ON THE COMPLEX MOMENTS OF SYMMETRIC POWER L-FUNCTIONS AT $s = 1$

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CONTENTS

1. INTRODUCTION

Investigations about the distribution of values of L-functions at $s = 1$ (in this paper, all L-functions are normalized so that the center of the critical strip is $s = 1/2$) began with the works of Chowla and Chowla/Erdös in the case of L-functions associated to the family of real Dirichlet characters. Via Dirichlet's class number formula these have implications to the study of the distribution and extreme values of class numbers of imaginary quadratic fields of large discriminants (see also the recent works by Duke on extreme values of class number of number fields of higher degree [2, 3]). The case of degree 1 L-functions was further investigated by several people including notably Barban, Lavrik, Eliott and more recently Montgomery/Vaughan [23] and Granville/Soundararajan [8].

The distribution of values at $s = 1$ of higher degree L-functions has been investigated only recently, in the work of Luo: motivated by problems in spectral deformation theory, he considered the case of symmetric square L-functions of Maass forms having large eigenvalue [21] to show that the set of values at $s = 1$ of such L-function is unbounded. One main difficulty here is precisely that the L-functions are Euler products of degree > 1 (degree 3 for symmetric square L-functions), so that the Dirichlet coefficients of the L-functions do not form a completely multiplicative function. The loss of complete multiplicativity then makes the combinatorial analysis in the asymptotics of the moments somewhat more complicated than in the degree 1 case. Luo's work was extended and developed further in a recent series of paper by Royer and his collaborators who considered the first two symmetric power L-functions attached to holomorphic cusp forms with large squarefree level [30, 31, 32, 9]. In these papers, Royer et al. found beautiful combinatorial interpretations of the asymptotic value of the integral moments in terms of the

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numbers of Riordan and Dyck paths. These interpretations enabled them to evaluate the corresponding generating series and to provide precise estimates for the extreme values at $s = 1$. However these methods look somewhat ad-hoc and seem difficult to generalize to complex moments or to higher symmetric power L-functions.

The purpose of the present work is to propose a hopefully more conceptual approach to analyzing the distribution of values at $s = 1$ of automorphic L-functions for appropriate families of automorphic forms. We illustrate this by computing arbitrary complex moments of arbitrary symmetric power L-functions of holomorphic forms of large (prime) level (assuming that the corresponding L-functions are automorphic, as is predicted by the Langlands functoriality conjectures and effectively proved for the symmetric powers up to 4 [5, 15, 16, 14]) and provide a natural probabilistic interpretation of these computations. It will be clear that such an approach will generalize to quite arbitrary L-functions of appropriate families of automorphic forms (higher symmetric power L-functions of Maass forms with large eigenvalues as in [21] for example).

1.1. **Modular forms and their symmetric power L-functions.** For $q \geqslant 1$ a square-free integer, we denote by $S_2^p(q)$ the set of arithmetically normalized primitive holomorphic cusp forms for $\Gamma_0(q)$ of weight 2 with trivial nebentypus: any $f \in S_2^p(q)$ has a Fourier expansion at infinity of the form

$$f(z) = \sum_{n \geqslant 1} \lambda_f(n) \sqrt{n} e(nz)$$

with $\lambda_f(1) = 1$ and $\lambda_f(n)$ denoting the n-th eigenvalue of the (normalized) Hecke operator T_n; in particular $\lambda_f(n)$ is a multiplicative function. By a well known recipe, to each $f \in S_2^p(q)$ is associated an automorphic cuspidal representation π_f of $GL_2(\mathbf{A}_{\mathbf{Q}})$. This representation factors as a restricted tensor product of local GL_2 representations $\pi_f = \otimes_v' \pi_{f,v}$ (v running over all places of $\mathbf{Q}$) and has the following properties

- π_f has trivial central character and conductor q.
- $\pi_{f,\infty} \simeq D_2$ the discrete series representation of $GL_2(\mathbf{R})$ of weight 2.
- If p divides q, then $\pi_{f,p} \simeq \chi_p \otimes St_{2,p}$ where χ_p is an unramified character of order at most 2 and $St_{2,p}$ is the Steinberg representation of $GL_2(\mathbf{Q}_p)$; one has $\lambda_f(p) = \chi_p(p) p^{-1/2}$.
- If $p \not| \, q$, $\pi_{f,p}$ is an unramified principal series representation, and one associates to it a semi-simple $SL_2(\mathbf{C})$-conjugacy class
$$g_f^\natural(p) = \begin{pmatrix} \beta_{f,p} & 0 \\ 0 & \beta_{f,p}^{-1} \end{pmatrix}^\natural,$$
such that for any $\alpha \geqslant 0$,
$$\lambda_f(p^\alpha) = \sum_{i=0}^{\alpha} \beta_{f,p}^i \beta_{f,p}^{-(\alpha - i)} = \mathrm{tr}(\mathrm{Sym}^\alpha(g_f(p))),$$
where Sym^α denote the symmetric α-th power representation of the standard representation of GL_2.
- Moreover, by a result of Deligne (Eichler/Igusa in this case) $\pi_{f,p}$ is tempered: one has $|\beta_{f,p}| = 1$ or in other words $g_f(p) \in SU(2)$.

For $k \geqslant 1$ an integer, the symmetric k-th power L-function associated to $f \in S_2^p(q)$ is the Euler product of degree $k + 1$ given by

$$L(s, \mathrm{Sym}^k f) = \prod_p L_p(s, \mathrm{Sym}^k f) = \sum_{n \geqslant 1} \frac{\lambda_f^k(n)}{n^s},$$

with local factors given by

$$L_p(s,\mathrm{Sym}^k f)=\det(I-p^{-s}\mathrm{Sym}^k(g_f(p)))^{-1},$$

if $p\not| q$ and, if $p|q$, by

$$L_p(s,\mathrm{Sym}^k f)=(1-(\lambda_f(p))^k p^{-s})^{-1}=(1-\lambda_f(p^k)p^{-s})^{-1}. \tag{1.1}$$

This L-function is absolutely convergent and non-vanishing for $\Re e(s)>>1$ (in fact for $\Re e(s)>2$) and is conjectured to have holomorphic continuation to $\mathbf{C}$ with a functional equation relating $L(s,\mathrm{Sym}^k f)$ to $L(1-s,\mathrm{Sym}^k f)$.

Such a conjecture is sufficient to define $L(s,\mathrm{Sym}^k f)$ at $s=1$, but in order to be able to study the distribution of the values $L(1,\mathrm{Sym}^k f)$ as f varies over $S_2^p(q)$, we need to consider some stronger assumptions which are discussed in the rest of this section.

Our main assumption is the following: given $k\geqslant 1$ and $f\in S_2^p(q)$ with q square-free,

Hypothesis $\mathrm{Sym}^k(f)$ **.** *There exists a automorphic cuspidal self-dual representation, denoted by $\mathrm{Sym}^k\pi_f=\otimes'\mathrm{Sym}^k\pi_{f,v}$, of $GL_{k+1}(\mathbf{A_Q})$ whose local L-factors $L(s,\mathrm{Sym}^k\pi_{f,p})$ agree with the local factors $L_p(s,\mathrm{Sym}^k f)$ given above; more precisely*

- *At the infinite place*[1]*, the local L-factor of $\mathrm{Sym}^k\pi_{f,\infty}$ is given by*

$$L(s,\mathrm{Sym}^k\pi_{f,\infty})=\begin{cases}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)2^m\prod_{j=1}^m(2\pi)^{-(s+j)}\Gamma\left(s+j\right) & m \textit{ even}\\[2ex] \pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)2^m\prod_{j=1}^m(2\pi)^{-(s+j)}\Gamma\left(s+j\right) & m \textit{ odd}\end{cases}$$
$$=: L_\infty(s,\mathrm{Sym}^k f)$$

  if $k=2m$ is even, and

$$L(s,\mathrm{Sym}^k\pi_{f,\infty})=2^{m+1}\prod_{j=0}^{m}(2\pi)^{-(s+j+\frac12)}\Gamma\left(s+j+\frac{1}{2}\right)=:L_\infty(s,\mathrm{Sym}^k f)$$

  if $k=2m+1$ is odd.
- *For $p\not| q$, $\mathrm{Sym}^k\pi_{f,p}$ is an unramified principal series associated to the conjugacy class $\mathrm{Sym}^k g^\natural{}_f(p)$.*
- *For $p|q$, the local component $\mathrm{Sym}^k\pi_{f,p}$ is isomorphic to $(\chi_p)^k\otimes St_{k+1,p}$ where χ_p is the unramified character of order at most 2 appearing in $\pi_{f,p}$ and $St_{k+1,p}$ denotes the Steinberg representation of GL_{k+1}. Consequently the conductor of $\mathrm{Sym}^k\pi_f$ equals q^k and $L(s,\mathrm{Sym}^k f)=L(s,\mathrm{Sym}^k\pi_f)$ satisfies the functional equation*

$$L_\infty(s,\mathrm{Sym}^k f)L(s,\mathrm{Sym}^k f)=\varepsilon(\mathrm{Sym}^k f)q^{k(1-2s)/2}L_\infty(1-s,\mathrm{Sym}^k f)L(1-s,\mathrm{Sym}^k f), \tag{1.2}$$

  with

$$\varepsilon(\mathrm{Sym}^k f)=\varepsilon(\tfrac12,\mathrm{Sym}^k\pi_{f,\infty},\psi_\infty)\prod_{p|q}\varepsilon(\tfrac12,\mathrm{Sym}^k\pi_{f,p},\psi_p)$$
$$=\pm1\prod_{p|q}\left[(-1)^k\chi_p(p)^{k^2}\right]=\pm1.$$

[1]In this paper, the very precise form of the local factors at ∞ and at $p|q$ is not strictly necessary but for reference we carry out the explicit computation of these local factors and root numbers — via the local Langlands correspondence— in Section 3.

Remark 1. Hypothesis $\mathrm{Sym}^k(f)$ is known unconditionally for k up to 4: it is tautological if $k = 1$ and follows from the work of Gelbart/Jacquet for $k = 2$ and from the more recent works of Kim/Shahidi and Kim when $k = 3, 4$ [5, 15, 16, 14].

Also by combining these results with Rankin/Selberg theory, Kim/Shahidi established the functional equation (1.2) and the meromorphic continuation of $L(s, \mathrm{Sym}^k f)$ to $\mathbf{C}$ for $k = 5, \dots, 9$ and the holomorphy and non-vanishing of $L(s, \mathrm{Sym}^k f)$ in the half-plane $\Re e(s) \geqslant 1$, for $k = 5, \dots, 8$. In fact, the standard zero free regions for L-functions (à la Hadamard/de la Vallée-Poussin) show that for $k = 5, \dots, 8$, $L(s, \mathrm{Sym}^k f)$ is analytic in the slightly larger domain $\Re e(s) \geqslant 1 - A/\log(q(|\Im m(s)| + 2))$ for some absolute $A > 0$, except for a possible simple real pole (a Landau/Siegel pole).

Clearly Hypothesis $\mathrm{Sym}^k(f)$ is carries much more information than the analytic continuation and functional equation of $L(s, \mathrm{Sym}^k f)$ alone. From the automorphy of the $L(s, \mathrm{Sym}^k f)$ one can deduce three analytic ingredients:

- *Individual upper bounds*. A first (mostly technical) consequence of Hypothesis $\mathrm{Sym}^k(f)$ is the standard but useful individual upper bound (see Lemma 4.1)

$$L(1, \mathrm{Sym}^k f) \ll_k (\log q)^{k+1}. \tag{1.3}$$

- *Individual lower bounds*. We will also need lower bounds for $L(1, \mathrm{Sym}^k f)$, and these are usually furnished by non-trivial zero-free regions for $L(s, \mathrm{Sym}^k f)$: when $L(s, \mathrm{Sym}^k f)$ is automorphic, standard methods show ([22] for instance) that $L(s, \mathrm{Sym}^k f)$ has at most one zero in the domain

$$\{s : \Re e(s) \geqslant 1 - A_k/\log(q(|\Im m(s)| + 2))\}; \tag{1.4}$$

  moreover, this zero, whenever it exists is simple, real and is usually called the exceptional (or Landau/Siegel) zero. In fact, the exceptional zero whenever it exists is *unique* amongst *all* $L(s, \mathrm{Sym}^k f)$ for $f \in S_2^p(q)$. This is the content of the following Landau/Page type result which is a direct consequence of the multiplicity one type result given in Corollary 5.2 and of Theorem A of [11]:

  Lemma 1.1. *Given q squarefree such that Hypothesis $\mathrm{Sym}^k(f)$ holds for any $f \in S_2^p(q)$, there exists $A_k > 0$, depending on k only, and a set $S_{2,ex}^p(q) \subset S_2^p(q)$ with at most one element, such that for any $f \in S_2^p(q) \backslash S_{2,ex}^p(q)$, $L(s, \mathrm{Sym}^k f)$ has no zeros on the real interval $[1 - A_k/\log q, 1]$.*

  The exceptional zero being very rare, it won't be too harmful; but on some occasions (in particular to get cleaner statements) we will also consider the additional assumption that there is no exceptional zero at all:

  Hypothesis $\mathrm{LSZ}^k(q)$ **.** *There exists a constant A_k depending on k only such that $L(s, \mathrm{Sym}^k f)$ has no zeros on the real interval $[1 - A_k/\log q, 1]$. In other words $S_{2,ex}^p(q) = \emptyset$.*

  Remark 2. Remarkably (by comparison with the exceptional zero problem for L-functions of quadratic characters) Hypothesis $\mathrm{LSZ}^k(q)$ is known unconditionally for a few k's: for $k = 1, 2$ by the works of Hoffstein/Lockhart, Goldfeld/Hoffstein/Lieman and Hoffstein/-Ramakrishnan [10, 6, 11] and for $k = 4$ by the work of Ramakrishnan/Wang [27]. In fact, all their proofs use, in one way or another, some consequences of the automorphy of $\mathrm{Sym}^{k'}(f)$ for several $k' > k$ and so Hypothesis $\mathrm{LSZ}^k(q)$ is related to Hypothesis $\mathrm{Sym}^k(f)$. For instance, using the method of [11], it is not difficult to see that for $k > 2$, Hypothesis

$\mathrm{LSZ}^k(q)$ follows from the Hypotheses $\mathrm{Sym}^{k'}(f)$ for all $k' \leqslant k+2$ (even weaker assumptions are sufficient). The case $k = 3$, however, seems to escape a purely unconditional treatment; we explain below (Proposition 4.5) how to conclude the non-existence of an exceptional zero from the absence of a real pole too close to 1 for $L(s, \mathrm{Sym}^5 f)$ (which is a weak consequence of Hypothesis $\mathrm{Sym}^5(f)$).

In the present paper, the most useful consequence of $L(s, \mathrm{Sym}^k f)$ having no exceptional zero is the lower bound (see Lemma 4.2)

$$L(1, \mathrm{Sym}^k f) \gg_k (\log q)^{-C_k}, \tag{1.5}$$

for some constant $C_k > 0$ depending on k only.

- *Good approximation of* $L(1, \mathrm{Sym}^k f)$ *on average*. The last analytic ingredient needed is the most serious: to evaluate successfully the distribution of $L(1, \mathrm{Sym}^k f)$ on average, we require approximations of these L-values by very short Dirichlet polynomials. Such approximations cannot be reached with standard zero free regions like (1.4), but could be obtained with the Generalized Riemann Hypothesis. Fortunately, we will not need to make such a strong assumption: classical methods from analytic number theory are capable of providing unconditional substitutes which are as strong as GRH (for the present purpose at least). These methods build on density estimates for zeros in families of L-functions (see Proposition 5.3) and use Rankin/Selberg theory for pairs of automorphic representations in $\{\mathrm{Sym}^k \pi_f \mid f \in S_2^p(q)\}$; so the automorphy of the $L(s, \mathrm{Sym}^k f)$ is used very strongly at this point.

1.2. **Moments of symmetric power L-functions.** For simplicity, we now restrict to forms of prime level q. Our goal will be to evaluate $L(1, \mathrm{Sym}^k f)^z$, for some $z \in \mathbf{C}$, on average over $f \in S_2^p(q)$.

Given a sequence $(\alpha_f)_{f \in S_2^p(q)}$, the *harmonic* average is defined as the sum

$$\sum_{f \in S_2^p(q)}^{h} \alpha_f = \sum_{f \in S_2^p(q)} \frac{\alpha_f}{4\pi \langle f, f \rangle}$$

and if $S \subset S_2^p(q)$ then we will let $|S|_h$ denote the "harmonic measure" of S, that is,

$$|S|_h = \sum_{f \in S}^{h} 1.$$

Such averaging is natural in view of the following consequence of the Petersson trace formula (Proposition 1.4 below)

$$\sum_{f \in S_2^p(q)}^{h} 1 = |S_2^p(q)|_h = 1 + O\left(\frac{\log(q)}{q^{3/2}}\right).$$

This shows that the weights $1/(4\pi\langle f, f\rangle)$ define asymptotically a probability measure on $S_2^p(q)$ when q is prime.

Remark 3. In fact from the well known relations between $\langle f, f\rangle$ and $L(1, \mathrm{Sym}^2 f)$ [12] and from the upper and lower bounds (1.3) and (1.5) for $L(1, \mathrm{Sym}^2 f)$ (with $C_2 = 1$ [6]), one has

$$q(\log q)^{-1} \ll 4\pi\langle f, f\rangle \ll q(\log q)^3 \tag{1.6}$$

so that the harmonic weight $1/(4\pi\langle f, f\rangle)$ is not far from the natural weight $1/|S_2^p(q)|$. In fact there is a procedure to convert a harmonic type average into a uniform one (see [18, 12, 30]), however for simplicity we consider only the harmonic averaging.

We consider now the probability space $S_2^p(q)$, with each f weighted by $(4\pi\langle f,f\rangle |S_2^p(q)|_h)^{-1}$, and the Random variable $f\to L(1,\mathrm{Sym}^k f)$. Given $z\in\mathbf{C}$, our main result (Theorem 1.2 below) consists in computing asymptotically the z-moment of this variable (ie. the expectation of $f\to L(1,\mathrm{Sym}^k f)^z$),

$$L_q^z(1,\mathrm{Sym}^k) := \frac{1}{|S_2^p(q)|_h}\sum_{f\in S_2^p(q)}^h L(1,\mathrm{Sym}^k f)^z$$

(say) as $q\to+\infty$ over the primes.

To describe the answer, we consider the compact group $G=SU(2)$ endowed with its natural Haar measure μ_G; we then let $G^\natural$ be the set of conjugacy classes of G endowed with the Sato/Tate measure μ_{st} (i.e. the direct image of μ_G by the canonical projection). By Weyl's integration formula, the map

$$\theta\to g(\theta)^\natural = \begin{pmatrix} e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix}^\natural$$

identifies $G^\natural$ with the interval $[0,\pi]$ and μ_{st} with the measure

$$d\mu_{st}(\theta)=\frac{2}{\pi}\sin^2(\theta)d\theta.$$

For z,s in $\mathbf{C}$, such that $\Re e(s)>0$ we set

$$L^z(s,\mathrm{Sym}^k)=\prod_p L_p^z(s,\mathrm{Sym}^k),$$

with

$$\begin{aligned} L_p^z(s,\mathrm{Sym}^k) &= \int_{G^\natural}\det(I-p^{-s}\mathrm{Sym}^k(g^\natural))^{-z}d\mu_{st}(g^\natural)\\ &= \int_{[0,\pi]}\det(I-p^{-s}\mathrm{Sym}^k(g(\theta)))^{-z}d\mu_{st}(\theta). \end{aligned}$$

For $\Re e(s)>1/2$, this product is absolutely convergent and has the following probabilistic interpretation. Consider (Ω,μ) a probability space and $\{{g^\natural}_p(\omega)\}_{p\geqslant 2}$ a sequence of independent random variables indexed by the prime numbers, with values in $G^\natural$ and distributed according to the measure μ_{st}. For $\Re e(s)>1/2$, the Euler product

$$L(s,\mathrm{Sym}^k,\omega)^z=\prod_p \det(I-p^{-s}\mathrm{Sym}^k {g^\natural}_p(\omega))^{-z}$$

turns out to be absolutely convergent a.s. and then

$$\mathbf{E}(L(s,\mathrm{Sym}^k,\cdot)^z)=\int_\Omega L(s,\mathrm{Sym}^k,\omega)^z d\mu(\omega)=\prod_p \mathbf{E}(L_p(s,\mathrm{Sym}^k,\cdot)^z)=L^z(s,\mathrm{Sym}^k).$$

Theorem 1.2. *Let $k\geqslant 1$ be an integer and q be a prime such that Hypothesis $\mathrm{Sym}^k(f)$ holds for all $f\in S_2^p(q)$ and Hypothesis $\mathrm{LSZ}^k(q)$ holds. Then there exists $C=C(k)>0$ and $\delta=\delta(k)>0$ such that for any complex number z satisfying $|z|\leqslant C\log q/(\log_3 q\log_2 q)$, one has*

$$L_q^z(1,\mathrm{Sym}^k)=L^z(1,\mathrm{Sym}^k)+O_k\left(\exp\left(-\delta\frac{\log q}{\log_2 q}\right)\right), \tag{1.7}$$

the implied constant depending on k only.

Above and below we denote by $\log_2 q = \log(\log q)$ and more generally $\log_r(q) = \log(\log(\dots))$ the r-th iterated logarithm.

Remark 4. Observe that for $k = 1, 2, 4$, Hypothesis $\mathrm{Sym}^k(f)$ holds for all $f \in S_2^p(q)$ and that Hypothesis $\mathrm{LSZ}^k(q)$ holds as well (cf. Remarks 1 and 2) so (1.7) is unconditional for such k. Moreover, Hypothesis $\mathrm{LSZ}^k(q)$ is used only to cover the case when $\Re e(z)$ is negative and large, so if we restrict to z such that $\Re e z \geqslant 0$, (1.7) is unconditional for $k = 3$ as well. More generally, one can dispose completely of Hypothesis $\mathrm{LSZ}^k(q)$ at the expense of removing (at most) one exceptional f : replacing the use of Hypothesis $\mathrm{LSZ}^k(q)$ by Lemma 1.1, one has the following theorem.

Theorem 1.3. *Given $k \geqslant 1$ an integer and q a prime such that Hypothesis $\mathrm{Sym}^k(f)$ holds for any $f \in S_2^p(q)$, there exists $C = C(k) > 0$ and $\delta = \delta(k) > 0$ and a set $S_{2,ex}^p(q) \subset S_2^p(q)$ with at most one element such that, for any complex number z satisfying $|z| \leqslant C \log q/(\log_3 q \log_2 q)$, one has*

$$\frac{1}{|S_2^p(q)\backslash S_{2,ex}^p(q)|_h} \sum_{f\in S_2^p(q)\backslash S_{2,ex}^p(q)}^h L(1, \mathrm{Sym}^k f)^z = L^z(1, \mathrm{Sym}^k) + O_k\left(\exp\left(-\delta\frac{\log q}{\log_2 q}\right)\right),$$

the implied constant depending on k only.

In particular the latter result is unconditional also for $k = 3$. As it's proof is very similar to that of Theorem 1.2 we will not give it here.

Remark 5. One key ingredient of the proof of Theorems 1.2 and 1.3 is the Petersson trace formula (see [12]) and one of its consequences:

Proposition 1.4. *For q a prime and $n \geqslant 1$, one has*

$$\sum_{f\in S_2^p(q)}^h \lambda_f(n) = \sum_{f\in S_2^p(q)} \frac{\lambda_f(n)}{4\pi\langle f, f\rangle} = \delta_{n,1} + O\left(\frac{\log(qn)n^{1/2}}{q^{3/2}}\right).$$

In the sequel it will be useful to rewrite this formula in the normalized form

$$\frac{1}{|S_2^p(q)|_h} \sum_{f\in S_2^p(q)}^h \lambda_f(n) = \delta_{n,1} + O\left(\frac{\log(qn)n^{1/2}}{q^{3/2}}\right). \tag{1.8}$$

The formula (1.8) can be interpreted as follows: writing the prime factorization of n as $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, one has the identity

$$\lambda_f(n) = \lambda_f(p_1^{\alpha_1}) \cdots \lambda_f(p_r^{\alpha_r}) = \mathrm{tr}(\mathrm{Sym}^{\alpha_1}(g_f(p_1))) \cdots \mathrm{tr}(\mathrm{Sym}^{\alpha_r}(g_f(p_r))).$$

Fix now $r \geqslant 1$ and $p_1, \dots, p_r$, r distinct prime numbers. Then by the identity above, by the Peter/Weyl theorem and by Weyl's equidistribution criterion, the equality (1.8) applied to integers n divisible only by primes in $\{p_1, \dots, p_r\}$ yields the equidistribution of the r-tuple of conjugacy classes $\{(g_f^\natural(p_1), \dots, g_f^\natural(p_r))\}_{f\in S_2^p(q)}$ (appropriately weighted by $1/4\pi\langle f, f\rangle$) into the product of r copies of $G^\natural$ as $q \to +\infty$ over the primes. This is a variant of the "Vertical" Sato/Tate Law; we refer to [34, 1, 29] for other applications of this law. Varying r and the set of primes $\{p_1, \dots, p_r\}$, one can even view this as the equidistribution (in an appropriate sense), as $q \to +\infty$, of the family of tuples of conjugacy classes

$$\{(g_f^\natural(2), g_f^\natural(3), \ \dots, \ g_f^\natural(p), \dots)\}_{f\in S_2^p(q)}$$

inside the "infinite solenoid" $(SU(2)^\natural)^{\mathcal{P}}$ indexed by the set of all primes $\mathcal{P}$.

Now, $s=1$ being at the limit of the zone of absolute convergence, the following factorization is "almost" valid
$$L(1,\mathrm{Sym}^k f)^z=\prod_p \det(I-p^{-1}\mathrm{Sym}^k(g^\natural{}_f(p)))^{-z},$$
and then, by the equidistribution law described above, it is natural to expect that the average of the $L(1,\mathrm{Sym}^k f)^z$ converges to $L^z(1,\mathrm{Sym}^k)$.

Remark 6. From the definition of the symmetric k-th power of a diagonal matrix, the local factor $L_p^z(s,\mathrm{Sym}^k)$ has an elementary expression
$$L_p^z(s,\mathrm{Sym}^k)=\left(1-\frac{1}{p^s}\right)^{-z\delta_{2|k}}\int_{[0,\pi]}\prod_{0\leqslant j<k/2}\left(1-\frac{2\cos((k-2j)\theta)}{p^s}+\frac{1}{p^{2s}}\right)^{-z}\frac{2}{\pi}\sin^2\theta d\theta,$$
where $\delta_{2|k}=1$ if k is even and 0 otherwise. In particular, for z an integer, one has
$$L_p^z(s,\mathrm{Sym}^k)=F_{k,z}(p^{-s})$$
where $F_{k,z}(X)\in\mathbf{Q}(X)$ is a rational function (depending on k and z) with rational coefficients, satisfying $F_{k,z}(0)=1$ and with its poles located at k-th roots of unity; moreover, if z is a negative integer, $F_{k,z}(X)$ a self-reciprocal (palindromic) polynomial. These facts are easy consequence of the change of variable $u=e^{i\theta}$ and of the residue theorem. This generalizes to any integer k (for which Hypothesis $\mathrm{Sym}^k(f)$ holds for all $f\in S_2^p(q)$) results of Royer and his collaborators [30, 31, 32, 9], obtained in the case $k=1,2$ by somewhat ad-hoc methods.

Remark 7. Specializing to $k=1,2$ one has
$$\begin{aligned}L_p^z(s,\mathrm{Sym}^1)&=\int_{[0,\pi]}\left(1-\frac{2\cos(\theta)}{p^s}+\frac{1}{p^{2s}}\right)^{-z}\frac{2}{\pi}\sin^2\theta d\theta\\&=\frac{16}{\pi}\int_{[0,\pi/2]}\left(1-\frac{2(1-2\sin^2\theta)}{p^s}+\frac{1}{p^{2s}}\right)^{-z}\sin^2\theta\cos^2\theta d\theta\end{aligned}$$
on making the change of variable $\theta'=\theta/2$ and
$$\begin{aligned}L_p^z(s,\mathrm{Sym}^2)&=\left(1-\frac{1}{p^s}\right)^{-z}\int_{[0,\pi/2]}\left(1-\frac{2\cos(2\theta)}{p^s}+\frac{1}{p^{2s}}\right)^{-z}\frac{4}{\pi}\sin^2\theta d\theta\\&=\left(1-\frac{1}{p^s}\right)^{-z}\frac{4}{\pi}\int_{[0,\pi/2]}\left(1-\frac{2(1-2\sin^2\theta)}{p^s}+\frac{1}{p^{2s}}\right)^{-z}\sin^2\theta d\theta.\end{aligned}$$
Changing the variable to $u=\sin^2\theta$ one gets
$$\begin{aligned}L_p^z(s,\mathrm{Sym}^1)&=(1-X)^{-2z}\frac{8}{\pi}\int_{[0,1]}\left(1+\frac{4X}{(1-X)^2}u\right)^{-z}u^{1/2}(1-u)^{1/2}du\\&=(1-X)^{-2z}F\left(z,\frac{3}{2},3;-\frac{4X}{(1-X)^2}\right),\end{aligned}$$
and
$$\begin{aligned}L_p^z(s,\mathrm{Sym}^2)&=(1-X)^{-3z}\frac{2}{\pi}\int_{[0,1]}\left(1+\frac{4X}{(1-X)^2}u\right)^{-z}u^{1/2}(1-u)^{-1/2}du\\&=(1-X)^{-3z}F\left(z,\frac{3}{2},2;-\frac{4X}{(1-X)^2}\right),\end{aligned}$$

where $X=p^{-s}$ and

$$F(a,b,c;d)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_{[0,1]}x^{b-1}(1-u)^{c-b-1}(1-du)^{-a}du$$

denotes the hypergeometric function. Again, we retrieve the results of [29, 32, 9] which where established for z an integer.

We prove two estimates concerning the behavior of $L^z(1,\mathrm{Sym}^k)$ as $|z|\to+\infty$.

Theorem 1.5. *For real $r>0$, one has for $r\to+\infty$,*

$$\log L^{\pm r}(1,\mathrm{Sym}^k)=\mathrm{Sym}^k_{\pm}r\log\log r+\mathrm{Sym}^{k,1}_{\pm}r+O(r/\log r), \tag{1.9}$$

where $\mathrm{Sym}^k_{\pm},\ \mathrm{Sym}^{k,1}_{\pm}$ are constants depending on k only. More precisely one has

$$\mathrm{Sym}^k_{+}=k+1\quad\textit{and}\quad\mathrm{Sym}^{k,1}_{+}=(k+1)\gamma;$$

moreover, if k is odd

$$\mathrm{Sym}^k_{-}=k+1\quad\textit{and}\quad\mathrm{Sym}^{k,1}_{-}=(k+1)(\gamma-\log\zeta(2)),$$

while

$$\mathrm{Sym}^2_{-}=1\quad\textit{and}\quad\mathrm{Sym}^{2,1}_{-}=(\gamma-2\log\zeta(2)).$$

More generally one has, for any k,

$$\mathrm{Sym}^k_{\pm}=\max_{g\in SU(2)}\pm\mathrm{tr}(\mathrm{Sym}^k g)>0.$$

For $t\in\mathbf{R}$, one has, as $|t|\to+\infty$,

$$L^{it}(1,\mathrm{Sym}^k)\leqslant\exp\left(-c_k\frac{|t|}{\log^2|t|}\right) \tag{1.10}$$

for some constant $c_k>0$ depending on k only.

Taking $r=C(k)\frac{\log q}{\log_2 q\log_3 q}$ in (1.9), we infer the following

Corollary 1.6. *Under the assumptions of Theorem 1.2, there exists $f,g\in S_2^p(q)$ such that*

$$L(1,\mathrm{Sym}^k f)\geqslant(e^{\gamma})^{k+1}(\log\log q)^{k+1}(1+o_k(1)),$$
$$L(1,\mathrm{Sym}^k g)\leqslant e^{-\mathrm{Sym}^{k,1}_{-}}(\log\log q)^{-\mathrm{Sym}^k_{-}}(1+o_k(1)).$$

Remark 8. These bounds are (probably) best possible (up to the multiplicative constant): under GRH for $L(s,\mathrm{Sym}^k f)$, one has

$$(\log\log q)^{-\mathrm{Sym}^k_{-}}\ll_k L(1,\mathrm{Sym}^k f)\ll_k(\log\log q)^{k+1};$$

indeed by a standard argument of Littlewood (see [2] Proposition 5), one has (under GRH)

$$\log\left(L(1,\mathrm{Sym}^k f)^{-1}\right)=\sum_{p\leqslant\log^{1/2}(q^k)}\frac{\lambda_f^k(p)}{p}+O_k(1)$$

and the bounds follows since

$$-\mathrm{Sym}^k_{-}\leqslant\lambda_f^k(p)=\mathrm{tr}(\mathrm{Sym}^k g_f(p))\leqslant\mathrm{Sym}^k_{+}=k+1.$$

Remark 9. In view of Remark 4 and Theorem 1.3 the Corollary 1.6 is unconditional if $k=1,2,3$ or 4.

For $t \in \mathbf{R}$, the map $t \to L^{it}(1, \mathrm{Sym}^k)$ is the characteristic function of the random variable $\omega \to \log L(1, \mathrm{Sym}^k, \omega)$. We denote by $F(\mathrm{Sym}^k, x)$ the distribution function of $\log L(1, \mathrm{Sym}^k, \omega)$:

$$F(\mathrm{Sym}^k, x) := Prob(\{L(1, \mathrm{Sym}^k, \omega)\} \leqslant e^x\}).$$

Similarly $t \to L_q^{it}(1, \mathrm{Sym}^k)$ can be interpreted as the characteristic function of the random variable $f \to \log L(1, \mathrm{Sym}^k f)$ on $S_2^p(q)$, each f being weighted by $(4\pi\langle f, f\rangle |S_2^p(q)|_h)^{-1}$. We denote by

$$F_q(\mathrm{Sym}^k, x) := \frac{1}{|S_2^p(q)|_h} \sum_{\substack{f \in S_2^p(q) \\ L(1, \mathrm{Sym}^k f) \leqslant e^x}}^h 1$$

the corresponding distribution function.

The rapid decay of $L^{it}(1, \mathrm{Sym}^k)$ given in (1.10) implies that $F(\mathrm{Sym}^k, x)$ is smooth with uniformly bounded derivative, hence by Theorem 1.2 and the Berry-Esseen inequality, we obtain that the distribution function $F_q(\mathrm{Sym}^k, x)$ converges to $F(\mathrm{Sym}^k, x)$ uniformly for $x \in \mathbf{R}$; more precisely

Corollary 1.7. *Given q a prime such that Hypothesis $\mathrm{Sym}^k(f)$ and Hypothesis $\mathrm{LSZ}^k(q)$ are satisfied for any $f \in S_2^p(q)$, one has, uniformly for $x \in \mathbf{R}$,*

$$F_q(\mathrm{Sym}^k, x) = F(\mathrm{Sym}^k, x) + O_k\left(\frac{\log_3 q \log_2 q}{\log q}\right).$$

1.3. Further comments.

- We have seen from Remark 5 that the main reason explaining the asymptotic formula (1.7) is the equidistribution property of the conjugacy classes $\{g_f^\natural(p)\}_{f \in S_2^p(q)}$ in $G^\natural$ as $q \to +\infty$ which is proved via the Petersson formula (or via the Selberg trace formula as in [34] if one is interested in the natural averaging). Of course, equidistribution makes sense only under the Ramanujan/Petersson conjecture. Using this conjecture as a guide one sees that, in the present case, the Petersson/Kuznetzov formula is sufficient to prove (1.7) without assuming the Ramanujan/Petersson Conjecture. In particular the above is valid for the case treated by Luo [21] of Maass forms with large Laplace eigenvalue; however, the price to pay to avoid the Ramanujan/Petersson conjecture is that, apparently, the analog of (1.7) is valid only for $z \in \mathbf{C}$ *fixed* (i. e. without uniformity with respect to the parameters). More generally, we see that the method presented here would enable one to compute the complex moments at $s = 1$ of general families of automorphic L-functions. For this one requires a "trace" formula for the family expressing the equidistribution of the conjugacy classes of the corresponding automorphic forms inside some $G^\natural$, where G is a compact group whose representation theory is sufficiently well understood.
- In term of uniformity in q, our Theorem 1.2 is an (slightly stronger) analog of [8] Theorem 2, which was obtained for the family of L-functions of quadratic characters. In this case the corresponding underlying group is $\mathbf{Z}/2\mathbf{Z}$ whose representation theory is trivial. In fact, for this peculiar setting, Granville and Soundararajan obtained several results which are far more precise than the ones presented here:

  The first ones concerns the study of the behavior of the corresponding distribution function $F(\mathbf{Z}/2\mathbf{Z}, x)$ when $|x| \to +\infty$. In our present context it would probably be interesting to have more precise results about the behavior of $F(\mathrm{Sym}^k, x)$: this amounts to evaluating more precisely $L^z(1, \mathrm{Sym}^k)$ as $|z| \to +\infty$. The latter can be done by a more sophisticated use of the stationary phase method along with some basic facts from the representation theory of $SU(2)$ (or for $k = 1, 2$ by using the closed formulas of Remark 7). In fact, one should even be able to develop general arguments valid when $(\mathrm{Sym}^k, SU(2))$ is replaced by any (ρ, G) where G is a general compact connected Lie group and ρ is an irreducible finite dimensional representation of G (for instance Theorem 1.5 can be extended to this more general setting without any particular difficulty).

The second type of sharp results obtained in [8] concern the study of extreme values of the corresponding L-functions at $s = 1$ (in some sense the precise shape of the analog of the term $(1 + o_k(1))$ in Corollary 1.6). To proceed, the authors use quite delicate techniques which exploit several peculiar features of the quadratic characters: the quadratic reciprocity law, Graham/Ringrose estimates for short characters sums of highly composite moduli. We do not see the analog of these features in the case of modular forms; however there is certainly room for improvements and it would be interesting to try to get, as close as possible, the analogs of the conjectures of Montgomery/Vaughan on extreme values at $s = 1$ of families of quadratic characters L-functions [23] (which have been partially proven in [8]). For the interested reader, we note that at least one feature might be transposed to the context of symmetric power L-functions of modular forms: namely the large sieve type inequalities for large powers of sums of Hecke eigenvalues of the form

$$\sum_{f \in S_2^p(q)}^{h} \Big| \sum_{n \leqslant N} a_n \frac{\lambda_f^k(n)}{n} \Big|^{2l}$$

where the (a_n) are arbitrary complex coefficients and $l \geqslant 1$ is an integer. In the case of characters, the complete multiplicativity readily converts such sums into a standard quadratic type sum. The case of coefficient attached to symmetric power lifts seem more problematic, however for the symmetric second power, analog of such large sieve inequalities have been developed in [18] for different yet somewhat related purposes.

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2. Remarks on representations of compact groups

Let G be a compact group and ρ be an irreducible finite dimensional representation of G of dimension d. Since G is compact, for any $g \in G$, the eigenvalues of $\rho(g)$ have modulus 1; in particular

$$D(X, \rho, g) := \det(I - X.\rho(g))^{-1}$$

(the inverse of the characteristic polynomial of $\rho(g)$) is holomorphic and non-vanishing in the complex disc $D_{<1} := \{X \in \mathbf{C},\ |X| < 1\}$. Consequently, for any complex number z,

$$D(X, \rho, g)^z := \det(I - X.\rho(g))^{-z} = \exp(z \log D(X, \rho, g))$$

is holomorphic for $|X| < 1$. We consider the Taylor expansion of $D(X, \rho, g)^z$ near $X = 0$:

$$D(X, \rho, g)^z = \sum_{\alpha \geqslant 0} \lambda_\rho^{z,\alpha}(g) X^\alpha. \tag{2.1}$$

From the standard identity

$$\log\det(I-X.\rho(g))^{-1}=\sum_{\alpha\geqslant 1}\frac{\mathrm{tr}(\rho(g^{\alpha}))}{\alpha}X^{\alpha}$$

and the trivial bound

$$|\mathrm{tr}(\rho(g^{\alpha}))|\leqslant d=\mathrm{tr}(\rho(e_G))$$

it follows that $|\lambda_{\rho}^{z,\alpha}(g)|$ is bounded by the α-th coefficient in the Taylor expansion at $X=0$ of

$$\exp\left(d|z|\sum_{\alpha\geqslant 1}\frac{X^{\alpha}}{\alpha}\right)=D(X,\rho,e_G)^{|z|}=(1-X)^{-d|z|};\tag{2.2}$$

in other words one has

$$|\lambda_{\rho}^{z,\alpha}(g)|\leqslant\frac{\Gamma(d|z|+\alpha)}{\Gamma(d|z|)(\alpha!)}=:\lambda_{\rho}^{|z|,\alpha}(e_G)=\lambda_{\mathbf{1}}^{d|z|,\alpha}(e_G),\tag{2.3}$$

(here $\mathbf{1}$ denotes the trivial representation). One deduces from these bounds that the Taylor expansion (2.1) is convergent for all $X\in D_{<1}$, that for such X one has

$$|D(X,\rho,g)^{z}|\leqslant(1-|X|)^{-d|z|},$$

and that for any $A\geqslant -1$

$$D(X,\rho,g)^{z}=\sum_{\alpha\leqslant A}\lambda_{\rho}^{z,\alpha}(g)X^{\alpha}+O_A\big((d|z||X|)^{A+1}\big)\tag{2.4}$$

uniformly for $|X|\leqslant 1/(d|z|)$.

We denote by $R\langle G\rangle$ the (sub-)ring of central functions on G generated by the characters of G; abusing notation we will consider freely such functions as functions on the set of conjugacy classes $G^{\natural}$. $R\langle G\rangle$ is equipped with the inner product

$$\langle F,G\rangle=\int_G F(g)\overline{G(g)}dg=\int_{G^{\natural}}F(g^{\natural})\overline{G(g^{\natural})}dg^{\natural}.$$

From the identity

$$D(X,\rho,g)^{z}=\exp\left(z\log\Big(\sum_{\alpha\geqslant 0}\mathrm{tr}(\mathrm{Sym}^{\alpha}\rho(g))X^{\alpha}\Big)\right),$$

where we denote by Sym^{α} the α-th symmetric power representation of $GL_d(\mathbf{C})$, it is clear that

$$\lambda_{\rho}^{z,\alpha}(\cdot)\in R\langle G\rangle\otimes_{\mathbf{Z}}\mathbf{Q}[z].$$

For instance

$$\lambda^{z,0}=1,\ \lambda_{\rho}^{z,1}(g)=z\mathrm{tr}\rho(g),\ \lambda_{\rho}^{z,2}(g)=\tfrac{1}{2}z(z-1)\mathrm{tr}\rho^{\otimes 2}(g)+z\mathrm{tr}(\mathrm{Sym}^{2}\rho(g));\tag{2.5}$$

more generality we define the coefficients $\mu_{\rho,\rho'}^{z,\alpha}$ by the formulas

$$\lambda_{\rho}^{z,\alpha}(g)=\sum_{\rho'}\mu_{\rho,\rho'}^{z,\alpha}\mathrm{tr}\rho'(g)$$

where ρ' ranges over the irreducible representations of G (note that the above sum is finite); we can view these quantities as the multiplicities of ρ' in the virtual representation with character

$g \to \lambda_\rho^{z,\alpha}(g)$. These multiplicities (which are polynomial in z with rational coefficients) are given by

$$\mu^{z,\alpha}_{\rho,\rho'} = \langle \lambda^{z,\alpha}_\rho, \mathrm{tr}\rho' \rangle = \int_G \lambda_\rho^{z,\alpha}(g)\overline{\mathrm{tr}\rho'(g)}dg.$$

We have the following bound on the multiplicities

Proposition 2.1. *Let G be either a finite or a compact connected Lie group and let ρ be an irreducible d dimensional representation of G. For any $\alpha \geqslant 0$ and $z \in \mathbf{C}$ one has,*

$$\sum_{\rho'} |\mu^{z,\alpha}_{\rho,\rho'}| \leqslant A(\alpha+1)^{r/2}\lambda_{\mathbf{1}}^{d|z|,\alpha}(e_G);$$

here $A > 0$ is a constant depending on the pair (G, ρ) only and r denotes the rank of G (if G is finite, one sets $r = 0$).

Proof. By Cauchy's inequality

$$\sum_{\rho'} |\mu^{z,\alpha}_{\rho,\rho'}| \leqslant (\sum_{\rho'} |\mu^{z,\alpha}_{\rho,\rho'}|^2)^{1/2}(\sum_{\rho'} \delta(\mu^{z,\alpha}_{\rho,\rho'} \neq 0))^{1/2},$$

with $\delta(\mu^{z,\alpha}_{\rho,\rho'} \neq 0) = 1$ if $\mu^{z,\alpha}_{\rho,\rho'} \neq 0$ and 0 otherwise. By Plancherel, one has

$$\sum_{\rho'} |\mu^{z,\alpha}_{\rho,\rho'}|^2 = \int_G |\lambda_\rho^{z,\alpha}(g)|^2 dg \leqslant \left|\lambda_\rho^{|z|,\alpha}(e_G)\right|^2 = \left|\lambda_{\mathbf{1}}^{d|z|,\alpha}(e_G)\right|^2. \tag{2.6}$$

It remains to bound $\sum_{\rho'} \delta(\mu^{z,\alpha}_{\rho,\rho'} \neq 0)$. If G is finite, this is bounded by the number (A say) of irreducible representations of G. From now on, we assume that G is a connected compact Lie group of rank r; we denote by $(\Phi, V, \langle\ ,\ \rangle)$ the root system of G with its real inner product space (we set $|\lambda| = \langle \lambda, \lambda\rangle^{1/2}$); the choice of a lexicographic ordering on a basis of V determines a set of positive roots and a system of simple roots, Δ say. We denote by $L \subset V$ the root lattice of V (the lattice of algebraically integral weights), and $\lambda_\rho \in L$ the highest weight of ρ with respect to Δ.

By construction $\lambda_\rho^{z,\alpha}$ is given by a linear combination (with coefficients in $\mathbf{Q}[z]$) of the characters of representations of the form

$$\mathrm{Sym}^{\alpha_1}(\rho) \otimes \cdots \otimes \mathrm{Sym}^{\alpha_k}(\rho)$$

with $\alpha_1 + \alpha_2 + \cdots + \alpha_k = \alpha$. Such representations are subrepresentations of the α-th tensor product representation $(\rho)^{\otimes^\alpha}$, hence $\sum_{\rho'} \delta(\mu^{z,\alpha}_{\rho,\rho'} \neq 0)$ is bounded by the number of irreducible representations occurring in $(\rho)^{\otimes^\alpha}$.

To bound this last number, we observe that the weights of $(\rho)^{\otimes^\alpha}$ are the linear integral combinations of the form

$$\sum_{\lambda'} \alpha_{\lambda'}\lambda'$$

where λ' ranges over the weights of ρ and the $\alpha_{\lambda'}$ are non-negative integers such that $\sum_{\lambda'} \alpha_{\lambda'} = \alpha$. Moreover, one has for any λ'

$$|\lambda'| \leqslant |\lambda_\rho|.$$

It follows that the norm of any weight of $(\rho)^{\otimes^\alpha}$ is bounded by $\alpha|\lambda_\rho|$; in particular, this is the case of any of the highest weights of the irreducible representations occurring in $(\rho)^{\otimes^\alpha}$. Hence the number of irreducible representations occurring in $\alpha\lambda_\rho$, is bounded by the number of points in L with norm bounded by[2] $\alpha|\lambda_\rho|$. Since the number of such lattice points in bounded by

$$A(\alpha|\lambda_\rho| + 1)^r$$

[2]in fact such points are also contained in the positive cone defined by Δ but we won't need this here

for some A depending on L, the proof of the proposition follows. $\square$

Finally, we define for $|X|<1$ the following z-moment

$$D^z(X,\rho):=\int_G D(X,\rho,g)^z dg=\int_{G^\natural} D(X,\rho,g^\natural)^z dg^\natural;$$

taking the Taylor expansion (2.4), we have

$$D^z(X,\rho)=:\sum_{\alpha\geqslant 0}\lambda_\rho^{z,\alpha}X^\alpha$$

say with

$$\lambda_\rho^{z,\alpha}=\int_G \lambda_\rho^{z,\alpha}(g)dg=\mu_{\rho,\mathbf{1}}^{z,\alpha}.$$

In particular, when ρ is not the trivial representation, one has from (2.5)

$$\lambda_\rho^{z,0}=1,\ \lambda_\rho^{z,1}=0,\ \lambda_\rho^{z,2}=\tfrac{1}{2}\left[(\mathrm{FrSc}(\rho)z)^2+\mathrm{FrSc}(\rho)z\right] \tag{2.7}$$

where

$$\mathrm{FrSc}(\rho)=\int_G[\mathrm{tr}(\mathrm{Sym}^2\rho(g))-\mathrm{tr}(\wedge^2\rho(g))]dg$$

is the Frobenius-Schur indicator of ρ, i.e.,

$$\mathrm{FrSc}(\rho)=\begin{cases}0 & \text{if } \rho \text{ is not self-dual}\\ 1 & \text{if } \rho \text{ is orthogonally self-dual}\\ -1 & \text{if } \rho \text{ is symplectically self-dual}\end{cases}.$$

From these expressions and (2.3) , one has

$$|\lambda_\rho^{z,\alpha}|\leqslant\frac{\Gamma(d|z|+\alpha)}{\Gamma(d|z|)(\alpha!)}=\lambda_{\mathbf{1}}^{d|z|,\alpha}, \tag{2.8}$$

so that one has

$$D^z(X,\rho)=\sum_{\alpha\leqslant A}\lambda_\rho^{z,\alpha}X^\alpha+O_A((d|z||X|)^{A+1}) \tag{2.9}$$

uniformly for $d|z|X\leqslant 1$.

2.1. **Euler products.** We define several arithmetic multiplicative functions by their values on primes powers p^α:

$$\lambda_{\rho,g}^z(p^\alpha)=\lambda_\rho^{z,\alpha}(g),\ \lambda_\rho^z(p^\alpha)=\lambda_\rho^{z,\alpha},$$
$$\mu_\rho^z(p^\alpha)=\mu_{\rho,\mathbf{1}}^{z,\alpha}=\lambda_\rho^z(p^\alpha).$$

When $\rho=\mathbf{1}$ is the trivial representation we use the notation

$$\lambda_{\mathbf{1}}^z(n)=\lambda_{\mathbf{1},g}^z(n)=:d_z(n)$$

since for z a positive integer, $d_z(n)$ equals the standard z-th divisor function $\sum_{d_1\dots d_z=n}1$. With these notations (2.3) and (2.8) can be rewritten as

$$|\lambda_{\rho,g}^z(n)|\leqslant d_{d|z|}(n)\quad\text{and}\quad|\lambda_\rho^z(n)|\leqslant d_{d|z|}(n). \tag{2.10}$$

For $\Re e(s) > 0$, we also define the local factors

$$L_p(s,\rho,g)^z = \sum_{\alpha\geqslant 0} \lambda^z_{\rho,g}(p^\alpha)p^{-\alpha s} = D(p^{-s},\rho,g)^z$$
$$L^z_p(s,\rho) = \sum_{\alpha\geqslant 0} \lambda^z_{\rho}(p^\alpha)p^{-\alpha s} = D^z(p^{-s},\rho)$$

and when $\rho = \mathbf{1}$ these are simply given by $(1-p^{-s})^{-z}$. Consider Ω a space and for each prime $p \in \mathcal{P}$ a map

$$g(p,\cdot):\ \omega \in \Omega \to g(p,\omega) \in G.$$

We define the Euler products

$$L(s,\rho,\omega)^z := \prod_p L_p(s,\rho,g(p,\omega))^z = \sum_{n\geqslant 1} \lambda^z_{\rho,\omega}(n)n^{-s},$$
$$L^z(s,\rho) := \prod_p L^z_p(s,\rho) = \sum_{n\geqslant 1} \lambda^z_{\rho}(n)n^{-s}.$$

Note that for $\rho = \mathbf{1}$ we have $L^z(s,\mathbf{1}) = \zeta(s)^z$. In view of the bound (2.3), one sees easily that $L(s,\rho,\omega)^z$ and $L^z(s,\rho)$ are absolutely convergent for $\Re e(s) > 1$. Suppose now that ρ is non-trivial and irreducible. Then one has, by (2.9) and (2.7),

$$L^z_p(1,\rho) = 1 + \frac{\big[(\mathrm{FrSc}(\rho)z)^2 + \mathrm{FrSc}(\rho)z\big]/2}{p^2} + O\left(\left(\frac{d|z|}{p}\right)^3\right),$$

uniformly for $p \geqslant d|z|+1$; hence $L^z(s,\rho)$ is an absolutely convergent Dirichlet series for $\Re e(s) > 1/2$. Notice that $L^z_p(s,\rho)$ may vanish for $\Re e(s) > 0$; however given a fixed compact K, and some fixed $\delta > 0$, for p sufficiently large (depending on K) one has $L^z_p(s,\rho) \neq 0$ and the Euler product $L^z(s,\rho) := \prod_p L^z_p(s,\rho)$, is absolutely convergent uniformly for $\Re e(s) \geqslant 1/2+\delta$ and $z \in K$. In particular $L^z(1,\mathrm{Sym}^k)$ is a continuous function of z.

In fact $L^z(s,\mathrm{Sym}^k)$ has the following probabilistic interpretation; suppose that Ω is a probability space with measure μ (say) and that the functions $\{g(p,\omega)\}_{p\in\mathcal{P}}$ are independent random variables distributed following the Haar measure on G (i.e. $\mu(\{\omega,\ g(p,\omega)\in A\}) = \mu_G(A)$ for any measurable subset $A \subset G$). For $\Re es > 1/2$, one has

$$\log L_p(s,\rho,\omega) = \frac{\mathrm{tr}(\rho(g(p,\omega)))}{p^s} + O\left(\frac{1}{p^{2s}}\right)$$

and since the variables $g(p,.)$ are independent

$$\mathbf{E}(|\sum_{p\leqslant X} \frac{\mathrm{tr}(\rho(g(p,\omega)))}{p^s}|^2) = \sum_{p\leqslant X} \frac{\mathbf{E}(|\mathrm{tr}(\rho(g(p,\omega)))|^2)}{p^{2\Re e(s)}}$$
$$+ 2\Re e \sum_{p<p'\leqslant X} \frac{\mathbf{E}(\mathrm{tr}(\rho(g(p,\omega))))\mathbf{E}(\overline{\mathrm{tr}}(\rho(g(p',\omega'))))}{p^s p'^{\overline{s}}}.$$

By assumption (since ρ is non-trivial and irreducible)

$$\mathbf{E}(\mathrm{tr}(\rho(g(p,\omega)))) = \int_G \mathrm{tr}(\rho(g))d\mu_G(g) = 0$$
$$\mathbf{E}(|\mathrm{tr}(\rho(g(p,\omega)))|^2) = \int_G |\mathrm{tr}(\rho(g))|^2 d\mu_G(g) = 1$$

so that

$$\mathbf{E}(|\sum_{p\leqslant X}\frac{\mathrm{tr}(\rho(g(p,\omega)))}{p^s}|^2)=\sum_{p\leqslant X}\frac{1}{p^{2\Re e(s)}};$$

this show that the Euler product $L(s,\rho,\omega)$ is a.e. convergent for $\Re e(s)>1/2$, and then $L^z(s,\rho)$ is interpreted as

$$\mathbf{E}(L(s,\rho,\omega)^z)=\prod_p\mathbf{E}(L_p(s,\rho,\omega)^z)=\prod_p\int_G L_p(s,\rho,g)^z d\mu_G(g)=L^z(s,\rho);$$

in particular the function of the real variable $t\to L^{it}(s,\rho)$ is interpreted the characteristic function of the random variable $\log L(s,\rho,\omega)$.

2.2. **Representations of SU(2).** We restrict here to the case where $G=SU(2)$, the maximal compact subgroup of $SL(2,\mathbf{C})$. In that case the irreducible representations of G are the symmetric power representations Sym^k of the standard representation and are of dimension $d=k+1$. The space of conjugacy classes $G^\natural$ is identified in this case with the interval $[0,\pi]$ through the map

$$\theta\to g^\natural(\theta)=\begin{pmatrix}e^{i\theta}&0\\0&e^{-i\theta}\end{pmatrix}^\natural,$$

and the Sato/Tate measure $dg^\natural$ is identified with $\frac{2}{\pi}\sin^2\theta d\theta$, moreover

$$\mathrm{Sym}^k g^\natural(\theta)=\begin{pmatrix}e^{ik\theta}&&&\\&e^{i(k-2)\theta}&&\\&&\ddots&\\&&&e^{-ik\theta}\end{pmatrix}^\natural,$$

and

$$\mathrm{trSym}^k(g^\natural(\theta))=\mathrm{sym}_k(\theta):=\frac{\sin((k+1)\theta)}{\sin\theta}.$$

We set $\rho=\mathrm{Sym}^k$ and we consider for $z\in\mathbf{C}$ and $\alpha\geqslant 1$, the decomposition of the character

$$\lambda^{z,\alpha}_{\mathrm{Sym}^k}(\cdot)=\sum_{k'}\mu^{z,\alpha}_{\mathrm{Sym}^k,\mathrm{Sym}^{k'}}\mathrm{tr}(\mathrm{Sym}^{k'}(\cdot))\in R\langle G\rangle\otimes_{\mathbf{Z}}\mathbf{Q}[z];$$

then it is not difficult to see that $\mu^{z,\alpha}_{\mathrm{Sym}^k,\mathrm{Sym}^{k'}}=0$ whenever $k'>k\alpha$ so that

$$\lambda^{z,\alpha}_{\mathrm{Sym}^k}(\cdot)=\sum_{k'\leqslant\alpha k}\mu^{z,\alpha}_{\mathrm{Sym}^k,\mathrm{Sym}^{k'}}\mathrm{tr}(\mathrm{Sym}^{k'}(\cdot)).\tag{2.11}$$

and the bound of Proposition 2.1 becomes

$$\sum_{k'\leqslant\alpha k}|\mu^{z,\alpha}_{\mathrm{Sym}^k,\mathrm{Sym}^{k'}}|\leqslant(\alpha k+1)^{1/2}\lambda_{\mathbf{1}}^{(k+1)|z|,\alpha}(e_G)\tag{2.12}$$

2.2.1. *Proof of Theorem 1.5.*

Proof. Since Sym^k is a self-dual representation of $G=SU(2)$ one has for $g\in G$

$$(1+|X|)^{-(k+1)} \leqslant D(X,\mathrm{Sym}^k,g) \leqslant (1-|X|)^{-(k+1)}$$

for $-1<X<1$ so that for any $p\geqslant 2$

$$\left(1\pm\frac{1}{p}\right)^{\mp(k+1)r} \leqslant L_p^{\pm r}(1,\mathrm{Sym}^k) \leqslant \left(1\mp\frac{1}{p}\right)^{\mp(k+1)r},$$

hence

$$\log L_p^{\pm r}(1,\mathrm{Sym}^k) = O_k\left(\frac{r}{p}\right). \tag{2.13}$$

One has

$$\log L^{\pm r}(1,\mathrm{Sym}^k) = \sum_{p\leqslant(k+1)r}\log L_p^{\pm r}(1,\mathrm{Sym}^k) + \sum_{p>(k+1)r}\log L_p^{\pm r}(1,\mathrm{Sym}^k).$$

For $p>(k+1)r$ one has, by (2.9) and (2.7),

$$L_p^{\pm r}(1,\mathrm{Sym}^k) = 1+\frac{(r^2\pm(-1)^k r)/2}{p^2} + O\left(\left(\frac{(k+1)r}{p}\right)^3\right),$$

so that by (2.13) and the above estimate

$$\sum_{p\geqslant(k+1)r}\log L_p^{\pm r}(1,\mathrm{Sym}^k) = O_k\left(\sum_{p\geqslant(k+1)r}\frac{r^2}{p^2}\right) = O_k\left(\frac{r}{\log r}\right).$$

We consider the case where $p\leqslant(k+1)r$. We set

$$L_{p,+}(1,\mathrm{Sym}^k) := \max_{g\in G} L_p(1,\mathrm{Sym}^k,g)$$

$$L_{p,-}(1,\mathrm{Sym}^k) := \min_{g\in G} L_p(1,\mathrm{Sym}^k,g) \geqslant \left(1+\frac{1}{p}\right)^{-(k+1)}.$$

We also denote by $\theta_{p,\pm}\in[0,\pi]$ points where these extremes are achieved:

$$L_{p,\pm}(1,\mathrm{Sym}^k) = L_p(1,\mathrm{Sym}^k,g(\theta_{p,\pm})).$$

Set $\eta_p = p/(k+1)^2 r \leqslant 1/(k+1)$, then for $\theta\in I(\theta_{p,\pm},\eta_p)=[0,\pi]\cap[\theta_{p,\pm}-\eta_p,\theta_{p,\pm}+\eta_p]$ one has

$$\log L_p(1,\mathrm{Sym}^k,g(\theta)) = \log L_{p,\pm}(1,\mathrm{Sym}^k) + O_k\left(\frac{\eta_p^2}{p}\right)$$

since

$$\frac{\partial}{\partial\theta}\log L_p(1,\mathrm{Sym}^k,g(\theta_{p,\pm}))=0 \quad\text{and}\quad \frac{\partial^2}{\partial\theta^2}\log L_p(1,\mathrm{Sym}^k,g(\theta)) \ll_k \frac{1}{p}$$

for $\theta\in[0,\pi]$. Hence

$$L_{p,\pm}(1,\mathrm{Sym}^k)^{\pm r}\left(1+O_k\left(\frac{\eta_p^2}{p}\right)\right)^{\pm r}\mu_{st}(I(\theta_{p,\pm},\eta_p)) \leqslant L_p^{\pm r}(1,\mathrm{Sym}^k) \leqslant L_{p,\pm}(1,\mathrm{Sym}^k)^{\pm r}.$$

Since

$$|\log\mu_{st}(I(\theta_{p,\pm},\eta_p))| \ll -\log(\eta_p)+1$$

we conclude

$$\sum_{p\leqslant(k+1)r}\log L_p^{\pm r}(1,\mathrm{Sym}^k)=r\sum_{p\leqslant(k+1)r}\log L_{p,\pm}(1,\mathrm{Sym}^k)^{\pm 1}+O\left(\frac{r}{\log r}\right);$$

in the above we have used the following consequences of the Prime Number Theorem

$$\sum_{p\leqslant(k+1)r}\log\left(\frac{r(k+1)}{p}\right)+1+\frac{p}{r(k+1)}=O\left(\frac{r}{\log r}\right).$$

It remains to compute $\sum_{p\leqslant(k+1)r}\log L_{p,\pm}(1,\mathrm{Sym}^k)^{\pm 1}$. For $g=g(\theta)$

$$L_p(1,\mathrm{Sym}^k,g)=\prod_{i=0}^{k}\left(1-\frac{\mathrm{e}^{(k-2i)\theta}}{p}\right)^{-1}=\left(1-\frac{1}{p}\right)^{-\delta_{2|k}}\prod_{0\leqslant i<k/2}^{k}\left(1-\frac{2\cos(k-2i)\theta}{p}+\frac{1}{p^2}\right)^{-1}.$$

Clearly, one has for any k

$$L_{p,+}(1,\mathrm{Sym}^k)=L_p(1,\mathrm{Sym}^k,g(0))=\left(1-\frac{1}{p}\right)^{-(k+1)}=\zeta_p(1)^{k+1}$$

and for k odd

$$L_{p,-}(1,\mathrm{Sym}^k)=L_p(1,\mathrm{Sym}^k,g(\pi))=\left(1+\frac{1}{p}\right)^{-(k+1)}=(\zeta_p(2)/\zeta_p(1))^{k+1}.$$

Hence by Mertens' Theorem, one has for any $k\geqslant 1$

$$\sum_{p\leqslant(k+1)r}\log L_{p,+}(1,\mathrm{Sym}^k)=(k+1)\log\log r+(k+1)\gamma+O_k\left(\frac{1}{\log r}\right)$$

and for k odd

$$\sum_{p\leqslant(k+1)r}\log L_{p,-}(1,\mathrm{Sym}^k)^{-1}=(k+1)\log\log r+(k+1)(\gamma-\log\zeta(2))+O_k\left(\frac{1}{\log r}\right).$$

When $k=2$, one has

$$L_{p,-}(1,\mathrm{Sym}^2)=L_p(1,\mathrm{Sym}^2,g(\pi/2))=\left(1-\frac{1}{p}\right)^{-1}\left(1+\frac{1}{p}\right)^{-2}=\zeta_p(2)^2/\zeta_p(1),$$

and

$$\sum_{p\leqslant 3r}\log L_{p,-}(1,\mathrm{Sym}^2)^{-1}=\log\log r+(\gamma-2\log\zeta(2))+O\left(\frac{1}{\log r}\right).$$

In the remaining case, we set

$$\mathrm{Sym}^k_-:=\max_\theta\Big(-\mathrm{tr}(\mathrm{Sym}^k g(\theta))\Big)>0$$

and denote by θ^{tr}_- any solution of the equation $\mathrm{tr}(\mathrm{Sym}^k g(\theta))=-\mathrm{Sym}^k_-$. For any $g\in G$, one has $L_p(1,\mathrm{Sym}^k,g)=1+\frac{\mathrm{tr}(\mathrm{Sym}^k(g))}{p}+O_k(p^{-2})$; now, by the definition of $L_{p,-}(1,\mathrm{Sym}^k)$ and Sym^k_- one has

$$0 \leqslant L_p(1, \mathrm{Sym}^k(g(\theta_-^{\mathrm{tr}}))) - L_{p,-}(1, \mathrm{Sym}^k)$$
$$= \frac{\mathrm{tr}(\mathrm{Sym}^k(g(\theta_-^{\mathrm{tr}}))) - \mathrm{tr}(\mathrm{Sym}^k(g(\theta_{p,-})))}{p} + O_k\left(\frac{1}{p^2}\right) \leqslant O_k\left(\frac{1}{p^2}\right)$$

since

$$\mathrm{tr}(\mathrm{Sym}^k(g(\theta_-^{\mathrm{tr}}))) - \mathrm{tr}(\mathrm{Sym}^k(g(\theta_{p,-}))) \leqslant 0.$$

Hence

$$L_{p,-}(1, \mathrm{Sym}^k) = L_p(1, \mathrm{Sym}^k g(\theta_-^{\mathrm{tr}}))\left(1 + O_k\left(\frac{1}{p^2}\right)\right),$$

from which we conclude that

$$\sum_{p\leqslant(k+1)r} \log L_{p,-}(1, \mathrm{Sym}^k)^{-1} = -\mathrm{Sym}_-^k \log\log r - \mathrm{Sym}_-^{k,1} + O_k\left(\frac{1}{\log r}\right)$$

for some constant $\mathrm{Sym}_-^{k,1}$.

We turn now to the proof of (1.10). One has for any prime p and $t \in \mathbf{R}$,

$$|L_p^{it}(1, \mathrm{Sym}^k)| = \Big|\int_G \det(I - p^{-1}\mathrm{Sym}^k(g))^{-it} dg\Big| \leqslant 1,$$

hence

$$\log|L^{it}(1, \mathrm{Sym}^k)| \leqslant \sum_{p\geqslant|t|\log|t|} \log|L_p^{it}(1, \mathrm{Sym}^k)|$$
$$= \sum_{p\geqslant|t|\log|t|} \log\left(1 + \frac{-t^2 + (-1)^k it}{p^2} + O_k\left(\frac{|t|^3}{p^3}\right)\right) = -\frac{|t|}{\log^2|t|}(1 + o_k(1))$$

as $|t| \to +\infty$. □

3. Symmetric power L-functions, computation of the local factors

In this section we compute the local factor $L_v(s, \mathrm{Sym}^k f) = L(s, \mathrm{Sym}^k \pi_{f,v})$ of the symmetric k-th power lifting by using the local Langlands correspondence in the following cases:

– $v = \infty$ and $\pi_{f,\infty} \simeq D_\ell$ is the discrete series of of $GL_2(\mathbf{R})$ weight $\ell \geqslant 2$. (In our situation $\ell = 2$.)

– $v = p$ is non-archimedean and $\pi_{f,p} = \chi_p \otimes St_{2,p}$ is an unramified twist of the Steinberg representation $St_{2,p}$ of $GL_2(\mathbf{Q}_p)$. (In our situation, χ_p will also be quadratic.)

These are the local ramified local factors that occur in our problem.

The results in this section are surely well known. However, since we do not know of a suitable reference we present their derivation here.

3.1. **Symmetric powers of the discrete series.** Let D_ℓ be the discrete series representation of $GL_2(\mathbf{R})$ of weight $\ell \geqslant 2$. We index the discrete series so that the representation D_ℓ corresponds to the infinite component of the automorphic representation afforded by a classical cusp form of weight ℓ. In this paper we are primarily interested in the case $\ell = 2$ but we carry out the computation in general for possible future use.

3.1.1. *The Langlands parameters.* The Langlands parameter for the discrete series D_ℓ with $\ell \geqslant 2$ is the following. The Weil group $W_{\mathbf{R}}$ of $\mathbf{R}$ can be realized as $W_{\mathbf{R}} = \mathbf{C}^\times \cup j\mathbf{C}^\times$ with $j^2 = -1 \in \mathbf{C}^\times$ and $jzj^{-1} = \overline{z}$ for $z \in \mathbf{C}^\times$. Let μ , $\nu \in \mathbf{C}$ such that $\ell - 1 = \mu - \nu \in \mathbf{Z}$ and $2t = \mu + \nu \in \mathbf{C}$. Consider the two dimensional representation $\rho = \rho_{\ell,t}$ of $W_{\mathbf{R}}$ on the two dimensional vector space $V_2 = \langle e_0, e_1\rangle$ given by

$$\rho(z)e_0 = z^\mu \overline{z}^\nu e_0 \qquad \rho(z)e_1 = z^\nu \overline{z}^\mu e_1$$
$$\rho(j)e_0 = e_1 \qquad \rho(j)e_1 = (-1)^{\mu-\nu} e_0$$

or in matrix form, writing $z = re^{i\theta}$,

$$\rho(z) = \begin{pmatrix} r^{2t}e^{i(\ell-1)\theta} & \\ & r^{2t}e^{-i(\ell-1)\theta} \end{pmatrix} \qquad \rho(j) = \begin{pmatrix} & (-1)^{\ell-1} \\ 1 & \end{pmatrix}.$$

Then under the local Langlands correspondence $\rho_{\ell,t}$ corresponds to $D_\ell \otimes |\det|^t$ [17] (note that our numbering of the discrete series differs from that of [17] by a shift of one). So to obtain D_ℓ we take $t = 0$ and write $\rho = \rho_\ell$ so that

$$\rho_\ell(re^{i\theta}) = \begin{pmatrix} e^{i(\ell-1)\theta} & \\ & e^{-i(\ell-1)\theta} \end{pmatrix} \qquad \rho_\ell(j) = \begin{pmatrix} & (-1)^{\ell-1} \\ 1 & \end{pmatrix}.$$

3.1.2. *The symmetric powers.* One can easily compute the symmetric powers of the discrete series representations. By the local Langlands correspondence, we may do the calculation in terms of the representation ρ_ℓ of $W_{\mathbf{R}}$.

Proposition 3.1. *The symmetric powers of the representation ρ_ℓ are given by*

$$\mathrm{Sym}^{2m+1}(\rho_\ell) = \bigoplus_{a=0}^{m} \rho_{(2a+1)(\ell-1)+1}$$

and

$$\mathrm{Sym}^{2m}(\rho_\ell) = \rho_0^{\pm} \oplus \bigoplus_{a=1}^{m} \rho_{2a(\ell-1)+1}$$

where $\rho_0^\pm$ are the one dimensional representations of $W_{\mathbf{R}}$ defined by $\rho_0^\pm(z) = 1$, $\rho_0^+(j) = 1$, and $\rho_0^-(j) = -1$ and the choice of representation occurring in the decomposition is determined by $\rho_0^\pm(j) = (-1)^{m(\ell-1)}$.

Proof: For any positive integer k let S_k denote the symmetric group on k letters. For any complex vector space V of finite dimension let $s_k : V^{\otimes k} \to \mathrm{Sym}^k(V)$ denote projection given by symmetrization

$$s_k(x_1 \otimes \cdots \otimes x_k) = \frac{1}{k!}\sum_{\sigma\in S_k} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}.$$

In $V_2^{\otimes k}$ we let e_k^b denote the tensor

$$e_k^b = e_0 \otimes \cdots \otimes e_0 \otimes e_1 \otimes \cdots \otimes e_1$$

where e_0 is repeated $k-b$ times and e_1 is repeated b times. For example

$$e_3^0 = e_0 \otimes e_0 \otimes e_0 \quad \text{and} \quad e_3^1 = e_0 \otimes e_0 \otimes e_1.$$

Then a natural basis for $\mathrm{Sym}^k(V_2)$ consists of the symmetrization of these $k+1$ vectors, that is, the $f_k^b = s_k(e_k^b)$ for $b = 0, \ldots, k$. For example, for the symmetric cube we have the basis

$$\begin{aligned}
f_3^0 &= e_0 \otimes e_0 \otimes e_0 \\
f_3^1 &= \tfrac{1}{3}(e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0) \\
f_3^2 &= \tfrac{1}{3}(e_0 \otimes e_1 \otimes e_1 + e_1 \otimes e_0 \otimes e_1 + e_1 \otimes e_1 \otimes e_0) \\
f_3^3 &= e_1 \otimes e_1 \otimes e_1
\end{aligned}$$

It is then easy to calculate that with respect to this basis we have

$$\mathrm{Sym}^k(\rho_\ell)(re^{i\theta}) f_k^b = e^{i(k-2b)(\ell-1)\theta} f_k^b \quad \text{and} \quad \mathrm{Sym}^k(\rho_\ell)(j) f_k^b = (-1)^{b(\ell-1)} f_k^{k-b}.$$

Hence each subspace $\langle f_k^b, f_k^{k-b} \rangle$, for $b = 0, \ldots, \left[\frac{k}{2}\right]$, is stable under $\mathrm{Sym}^k(\rho_\ell)$. If $\dim\langle f_k^b, f_k^{k-b}\rangle = 2$ then this affords a copy of the irreducible representation $\rho_{(k-2b)(\ell-1)+1}$. If $\dim\langle f_k^b, f_k^{k-b}\rangle = 1$, so that $k = 2b$ is even, then this is a one dimensional representation of $W_{\mathbf{R}}$. If $b(\ell-1)$ is even this is the trivial representation ρ_0^+ , so $\rho_0^+(z) = \rho_0^+(j) = 1$. If $b(\ell-1)$ is odd then this is the representation ρ_0^- defined by $\rho_0^-(z) = 1$ and $\rho_0^-(j) = -1$.

Hence if $k = 2m+1$ is odd we have

$$\mathrm{Sym}^{2m+1}(\rho_\ell) = \bigoplus_{b=0}^{m} \rho_{(2m+1-2b)(\ell-1)+1}$$

which gives the first formula in the proposition, and if $k = 2m$ is even, then

$$\mathrm{Sym}^{2m}(\rho_\ell) = \rho_0^\pm \oplus \bigoplus_{b=0}^{m-1} \rho_{(2m-2b)(\ell-1)+1},$$

with the choice of $\rho_0^\pm$ indicated above, which gives the second formula. □

If we interpret this result in terms of the local functorial lift, then the direct sum of Weil group representations correspond to the isobaric sums of representations of $GL_d(\mathbf{R})$ [35]. Thus if we let $\mathrm{Sym}^k(D_\ell)$ denote the symmetric k-th power lift of the discrete series of weight ℓ with $\ell \geqslant 2$ and let D_0^+ denote the trivial representation of $GL_1(\mathbf{R})$ and D_0^- denote the sign character sgn of $GL_1(\mathbf{R})$ then we have the following corollary.

Corollary 3.2. *The symmetric powers of the discrete series representation D_ℓ with weight $\ell \geqslant 2$ are given by*

$$\mathrm{Sym}^{2m+1}(D_\ell) = \boxplus_{a=0}^{m} D_{(2a+1)(\ell-1)+1}$$

and

$$\mathrm{Sym}^{2m}(D_\ell) = D_0^\pm \boxplus \left[\boxplus_{a=1}^{m} D_{2a(\ell-1)+1}\right]$$

where again the sign of $D_0^\pm$ is equal to the sign of $(-1)^{m(\ell-1)}$.

3.1.3. *The L-function.* Let us set

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \quad \text{and} \quad \Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

Then one can compute [17, 35]

$$L(s, D_\ell) = L(s, \rho_\ell) = \Gamma_{\mathbf{C}}\left(s + \frac{\ell-1}{2}\right)$$

for $\ell \geqslant 2$,

$$L(s, D_0^+) = L(s, \rho_0^+) = \Gamma_{\mathbf{R}}(s),$$

and

$$L(s, D_0^-) = L(s, \rho_0^-) = \Gamma_{\mathbf{R}}(s+1).$$

Then from the above calculation we arrive at

$$\begin{aligned} L(s, \mathrm{Sym}^{2m+1}(D_\ell)) &= \prod_{a=0}^{m} L(s, D_{(2a+1)(\ell-1)+1}) \\ &= \prod_{a=0}^{m} \Gamma_{\mathbf{C}}\left(s + \frac{(2a+1)(\ell-1)}{2}\right) \end{aligned}$$

and similarly

$$\begin{aligned} L(s, \mathrm{Sym}^{2m}(D_\ell)) &= L(s, D_0^{\pm}) \prod_{a=1}^{m} L(s, D_{2a(\ell-1)+1}) \\ &= \begin{cases} \Gamma_{\mathbf{R}}(s) \prod_{a=1}^{m} \Gamma_{\mathbf{C}}\left(s + a(\ell-1)\right) & m(\ell-1) \text{ even} \\ \\ \Gamma_{\mathbf{R}}(s+1) \prod_{a=1}^{m} \Gamma_{\mathbf{C}}\left(s + a(\ell-1)\right) & m(\ell-1) \text{ odd} \end{cases}. \end{aligned}$$

If we now specialize these to the case $\ell = 2$ and use the definitions of $\Gamma_{\mathbf{R}}$ and $\Gamma_{\mathbf{C}}$ we arrive at the formulas in Hypothesis $\mathrm{Sym}^k(f)$.

3.1.4. *The root numbers.* We would now like to compute the local root numbers for the symmetric powers of the discrete series representations. Let $\psi(x)$ be the standard additive character for $\mathbb{R}$, namely $\psi(x) = e(x) = e^{2\pi i x}$. Then one knows, from [17] for example, that

$$\varepsilon(s, D_\ell, \psi) = i^\ell \quad \text{for} \quad \ell \geqslant 2,$$

while

$$\varepsilon(s, D_0^+, \psi) = 1 \quad \text{and} \quad \varepsilon(s, D_0^-, \psi) = i.$$

Note that these are in fact independent of s, and so equal to the local root number (the value at $s = 1/2$) as well. Then from the calculation of the symmetric powers and the multiplicativity of the local ε–factors over isobaric sums ($\boxplus$) we arrive at

$$\begin{aligned} \varepsilon(s, \mathrm{Sym}^{2m+1}(D_\ell), \psi) &= \prod_{a=0}^{m} \varepsilon(s, D_{(2a+1)(\ell-1)+1}, \psi) = \prod_{a=0}^{m} i^{(2a+1)(\ell-1)+1} \\ &= \begin{cases} i^\ell & m \equiv 0 \pmod 4 \\ -1 & m \equiv 1 \pmod 4 \\ -i^\ell & m \equiv 2 \pmod 4 \\ 1 & m \equiv 3 \pmod 4 \end{cases}. \end{aligned}$$

and similarly

$$\varepsilon(s, \mathrm{Sym}^{2m}(D_\ell), \psi) = \varepsilon(s, D_0^{\pm}, \psi) \prod_{a=1}^{m} \varepsilon(s, D_{2a(\ell-1)+1}, \psi) = \varepsilon(s, D_0^{\pm}, \psi) i^{m(m+1)(\ell-1)+m}$$
$$= \begin{cases} 1 & \ell \text{ even} \\ i^m & \ell \text{ odd} \end{cases}.$$

To obtain the formula for $\varepsilon(\frac{1}{2}, \mathrm{Sym}^k \pi_{f,\infty}, \psi_\infty)$ used in Hypothesis $\mathrm{Sym}^k(f)$ we specialize these to the case $\ell = 2$ and find

$$\varepsilon(\tfrac{1}{2}, \mathrm{Sym}^k \pi_{f,\infty}, \psi_\infty) = \varepsilon(\tfrac{1}{2}, \mathrm{Sym}^k(D_2), \psi) = \begin{cases} -1 & k \equiv 1, 3 \pmod 8 \\ 1 & k \not\equiv 1, 3 \pmod 8 \end{cases}.$$

3.2. **Symmetric powers of the Steinberg representation.** Now consider the Steinberg representation of $GL_2(\mathbf{Q}_p)$. Representation theoretically this is the irreducible quotient of the induced representation $\mathrm{Ind}_{B_2}^{GL_2}(|\ |^{-1/2}, |\ |^{1/2})$ associated to the segment $\Delta = [\nu^{-1/2}, \nu^{1/2}]$ where, as is common, we have used $\nu(x) = |\det(x)|$ to denote the absolute value character of any GL_d. This is commonly denoted $\delta(\mathbf{1}, 2)$. In general if ρ is a supercuspidal representation of some $GL_d(\mathbf{Q}_p)$ then $\delta(\rho, a)$ is the discrete series representation of $GL_{da}(\mathbf{Q}_p)$ associated to the segment $\Delta = [\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]$ which is the irreducible quotient of the induced representation

$$\mathrm{Ind}(\nu^{-(a-1)/2}\rho \otimes \nu^{((-(a-1)/2)+1)}\rho \otimes \cdots \otimes \nu^{(a-1)/2}\rho).$$

Note that, in our notation, $St_{2,p} = \delta(\mathbf{1}, 2)$ and $St_{n,p} = \delta(\mathbf{1}, n)$ is the Steinberg representation of $GL_n(\mathbf{Q}_p)$ which is the irreducible quotient of $\mathrm{Ind}_{B_n}^{GL_n}(\nu^{-(n-1)/2} \otimes \cdots \otimes \nu^{(n-1)/2})$.

If we consider the twisted Steinberg $\chi_p \otimes St_{n,p}$ for a character χ_p of $\mathbf{Q}_p^\times$ then $\chi_p \otimes St_{n,p} = \delta(\chi_p, n)$ is the representation of $GL_n(\mathbf{Q}_p)$ given by the irreducible quotient of

$$\mathrm{Ind}_{B_n}^{GL_n}(\chi_p\nu^{-(n-1)/2} \otimes \cdots \otimes \chi_p\nu^{(n-1)/2}).$$

3.2.1. *The special representation of the Weil-Deligne group.* Under the local Langlands correspondence for GL_2, the Steinberg representation of $GL_2(\mathbf{Q}_p)$ corresponds to a twist of the special representation $sp(2)$ of the Weil-Deligne group W_p' of $\mathbf{Q}_p$ [20, 35]. Recall that a representation of the Weil-Deligne group corresponds to a pair (ρ, N) consisting of a representation ρ of the Weil group $W_{\mathbf{Q}_p}$ of $\mathbf{Q}_p$ on a complex vector space V and a nilpotent endomorphism N of V such that $\rho(w)N\rho(w)^{-1} = ||w||N$.

In general, the special representation $sp(n) = (\rho_n, N_n)$ is the n-dimensional representation of $W_{\mathbf{Q}_p}'$ realized on the space $V_n = \langle e_0, e_1, \ldots, e_{n-1}\rangle$ by

$$\rho_n(w)e_i = ||w||^i e_i \quad \text{and} \quad N_n e_i = e_{i+1}$$

with the convention that $e_n = 0$. As is now common, we normalize the local class field theory isomorphism so that the geometric Frobenius $\Phi_p = \mathrm{Frob}_p^{-1}$ corresponds to the uniformizer $\varpi_p = p$, so that $||\Phi_p|| = |p| = p^{-1}$. Then the L-function of $sp(n)$ is given by

$$L(s, sp(n)) = \det(I - p^{-s}\rho_n(\Phi_p)|Ker(N_n)^{I_p})^{-1} = (1 - p^{-(n-1)}p^{-s})^{-1} = \zeta_p(s + n - 1).$$

In particular, $sp(2)$ is realized on $V_2 = \langle e_0, e_1\rangle$ by

$$\rho_2(w)e_0 = e_0, \quad \rho_2(w)e_1 = ||w||e_1, \quad N_2 e_0 = e_1, \quad \text{and} \quad N_2 e_1 = 0$$

and $L(s, sp(2)) = \zeta_p(s + 1)$.

If we let ω_p be a character of $W'_{\mathbf{Q}_p}$, which we also view as a character of $\mathbf{Q}_p^\times$ via local class field theory, then we may twist the special representation by ω_p to obtain $\omega_p \otimes sp(n) = (\omega_p \otimes \rho_n, N_n)$. In this case, we have

$$L(s, \omega_p \otimes sp(n)) = \det(I - p^{-s}\omega_p(\Phi_p)\rho_n(\Phi_p)|Ker(N_n)^{I_p})^{-1} = L(s+n-1, \omega_p).$$

3.2.2. *The Langlands parameters.* The special representation $sp(n)$ corresponds to the irreducible quotient of the induced representation $\mathrm{Ind}(\mathbf{1} \otimes \nu \otimes \cdots \otimes \nu^{n-1})$. Thus $St_{n,p} = \delta(\mathbf{1}, n)$, which is the irreducible quotient of

$$\mathrm{Ind}(\nu^{-(n-1)/2} \otimes \cdots \otimes \nu^{(n-1)/2}) = \nu^{-(n-1)/2} \otimes \mathrm{Ind}(\mathbf{1} \otimes \nu \otimes \cdots \otimes \nu^{n-1}),$$

corresponds to the representation $||\ ||^{-(n-1)/2} \otimes sp(n)$ under the local Langlands correspondence and the twisted Steinberg $\chi_p \otimes St_{n,p}$ corresponds to $\chi_p||\ ||^{-(n-1)/2} \otimes sp(n)$ [20, 35]. Then

$$L(s, St_{n,p}) = L(s, ||\ ||^{-(n-1)/2} \otimes sp(n)) = L\left(s - \frac{n-1}{2}, sp(n)\right) = \zeta_p\left(s + \frac{n-1}{2}\right)$$

and similarly

$$L(s, \chi_p \otimes St_{n,p}) = L\left(s + \frac{n-1}{2}, \chi_p\right).$$

In particular

$$L(s, St_{2,p}) = \zeta_p\left(s + \frac{1}{2}\right) \quad \text{and} \quad L(s, \chi_p \otimes St_{2,p}) = L\left(s + \frac{1}{2}, \chi_p\right).$$

3.2.3. *The symmetric powers.* In order to compute the symmetric powers of the Steinberg $St_{2,p}$ or its twist $\chi_p \otimes St_{2,p}$ we will use the local Langlands correspondence. Thus our first task is to compute the symmetric powers of the special representation $sp(2)$ of $W'_{\mathbf{Q}_p}$. Again, this must be well known.

Proposition 3.3. $\mathrm{Sym}^k(sp(2)) = sp(k+1)$.

Proof: First, note that the tensor product of two Weil-Deligne representations is given by

$$(\rho, N) \otimes (\rho', N') = (\rho \otimes \rho', N \otimes 1 + 1 \otimes N').$$

Thus the k-th tensor power of a Weil-Deligne representation (ρ, N) is given by $(\rho^{\otimes k}, N^{\otimes k})$ where we have set

$$N^{\otimes k} = N \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes N \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes N.$$

As before, for any complex vector space V of finite dimension let $s_k : V^{\otimes k} \to \mathrm{Sym}^k(V)$ denote projection given by symmetrization

$$s_k(x_1 \otimes \cdots \otimes x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}.$$

In $V_2^{\otimes k}$ if we let e_k^i denote the tensor

$$e_k^i = e_0 \otimes \cdots \otimes e_0 \otimes e_1 \otimes \cdots \otimes e_1$$

where e_0 is repeated $k-i$ times and e_1 is repeated i times, then a natural basis for $\mathrm{Sym}^k(V_2)$ consists of the symmetrization of these $k+1$ vectors, that is, the $f_k^i = s_n(e_k^i)$for $i = 0, \ldots, k$.

In this basis it is elementary to compute that

$$\mathrm{Sym}^k(\rho_2)(w) f_k^i = ||w||^i f_k^i$$

and only slightly more difficult to compute that

$$\mathrm{Sym}^k(N_2)f_k^i = N_2^{\otimes k} f_k^i = (k-i)f_k^{i+1}.$$

Hence by scaling the f_k^i appropriately we find

$$\mathrm{Sym}^k(sp(2)) = (\mathrm{Sym}^k(\rho_2), \mathrm{Sym}^k(N_2)) \simeq (\rho_{k+1}, N_{k+1}) = sp(k+1)$$

which completes the proof of the proposition. □

If we now apply this to the computation of the local symmetric powers of $St_{2,p}$ or $\chi_p \otimes St_{2,p}$ via the local Langlands correspondence, we have the following corollary.

Corollary 3.4. $\mathrm{Sym}^k(St_{2,p}) = St_{k+1,p}$ *and* $\mathrm{Sym}^k(\chi_p \otimes St_{2,p}) = \chi_p^k \otimes St_{k+1,p}$.

Proof: From the fact that $\mathrm{Sym}^k(sp(2)) = sp(k+1)$ it follows that for any character ω_p of $W'_{\mathbf{Q}_p}$, which we also view as a character of $\mathbf{Q}_p^\times$, we have

$$\mathrm{Sym}^k(\omega_p \otimes sp(2)) = \omega_p^k \otimes \mathrm{Sym}^k(sp(2)) = \omega_p^k \otimes sp(k+1).$$

Since $|| \, ||^{-1/2} \otimes sp(2)$ corresponds to $St_{2,p}$ under the local Langlands correspondence and $|| \, ||^{-k/2} \otimes sp(k+1)$ corresponds precisely to $St_{k+1,p}$ under the local Langlands correspondence, this gives both $\mathrm{Sym}^k(St_{2,p}) = St_{k+1,p}$ and $\mathrm{Sym}^k(\chi_p \otimes St_{2,p}) = \chi_p^k \otimes St_{k+1,p}$. □

Since we know the L-function associated to the special representations of $W'_{\mathbf{Q}_p}$ and the Steinberg representations of $GL_n(\mathbf{Q}_p)$ we also obtain the following corollary.

Corollary 3.5. *We have*

$$L(s, \mathrm{Sym}^k(St_{2,p})) = L(s, St_{k+1,p}) = \zeta_p\left(s + \frac{k}{2}\right)$$

and in general

$$L(s, \mathrm{Sym}^k(\chi_p \otimes St_{2,p})) = L(s, \chi_p^k \otimes St_{k+1,p}) = L\left(s + \frac{k}{2}, \chi_p^k\right).$$

3.2.4. *The root number and conductor of the Steinberg representation.* We now want to compute the root number and conductor of either the symmetric powers of the Steinberg representation $\mathrm{Sym}^k(St_{2,p}) = St_{k+1,p}$ or the twisted version $\mathrm{Sym}^k(\chi_p \otimes St_{2,p}) = \chi_p^k \otimes St_{k+1,p}$. In order to consider both situations simultaneously, we will compute the conductor of a simple twist $\omega_p \otimes St_{k+1,p}$ which we can then specialize to $\omega_p = \mathbf{1}$ or $\omega_p = \chi_p^k$. In the situation in which we are interested, χ_p is quadratic and unramified, as is $\mathbf{1}$. Hence we can, and will, assume that ω_p is unramified. In order to simplify notation, let us also write, as is common, $\omega_p \otimes St_{k+1,p} = \omega_p St_{k+1,p}$.

The root number and conductor are defined in terms of the ε-factor of a representation. In general, for an irreducible admissible generic representation π_p of $GL_n(\mathbf{Q}_p)$ and for a normalized unramified additive character ψ_p of $\mathbf{Q}_p$ we have,

$$\varepsilon(s, \pi_p, \psi_p) = \varepsilon(\tfrac{1}{2}, \pi_p, \psi_p) p^{-f(\pi_p)(s-1/2)}$$

with $\varepsilon(\frac{1}{2}, \pi_p, \psi_p)$ the local root number and $f(\pi_p)$ the conductor.

For the Steinberg and its twist, if we compute the associated γ-factor, which we can do by multiplicativity [13], then we have

$$\gamma(s, \omega_p St_{k+1,p}, \psi_p) = \frac{\varepsilon(s, \omega_p St_{k+1,p}, \psi_p) L(1-s, \omega_p^{-1} St_{k+1,p})}{L(s, \omega_p St_{k+1,p})}$$

and since we know the formula for the L-function we can then compute the ε-factor and hence the root number and conductor.

Since $\omega_p St_{k+1,p}$ is the irreducible quotient of $\mathrm{Ind}(\omega_p\nu^{-k/2}\otimes\cdots\otimes\omega_p\nu^{k/2})$ then by the multiplicativity of the local γ-factor [13], and assuming ψ_p is a normalized unramified character of $\mathbf{Q}_p$, we have

$$\begin{aligned}\gamma(s,\omega_p St_{k+1,p},\psi_p) &= \prod_{j=0}^{k}\gamma(s,\omega_p\nu^{j-(k/2)},\psi_p) = \prod_{j=0}^{k}\frac{\varepsilon(s,\omega_p\nu^{j-(k/2)},\psi_p)L(1-s,\omega_p^{-1}\nu^{j-(k/2)})}{L(s,\omega_p\nu^{j-(k/2)})}\\ &= \prod_{j=0}^{k}\frac{L\left(1-s+j-\frac{k}{2},\omega_p^{-1}\right)}{L\left(s+j-\frac{k}{2},\omega_p\right)}\\ &= \frac{L(1-s,\omega_p^{-1}St_{k+1,p})}{L(s,\omega_p St_{k+1,p})}\prod_{j=0}^{k-1}\frac{L\left(1-s+j-\frac{k}{2},\omega_p^{-1}\right)}{L\left(s+j-\frac{k}{2},\omega_p\right)}\end{aligned}$$

and thus

$$\varepsilon(s,\omega_p St_{k+1,p},\psi_p) = \prod_{j=0}^{k-1}\frac{L\left(1-s+j-\frac{k}{2},\omega_p^{-1}\right)}{L\left(s+j-\frac{k}{2},\omega_p\right)}.$$

For each index $0\leqslant j\leqslant k-1$ we have

$$\begin{aligned}L\left(1-s+j-\frac{k}{2},\omega_p^{-1}\right) &= \left(1-\omega_p^{-1}(p)p^{-(1-s+j-\frac{k}{2})}\right)^{-1}\\ &= \left[-\omega_p(p)p^{(1-s+j-\frac{k}{2})}\right]\left(1-\omega_p(p)p^{-s+(j+1)-\frac{k}{2}}\right)^{-1}\\ &= \left[-\omega_p(p)p^{(1-s+j-\frac{k}{2})}\right]L\left(s+\frac{k}{2}-(j+1),\omega_p\right).\end{aligned}$$

Since

$$\prod_{j=0}^{k-1}L\left(s+j-\frac{k}{2},\omega_p\right) = \prod_{j=0}^{k-1}L\left(s+\frac{k}{2}-(j+1),\omega_p\right)$$

we see that

$$\varepsilon(s,\omega_p St_{k+1,p}\psi_p) = \prod_{j=0}^{k-1}\left[-\omega_p(p)p^{(1-s+j-\frac{k}{2})}\right]$$

which simplifies to

$$\varepsilon(s,\omega_p St_{k+1,p},\psi_p) = (-\omega_p(p))^k p^{-k(s-1/2)}.$$

From here, we see that for the twisted Steinberg $\omega_p St_{k+1,p}$ the local root number and the conductor are

$$\varepsilon(\tfrac{1}{2},\omega_p St_{k+1,p},\psi_p) = (-\omega_p(p))^k \quad\text{and}\quad f(\omega_p St_{k+1,p}) = k.$$

If we specialize this to the cases of interest, we have the following proposition.

Proposition 3.6. *(i) The local root number and conductor for* $\mathrm{Sym}^k(St_{2,p}) = St_{k+1,p}$ *are*

$$\varepsilon(\tfrac{1}{2},\mathrm{Sym}^k(St_{2,p}),\psi_p) = (-1)^k \quad\textit{and}\quad f(\mathrm{Sym}^k(St_{2,p})) = k.$$

(ii) If χ_p *is the non-trivial unramified quadratic character of* $\mathbf{Q}_p^\times$*, so* $\chi_p(p) = -1$*, then the root number and conductor of* $\mathrm{Sym}^k(\chi_p St_{2,p}) = \chi_p^k St_{k+1,p}$ *are*

$$\varepsilon(\tfrac{1}{2},\mathrm{Sym}^k(\chi_p St_{2,p}),\psi_p) = (-1)^k\chi_p(p)^{k^2} = 1 \quad\textit{and}\quad f(\mathrm{Sym}^k(\chi_p St_{2,p})) = k.$$

4. Bounds for symmetric power L-functions

In this section we quote a few results about the size the individual values $L(s, \mathrm{Sym}^k f)$ when $\Re e(s)$ is close to 1. The first is the upper bound (1.3).

Lemma 4.1. *For $f \in S_2^p(q)$ satisfying Hypothesis $\mathrm{Sym}^k(f)$, one has*

$$L(s, \mathrm{Sym}^k f) \ll_k (\log q(|s|+2))^{k+1},$$

uniformly for $\Re e(s) \geqslant 1 - 1/\log(q(|s|+2))$.

Proof. It suffices to consider s such that $\frac{3}{2} > \Re e(s) \geqslant 1 - 1/\log(q(|s|+2))$, the complementary range being understood. Then by standard contour shifts one has for any $\varepsilon > 0$

$$\begin{aligned}\sum_{n\geqslant 1} \frac{\lambda_f^k(n)}{n^s} e^{-n/X} &= \frac{1}{2\pi i} \int_{(2)} L(u+s, \mathrm{Sym}^k f)\Gamma(u) X^u du \\ &= L(s, \mathrm{Sym}^k f) + \frac{1}{2\pi i} \int_{(1/2-\Re e(s))} L(u+s, \mathrm{Sym}^k f)\Gamma(u) X^u du \\ &= L(s, \mathrm{Sym}^k f) + O_{k,\varepsilon}\left((q|s|)^{\varepsilon} q^{k/4} |s|^{(k+1)/4} X^{1/2-\Re e(s)}\right);\end{aligned}$$

the last estimate coming from the convexity bound (see [22] for instance): for $\Re e(u) = 1/2$

$$L(u, \mathrm{Sym}^k f) \ll_{k,\varepsilon} q^{k/4+\varepsilon} |u|^{(k+1)/4+\varepsilon}.$$

The bound follows by taking $X = q^{k/2+1}$ and by using $|\lambda_f^k(n)/n^s| \leqslant d_{k+1}(n)/n^{\Re e(s)}$. □

Under the assumption that $L(s, \mathrm{Sym}^k f)$ has no zeros in the standard zero-free region, we derive the following lower bounds —these are certainly know to other people but we have not found it in the literature—

Lemma 4.2. *Let $f \in S_2^p(q)$ satisfying Hypothesis $\mathrm{Sym}^k(f)$. Suppose that there exists $A_k > 0$ such that $L(s, \mathrm{Sym}^k f)$ does not vanish in the interval $[1 - A_k/\log q, 1]$. Then there is a constant $C_k > 0$ such that*

$$L(s, \mathrm{Sym}^k f) \gg_k (\log q(|s|+2))^{-C_k},$$

uniformly for $\Re e(s) = 1$. Here the implied constant and C_k depend only on k and A_k.

Proof. For simplicity, we give the details of the arguments for $s = 1$ and present the main modification needed to get the general case.

The proof is a variant of Proposition 1.1 of [10]. Since $L(s, \mathrm{Sym}^k f)$ is automorphic, the standard Hadamard/de la Vallée-Poussin method (see [22] for instance) and our assumptions show that $L(s, \mathrm{Sym}^k f)$ does not vanish in the domain $\Re e(s) \geqslant 1 - A_k/\log(q(|s|+3))$ (up to a change in the definition of A_k).

Let $L^{1/(k+1)}(s, \mathrm{Sym}^k f)$ be the $(k+1)$-th root of $L(s, \mathrm{Sym}^k f)$ which takes the value 1 as $s \to +\infty$, one sees that $D(s) = \zeta(s) L^{1/(k+1)}(s, \mathrm{Sym}^k f)$ is holomorphic, except for a simple pole at $s = 1$, and, by Lemma 4.1, is bounded by by

$$D(s) \ll_k \log|s| \log(q(|s|+1))/|s-1| \tag{4.1}$$

in this domain. Moreover, by using the bound (which follows from the truth of the Ramanujan/Petersson conjecture) $|\mathrm{tr}(\mathrm{Sym}^k(g_f(p)^\alpha))| \leqslant k+1$ one sees that, for $\Re e(s) > 1$, $D(s)$ can be written as an absolutely convergent Dirichlet series

$$D(s)=\sum_{n\geqslant 1}\eta_f^k(n)n^{-s}$$

where the coefficients $\eta_f^k(n)$ are non-negative, bounded by the divisor function $d(n)=d_2(n)$ and such that $\eta_f^k(1)=1$. We set $\beta=1-1/\log^2(3q)$ and $X=\exp(\log^2 q)$ and we consider

$$\frac{1}{2\pi i}\int\limits_{(2)}D(s+\beta)\Gamma(s)X^s ds=\sum_{n\geqslant 1}\frac{\eta_f^k(n)}{n^\beta}e^{-n/X}\geqslant e^{-1/X}\gg 1.$$

We shift the line of integration to the left, up to the path given by

$$\begin{aligned}\mathcal{C}:=[-i\infty,-i\log^2 q]\cup[-i\log^2 q,-\frac{A_k'}{4\log(2q)}-i\log^2 q]\\ \cup[-\frac{A_k'}{4\log(2q)}-i\log^2 q,-\frac{A_k'}{4\log(2q)}+i\log^2 q]\\ \cup[-\frac{A_k'}{4\log(2q)}+i\log^2 q,+i\log^2 q]\cup[+i\log^2 q,+i\infty],\end{aligned}$$

and in doing so we hit a pole at $s=1-\beta$ and at $s=0$ getting

$$1\ll L^{1/(k+1)}(1,\mathrm{Sym}^k f)\Gamma(1-\beta)X^{1-\beta}+D(\beta)+\frac{1}{2\pi i}\int\limits_{\mathcal{C}}D(s+\beta)\Gamma(s)X^s ds.$$

Note that $D(\beta)<0$ (since $D(s)$ has a simple pole at $s=1$, takes positive values for $s>1$ and does not vanish on $[1-A_k/\log q,1]$) and moreover, by using (4.1) and Stirling's formula, the integral along $\mathcal{C}$ is bounded by

$$O_k\left(\log^4 q\exp\left(-\frac{A_k}{10}\log q\right)\right)=o_k(1).$$

Consequently, one gets

$$1\ll\frac{L^{1/(k+1)}(1,\mathrm{Sym}^k f)X^{1-\beta}}{1-\beta}\ll\frac{L^{1/(k+1)}(1,\mathrm{Sym}^k f)}{1-\beta}$$

which gives (1.5) with $C_k=2(k+1)$.

Given $s_0=1+i\tau_0$, the proof of the lower bound for $L(s_0,\mathrm{Sym}^k f)$ follows the same method as above, but with a different choice for the function $D(s)$: one considers the function

$$D(s)=\zeta(s)\left(L(s+i\tau_0,\mathrm{Sym}^k)L(s-i\tau_0,\mathrm{Sym}^k)\right)^{1/2(k+1)}$$

which is holomorphic for $\Re e(s)\geqslant 1-A_k/\log(q(1+|\tau_0|+|s|))$ and has non-negative Dirichlet coefficients. Taking $\beta=1-1/\log^2(q(2+|\tau_0|))$, the same method shows that

$$|L(1+i\tau_0,\mathrm{Sym}^k)|^2\gg_k 1/\log^{4(k+1)}(q(|\tau_0|+2)).$$

□

Remark 10. The proof of Lemma 4.2 uses an auxiliary Dirichlet series $D(s)$ with non-negative coefficients. In our case, we have taken advantage of the Ramanujan/Petersson conjecture to exhibit a simple choice of $D(s)$; however other choices are possible which enables one to avoid this conjecture (but then one has to assume other instances of functoriality): for example, one may consider

$$D(s) = L(s, \mathrm{Sym}^{k/2} f \times \mathrm{Sym}^{k/2} f) = \prod_{\substack{i=0,\dots,k\\ i \text{ even}}} L(s, \mathrm{Sym}^i f)$$

if k is even and if k is odd

$$D(s) = \Big(L\Big(s, (\mathrm{Sym}^{(k-1)/2} f \boxplus \mathrm{Sym}^{(k+1)/2} f) \times (\mathrm{Sym}^{(k-1)/2} f \boxplus \mathrm{Sym}^{(k+1)/2} f)\Big)\Big)^{1/2}.$$

One can improve substantially the above upper and lower bound when the L-function has a large zero-free region; this is the content in the following lemma. It is proved in [7] (Lemmas 8.1 and 8.2) in the case of L-functions of Dirichlet characters; the proof in our case is entirely similar (because the Ramanujan/Petersson conjecture holds in our case) so we omit it.

Lemma 4.3. *Given q squarefree and $f \in S_2^p(q)$ such Hypothesis $\mathrm{Sym}^k(f)$ is satisfied, let $s = \sigma + it$ with $\sigma > 1/2$ and $|t| \leqslant 2q$. Let $y \geqslant 2$ be a real number, and let $1/2 \leqslant \sigma_0 < \sigma$. Suppose that there are no zeros of $L(u, \mathrm{Sym}^k f)$ inside the rectangle $\{u :\ \sigma_0 \leqslant \Re e(u) \leqslant 1,\ |\Im m(u) - t| \leqslant y + 3\}$. Put $\sigma_1 = \min(\frac{\sigma+\sigma_0}{2}, \sigma_0 + \frac{1}{\log y})$. Then*

$$\log L(s, \mathrm{Sym}^k f) = \sum_{n=2}^{y} \frac{\Lambda_f^k(n)}{n^s \log n} + O\left(\frac{\log q}{(\sigma - \sigma_0)^2} y^{-(\sigma-\sigma_1)}\right).$$

Here the implied constant depends on k only and $\Lambda_f^k(n)$ is the n-th Dirichlet coefficient of minus the logarithmic derivative of $L(s, \mathrm{Sym}^k f)$: if n is not a prime power, $\Lambda_f^k(n) = 0$, otherwise

$$\Lambda_f^k(p^\alpha) = \mathrm{tr}(\mathrm{Sym}^k(g_f^\alpha(p))) \log p.$$

Corollary 4.4. *Given $\eta > 2(\log_2 q)^{-1}$ and $f \in S_2^p(q)$ satisfying Hypothesis $\mathrm{Sym}^k(f)$ and such that $L(s, \mathrm{Sym}^k f)$ has no zero in the rectangle $\{s :\ \Re e(s) \in [1-\eta, 1],\ |\Im m(s)| \leqslant \log^{2004/\eta} q\}$ then*

$$|\log L(s, \mathrm{Sym}^k f)| \ll_{k,\eta} \log_3(q)$$

uniformly for $\Re e(s) \geqslant 1 - 1/\log_2 q$ and $|\Im m(s)| \leqslant \log^{10} q$, the implied constant depending on k, η.

Proof. We apply Lemma 4.3 with $\sigma_0 = 1 - \eta$, $\sigma = \Re e(s) \geqslant 1 - 1/\log_2 q$, $\sigma - \sigma_0 \geqslant \sigma - \sigma_1 \geqslant \eta/2$ and $y = \log^{10/\eta} q$. Since $|\Lambda_f^k(p^\alpha)| \leqslant (k+1)\log p$ we have

$$\begin{aligned}|\log L(s, \mathrm{Sym}^k f)| &= \Big|\sum_{p=2}^{y} \frac{\Lambda_f^k(p)}{p^s \log p}\Big| + O_{\eta,k}(1)\\ &\leqslant (k+1) \sum_{p=2}^{\log^{10/\eta} q} \frac{1}{p^{1-1/\log_2 q}} + O_{\eta,k}(1) \ll_{k,\eta} \log_3 q.\end{aligned}$$

□

Next, we consider the problem of the existence of an exceptional zero for $L(s, \mathrm{Sym}^k f)$. As we have said, for $k = 1, 2, 4$, $L(s, \mathrm{Sym}^k f)$ has no exceptional zero. We consider the case $k = 3$ and prove

Lemma 4.5. *Given $f \in S_2^p(q)$, there exist an absolute constant $A > 0$ such that $L(s, \mathrm{Sym}^3 f)$ has a zero (β say) in the interval $[1 - A/\log q, 1]$ if and only if $L(s, \mathrm{Sym}^5 f)$ has a pole at β. Moreover if this zero (or pole) exists it is necessarily simple.*

Proof. Since $L(s, \mathrm{Sym}^3 f)$ is automorphic cuspidal, if it has an exceptional zero β, this zero is necessarily simple. By Rankin/Selberg theory, the L-function $L(s, \mathrm{Sym}^2 f \times \mathrm{Sym}^3 f)$ is holomorphic everywhere and factors as $L(s, f)L(s, \mathrm{Sym}^3 f)L(s, \mathrm{Sym}^5 f)$; this proves the "if" part since $L(s, f)$ has no exceptional zero.

To prove the "only if" part we follow the general strategy of [11], Section 4, and we consider the representation $\Pi = 1 \boxplus \mathrm{Sym}^2\pi_f \boxplus \mathrm{Sym}^3\pi_f$ and the L-function $L(s, \Pi \times \Pi)$, which has non-negative Dirichlet coefficients and factors as

$$\zeta(s)L(s, \mathrm{Sym}^2 f \times \mathrm{Sym}^2 f)L(s, \mathrm{Sym}^3 f \times \mathrm{Sym}^3 f)L(s, \mathrm{Sym}^2 f)^2 L(s, \mathrm{Sym}^3 f)^2 L(s, \mathrm{Sym}^2 f \times \mathrm{Sym}^3 f)^2$$
$$= \zeta(s)^3 L(s, f)^2 L(s, \mathrm{Sym}^2 f)^4 L(s, \mathrm{Sym}^3 f)^4 L(s, \mathrm{Sym}^4 f)^2 L(s, \mathrm{Sym}^5 f)^2 L(s, \mathrm{Sym}^6 f)^3.$$

We appeal to Theorem D of [27] which shows that $L(s, \mathrm{Sym}^6 f)$ is holomorphic on the interval $[1-A/\log q, 1]$ for some $A > 0$; then Lemma c of [11] concludes the "only if" part since $L(s, \Pi\times\Pi)$ has a pole of order 3, and an exceptional zero of $L(s, \mathrm{Sym}^3 f)$ would provide a zero of order 4 near 1 (the latter is not canceled by any exceptional pole by assumption). □

5. A ZERO DENSITY RESULT AND SOME CONSEQUENCES

5.1. **Multiplicity one for Symmetric powers.** A principal tool for our purpose is the zero density result below (Proposition 5.3) which shows that on average over $f \in S_2^p(q)$, the $L(s, \mathrm{Sym}^k f)$ have very few zeros close to 1. But first we need to show that any $L(s, \mathrm{Sym}^k f_0)$ cannot occurs with too high a multiplicity in the multiset

$$\{L(s, \mathrm{Sym}^k f) : \ f \in S_2^p(q)\};$$

in other words we need to be able to distinguish two modular forms f and g by their local symmetric k-th power lifts. This is the content of the next proposition and its corollary.

Proposition 5.1. *Let f and g be two holomorphic primitive forms of even weight $2l$ and trivial nebentypus, one of them, f say, not of CM type* [28]. *Suppose that, for some given $k \geqslant 1$, one has*

$$L_p(s, \mathrm{Sym}^k f) = L_p(s, \mathrm{Sym}^k g) \tag{5.1}$$

for every prime p outside a set of density 0. Then there exist a character χ, of order at most 2, such that $g = f \otimes \chi$. (If k is odd then χ is trivial.)

Such statement is certainly known to other people; in fact D. Ramakrishnan kindly informed us that he had a proof of this result. For squarefree levels one has the following immediate consequence.

Corollary 5.2. *Suppose moreover that the level of f and g are squarefree, then $f = g$.*

Proof. Our proof is by recursion on k. The case $k = 1$ is classical and the case $k = 2$ (from which the case $k = 1$ follows) was proved in a stronger form by Ramakrishnan [24, 25]. Our assumption is that outside a set of primes of density 0, the conjugacy classes $\mathrm{Sym}^k(g_f(p))^\natural$ and $\mathrm{Sym}^k(g_g(p))^\natural$ coincide, or in other terms one has equality of the multisets

$$\{\alpha_f^k(p), \alpha_f^{k-2}(p), \dots, \alpha_f^{-(k-2)}(p), \alpha_f^{-k}(p)\} = \{\alpha_g^k(p), \alpha_g^{k-2}(p), \dots, \alpha_g^{-(k-2)}(p), \alpha_g^{-k}(p)\}$$

We distinguish between two cases

(1) $\alpha_f^k(p) = \alpha_g^{\pm k}(p)$.
(2) $\alpha_f^k(p) = \alpha_g^{k'}(p)$ for some $k' \equiv k(2)$ with $|k'| < k$.

Suppose we are in Case 2: we may assume that $k'>0$. Suppose that k is odd. Then $\alpha_f(p)=\alpha_g^{k''}(p)$ for some $k''\equiv k(2)$ and $1<|k''|\leqslant k$ (otherwise we would be in the first case). Then we have

$$\alpha_g^{kk''-k'}(p)=1$$

and since $|k''|>1$ and $|k'|<k$, we have $kk''-k'\neq 0$ and hence $\alpha_f(p),\ \alpha_g(p)$ are roots of unity of order at most k^2+k. If k is even the same conclusion follows (more precisely $\alpha_f^2(p),\ \alpha_g^2(p)$ are roots of unity of order at most $(k/2)^2+(k/2)$). Next we show that Case 2 almost never happens: recall that for ℓ a prime, there exists a finite extension E_λ of $\mathbf{Q}_\ell$ and a Galois representation $\rho_f:\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})\to GL_2(E_\lambda)$, unramified at primes not dividing ℓq_f, such that for such p

$$\mathrm{tr}(\rho_f(\mathrm{Frob}_p))=(\sqrt{p})^{2l-1}(\alpha_f(p)+\alpha_f^{-1}(p))\quad\text{and}\quad\det(\rho_f(\mathrm{Frob}_p))=p^{2l-1}.$$

Since f is not of CM type, by a theorem of Ribet [28], $\mathrm{Im}(\rho_f)$ is open in $GL_2(E_\lambda)$. We consider the polynomial in two variables with coefficients in **Z**:

$$H(X,T)=\prod_{j\leqslant k^2+k}\prod_{\zeta\in\mu_j}\left(X^2-(\zeta+\zeta^{-1})^2T\right)=\prod_{j\leqslant k^2+k}\prod_{m=1}^{j}\left(X^2-4\cos^2\left(\frac{2\pi im}{j}\right)T\right)$$

where μ_j denotes the group of j-th roots of unity. If p is a prime satisfying Case 2 one has

$$H(\mathrm{tr}(\rho_f(\mathrm{Frob}_p)),\det(\rho_f(\mathrm{Frob}_p)))=0,$$

and by the method of Serre ([33], Section 7.3), the set of such primes has zero density. Thus outside a set of prime of density 0, we are in Case 1. Hence for such primes, one has the equality of the multisets

$$\{\alpha_f^{k-2}(p),\dots,\alpha_f^{-(k-2)}(p)\}=\{\alpha_g^{k-2}(p),\dots,\alpha_g^{-(k-2)}(p)\},$$

i.e., $L_p(s,\mathrm{Sym}^{k-2}f)=L_p(s,\mathrm{Sym}^{k-2}g)$, and we conclude by recursion. □

5.2. **Approximation of $L(s,\mathrm{Sym}^k f)^z$ by short sums of Hecke eigenvalues.** We can now state the following zero density result which is a immediate consequence of Theorem 2 of [19] and of Corollary 5.2.

Proposition 5.3. *For q a square-free integer, we assume that Hypothesis $\mathrm{Sym}^k(f)$ hold for all $f\in S_2^p(q)$. For $T\geqslant 1$ and $\sigma>1/2$, we denote by $N(\mathrm{Sym}^k f;\sigma,T)$ the number of zeros of $L(s,\mathrm{Sym}^k f)$ in the half-strip $R(\sigma,T):=\{s:\ \Re e(s)\geqslant\sigma,\ |\Im m(s)|\leqslant T\}$. Then for $\sigma>3/4$ there exist constants $A,B>0$ depending on k only such that*

$$\sum_{f\in S_2^p(q)}N(\mathrm{Sym}^k f;\sigma,T)\ll_k T^B q^{A(1-\sigma)/(2\sigma-1)}.$$

For instance one can take $A=5k(k+1)+2$.

Given a fixed $0<\eta<1/4$, we denote by $S_2^{p,+}(q)(\eta)$ (or $S_2^{p,+}(q)$ if the context is clear) the subset of $f\in S_2^p(q)$ such that $L(s,\mathrm{Sym}^k f)$ has no zeros in the half-strip $R(1-\eta,(\log q)^{2004/\eta})$ and denote by $S_2^{p,-}(q)(\eta)$ (or $S_2^{p,-}(q)$) the complementary subset. Hence we have

Corollary 5.4. *Let q be a square-free integer such that Hypothesis $\mathrm{Sym}^k(f)$ hold for all $f\in S_2^p(q)$. For any $0<\eta\leqslant 1/(2A+2)$ there exist $\delta=\delta(\eta)>0$ such that the number of $f\in S_2^p(q)$ for which $L(s,\mathrm{Sym}^k f)$ has at least one zero in the half-strip $R(1-\eta,(\log q)^{2004/\eta})$ is bounded by $\ll_{k,\eta} q^{1-\delta}$, i.e., $|S_2^{p,-}(q)|\ll_{k,\eta} q^{1-\delta}$.*

Remark 11. We would like to point out that the multiplicity one result for symmetric powers, Corollary 5.2, is important in establishing the zero density result above and the Corollary 5.4: this is tied in with the duality method used in the proof of the large sieve inequality leading to Theorem 2 of [19]. This issue seem to have been missed in [21] (see *loc. cit.* p. 225) in the case of symmetric square lifts of Maass forms; fortunately, multiplicity one for symmetric squares of Maass forms had been established later by Ramakrishnan [25]. In fact, for the proof of Corollary 5.4, it is sufficient —instead of Corollary 5.2— to have a bound of the form $O(q^{1-\delta})$ (for some $\delta > 0$ possibly depending on k) for the number of $f \in S_2^p(q)$ having a given symmetric k-th power lift. Such (much) weaker statement can be established easily by means of large sieve techniques (see [22] Lecture 3 for instance) : an amplifier can be built out of the following relation, valid for $f, g \in S_2^p(q)$ and any prime $p \not| q$ for which $L_p(s, \mathrm{Sym}^k f) = L_p(s, \mathrm{Sym}^k g)$,

$$\begin{aligned}\mathrm{tr}(\mathrm{Sym}^k(g_g(p)))\mathrm{tr}(\mathrm{Sym}^k(g_f(p))) + \sum_{j=1}^{k}(-1)\mathrm{tr}(\mathrm{Sym}^{2j}(g_f(p))) = \\ \lambda_g(p^k)\lambda_f(p^k) + \sum_{j=1}^{k}(-1)\lambda_f(p^{2j}) = 1\end{aligned} \tag{5.2}$$

(under the above assumptions, one has $\mathrm{tr}(\mathrm{Sym}^k(g_g(p))) = \mathrm{tr}(\mathrm{Sym}^k(g_f(p)))$, and (5.2) follows from the well known identity between $SL(2)$-representations

$$(\mathrm{Sym}^k)^{\otimes 2} = \bigoplus_{j=0}^{k} \mathrm{Sym}^{2j} \text{).}$$

This latter method is interesting as it can be applied to Maass forms as well, thus enabling one to generalize Theorem 1.2 to symmetric power lifts of Maass forms with large level or with large eigenvalues (as in [21]). Indeed, in the case of Maass forms, an unconditional proof of Proposition 5.1 seems problematic.

For the complementary subset $S_2^{p,+}(q)$, we have from Corollary 4.4,

$$|\log L(s, \mathrm{Sym}^k f)| \ll_{k,\eta} \log_3(q),$$

uniformly for $\Re e(s) \geqslant 1 - 1/\log_2 q$ and $|\Im m(s)| \leqslant \log^{10} q$.

Given $z \in \mathbf{C}$, we consider $L(s, \mathrm{Sym}^k f)^z$ which we write as a Dirichlet series

$$L(s, \mathrm{Sym}^k f)^z = \sum_{n \geqslant 1} \frac{\lambda_f^{k,z}(n)}{n^s},$$

say where the $\lambda_f^{k,z}(n)$ are the multiplicative functions given on prime powers by

$$\lambda_f^{k,z}(p^\alpha) = \lambda^{z,\alpha}_{\mathrm{Sym}^k}(g_f^\natural(p))$$

if $p \not| q$ and by

$$\lambda_f^{k,z}(p^\alpha) = d_z(p^\alpha)\lambda_f(p^{k\alpha}) \tag{5.3}$$

if $p|q$ (cf. (1.1)).

Proposition 5.5. *Under the assumptions of Proposition 5.3, there exist* $0 < \eta \leqslant 1/(2A+2)$ *such that for any* $z \in \mathbf{C}$*, any* $X \geqslant 1$*, and any* $f \in S_2^{p,+}(q)$*, one has*

$$L(1, \mathrm{Sym}^k f)^z = L_f^z(X) + R_f^z(X)$$

where

$$L_f^z(X)=\sum_{n\geqslant 1}\frac{\lambda_f^{k,z}(n)}{n}e^{-n/X} \tag{5.4}$$

and

$$R_f^z(X)\ll_{k,\eta}(\log^2(q)\log_2 q)\exp\left(O_k(|z|\log_3 q)-\frac{\log X}{\log_2 q}\right)+\exp\left(O_k(|z|\log_2 q)-\pi\frac{\log^2 q}{8}\right). \tag{5.5}$$

Proof. We use the integral representation

$$L_f^z(X)=\frac{1}{2\pi i}\int_{(2)}L(s+1,\mathrm{Sym}^k f)^z\Gamma(s)X^s ds.$$

Shifting the contour to $\mathcal{C}$, where $\mathcal{C}$ is the path joining

$$-i\infty,\ -i(\log q)^2,\ -\eta'-i(\log q)^2,\ -\eta'+i(\log q)^2,\ +i(\log q)^2,\ +i\infty$$

with $\eta'=1/\log_2(q)$, we obtain

$$L(1,\mathrm{Sym}^k f)^z=L_f^z(X)+R_f^z(X)$$

with

$$R_f^z(X)=\frac{1}{2\pi i}\int_{(\mathcal{C})}L(s+1,\mathrm{Sym}^k f)^z\Gamma(s)X^s ds.$$

To conclude the proof of (5.5), we use Stirling's formula together with Lemmas 4.1 and 4.2 when s is on the segments $[-i\infty,\ -i(\log q)^2]\cup[+i(\log q)^2,\ +i\infty]$ and Corollary 4.4 when s is on

$$[-\eta'-i(\log q)^2,-i(\log q)^2]\cup[-\eta'-i(\log q)^2,-\eta'+i(\log q)^2]\cup[-\eta'+i(\log q)^2,i(\log q)^2].$$

□

Finally we recall some easy bounds about the divisor functions $d_z(n)$ borrowed from [8]:

$$|d_z(n)|\leqslant d_{|z|}(n)\leqslant d_l(n) \tag{5.6}$$

for any integer $l\geqslant|z|$. For any positive numbers a,b one has $d_a(n)d_b(n)\leqslant d_{a+b}(n)$. Moreover one has

$$\sum_{n\leqslant X}d_l(n)/n\leqslant\left(\sum_{n\leqslant X}1/n\right)^l\leqslant(\log 3X)^l \tag{5.7}$$

and for any $r\geqslant 0$,

$$\begin{aligned}\sum_{n\geqslant 1}d_l(n)n^r e^{-n/X}\ll_r(2(r+1)X\log X)^{r+1}&\sum_{n\leqslant 2(r+1)X\log X}\frac{d_l(n)}{n}e^{-n/X}\\&+\sum_{n\geqslant 2(r+1)X\log X}\frac{d_l(n)}{n}e^{-n/(2X)}\end{aligned}$$

which gives

$$\sum_{n\geqslant 1}d_l(n)n^r e^{-n/X}\ll_r(X\log X)^{r+1}(2\log X)^l. \tag{5.8}$$

6. Computation of the moments

6.1. **Proof of Theorem 1.2.** Our goal is to evaluate

$$M_k^z(q)=\sum_{f\in S_2^p(q)}^h L(1,\mathrm{Sym}^k f)^z.$$

The strategy is well known and as follows: we split the summation between the subset $S_2^{p,+}(q)$ and $S_2^{p,-}(q)$. The latter sum is estimated trivially using (1.3) or (1.5) and Corollary 5.4. For $f\in S_2^{p,+}(q)$, Proposition 5.5 shows that $L(1,\mathrm{Sym}^k f)^z$ is well approximated by a very short Dirichlet polynomial. We then average the corresponding short sums over the whole of $S_2^p(q)$ (the corresponding contribution from $S_2^{p,-}(q)$ being negligible) and use Petersson's formula to compute the main term.

We have

$$M_k^z(q)=M_k^{z,+}(q)+M_k^{z,-}(q)$$

with

$$M_k^{z,\pm}(q)=\sum_{f\in S_2^{p,\pm}(q)}^h L(1,\mathrm{Sym}^k f)^z.$$

By (1.5), Corollary 5.4 and (1.6) one find that

$$M_k^{z,-}(q)\ll q^{-\delta}(\log q)^{O_k(|\Re e(z)|)}$$

which is smaller than the error term in (1.7) if $|\Re e(z)|\leqslant|z|\ll_k\frac{\log q}{\log_2 q}$. To evaluate $M_k^{z,+}(q)$ we use Proposition 5.5. We choose X to be some power of q (depending on k only) getting

$$M_k^{z,+}(q)=\sum_{f\in S_2^{p,+}(q)}^h L_f^z(X)+O\Bigg((\log^2 q\log_2 q)\exp\left(O_k(|z|\log_3 q)-\frac{\log X}{\log_2 q}\right)$$
$$+\exp\left(O_k(|z|\log_2 q)-\pi\frac{\log^2 q}{8}\right)\Bigg);$$

for $|z|\ll_k \log q/(\log_3 q\log_2 q)$ we see that the error term is bounded by $O_k\left(\exp(-\eta\frac{\log q}{\log_2 q})\right)$ for some $\eta>0$ depending on k only. In the main term of the right hand side above, we add back the contribution of the $f\in S_2^{p,-}(q)$ at the cost of an error bounded by

$$\sum_{f\in S_2^{p,-}(q)}^h|L_f^z(X)|\leqslant\sum_{f\in S_2^{p,-}(q)}^h\sum_{n\geqslant 1}\frac{d_{(k+1)|z|}(n)}{n}e^{-n/X}\ll\exp\left(O\left((k|z|+1)\log\log X-\frac{\delta}{2}\log q\right)\right)$$

on using (2.8), (5.6) and (5.7). Again this term is smaller than the error term of (1.7) if $|z|\ll_k \log q(\log_3 q\log_2 q)^{-1}$. We are left with evaluation the complete sum

$$\sum_{f\in S_2^p(q)}^h L_f^z(X)=\sum_n\frac{e^{-n/X}}{n}\sum_{f\in S_2^p(q)}^h\lambda_f^{k,z}(n).$$

We have by (2.11) and (5.3)

$$\lambda_f^{k,z}(n)=d_z(q^{v_q(n)})\lambda_f(q^{kv_q(n)})\prod_{\substack{p^\alpha||n\\ p\neq q}}\left(\sum_{k'\leqslant k\alpha}\mu^{z,\alpha}_{\mathrm{Sym}^k,\mathrm{Sym}^{k'}}\lambda_f(p^{k'})\right).$$

Hence by applying Petersson's formula (1.8) a main term arises from the n's coprime with q and is given by

$$\prod_{p^\alpha \| n} \mu^{z,\alpha}_{\mathrm{Sym}^k,\mathbf{1}} = \lambda^z_{\mathrm{Sym}^k}(n) =: \lambda^{k,z}(n)$$

and we have

$$\begin{aligned}\sum_{f\in S_2^p(q)}^h \lambda_f^{k,z}(n) =& \delta_{(n,q)=1} |S_2^p(q)|_h \lambda^{k,z}(n) \\ &+ O_k\left(\log(qn)\frac{n^{k/2}}{q^{3/2}} d_{|z|}(q^{v_q(n)}) \prod_{\substack{p^\alpha \| n\\ (p,q)=1}} \left(\sum_{k'\leqslant k\alpha} |\mu^{z,\alpha}_{\mathrm{sym}^k,\mathrm{sym}^{k'}}|\right)\right).\end{aligned}$$

By (2.12) one has

$$\sum_{k'\leqslant k\alpha} |\mu^{z,\alpha}_{\mathrm{Sym}^k,\mathrm{Sym}^{k'}}| \leqslant (k\alpha+1)^{1/2} d_{(k+1)|z|}(p^\alpha) \leqslant d_{(k+1)(|z|+1)}(p^\alpha)$$

so that

$$\begin{aligned}\sum_{f\in S_2^p(q)}^h L_f^z(X) &= |S_2^p(q)|_h \sum_{(n,q)=1} \frac{\lambda^{k,z}(n)}{n} e^{-n/X} + O_k\left(\sum_n \frac{e^{-n/X}}{n}\log(qn)\frac{n^{k/2}}{q^{3/2}} d_{(k+1)(|z|+1)}(n)\right) \\ &= |S_2^p(q)|_h \sum_{(n,q)=1} \frac{\lambda^{k,z}(n)}{n} e^{-n/X} + O_{k,\varepsilon}\left((qX)^\varepsilon \frac{X^{k/2+1}}{q^{3/2}} (\log X)^{O_k(1+|z|)}\right)\end{aligned}$$

for any $\varepsilon > 0$. In the first sum, we remove the constraint $(n,q)=1$ at the cost of an error term bounded by

$$\ll \sum_{q|n} \frac{d_{|z|(k+1)}(n)}{n} e^{-n/X} \ll_k \frac{(2\log X)^{|z|(k+1)+1}}{q}.$$

We now choose $X = q^\eta$ for some fixed $\eta \in]0, 3/(k+2)[$. Since $|z| \ll_k \log q/(\log_3 q \log_2 q)$ we see that these error terms are bounded by $O_k(q^{-\eta'})$ for some $\eta' > 0$ depending on k only. Now by a contour shift

$$\begin{aligned}\sum_{n\geqslant 1} \frac{\lambda^{k,z}(n)}{n} e^{-n/X} &= \frac{1}{2\pi i}\int_{(2)} L^z(1+s,\mathrm{Sym}^k)\Gamma(s)X^s ds \\ &= L^z(1,\mathrm{Sym}^k) + \frac{1}{2\pi i}\int_{(-\sigma)} L^z(1+s,\mathrm{Sym}^k)\Gamma(s)X^s ds.\end{aligned}$$

for $\sigma = 1/\log_2 q$. For $\Re e(s) = -\sigma$ one has

$$\begin{aligned}\log|L^z(1+s,\mathrm{Sym}^k)| &\leqslant |z| \sum_{p\leqslant(|z|+2)^{1/(1-\sigma)}} \log(1-\frac{1}{p^{1-\sigma}})^{-(k+1)} + \sum_{p>(|z|+2)^{1/(1-\sigma)}} \log\left(1+O\left(\frac{|z|^2}{p^{2-2\sigma}}\right)\right) \\ &\ll_k \frac{|z|^{1/(1-\sigma)}}{\sigma\log(2+|z|)},\end{aligned}$$

so that, for $|z| \ll_k \log q(\log_3 q \log_2 q)^{-1}$, one has

$$\frac{1}{2\pi i}\int_{(-\sigma)} L^z(1+s,\mathrm{Sym}^k)\Gamma(s)X^s ds \ll \exp\left(-\frac{\log X}{\log_2 q}+O_k\left(\frac{|z|^{1/(1-\sigma)}}{\sigma\log(2+|z|)}\right)\right)$$
$$= O_k\left(\exp\left(-\eta'\frac{\log q}{\log_2 q}\right)\right)$$

for some $\eta' > 0$ depending on k.

Collecting the above estimates, one sees that for $|z| \ll_k \log q(\log_3 q \log_2 q)^{-1}$ (where the constant implied is sufficiently small) there is $\delta = \delta(k) > 0$ such that

$$\sum_{f\in S_2^p(q)}^h L^z(1,\mathrm{Sym}^k f) = |S_2^p(q)|_h L^z(1,\mathrm{Sym}^k) + O_k\left(\exp\left(-\delta\frac{\log q}{\log_2 q}\right)\right). \quad \square$$

6.2. **Proof of Corollary 1.7.** Recall that $t \to L^{it}(1,\mathrm{Sym}^k)$ is the characteristic function of the random variable $\omega \to \log L(1,\mathrm{Sym}^k,\omega)$. In particular, the rapid decay of $L^{it}(1,\mathrm{Sym}^k)$ as $|t| \to +\infty$ given in (1.10), implies that the distribution function $x \to F(\mathrm{Sym}^k, x)$ is smooth with uniformly bounded derivative: indeed one has

$$F'(\mathrm{Sym}^k,x) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{-itx}L^{it}(1,\mathrm{Sym}^k)dt \ll_k 1.$$

We can now apply the Berry/Esseen inequality (see [36], §II.7.6) which gives

$$||F_q(\mathrm{Sym}^k) - F(\mathrm{Sym}^k)||_\infty \ll \frac{||F'(\mathrm{Sym}^k)||_\infty}{T} + \int_{[-T,T]}\Big|\frac{L_q^{it}(1,\mathrm{Sym}^k) - L^{it}(1,\mathrm{Sym}^k)}{t}\Big|dt$$

for any $T > 0$ and where the constant implied is absolute. We choose $T = c_k \log q(\log_3 q \log_2 q)^{-1}$ for some $c_k > 0$ small enough, set $T_0 = 1/\log^2 q$ and split the integral into

$$\int_{[-T,T]}\cdots = \int_{[-T_0,T_0]}\cdots + \int_{[-T,T]\setminus[-T_0,T_0]}\cdots$$

By Theorem 1.2, we obtain

$$||F_q(\mathrm{Sym}^k) - F(\mathrm{Sym}^k)||_\infty \ll_k \frac{\log_3 q\log_2 q}{\log q} + \exp\left(-\delta\frac{\log q}{\log_2 q}\right)$$
$$+ \int_{[-T_0,T_0]}\Big|\frac{L_q^{it}(1,\mathrm{Sym}^k) - L^{it}(1,\mathrm{Sym}^k)}{t}\Big|dt$$

for some $\delta = \delta(k) > 0$. We need to bound the remaining integral; to do this we repeat the argument leading to Theorem 1.2 but now taking into account the fact that $z = it$ is very small.

Lemma 6.1. *Given q a prime such that Hypothesis $\mathrm{Sym}^k(f)$ holds for every $f \in S_2^p(q)$ and such that Hypothesis $\mathrm{LSZ}^k(q)$ holds, one has uniformly for $|z| \leqslant 1/\log^2 q$*

$$L_q^z(1,\mathrm{Sym}^k) - L^z(1,\mathrm{Sym}^k) = O_k\left(\frac{|z|}{\log q}\right).$$

Proof. By Lemmas 4.1 and 4.2, one has for $|z| \leqslant 1/\log^2 q$

$$L^z(1,\mathrm{Sym}^k f) = 1 + z\log L(1,\mathrm{Sym}^k f) + O_k(|z|^2\log_2 q)$$

hence

$$\sum_{f\in S_2^p(q)}^h L^z(1,\mathrm{Sym}^k f) = |S_2^p(q)|_h + z \sum_{f\in S_2^p(q)}^h \log L(1,\mathrm{Sym}^k f) + O_k(|z|^2 \log_2 q).$$

As in Section 6.1, we split the middle sum on the left as

$$\sum_{f\in S_2^p(q)}^h \log L(1,\mathrm{Sym}^k f) = \sum_{f\in S_2^{p,+}(q)}^h \log L(1,\mathrm{Sym}^k f) + \sum_{f\in S_2^{p,-}(q)}^h \log L(1,\mathrm{Sym}^k f).$$

By Corollary 5.4, the last sum is bounded by $(\log_2 q)q^{-\delta}$ for some $\delta>0$. On the other hand, in the first sum, we use Lemma 4.3 to obtain

$$\sum_{f\in S_2^{p,+}(q)}^h \log L(1,\mathrm{Sym}^k f) = \sum_{f\in S_2^{p,+}(q)}^h \sum_{2\leqslant p^\alpha\leqslant y} \frac{1}{\alpha p^\alpha}\mathrm{tr}\left(\mathrm{Sym}^k(g_f^\alpha(p))\right) + O_k\left(\log^{-4} q\right)$$

with $y=\log^{10/\eta} q$. For $\alpha\geqslant 1$ we decompose the central function on G given by $g\to \mathrm{tr}(\mathrm{Sym}^k(g^\alpha))$ as a sum of irreducible characters

$$\mathrm{tr}\left(\mathrm{Sym}^k(g^\alpha)\right) = \sum_{k'\leqslant k\alpha} \eta^\alpha_{\mathrm{Sym}^k,\mathrm{Sym}^{k'}}\mathrm{tr}\left(\mathrm{Sym}^{k'}(g)\right) \tag{6.1}$$

with

$$\eta^\alpha_{\mathrm{Sym}^k,\mathrm{Sym}^{k'}} = \int_G \mathrm{tr}\left(\mathrm{Sym}^k(g^\alpha)\right)\overline{\mathrm{tr}\left(\mathrm{Sym}^{k'}(g)\right)}dg.$$

In particular

$$|\eta^\alpha_{\mathrm{Sym}^k,\mathrm{Sym}^{k'}}| \leqslant (k+1)(k'+1)\leqslant (k+1)(k\alpha+1).$$

We add back the corresponding contribution from $S_2^{p,-}(q)$ at the cost of a small error term:

$$\sum_{f\in S_2^{p,+}(q)}^h \sum_{2\leqslant p^\alpha\leqslant y} \frac{1}{\alpha p^\alpha}\mathrm{tr}\left(\mathrm{Sym}^k(g_f^\alpha(p))\right) = \sum_{f\in S_2^p(q)}^h \cdots - \sum_{f\in S_2^{p,-}(q)}^h \cdots$$

with, on using the bound $|\mathrm{tr}(\mathrm{Sym}^k(g_f^\alpha(p)))|\leqslant k+1$,

$$\sum_{f\in S_2^{p,-}(q)}^h \cdots = O_k\left((\log_3 q)q^{-\delta}\right)$$

for some $\delta>0$. From (6.1), we have

$$\mathrm{tr}\left(\mathrm{Sym}^k(g_f^\alpha(p))\right) = \sum_{k'\leqslant k\alpha} \eta^\alpha_{\mathrm{Sym}^k,\mathrm{Sym}^{k'}}\lambda_f(p^{k'})$$

and we are in position to apply Petersson's formula to obtain

$$\sum_{f\in S_2^p(q)}^h \sum_{2\leqslant p^\alpha\leqslant y} \frac{1}{\alpha p^\alpha}\mathrm{tr}\left(\mathrm{Sym}^k(g_f^\alpha(p))\right) = \sum_{2\leqslant p^\alpha\leqslant y} \frac{1}{\alpha p^\alpha}\eta^\alpha_{\mathrm{Sym}^k,\mathbf{1}} + O_k\left(\frac{y^{k/2}\log^3(qy)}{q^{3/2}}\right).$$

Notice that $\eta^1_{\mathrm{Sym}^k,\mathbf{1}}=0$, so that the main term can rewritten as

$$\begin{aligned}\sum_{2\leqslant p^\alpha}\frac{1}{\alpha p^\alpha}\eta^\alpha_{\mathrm{Sym}^k,\mathbf{1}}+O_k(y^{-1})&=\sum_{p\geqslant 2}\int_G\sum_{\alpha\geqslant 1}\frac{1}{\alpha p^\alpha}\mathrm{tr}\left(\mathrm{Sym}^k(g^\alpha)\right)dg+O_k(y^{-1})\\&=\sum_{p\geqslant 2}\int_G\log\left(\det\left(I-p^{-1}\mathrm{Sym}^k(g)\right)^{-1}\right)dg+O_k(y^{-1}).\end{aligned}$$

On the other hand, one has

$$\begin{aligned}L^z(1,\mathrm{Sym}^k)&=\exp\left(\log\left(\prod_p\int_G\det\left(I-p^{-1}\mathrm{Sym}^k(g)\right)^{-z}dg\right)\right)\\&=\exp\left(\sum_p\log\left(\int_G\det\left(I-p^{-1}\mathrm{Sym}^k(g)\right)^{-z}dg\right)\right)\end{aligned}$$

and

$$\begin{aligned}\int_G\det\left(I-p^{-1}\mathrm{Sym}^k(g)\right)^{-z}dg&=\int_G\left(1+z\log\left(\det\left(I-p^{-1}\mathrm{Sym}^k(g)\right)^{-1}\right)+O_k\left(\frac{|z|^2}{p^2}\right)\right)dg\\&=1+z\int_G\log\left(\det\left(I-p^{-1}\mathrm{Sym}^k(g)\right)^{-1}\right)dg+O_k\left(\frac{|z|^2}{p^2}\right),\end{aligned}$$

hence

$$\begin{aligned}L^z(1,\mathrm{Sym}^k)&=\exp\left(z\sum_p\int_G\log\left(\det\left(I-p^{-1}\mathrm{Sym}^k(g)\right)^{-1}\right)dg\right)+O_k(|z|^2)\\&=1+z\sum_p\int_G\log\left(\det\left(I-p^{-1}\mathrm{Sym}^k(g)\right)^{-1}\right)dg+O_k(|z|^2).\end{aligned}$$

Collecting the above estimates, we have

$$\begin{aligned}\frac{1}{|S_2^p(q)|_h}\sum_{f\in S_2^p(q)}^h L(1,\mathrm{Sym}^k f)^z&=L^z(1,\mathrm{Sym}^k)+|z|O_k\left(\frac{1}{y}+\frac{\log_2 q}{q^\delta}+\frac{y^{k/2}\log^3(qy)}{q^{3/2}}+|z|\right)\\&=L^z(1,\mathrm{Sym}^k)+O_k\left(\frac{|z|}{\log q}\right).\qquad\square\end{aligned}$$

This final estimate then concludes the proof of Corollary 1.7. □

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