GL_1 twists would be

$$\int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \varphi \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} \varphi'(h) |\det(h)|^{s-(n-m)/2} \, dh.$$

This family of integrals would have all the nice analytic properties as before (entire functions of finite order satisfying a functional equation), but they would not be Eulerian except in the case $m = n-1$, which proceeds exactly as in the GL_2 case. The problem is that the restriction of the form φ to GL_m is too brutal to allow a nice unfolding when the Fourier expansion of φ is inserted. Instead we will introduce projection operators from cusp forms on $\mathrm{GL}_n(\mathbb{A})$ to cuspidal functions on on $P_{m+1}(\mathbb{A})$ which are given by part of the unipotent integration through which the Whittaker function is defined.

In GL_n, let $\mathrm{Y}_{n,m}$ be the unipotent radical of the standard parabolic subgroup attached to the partition $(m+1, 1, \dots, 1)$. If ψ is our standard additive character of $k\backslash\mathbb{A}$, then ψ defines a character of $\mathrm{Y}_{n,m}(\mathbb{A})$ trivial on $\mathrm{Y}_{n,m}(k)$ since $\mathrm{Y}_{n,m} \subset \mathrm{N}_n$. The group $\mathrm{Y}_{n,m}$ is normalized by $\mathrm{GL}_{m+1} \subset \mathrm{GL}_n$ and the mirabolic subgroup $\mathrm{P}_{m+1} \subset \mathrm{GL}_{m+1}$ is the stabilizer in GL_{m+1} of the character ψ.

If $\varphi(g)$ is a cusp form on $\mathrm{GL}_n(\mathbb{A})$ define the projection operator $\mathbb{P}^n_m$ from cusp forms on $\mathrm{GL}_n(\mathbb{A})$ to cuspidal functions on $\mathrm{P}_{m+1}(\mathbb{A})$ by

$$\mathbb{P}^n_m \varphi(p) = |\det(p)|^{-\left(\frac{n-m-1}{2}\right)} \int_{\mathrm{Y}_{n,m}(k)\backslash \mathrm{Y}_{n,m}(\mathbb{A})} \varphi\left(y \begin{pmatrix} p & \\ & I_{n-m-1} \end{pmatrix}\right) \psi^{-1}(y) \, dy$$

for $p \in \mathrm{P}_{m+1}(\mathbb{A})$. As the integration is over a compact domain, the integral is absolutely convergent. One can easily check that $\mathbb{P}^n_m\varphi(p)$ is indeed cuspidal on $\mathrm{P}_{m+1}(\mathbb{A})$. From Section 1, we know that cuspidal functions on $\mathrm{P}_{m+1}(\mathbb{A})$ have a Fourier expansion summed over $\mathrm{N}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})$. Applying this expansion to our projected cusp form on $\mathrm{GL}_n(\mathbb{A})$ we find that for $h \in \mathrm{GL}_m(\mathbb{A})$, $\mathbb{P}^n_m\varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix}$ has the Fourier expansion

$$\mathbb{P}^n_m\varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} = |\det(h)|^{-\left(\frac{n-m-1}{2}\right)} \sum_{\gamma \in \mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & I_{n-m} \end{pmatrix} \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix}\right)$$

with convergence absolute and uniform on compact subsets.

We now have the prerequisites for writing down a family of Eulerian integrals for cusp forms φ on GL_n twisted by automorphic forms on GL_m for $m < n$. Let $\varphi \in V_\pi$ be a cusp form on $\mathrm{GL}_n(\mathbb{A})$ and $\varphi' \in V_{\pi'}$ a cusp form on $\mathrm{GL}_m(\mathbb{A})$. (Actually, we could take φ' to be an arbitrary automorphic form on $\mathrm{GL}_m(\mathbb{A})$.) Consider the integrals

$$I(s;\varphi,\varphi') = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \mathbb{P}^n_m\varphi \begin{pmatrix} h & 0 \\ 0 & I_{n-m} \end{pmatrix} \varphi'(h)|\det(h)|^{s-1/2}\ dh.$$

The integral $I(s;\varphi,\varphi')$ is absolutely convergent for all values of the complex parameter s, uniformly in compact subsets, since the cusp forms are rapidly decreasing. Hence it is entire and bounded in any vertical strip as before.

Let us now investigate the Eulerian properties of these integrals. We first replace $\mathbb{P}^n_m\varphi$ by its Fourier expansion to obtain

$$I(s;\varphi,\varphi') = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \sum_{\gamma\in \mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)} W_\varphi \begin{pmatrix} \gamma h & 0 \\ 0 & I_{n-m} \end{pmatrix} \varphi'(h)|\det(h)|^{s-(n-m)/2}\ dh.$$

Since $\varphi'(h)$ is automorphic on $\mathrm{GL}_m(\mathbb{A})$ and $|\det(\gamma)| = 1$ for $\gamma\in \mathrm{GL}_m(k)$ we may interchange the order of summation and integration for $\mathrm{Re}(s) >> 0$ and then recombine to obtain

$$I(s;\varphi,\varphi') = \int_{\mathrm{N}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} W_\varphi \begin{pmatrix} h & 0 \\ 0 & I_{n-m} \end{pmatrix} \varphi'(h)|\det(h)|^{s-(n-m)/2}\ dh.$$

This integral is absolutely convergent for $\mathrm{Re}(s) >> 0$ by the gauge estimates of [22, Section 13] and this justifies the interchange. Let us now integrate first over $\mathrm{N}_m(k)\backslash \mathrm{N}_m(\mathbb{A})$. Recall that for $n\in \mathrm{N}_m(\mathbb{A})\subset \mathrm{N}_n(\mathbb{A})$ we have $W_\varphi(ng) = \psi(n)W_\varphi(g)$. Hence we obtain

$$\begin{aligned} I(s;\varphi,\varphi') &= \int_{\mathrm{N}_m(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A})} W_\varphi \begin{pmatrix} h & 0 \\ 0 & I_{n-m} \end{pmatrix} W'_{\varphi'}(h)|\det(h)|^{s-(n-m)/2}\ dh \\ &= \Psi(s;W_\varphi,W'_{\varphi'}) \end{aligned}$$

where $W'_{\varphi'}(h)$ is the ψ^{-1}-Whittaker function on $\mathrm{GL}_m(\mathbb{A})$ associated to φ', i.e.,

$$W'_{\varphi'}(h) = \int_{\mathrm{N}_m(k)\backslash \mathrm{N}_m(\mathbb{A})} \varphi'(nh)\psi(n)\ dn,$$

and we retain absolute convergence for $\mathrm{Re}(s) >> 0$.

From this point, the fact that the integrals are Eulerian is a consequence of the uniqueness of the Whittaker model for GL_n. Take φ a smooth cusp form in a cuspidal representation π of $\mathrm{GL}_n(\mathbb{A})$. Assume in addition that φ is factorizable, i.e., in the decomposition $\pi = \otimes'\pi_v$ of π into a restricted tensor product of local representations, $\varphi = \otimes\varphi_v$ is a pure tensor. Then as we have seen there is a choice of local Whittaker models so that $W_\varphi(g) = \prod W_{\varphi_v}(g_v)$. Similarly for decomposable φ' we have the factorization $W'_{\varphi'}(h) = \prod W'_{\varphi'_v}(h_v)$. If we substitute these factorizations into our integral expression, then since the domain of integration factors $\mathrm{N}_m(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A}) = \prod \mathrm{N}_m(k_v)\backslash \mathrm{GL}_m(k_v)$ we see that our integral factors into a product of local integrals as

$$I(s;\varphi,\varphi') = \Psi(s;W_\varphi,W'_{\varphi'}) = \prod_v \Psi_v(s;W_{\varphi_v},W'_{\varphi'_v})$$

where the local integrals are given by

$$\Psi_v(s;W_{\varphi_v},W'_{\varphi'_v}) = \int_{\mathrm{N}_m(k_v)\backslash \mathrm{GL}_m(k_v)} W_{\varphi_v} \begin{pmatrix} h_v & 0 \\ 0 & I_{n-m} \end{pmatrix} W'_{\varphi'_v}(h_v)|\det(h_v)|_v^{s-(n-m)/2}\ dh_v.$$

The individual local integrals converge for $\mathrm{Re}(s) >> 0$ by the gauge estimate of [22, Prop. 2.3.6]. We now see that we now have constructed a family of Eulerian integrals.

Now let us return to the question of a functional equation. As in the case of GL_2, the functional equation is essentially a consequence of the existence of the outer automorphism $g \mapsto \iota(g) = g^\iota = {}^t g^{-1}$ of GL_n. If we define the action of this automorphism on automorphic forms by setting $\widetilde{\varphi}(g) = \varphi(g^\iota) = \varphi(w_n g^\iota)$ and let $\widetilde{\mathbb{P}}^n_m = \iota \circ \mathbb{P}^n_m \circ \iota$ then our integrals naturally satisfy the functional equation

$$I(s;\varphi,\varphi') = \widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}')$$

where

$$\widetilde{I}(s;\varphi,\varphi') = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \widetilde{\mathbb{P}}^n_m \varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) |\det(h)|^{s-1/2} \, dh.$$

We have established the following result.

Theorem 2.1. *Let $\varphi \in V_\pi$ be a cusp form on $\mathrm{GL}_n(\mathbb{A})$ and $\varphi' \in V_{\pi'}$ a cusp form on $\mathrm{GL}_m(\mathbb{A})$ with $m < n$. Then the family of integrals $I(s;\varphi,\varphi')$ define entire functions of s, bounded in vertical strips, and satisfy the functional equation*

$$I(s;\varphi,\varphi') = \widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}').$$

Moreover the integrals are Eulerian and if φ and φ' are factorizable, we have

$$I(s;\varphi,\varphi') = \prod_v \Psi_v(s;W_{\varphi_v},W'_{\varphi'_v})$$

with convergence absolute and uniform for $\mathrm{Re}(s) >> 0$.

The integrals occurring in the right hand side of our functional equation are again Eulerian. One can unfold the definitions to find first that

$$\widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}') = \widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W}_\varphi,\widetilde{W}'_{\varphi'})$$

where the unfolded global integral is

$$\widetilde{\Psi}(s;W,W') = \int\int W \begin{pmatrix} h & & \\ x & I_{n-m-1} & \\ & & 1 \end{pmatrix} \, dx \; W'(h) |\det(h)|^{s-(n-m)/2} \, dh$$

with the h integral over $\mathrm{N}_m(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A})$ and the x integral over $M_{n-m-1,m}(\mathbb{A})$, the space of $(n-m-1)\times m$ matrices, ρ denoting right translation, and $w_{n,m}$ the Weyl element $w_{n,m} = \begin{pmatrix} I_m & \\ & w_{n-m} \end{pmatrix}$ with $w_{n-m} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ the standard long Weyl element in GL_{n-m}. Also, for $W \in \mathcal{W}(\pi,\psi)$ we set $\widetilde{W}(g) = W(w_n g^\iota) \in \mathcal{W}(\widetilde{\pi},\psi^{-1})$. The extra unipotent integration is the remnant of $\widetilde{\mathbb{P}}^n_m$. As before, $\widetilde{\Psi}(s;W,W')$ is absolutely convergent for $\mathrm{Re}(s) >> 0$. For φ and φ' factorizable as before, these integrals $\widetilde{\Psi}(s;W_\varphi,W'_{\varphi'})$ will factor as well. Hence we have

$$\widetilde{\Psi}(s;W_\varphi,W'_{\varphi'}) = \prod_v \widetilde{\Psi}_v(s;W_{\varphi_v},W'_{\varphi'_v})$$

where

$$\widetilde{\Psi}_v(s;W_v,W'_v)=\int\int W_v\begin{pmatrix} h_v & & \\ x_v & I_{n-m-1} & \\ & & 1\end{pmatrix}\,dx_v\;W'_v(h_v)|\det(h_v)|^{s-(n-m)/2}\,dh_v$$

where now with the h_v integral is over $\mathrm{N}_m(k_v)\backslash\mathrm{GL}_m(k_v)$ and the x_v integral is over the matrix space $M_{n-m-1,m}(k_v)$. Thus, coming back to our functional equation, we find that the right hand side is Eulerian and factors as

$$\widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}')=\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W}_{\varphi},\widetilde{W}'_{\varphi'})=\prod_v\widetilde{\Psi}_v(1-s;\rho(w_{n,m})\widetilde{W}_{\varphi_v},\widetilde{W}'_{\varphi'_v}).$$

2.3. **Integral representations for** $\mathrm{GL}_n\times\mathrm{GL}_n$**.** The paradigm for integral representations of L-functions for $\mathrm{GL}_n\times\mathrm{GL}_n$ is not Hecke but rather the classical papers of Rankin [33] and Selberg [34]. These were first interpreted in the framework of automorphic representations by Jacquet for $\mathrm{GL}_2\times\mathrm{GL}_2$ [20] and then Jacquet and Shalika in general [25].

Let (π,V_π) and $(\pi',V_{\pi'})$ be two cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$. Let $\varphi\in V_\pi$ and $\varphi'\in V_{\pi'}$ be two cusp forms. The analogue of the construction above would be simply

$$\int_{\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})}\varphi(g)\varphi'(g)|\det(g)|^s\,dg.$$

This integral is essentially the L^2-inner product of φ and φ' and is not suitable for defining an L-function, although it will occur as a residue of our integral at a pole. Instead, following Rankin and Selberg, we use an integral representation that involves a third function: an Eisenstein series on $\mathrm{GL}_n(\mathbb{A})$. This family of Eisenstein series is constructed using the mirabolic subgroup once again.

To construct our Eisenstein series we return to the observation that $\mathrm{P}_n\backslash\mathrm{GL}_n\simeq k^n-\{0\}$. If we let $\mathcal{S}(\mathbb{A}^n)$ denote the Schwartz–Bruhat functions on $\mathbb{A}^n$, then each $\Phi\in\mathcal{S}$ defines a smooth function on $\mathrm{GL}_n(\mathbb{A})$, left invariant by $\mathrm{P}_n(\mathbb{A})$, by $g\mapsto\Phi((0,\dots,0,1)g)=\Phi(e_ng)$. Let η be a unitary idele class character. (For our application η will be determined by the central characters of π and π'.) Consider the function

$$F(g,\Phi;s,\eta)=|\det(g)|^s\int_{\mathbb{A}^\times}\Phi(ae_ng)|a|^{ns}\eta(a)\,d^\times a.$$

If we let $\mathrm{P}'_n=\mathrm{Z}_n\,\mathrm{P}_n$ be the parabolic of GL_n associated to the partition $(n-1,1)$ and extend η to a character of P'_n by $\eta(p')=\eta(d)$ for $p'=\begin{pmatrix} h & y\\ 0 & d\end{pmatrix}\in\mathrm{P}'_n(\mathbb{A})$ with $h\in\mathrm{GL}_{n-1}(\mathbb{A})$ and $d\in\mathbb{A}^\times$ we have that $F(g,\Phi;s,\eta)$ is a smooth section of the normalized induced representation $\mathrm{Ind}_{\mathrm{P}'_n(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})}(\delta_{\mathrm{P}'_n}^{s-1/2}\eta)$. Since the inducing character $\delta_{\mathrm{P}'_n}^{s-1/2}\eta$ of $\mathrm{P}'_n(\mathbb{A})$ is invariant under $\mathrm{P}'_n(k)$ we may form Eisenstein series from this family of sections by

$$E(g,\Phi;s,\eta)=\sum_{\gamma\in\mathrm{P}'_n(k)\backslash\mathrm{GL}_n(k)}F(\gamma g,\Phi;s,\eta).$$

This is absolutely convergent for $\mathrm{Re}(s)>1$ [25].

If we replace F in this sum by its definition and unfold we can rewrite this Eisenstein series as
$$E(g,\Phi;s,\eta)=|\det(g)|^s\int_{k^\times\backslash\mathbb{A}^\times}\Theta'_\Phi(a,g)|a|^{ns}\eta(a)\ d^\times a.$$
This second expression essentially gives the Eisenstein series as the Mellin transform of the Theta series
$$\Theta_\Phi(a,g)=\sum_{\xi\in k^n}\Phi(a\xi g),$$
where in the above we have written
$$\Theta'_\Phi(a,g)=\sum_{\xi\in k^n-\{0\}}\Phi(a\xi g)=\Theta_\Phi(a,g)-\Phi(0).$$
This allows us to obtain the analytic properties of the Eisenstein series from the Poisson summation formula for Θ_Φ. Poisson summation gives
$$\begin{aligned}E(g,\Phi,s,\eta)=|\det(g)|^s&\int_{|a|\geq 1}\Theta'_\Phi(a,g)|a|^{ns}\eta(a)\ d^\times a\\ &+|\det(g)|^{s-1}\int_{|a|\geq 1}\Theta'_{\hat{\Phi}}(a,{}^tg^{-1})|a|^{n(1-s)}\eta^{-1}(a)\ d^\times a+\delta(s)\end{aligned}$$
where the Fourier transform $\hat{\Phi}$ on $\mathcal{S}(\mathbb{A}^n)$ is defined by
$$\hat{\Phi}(x)=\int_{\mathbb{A}^\times}\Phi(y)\psi(y{}^tx)\ dy$$
and
$$\delta(s)=\begin{cases}0 & \text{if } \eta \text{ is ramified}\\ -c\Phi(0)\frac{|\det(g)|^s}{s+i\sigma}+c\hat{\Phi}(0)\frac{|\det(g)|^{s-1}}{s-1+i\sigma} & \text{if } \eta(a)=|a|^{in\sigma} \text{ with } \sigma\in\mathbb{R}\end{cases}$$
with c a non-zero constant.

From this we derive easily the basic properties of our Eisenstein series [25, Section 4]. The Eisenstein series $E(g,\Phi;s,\eta)$ has a meromorphic continuation to all of $\mathbb{C}$ with at most simple poles at $s=-i\sigma,1-i\sigma$ when η is unramified of the form $\eta(a)=|a|^{in\sigma}$. As a function of g it is smooth of moderate growth and as a function of s it is bounded in vertical strips (away from the possible poles), uniformly for g in compact sets. Moreover, we have the functional equation
$$E(g,\Phi;s,\eta)=E(g^\iota,\hat{\Phi};1-s,\eta^{-1})$$
where $g^\iota={}^tg^{-1}$. Note that under the center the Eisenstein series transforms by the central character η^{-1}.

Now let us return to our Eulerian integrals. Let π and π' be our irreducible cuspidal representations. Let their central characters be ω and ω'. Set $\eta=\omega\omega'$. Then for each pair of cusp forms $\varphi\in V_\pi$ and $\varphi'\in V_{\pi'}$ and each Schwartz-Bruhat function $\Phi\in\mathcal{S}(\mathbb{A}^n)$ set
$$I(s;\varphi,\varphi',\Phi)=\int_{\mathrm{Z}_n(\mathbb{A})\,\mathrm{GL}_n(k)\backslash\,\mathrm{GL}_n(\mathbb{A})}\varphi(g)\varphi'(g)E(g,\Phi;s,\eta)\ dg.$$
Since the two cusp forms are rapidly decreasing on $\mathrm{Z}_n(\mathbb{A})\,\mathrm{GL}_n(k)\backslash\,\mathrm{GL}_n(\mathbb{A})$ and the Eisenstein is only of moderate growth, we see that the integral converges absolutely for all s away from

the poles of the Eisenstein series and is hence meromorphic. It will be bounded in vertical strips away from the poles and satisfies the functional equation

$$I(s;\varphi,\varphi',\Phi)=I(1-s;\widetilde{\varphi},\widetilde{\varphi}',\hat{\Phi}),$$

coming from the functional equation of the Eisenstein series, where we still have $\widetilde{\varphi}(g)=\varphi(g^\iota)=\varphi(w_n g^\iota)\in V_{\widetilde{\pi}}$ and similarly for $\widetilde{\varphi}'$.

These integrals will be entire unless we have $\eta(a)=\omega(a)\omega'(a)=|a|^{in\sigma}$ is unramified. In that case, the residue at $s=-i\sigma$ will be

$$\operatorname*{Res}_{s=-i\sigma} I(s;\varphi,\varphi',\Phi)=-c\Phi(0)\int_{\mathrm{Z}_n(\mathbb{A})\,\mathrm{GL}_n(\mathbb{A})\backslash\,\mathrm{GL}_n(\mathbb{A})}\varphi(g)\varphi'(g)|\det(g)|^{-i\sigma}\,dg$$

and at $s=1-i\sigma$ we can write the residue as

$$\operatorname*{Res}_{s=1-i\sigma} I(s;\varphi,\varphi',\Phi)=c\hat{\Phi}(0)\int_{\mathrm{Z}_n(\mathbb{A})\,\mathrm{GL}_n(k)\backslash\,\mathrm{GL}_n(\mathbb{A})}\widetilde{\varphi}(g)\widetilde{\varphi}'(g)|\det(g)|^{i\sigma}\,dg.$$

Therefore these residues define $\mathrm{GL}_n(\mathbb{A})$ invariant pairings between π and $\pi'\otimes|\det|^{-i\sigma}$ or equivalently between $\widetilde{\pi}$ and $\widetilde{\pi}'\otimes|\det|^{i\sigma}$. Hence a residues can be non-zero only if $\pi\simeq\widetilde{\pi}'\otimes|\det|^{i\sigma}$ and in this case we can find φ, φ', and Φ such that indeed the residue does not vanish.

We have yet to check that our integrals are Eulerian. To this end we take the integral, replace the Eisenstein series by its definition, and unfold we find

$$I(s;\varphi,\varphi',\Phi)=\int_{\mathrm{P}_n(k)\backslash\,\mathrm{GL}_n(\mathbb{A})}\varphi(g)\varphi'(g)\Phi(e_n g)|\det(g)|^s\,dg.$$

We next replace φ by its Fourier expansion and unfold as before to find

$$I(s;\varphi,\varphi',\Phi)=\Psi(s;W_\varphi,W'_{\varphi'},\Phi)=\int_{\mathrm{N}_n(\mathbb{A})\backslash\,\mathrm{GL}_n(\mathbb{A})}W_\varphi(g)W'_{\varphi'}(g)\Phi(e_n g)|\det(g)|^s\,dg.$$

This expression converges for $\mathrm{Re}(s)>>0$.

To continue, we assume that φ, φ' and Φ are decomposable tensors under the isomorphisms $\pi\simeq\otimes'\pi_v$, $\pi'\simeq\otimes'\pi'_v$, and $\mathcal{S}(\mathbb{A}^n)\simeq\otimes'\mathcal{S}(k_v^n)$ so that we have $W_\varphi(g)=\prod_v W_{\varphi_v}(g_v)$, $W'_{\varphi'}(g)=\prod_v W'_{\varphi'_v}(g_v)$ and $\Phi(g)=\prod_v\Phi_v(g_v)$. Then, since the domain of integration also naturally factors we can decompose this last integral into an Euler product and now write

$$\Psi(s;W_\varphi,W'_{\varphi'},\Phi)=\prod_v\Psi_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v),$$

where

$$\Psi_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v)=\int_{\mathrm{N}_n(k_v)\backslash\,\mathrm{GL}_n(k_v)}W_{\varphi_v}(g_v)W'_{\varphi'_v}(g_v)\Phi_v(e_n g_v)|\det(g_v)|^s\,dg_v,$$

still with convergence for $\mathrm{Re}(s)>>0$ by the local gauge estimates. Once again we see that the Euler factorization is a direct consequence of the uniqueness of the Whittaker models.

Theorem 2.2. *Let $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ cusp forms on* $\mathrm{GL}_n(\mathbb{A})$ *and let* $\Phi \in \mathcal{S}(\mathbb{A}^n)$. *Then the family of integrals* $I(s;\varphi,\varphi',\Phi)$ *define meromorphic functions of* s, *bounded in vertical strips away from the poles. The only possible poles are simple and occur iff* $\pi \simeq \widetilde{\pi}' \otimes |\det|^{i\sigma}$ *with* σ *real and are then at* $s = -i\sigma$ *and* $s = 1 - i\sigma$ *with residues as above. They satisfy the functional equation*

$$I(s;\varphi,\varphi',\Phi) = I(1-s;\widetilde{W}_\varphi,\widetilde{W}'_{\varphi'},\hat{\Phi}).$$

Moreover, for φ, φ', *and* Φ *factorizable we have that the integrals are Eulerian and we have*

$$I(s;\varphi,\varphi',\Phi) = \prod_v \Psi_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v)$$

with convergence absolute and uniform for $\mathrm{Re}(s) >> 0$.

We remark in passing that the right hand side of the functional equation also unfolds as

$$\begin{aligned} I(1-s;\widetilde{\varphi},\widetilde{\varphi}',\hat{\Phi}) &= \int_{\mathrm{N}_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A})} \widetilde{W}_\varphi(g)\widetilde{W}'_{\varphi'}(g)\hat{\Phi}(e_n g)|\det(g)|^{1-s}\,dg \\ &= \prod_v \Psi_v(1-s;\widetilde{W}_\varphi,\widetilde{W}'_{\varphi'},\hat{\Phi}) \end{aligned}$$

with convergence for $\mathrm{Re}(s) << 0$.

3. Local L-functions

If (π, V_π) is a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ and $(\pi', V_{\pi'})$ is a cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$ we have associated to the pair (π,π') a family of Eulerian integrals $\{I(s;\varphi,\varphi')\}$ (or $\{I(s;\varphi,\varphi',\Phi)\}$ if $m = n$) and through the Euler factorization we have for each place v of k a family of local integrals $\{\Psi_v(s;W_v,W'_v)\}$ (or $\{\Psi_v(s;W_v,W'_v,\Phi_v)\}$) attached to the pair of local components (π_v,π'_v). In this section we would like to attach a local L-function (or local Euler factor) $L(s,\pi_v\times\pi'_v)$ to such a pair of local representations through the family of local integrals and analyze its basic properties, including the local functional equation.

3.1. **Non-archimedean local factors.** For this section let k denote a non-archimedean local field. We will let $\mathfrak{o}$ denote the ring of integers of k and $\mathfrak{p}$ the unique prime ideal of $\mathfrak{o}$. Fix a generator ϖ of $\mathfrak{p}$. We let q be the residue degree of k, so $q = |\mathfrak{o}/\mathfrak{p}| = |\varpi|^{-1}$. We fix a non-trivial continuous additive character ψ of k. (π, V_π) and $(\pi', V_{\pi'})$ will now be the smooth vectors in irreducible admissible unitary generic representations of $\mathrm{GL}_n(k)$ and $\mathrm{GL}_m(k)$ respectively, as is true for local components of cuspidal representations. We will let ω and ω' denote their central characters. The basic reference here is the paper of Jacquet, Piatetski-Shapiro, and Shalika [23].

For each pair of Whittaker functions $W \in \mathcal{W}(\pi,\psi)$ and $W' \in \mathcal{W}(\pi',\psi^{-1})$ and in the case $n = m$ each Schwartz-Bruhat function $\Phi \in \mathcal{S}(k^n)$ we have defined local integrals $\Psi(s;W,W')$, $\widetilde{\Psi}(s;W,W')$ in the case $m < n$ and $\Psi(s;W,W',\Phi)$ in the case $n = m$, both convergent for

$\mathrm{Re}(s) >> 0$. To make the notation more convenient for what follows, in the case $m < n$ for any $0 \le j \le n-m-1$ let us set

$$\Psi_j(s : W, W') = \int_{\mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)} \int_{M_{j,m}(k)} W \begin{pmatrix} h & & \\ x & I_j & \\ & & I_{n-m-j} \end{pmatrix} dx \; W'(h) |\det(h)|^{s-(n-m)/2} \, dh,$$

so that $\Psi(s; W, W') = \Psi_0(s; W, W')$ and $\widetilde{\Psi}(s; W, W') = \Psi_{n-m-1}(s; W, W')$, which is still absolutely convergent for $\mathrm{Re}(s) >> 0$.

We need to understand what type of functions of s these local integrals are. To this end, we need to understand the local Whittaker functions. So let $W \in \mathcal{W}(\pi, \psi)$. Since W is smooth there is a compact open subgroup K so that $W(gk) = W(g)$ for all $k \in K$. W transforms on the left under $\mathrm{N}_n(k)$ via ψ. So the Iwasawa decomposition on $\mathrm{GL}_n(k)$ gives that it suffices to understand a general Whittaker function on the torus. Let α_i, $i = 1, \ldots, n-1$, denote the standard simple roots of GL_n, so that if $a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in \mathrm{A}_n(k)$ then $\alpha_i(a) = a_i/a_{i+1}$. The fundamental result on the asymptotics of Whittaker functions [22] is that there is a finite set of finite functions $X(\pi) = \{\chi_i\}$ on $\mathrm{A}_n(k)$, depending only on π, so that for every $W \in \mathcal{W}(\pi, \psi)$ there are Schwartz –Bruhat functions $\phi_i \in \mathcal{S}(k^{n-1})$ such that for all $a \in \mathrm{A}_n(k)$ with $a_n = 1$ we have

$$W(a) = \sum_{X(\pi)} \chi_i(a) \phi_i(\alpha_1(a), \ldots, \alpha_{n-1}(a)).$$

By a finite function on $\mathrm{A}_n(k)$ we mean a continuous function whose translates span a finite dimensional vector space [21, 22, Section 2.2]. For the field $k^\times$ itself the finite functions are spanned by products of characters and powers of the valuation map. The finite set of finite functions $X(\pi)$ which occur in the asymptotics near 0 of the Whittaker functions come from analyzing the Jacquet module $\mathcal{W}(\pi, \psi)/\langle \pi(n)W - W | n \in \mathrm{N}_n \rangle$ which is naturally an $\mathrm{A}_n(k)$–module. Note that due to the Schwartz-Bruhat functions, the Whittaker functions vanish whenever any simple root $\alpha_i(a)$ becomes large.

Several nice consequences follow from inserting these formulas for W and W' into the local integrals $\Psi_j(s; W, W')$ or $\Psi(s; W, W', \Phi)$ [22, 23].

(1) Each integral converges for $\mathrm{Re}(s) >> 0$. For π and π' unitary, as we have assumed, they converge absolutely for $\mathrm{Re}(s) \ge 1$. For π and π' tempered, we have absolute convergence for $\mathrm{Re}(s) > 0$.
(2) Each integral defines a rational function in q^{-s} and hence meromorphically extends to all of $\mathbb{C}$.
(3) Each such rational function can be written with a common denominator which depends only on the finite functions $X(\pi)$ and $X(\pi')$ and hence only on π and π'.

Let $\mathcal{I}_j(\pi, \pi')$ denote the complex linear span of the local integrals $\Psi_j(s; W, W')$ if $m < n$ and $\mathcal{I}(\pi, \pi')$ the complex linear span of the $\Psi(s; W, W', \Phi)$ if $m = n$. In the case $m < n$ one can show by a rather elementary although somewhat involved manipulation of the integrals that all of the ideals $\mathcal{I}_j(\pi,\pi')$ are the same [23], so we will write this ideal as $\mathcal{I}(\pi,\pi')$. These are then subspaces of $\mathbb{C}(q^{-s})$ which have "bounded denominators" in the sense of (3). In fact, these subspaces have more structure – each $\mathcal{I}(\pi,\pi')$ is a fractional $\mathbb{C}[q^s,q^{-s}]$–ideal of $\mathbb{C}(q^{-s})$. Since $\mathbb{C}[q^s,q^{-s}]$ is a principal ideal domain each fractional ideal $\mathcal{I}(\pi,\pi')$ has a single generator and since each of these fractional ideals contain 1 we can always normalize our generator to be of the form $P_{\pi,\pi'}(q^{-s})^{-1}$ where the polynomial $P(X)$ satisfies $P(0)=1$. This gives us the definition of our local L-function:

$$L(s,\pi\times\pi') = P_{\pi,\pi'}(q^{-s})^{-1}$$

is the normalized generator of the fractional ideal $\mathcal{I}(\pi,\pi')$ formed by the family of local integrals. If $\pi'=\mathbf{1}$ is the trivial representation of $\mathrm{GL}_1(k)$ then we write $L(s,\pi)=L(s,\pi\times\mathbf{1})$.

One can show easily that the ideal $\mathcal{I}(\pi,\pi')$ is independent of the character ψ used in defining the Whittaker models, so that $L(s,\pi\times\pi')$ is independent of the choice of ψ. So it is not included in the notation. Also, note that for $\pi'=\chi$ an automorphic representation (character) of $\mathrm{GL}_1(\mathbb{A})$ we have the identity $L(s,\pi\times\chi)=L(s,\pi\otimes\chi)$ where $\pi\otimes\chi$ is the representation of $\mathrm{GL}_n(\mathbb{A})$ on V_π given by $\pi\otimes\chi(g)\xi=\chi(\det(g))\pi(g)\xi$.

We summarize the above in the following Theorem.

Theorem 3.1. *Let π and π' be as above. The family of local integrals form a $\mathbb{C}[q^s,q^{-s}]$–fractional ideal $\mathcal{I}(\pi,\pi')$ in $\mathbb{C}(q^{-s})$ with generator the local L-function $L(s,\pi\times\pi')$.*

Another useful way of thinking of the local L-function is the following. $L(s,\pi\times\pi')$ is the minimal (in terms of degree) function of the form $P(q^{-s})^{-1}$, with $P(X)$ a polynomial satisfying $P(0)=1$, such that the ratios $\dfrac{\Psi(s;W,W')}{L(s,\pi\times\pi')}$ (resp. $\dfrac{\Psi(s;W,W',\Phi)}{L(s,\pi\times\pi')}$) are entire for all $W\in\mathcal{W}(\pi,\psi)$ and $W'\in\mathcal{W}(\pi',\psi^{-1})$, and if necessary $\Phi\in\mathcal{S}(k^n)$. That is, $L(s,\pi\times\pi')$ is the standard Euler factor determined by the poles of the functions in $\mathcal{I}(\pi,\pi')$.

One should note that since the L-factor is a generator of the ideal $\mathcal{I}(\pi,\pi')$, then in particular it lies in $\mathcal{I}(\pi,\pi')$. Since this ideal is spanned by our local integrals then we can always write

$$L(s,\pi\times\pi')=\sum_i \Psi(s;W_i,W_i') \qquad \text{or} \qquad L(s,\pi\times\pi')=\sum_i \Psi(s;W_i,W_i',\Phi_i).$$

with a finite collection of $W_i\in\mathcal{W}(\pi,\psi)$, $W_i'\in\mathcal{W}(\pi',\psi^{-1})$, and if necessary $\Phi_i\in\mathcal{S}(k^n)$. This will be necessary for the global theory.

Either by analogy with Tate's thesis or from the corresponding global statement, we would expect our local integrals to satisfy a local functional equation. From the functional equations for our global integrals, we would expect these to relate the integrals $\Psi(s;W,W')$ and $\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W},\widetilde{W}')$ when $m<n$ and $\Psi(s;W,W',\Phi)$ and $\Psi(1-s;\widetilde{W},\widetilde{W}',\hat{\Phi})$ when $m=n$. This will indeed be the case. These functional equations will come from interpreting the local integrals as families (in s) of quasi-invariant bilinear forms on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})$ or trilinear forms on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})\times\mathcal{S}(k^n)$ depending on the case. One is able to analyze such functional using Bruhat's theory and one shows that, except for a finite number

if exceptional values of q^{-s} such bilinear functionals are unique [23]. Hence we obtain the following local functional equation.

Theorem 3.2. *There is a rational function $\gamma(s,\pi\times\pi',\psi)\in\mathbb{C}(q^{-s})$ such that we have*

$$\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W},\widetilde{W}')=\omega'(-1)^{n-1}\gamma(s,\pi\times\pi',\psi)\Psi(s;W,W') \quad \text{if } m<n$$

or

$$\Psi(1-s;\widetilde{W},\widetilde{W}',\hat{\Phi})=\omega'(-1)^{n-1}\gamma(s,\pi\times\pi',\psi)\Psi(s;W,W',\Phi) \quad \text{if } m=n$$

for all $W\in\mathcal{W}(\pi,\psi)$, $W'\in\mathcal{W}(\pi',\psi^{-1})$, and if necessary all $\Phi\in\mathcal{S}(k^n)$.

An equally important local factor, which occurs in the current formulations of the local Langlands correspondence [2, 16, 19], is the local ε-factor, defined as the ratio

$$\varepsilon(s,\pi\times\pi',\psi)=\frac{\gamma(s,\pi\times\pi',\psi)L(s,\pi\times\pi')}{L(1-s,\widetilde{\pi}\times\widetilde{\pi}')}.$$

With the local ε-factor the local functional equation can be written in the form

$$\frac{\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W},\widetilde{W}')}{L(1-s,\widetilde{\pi}\times\widetilde{\pi}')}=\omega'(-1)^{n-1}\varepsilon(s,\pi\times\pi',\psi)\frac{\Psi(s;W,W')}{L(s,\pi\times\pi')} \quad \text{if } m<n$$

or

$$\frac{\Psi(1-s;\widetilde{W},\widetilde{W}',\hat{\Phi})}{L(1-s,\widetilde{\pi}\times\widetilde{\pi}')}=\omega'(-1)^{n-1}\varepsilon(s,\pi\times\pi',\psi)\frac{\Psi(s;W,W',\Phi)}{L(s,\pi\times\pi')} \quad \text{if } m=n\ .$$

Analyzing the local functional equatin in this form allows one to prove that $\varepsilon(s,\pi\times\pi',\psi)$ is a monomial function of the form cq^{-fs}. If we consider a single π, take ψ unramified, and write $\varepsilon(s,\pi,\psi)=\varepsilon(0,\pi,\psi)q^{-f(\pi)s}$, then in [24] it is shown that $f(\pi)$ is a non-negative integer, $f(\pi)=0$ iff π is unramified, and in general the integer $f(\pi)$ is the *conductor of* π.

Let us now turn to the calculation of the local L-functions. The first case to consider is that where both π and π' are unramified. Since they are assumed generic, they are both full induced representations from unramified characters of the Borel subgroup [49]. So let us write $\pi\simeq\mathrm{Ind}_{\mathrm{B}_n}^{\mathrm{GL}_n}(\mu_1\otimes\cdots\otimes\mu_n)$ and $\pi'\simeq\mathrm{Ind}_{\mathrm{B}_m}^{\mathrm{GL}_m}(\mu_1'\otimes\cdots\otimes\mu_m')$ with the μ_i and μ_j' unramified characters of $k^\times$. The Satake parameterization of unramified representations associates to each of these representation the semi-simple conjugacy classes $[A_\pi]\in\mathrm{GL}_n(\mathbb{C})$ and $[A_{\pi'}]\in\mathrm{GL}_m(\mathbb{C})$ given by

$$A_\pi=\begin{pmatrix}\mu_1(\varpi)&&\\&\ddots&\\&&\mu_n(\varpi)\end{pmatrix} \qquad A_{\pi'}=\begin{pmatrix}\mu_1'(\varpi)&&\\&\ddots&\\&&\mu_m'(\varpi)\end{pmatrix}.$$

(Recall that ϖ is a uniformizing parameter for k, that is, a generator of $\mathfrak{p}$.) In the Whittaker models there will be unique normalized $\mathrm{K}=\mathrm{GL}(\mathfrak{o})$– fixed Whittaker functions, $W_\circ\in\mathcal{W}(\pi,\psi)$ and $W_\circ'\in\mathcal{W}(\pi',\psi^{-1})$, normalized by $W_\circ(e)=W_\circ'(e)=1$. There is an explicit formula for $W_\circ$ in terms of the Satake parameter A_π due to Shintani [43]. Utilizing this formula, one obtains the following explicit computation of the local L-factor in this case.

Theorem 3.3. *If π, π', and ψ are all unramified, then*

$$L(s,\pi\times\pi') = \det(I - q^{-s}A_\pi \otimes A_{\pi'})^{-1} = \begin{cases} \Psi(s;W_\circ,W'_\circ) & m<n \\ \Psi(s;W_\circ,W'_\circ,\Phi_\circ) & m=n \end{cases}$$

and $\varepsilon(s,\pi\times\pi',\psi)\equiv 1$.

The other basic case is when both π and π' are supercuspidal. For this calculation, one must analyze the local integrals in terms of the Kirillov models of the representations [11, 8].

Theorem 3.4. *If π and π' are both (unitary) supercuspidal, then $L(s,\pi\times\pi')\equiv 1$ if $m<n$ and if $m=n$ we have*

$$L(s,\pi\times\pi') = \prod (1-\alpha q^{-s})^{-1}$$

with the product over all $\alpha = q^{s_0}$ with $\widetilde{\pi} \simeq \pi' \otimes |\det|^{s_0}$.

In the other cases, we must rely on the Bernstein–Zelevinsky classification of generic representations of $\mathrm{GL}_n(k)$ [49]. All generic representations can be realized as prescribed constituents of representations parabolically induced from supercuspidals. One can proceed by analyzing the Whittaker functions of induced representations in terms of Whittaker functions of the inducing data as in [23] or by analyzing the poles of the local integrals in terms of quasi invariant pairings of derivatives of π and π' as in [8] to compute $L(s,\pi\times\pi')$ in terms of L-functions of pairs of supercuspidal representations. We refer you to those papers or [28] for the explicit formulas.

To conclude this section, let us mention two results on the γ-factors. One is used in the computations of L-factors in the general case. This is the *multiplicativity of γ-factors* [23]. The second is the *stability of γ-factors* [26]. Both of these results are necessary in applications of the Converse Theorem to liftings.

Multiplicativity of γ-factors: *If $\pi = \mathrm{Ind}(\pi_1\otimes\pi_2)$, with π_i and irreducible admissible representation of $\mathrm{GL}_{r_i}(k)$, then*

$$\gamma(s,\pi\times\pi',\psi) = \gamma(s,\pi_1\times\pi',\psi)\gamma(s,\pi_2\times\pi',\psi)$$

and similarly for π'. Moreover $L(s,\pi\times\pi')^{-1}$ divides $[L(s,\pi_1\times\pi')L(s,\pi_2\times\pi')]^{-1}$.

Stability of γ-factors: *If π_1 and π_2 are two irreducible admissible generic representations of $\mathrm{GL}_n(k)$, having the same central character, then for every sufficiently highly ramified character η of $\mathrm{GL}_1(k)$ we have*

$$\gamma(s,\pi_1\times\eta,\psi) = \gamma(s,\pi_2\times\eta,\psi)$$

and

$$L(s,\pi_1\times\eta) = L(s,\pi_2\times\eta) \equiv 1.$$

More generally, if in addition π' is an irreducible generic representation of $\mathrm{GL}_m(k)$ then for all sufficiently highly ramified characters η of $\mathrm{GL}_1(k)$ we have

$$\gamma(s,(\pi_1\otimes\eta)\times\pi',\psi) = \gamma(s,(\pi_2\otimes\eta)\times\pi',\psi)$$

and

$$L(s,(\pi_1\otimes\eta)\times\pi') = L(s,(\pi_2\otimes\eta)\times\pi') \equiv 1.$$

3.2. **The archimedean local factors.** The treatment of the archimedean local factors parallels that of the non-archimedean in many ways, but there are some significant differences. The major work on these factors is that of Jacquet and Shalika in [27], which we follow for the most part without further reference, and in the archimedean parts of [25].

One significant difference in the development of the archimedean theory is that the local Langlands correspondence was already in place when the theory was developed [30]. The correspondence is very explicit in terms of the usual Langlands classification. Thus to π is associated an n dimensional semi-simple representation $\rho=\rho_\pi$ of the Weil group W_k of k and to π' an m-dimensional semi-simple representation $\rho'=\rho'_\pi$ of W_k. Then $\rho\otimes\rho'$ is an nm dimensional representation of W_k and to this representation of the Weil group is attached Artin-Weil L– and ε–factors [47], denoted $L(s,\rho\otimes\rho')$ and $\varepsilon(s,\rho\otimes\rho',\psi)$. In essence, Jacquet and Shalika *define*

$$L(s,\pi\times\pi') = L(s,\rho_\pi\otimes\rho'_\pi) \quad \text{and} \quad \varepsilon(s,\pi\times\pi',\psi) = \varepsilon(s,\rho_\pi\otimes\rho'_\pi,\psi)$$

and then set

$$\gamma(s,\pi\times\pi',\psi) = \frac{\varepsilon(s,\pi\times\pi',\psi)L(1-s,\widetilde{\pi}\times\widetilde{\pi}')}{L(s,\pi\times\pi')}.$$

They then proceed to show that these functions have the expected relation to the local integrals. To this end, they define $\mathcal{M}(\pi\times\pi')$ to be the space of all meromorphic functions $\phi(s)$ with the property that if $P(s)$ is a polynomial function such that $P(s)L(s,\pi\times\pi')$ is holomorphic in a vertical strip $S[a,b]=\{s\ a\le \mathrm{Re}(s)\le b\}$ then $P(s)\phi(s)$ is bounded in $S[a,b]$. Note in particular that if $\phi\in\mathcal{M}(\pi\times\pi')$ then the quotient $\phi(s)L(s,\pi\times\pi')^{-1}$ is entire. One then analyzes the local integrals $\Psi_j(s;W,W')$ and $\Psi(s;W,W',\Phi)$, defined as in the non-archimedean case for $W\in\mathcal{W}(\pi,\psi)$, $W'\in\mathcal{W}(\pi',\psi^{-1})$, and $\Phi\in\mathcal{S}(k^n)$, using methods that are direct analogues of those used in [23] for the non-archimedean case.

Theorem 3.5. *The integrals $\Psi_j(s;W,W')$ or $\Psi(s;W,W',\Phi)$ extend to meromorphic functions of s which lie in $\mathcal{M}(\pi\times\pi')$. In particular, the ratios*

$$e_j(s;W,W') = \frac{\Psi_j(s;W,W')}{L(s,\pi\times\pi')} \quad \text{or} \quad e(s;W,W',\Phi) = \frac{\Psi(s;W,W',\Phi)}{L(s,\pi\times\pi')}$$

are entire and in fact are bounded in vertical strips.

This statement has more content than just the continuation and "bounded denominator" statements in the non-archimedean case. Since it prescribes the "denominator" to be the L factor $L(s,\pi\times\pi')^{-1}$ it is bound up with the actual computation of the poles of the local integrals. In fact, a significant part of the paper of Jacquet and Shalika [27] is taken up with the simultaneous proof of this and the local functional equations:

Theorem 3.6. *We have the local functional equations*

$$\Psi_{n-m-j-1}(1-s;\rho(w_{n,m})\widetilde{W},\widetilde{W}') = \omega'(-1)^{n-1}\gamma(s,\pi\times\pi',\psi)\Psi_j(s;W,W')$$

or

$$\Psi(1-s;\widetilde{W},\widetilde{W}',\hat{\Phi}) = \omega'(-1)^{n-1}\gamma(s,\pi\times\pi',\psi)\Psi(s;W,W',\Phi).$$

The one fact that we are missing is the statement of "minimality" of the L-factor. That is, we know that $L(s,\pi\times\pi')$ is a standard archimedean Euler factor (i.e., a product of Γ-functions of the standard type) and has the property that the poles of all the local integrals are contained in the poles of the L-factor, even with multiplicity. But we have not established that the L-factor cannot have extraneous poles. In particular, we do not know that we can achieve the local L-function as a finite linear combination of local integrals. Towards this end, Jacquet and Shalika enlarge the allowable space of local integrals. Let Λ and Λ' be the Whittaker functionals on V_π and $V_{\pi'}$ associated with the Whittaker models $\mathcal{W}(\pi,\psi)$ and $\mathcal{W}(\pi',\psi^{-1})$. Then $\hat{\Lambda}=\Lambda\otimes\Lambda'$ defines a continuous linear functional on the algebraic tensor product $V_\pi\hat{\otimes}V_{\pi'}$ which extends continuously to the topological tensor product $V_{\pi\otimes\pi'}=V_\pi\hat{\otimes}V_{\pi'}$, viewed as representations of $\mathrm{GL}_n(k)\times\mathrm{GL}_m(k)$. Now let

$$\mathcal{W}(\pi\hat{\otimes}\pi',\psi)=\{W(g,h)=\hat{\Lambda}(\pi(g)\otimes\pi'(h)\xi)|\xi\in V_{\pi\otimes\pi'}\}.$$

Then $\mathcal{W}(\pi\hat{\otimes}\pi',\psi)$ contains the algebraic tensor product $\mathcal{W}(\pi,\psi)\otimes\mathcal{W}(\pi',\psi^{-1})$ and is again equal to the topological tensor product. Now we can extend all our local integrals to the space $\mathcal{W}(\pi\hat{\otimes}\pi',\psi)$ by setting

$$\Psi_j(s;W)=\int\int W\left(\begin{pmatrix} h & & \\ x & I_j & \\ & & I_{n-m-j}\end{pmatrix},h\right)\,dx\,|\det(h)|^{s-(n-m)/2}\,dh$$

and

$$\Psi(s;W,\Phi)=\int W(g,g)\Phi(e_ng)|\det(g)|^s\,dh$$

for $W\in\mathcal{W}(\pi\hat{\otimes}\pi',\psi)$. Since the local integrals are continuous with respect to the topology on the topological tensor product, all of the above facts remain true, in particular the convergence statements, the local functional equations, and the fact that these integrals extend to functions in $\mathcal{M}(\pi\times\pi')$. At this point, let $\mathcal{I}_j(\pi,\pi')=\{\Psi_j(s;W)|W\in\mathcal{W}(\pi\otimes\pi')\}$ and let $\mathcal{I}(\pi,\pi')$ be the span of the local integrals $\{\Psi(s;W,\Phi)|W\in\mathcal{W}(\pi\hat{\otimes}\pi',\psi),\ \phi\in\mathcal{S}(k^n)\}$. Once again, in the case $m<n$ we have that the space $\mathcal{I}_j(\pi,\pi')$ is independent of j and we denote it also by $\mathcal{I}(\pi,\pi')$. These are still independent of ψ. So we know from above that $\mathcal{I}(\pi,\pi')\subset\mathcal{M}(\pi\times\pi')$. The remainder of [27] is then devoted to showing $\mathcal{I}(\pi,\pi')=\mathcal{M}(\pi\times\pi')$.

In the cases of $m=n-1$ or $m=n$, Stade [44, 45] and Jacquet and Shalika (see [9]) have shown that one can indeed get the local L-function as a finite linear combination of integrals involving only K-finite functions in $\mathcal{W}(\pi,\psi)$ and $\mathcal{W}(\pi',\psi^{-1})$, that is, without going to the completion of $\mathcal{W}(\pi,\psi)\otimes\mathcal{W}(\pi',\psi^{-1})$.

4. Global L-functions

Once again, we let k be a global field, $\mathbb{A}$ its ring of adeles, and fix a non-trivial continuous additive character $\psi = \otimes\psi_v$ of $\mathbb{A}$ trivial on k.

Let (π, V_π) be an cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ and $(\pi', V_{\pi'})$ a cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$. Since they are irreducible we have restricted tensor product decompositions $\pi \simeq \otimes'\pi_v$ and $\pi' \simeq \otimes'\pi'_v$ with (π_v, V_{π_v}) and $(\pi'_v, V_{\pi'_v})$ irreducible admissible smooth generic unitary representations of $\mathrm{GL}_n(k_v)$ and $\mathrm{GL}_m(k_v)$ [10, 13]. Let $\omega = \otimes'\omega_v$ and $\omega' = \otimes'\omega'_v$ be their central characters. These are both continuous characters of $k^\times\backslash\mathbb{A}^\times$. Let S be the finite set of places of k, containing the archimedean places S_∞, such that for all $v \notin S$ we have that π_v, π'_v, and ψ_v are unramified.

For each place v of k we have defined the local factors $L(s, \pi_v \times \pi'_v)$ and $\varepsilon(s, \pi_v \times \pi'_v, \psi_v)$. Then we can at least formally define

$$L(s, \pi \times \pi') = \prod_v L(s, \pi_v \times \pi'_v) \quad \text{and} \quad \varepsilon(s, \pi \times \pi') = \prod_v \varepsilon(s, \pi_v \times \pi'_v, \psi_v).$$

The product defining the L-function is absolutely convergent for $\mathrm{Re}(s) >> 0$. For the ε–factor, $\varepsilon(s, \pi_v \times \pi'_v, \psi_v) \equiv 1$ for $v \notin S$ so that the product is in fact a finite product and there is no problem with convergence. The fact that $\varepsilon(s, \pi \times \pi')$ is independent of ψ can either be checked by analyzing how the local ε–factors vary as you vary ψ, as is done in [5, Lemma 2.1], or it will follow from the global functional equation presented below.

4.1. **Basic analytic properties.**

Our first goal is to show that these L-functions have nice analytic properties.

Theorem 4.1. *The global L–functions $L(s, \pi \times \pi')$ are nice in the sense that*

(1) *$L(s, \pi \times \pi')$ has a meromorphic continuation to all of $\mathbb{C}$,*
(2) *the extended function is bounded in vertical strips (away from its poles),*
(3) *they satisfy the functional equation*

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1 - s, \widetilde{\pi} \times \widetilde{\pi}').$$

To do so, we relate the L-functions to the global integrals.

Let us begin with continuation. In the case $m < n$ for every $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ we know the integral $I(s; \varphi, \varphi')$ converges absolutely for all s. From the unfolding in Section 2 and the local calculation of Section 3 we know that for $\mathrm{Re}(s) >> 0$ and for appropriate choices of φ and φ' we have

$$I(s;\varphi,\varphi') = \prod_v \Psi_v(s;W_{\varphi_v},W_{\varphi'_v}) = \left(\prod_{v\in S}\Psi_v(s;W_{\varphi_v},W_{\varphi'_v})\right) L^S(s,\pi\times\pi')$$
$$= \left(\prod_{v\in S}\frac{\Psi_v(s;W_{\varphi_v},W_{\varphi'_v})}{L(s,\pi_v\times\pi'_v)}\right) L(s,\pi\times\pi') = \left(\prod_{v\in S} e_v(s;W_{\varphi_v},W_{\varphi'_v})\right) L(s,\pi\times\pi')$$

We know that each ratio $e_v(s;W_v,W'_v)$ is entire and hence $L(s,\pi\times\pi')$ has a meromorphic continuation. If $m=n$ then for appropriate $\varphi\in V_\pi$, $\varphi'\in V_{\pi'}$, and $\Phi\in\mathcal{S}(\mathbb{A}^n)$ we again have

$$I(s;\varphi,\varphi',\Phi) = \left(\prod_{v\in S} e_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v)\right) L(s,\pi\times\pi').$$

and so $L(s,\pi\times\pi')$ has a meromorphic continuation.

Let us next turn to the functional equation. This will follow from the functional equation for the global integrals and the local functional equations. We will consider only the case where $m<n$ since the other case is entirely analogous. The functional equation for the global integrals is simply

$$I(s;\varphi,\varphi') = \tilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}').$$

Once again we have for appropriate φ and φ'

$$I(s;\varphi,\varphi') = \left(\prod_{v\in S}\frac{\Psi(s;W_v,W'_v)}{L(s,\pi\times\pi')}\right) L(s,\pi\times\pi')$$

while on the other side

$$\tilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}') = \left(\prod_{v\in S}\frac{\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W}_v,\widetilde{W}'_v)}{L(1-s,\widetilde{\pi}\times\widetilde{\pi}')}\right) L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

However, by the local functional equations, for each $v\in S$ we have

$$\frac{\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W}_v,\widetilde{W}'_v)}{L(1-s,\widetilde{\pi}\times\widetilde{\pi}')} = \omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\frac{\Psi(s;W_v,W'_v)}{L(s,\pi\times\pi')}.$$

Combining these, we have

$$L(s,\pi\times\pi') = \left(\prod_{v\in S}\omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\right) L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

Now, for $v\notin S$ we know that π'_v is unramified, so $\omega'_v(-1)=1$, and also that $\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\equiv 1$. Therefore

$$\prod_{v\in S}\omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v) = \varepsilon(s,\pi\times\pi')$$

and we indeed have

$$L(s,\pi\times\pi') = \varepsilon(s,\pi\times\pi')L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

Note that this implies that $\varepsilon(s,\pi\times\pi')$ is independent of ψ as well.

Let us now turn to the boundedness in vertical strips. For the global integrals $I(s;\varphi,\varphi')$ or $I(s;\varphi,\varphi,\Phi)$ this simply follows from the absolute convergence. For the L-function itself, the paradigm is the following. For every finite place $v \in S$ we know that there is a choice of $W_{v,i}$, $W'_{v,i}$, and $\Phi_{v,i}$ if necessary such that

$$L(s,\pi_v \times \pi'_v) = \sum \Psi(s;W_{v,i},W'_{v'i}) \quad \text{or} \quad L(s,\pi_v \times \pi'_v) = \sum \Psi(s;W_{v,i},W'_{v'i},\Phi_{v,i}).$$

If $m = n-1$ or $m = n$ then by the work of Stade and Jacquet and Shalika we know that we have similar statements for $v \in S_\infty$. Hence if $m = n-1$ or $m = n$ there are global choices φ_i, φ'_i, and if necessary Φ_i such that

$$L(s,\pi \times \pi') = \sum I(s;\varphi_i,\varphi'_i) \quad \text{or} \quad L(s,\pi \times \pi') = \sum I(s;\varphi_i,\varphi'_i,\Phi_i).$$

Then the boundedness in vertical strips for the L-functions follows from that of the global integrals.

However, if $m < n-1$ then all we know at those $v \in S_\infty$ is that there is a function $W_v \in \mathcal{W}(\pi_v \hat{\otimes} \pi'_v,\psi_v) = \mathcal{W}(\pi_v,\psi_v)\hat{\otimes}\mathcal{W}(\pi'_v,\psi_v^{-1})$ or a finite collection of such functions $W_{v,i}$ and of $\Phi_{v,i}$ such that

$$L(s,\pi_v \times \pi'_v) = I(s;W_v) \quad \text{or} \quad L(s,\pi_v \times \pi'_v) = \sum I(s;W_{v,i},\Phi_{v,i}).$$

To make the above paradigm work for $m < n-1$ we would need to rework the theory of global Eulerian integrals for cusp forms in $V_\pi \hat{\otimes} V_{\pi'}$. This is naturally the space of smooth vectors in an irreducible unitary cuspidal representation of $\mathrm{GL}_n(\mathbb{A}) \times \mathrm{GL}_m(\mathbb{A})$. So we would need extend the global theory of integrals parallel to Jacquet and Shalika's extension of the local integrals in the archimedean theory. There seems to be no obstruction to carrying this out, but we have not done this.

We should point out that if one approaches these L-function by the method of constant terms and Fourier coefficients of Eisenstein series, then Gelbart and Shahidi have shown a wide class of automorphic L-functions, including ours, to be bounded in vertical strips [12].

4.2. **Poles of L-functions.** Let us determine where the global L-functions can have poles. The poles of the L-functions will be related to the poles of the global integrals. Recall from Section 2 that in the case of $m < n$ we have that the global integrals $I(s;\varphi,\varphi')$ are entire and that when $m = n$ then $I(s;\varphi,\varphi',\Phi)$ can have at most simple poles and they occur at $s = -i\sigma$ and $s = 1 - i\sigma$ for σ real when $\pi \simeq \widetilde{\pi}' \otimes |\det|^{i\sigma}$. As we have noted above, the global integrals and global L-functions are related, for appropriate φ, φ', and Φ, by

$$I(s;\varphi,\varphi') = \left(\prod_{v\in S} e_v(s;W_{\varphi_v},W'_{\varphi'_v})\right) L(s,\pi \times \pi')$$

or

$$I(s;\varphi,\varphi',\Phi) = \left(\prod_{v\in S} e_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v)\right) L(s,\pi \times \pi').$$

On the other hand, for any $s_0 \in \mathbb{C}$ and any v there is a choice of local W_v, W'_v, and Φ_v such that the local ratios $e_v(s_0;W_v,W'_v) \neq 0$ or $e_v(s_0;W_v,W'_v,\Phi_v) \neq 0$. So as we vary φ, φ' and Φ at the places $v \in S$ we see that division by these factors can introduce no extraneous poles in $L(s,\pi\times\pi')$, that is, in keeping with the local characterization of the L-factor in terms of poles of local integrals, globally the poles of $L(s,\pi\times\pi')$ are precisely the poles of the family of global integrals $\{I(s;\varphi,\varphi')\}$ or $\{I(s;\varphi,\varphi',\Phi)\}$. Hence from Theorems 2.1 and 2.2 we have.

Theorem 4.2. *If $m<n$ then $L(s,\pi\times\pi')$ is entire. If $m=n$, then $L(s,\pi\times\pi')$ has at most simple poles and they occur iff $\pi\simeq\widetilde{\pi}'\otimes|\det|^{i\sigma}$ with σ real and are then at $s=-i\sigma$ and $s=1-i\sigma$.*

There are two useful observationss that follow from this.

(1) $L(s,\pi\times\widetilde{\pi})$ has simple poles at $s=0$ and $s=1$.
(2) For π and π' cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$, $L(s,\pi\times\widetilde{\pi}')$ has a pole at $s=1$ iff $\pi\simeq\pi'$.

As a consequence of this, we obtain the analytic proof of the Strong Multiplicity One Theorem [31, 25].

Strong Multiplicity One: *Let (π,V_π) and $(\pi',V_{\pi'})$ be two cuspidal representations of $GL_n(\mathbb{A})$. Suppose there is a finite set of places S such that for all $v\notin S$ we have $\pi_v\simeq\pi'_v$. Then $\pi=\pi'$.*

5. Converse Theorems

Let us return first to Hecke. Recall that to a modular form

$$f(\tau)=\sum_{n-1}^{\infty}a_n e^{2\pi i n\tau}$$

for say $\mathrm{SL}_2(\mathbb{Z})$ Hecke attached an L function $L(s,f)$ and they were related via the Mellin transform

$$\Lambda(s,f)=(2\pi)^{-s}\Gamma(s)L(s,f)=\int_0^\infty f(iy)y^s\ d^\times y$$

and derived the functional equation for $L(s,f)$ from the modular transformation law for $f(\tau)$ under the modular transformation law for the transformation $\tau\mapsto-1/\tau$. In his fundamental paper [17] he inverted this process by taking a Dirichlet series

$$D(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

and assuming that it converged in a half plane, had an entire continuation to a function of finite order, and satisfied the same functional equation as the L-function of a modular form of weight k, then he could actually reconstruct a modular form from $D(s)$ by Mellin inversion

$$f(iy)=\sum_i a_n e^{-2\pi n y}=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}(2\pi)^{-s}\Gamma(s)D(s)y^s\ ds$$

and obtain the modular transformation law for $f(\tau)$ under $\tau \mapsto -1/\tau$ from the functional equation for $D(s)$ under $s \mapsto k - s$. This is Hecke's Converse Theorem.

In this Section we will present some analogues of Hecke's theorem in the context of L-functions for GL_n. Surprisingly, the technique is exactly the same as Hecke's, i.e., inverting the integral representation. This was first done in the context of automorphic representation for GL_2 by Jacquet and Langlands [21] and then extended and significantly strengthened for GL_3 by Jacquet, Piatetski-Shapiro, and Shalika [22]. This section is taken mainly from our survey [7]. Further details can be found in [5, 6].

Let k be a global field, $\mathbb{A}$ its adele ring, and ψ a fixed non-trivial continuous additive character of $\mathbb{A}$ which is trivial on k. We will take $n \geq 3$ to be an integer.

To state these Converse Theorems, we begin with an irreducible admissible representation Π of $\mathrm{GL}_n(\mathbb{A})$. In keeping with the conventions of these notes, we will assume that Π is unitary and generic, but this is not necessary. It has a decomposition $\Pi = \otimes' \Pi_v$, where Π_v is an irreducible admissible generic representation of $\mathrm{GL}_n(k_v)$. By the local theory of Section 3, to each Π_v is associated a local L-function $L(s, \Pi_v)$ and a local ε-factor $\varepsilon(s, \Pi_v, \psi_v)$. Hence formally we can form

$$L(s, \Pi) = \prod_v L(s, \Pi_v) \qquad \text{and} \qquad \varepsilon(s, \Pi, \psi) = \prod_v \varepsilon(s, \Pi_v, \psi_v).$$

We will always assume the following two things about Π:

(1) $L(s, \Pi)$ converges in some half plane $Re(s) >> 0$,
(2) the central character ω_Π of Π is automorphic, that is, invariant under $k^\times$.

Under these assumptions, $\varepsilon(s, \Pi, \psi) = \varepsilon(s, \Pi)$ is independent of our choice of ψ [5].

Our Converse Theorems will involve twists by cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A})$ for certain m. For convenience, let us set $\mathcal{A}(m)$ to be the set of automorphic representations of $\mathrm{GL}_m(\mathbb{A})$, $\mathcal{A}_0(m)$ the set of cuspidal representations of $\mathrm{GL}_m(\mathbb{A})$, and $\mathcal{T}(m) = \coprod_{d=1}^{m} \mathcal{A}_0(d)$. If we fix a finite set of S of finite places then we let $\mathcal{T}(S; m)$ denote the subset of $\mathcal{T}(m)$ consisting of representations that are *unramified* at all places $v \in S$.

Let $\pi' = \otimes' \pi'_v$ be a cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$ with $m < n$. Then again we can formally define

$$L(s, \Pi \times \pi') = \prod_v L(s, \Pi_v \times \pi'_v) \qquad \text{and} \qquad \varepsilon(s, \Pi \times \pi') = \prod_v \varepsilon(s, \Pi_v \times \pi'_v, \psi_v)$$

since again the local factors make sense whether Π is automorphic or not. A consequence of (1) and (2) above and the cuspidality of π' is that both $L(s, \Pi \times \pi')$ and $L(s, \widetilde{\Pi} \times \widetilde{\pi}')$ converge absolutely for $Re(s) >> 0$, where $\widetilde{\Pi}$ and $\widetilde{\pi}'$ are the contragredient representations, and that $\varepsilon(s, \Pi \times \pi')$ is independent of the choice of ψ.

We say that $L(s,\Pi\times\pi')$ is *nice* if it satisfies the same analytic properties it would if Π were cuspidal, i.e.,

(1) $L(s,\Pi\times\pi')$ and $L(s,\widetilde{\Pi}\times\widetilde{\pi}')$ have analytic continuations to entire functions of s,
(2) these entire continuations are bounded in vertical strips of finite width,
(3) they satisfy the standard functional equation

$$L(s,\Pi\times\pi')=\varepsilon(s,\Pi\times\pi')L(1-s,\widetilde{\Pi}\times\widetilde{\pi}').$$

The basic Converse Theorem for GL_n is the following [5, 4].

Theorem 5.1. *Let Π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ as above. Let S be a finite set of finite places. Suppose that $L(s,\Pi\times\pi')$ is nice for all $\pi'\in\mathcal{T}(S;n-1)$. Then Π is quasi-automorphic in the sense that there is an automorphic representation Π' such that $\Pi_v\simeq\Pi'_v$ for all $v\notin S$. If S is empty, then in fact Π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$.*

This result is of course valid for $n=2$ as well.

For applications [3], it is desirable to twist by as little as possible. There are essentially two ways to restrict the twisting. One is to restrict the rank of the groups that the twisting representations live on. The other is to restrict ramification.

When we restrict the rank of our twists, we can obtain the following result [6].

Theorem 5.2. *Let Π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ as above. Let S be a finite set of finite places. Suppose that $L(s,\Pi\times\pi')$ is nice for all $\pi'\in\mathcal{T}(S;n-2)$. Then Π is quasi-automorphic in the sense that there is an automorphic representation Π' such that $\Pi_v\simeq\Pi'_v$ for all $v\notin S$. If S is empty, then in fact Π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$.*

This result is stronger than Theorem 5.1, but its proof is more delicate.

In order to apply these theorems, one uses a good choice of the set S in conjunction with twisting by a highly ramified character η. The set S usually consists of the finite places where Π_v is ramified. η is used in conjunction with the local stability of γ-factors mentioned above. Then the following observation is a key ingredient in applying either of the above theorems [7].

Observation . *Let Π be as in Theorem 5.1 or 5.2. Suppose that η is a fixed (highly ramified) character of $k^\times\backslash\mathbb{A}^\times$. Suppose that $L(s,\Pi\times\pi')$ is nice for all $\pi'\in\mathcal{T}\otimes\eta$, where $\mathcal{T}$ is either of the twisting sets of Theorem 5.1 or 5.2. Then Π is quasi-automorphic as in those theorems.*

The only thing to observe, say by looking at the local or global integrals, is that if $\pi'\in\mathcal{T}$ then $L(s,\Pi\times(\pi'\otimes\eta))=L(s,(\Pi\otimes\eta)\times\pi')$ so that applying the Converse Theorem for Π with twisting set $\mathcal{T}\otimes\eta$ is equivalent to applying the Converse Theorem for $\Pi\otimes\eta$ with the twisting set $\mathcal{T}$. So, by either Theorem 5.1 or 5.2, whichever is appropriate, $\Pi\otimes\eta$ is quasi-automorphic and hence Π is as well.

The second way to restrict our twists is to restrict the ramification at all but a finite number of places [5]. Now fix a non-empty finite set of places S which in the case of a number field contains the set S_∞ of all archimedean places. Let $\mathcal{T}_S(m)$ denote the subset consisting of all representations π' in $\mathcal{T}(m)$ which are *unramified for all* $v \notin S$. Note that we are placing a grave restriction on the ramification of these representations.

Theorem 5.3. *Let Π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ as above. Let S be a non-empty finite set of places, containing S_∞, such that the class number of the ring $\mathfrak{o}_S$ of S-integers is one. Suppose that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}_S(n-1)$. Then Π is quasi-automorphic in the sense that there is an automorphic representation Π' such that $\Pi_v \simeq \Pi'_v$ for all $v \in S$ and all $v \notin S$ such that both Π_v and Π'_v are unramified.*

There are several things to note here. First, there is a class number restriction. However, if $k = \mathbb{Q}$ then we may take $S = S_\infty$ and we have a Converse Theorem with "level 1" twists. As a practical consideration, if we let S_Π be the set of finite places v where Π_v is ramified, then for applications we usually take S and S_Π to be disjoint. Once again, we are losing all information at those places $v \notin S$ where we have restricted the ramification unless Π_v was already unramified there.

The proof of Theorem 5.1 with S empty essentially follows the lead of Hecke, Weil, and Jacquet–Langlands. It is based on the integral representations of L-functions, Fourier expansions, Mellin inversion, and finally a use of the weak form of Langlands spectral theory. For Theorems 5.1, 5.2, and 5.3, where we have restricted our twists, we must impose certain local conditions to compensate for our limited twists. For Theorems 5.1 and 5.2 there are a finite number of local conditions and for Theorem 5.3 an infinite number of local conditions. We must then work around these by using results on generation of congruence subgroups and either weak or strong approximation.

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