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On Stability of Root Numbers

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To the Memory of Ilya Piatetski-Shapiro

Abstract. The purpose of this article is a brief review of the progress made on the question of stability of root numbers under twists by highly ramified characters. In particular, we discuss the problem for the cases of exterior and symmetric square factors, attached to irreducible admissible representations of $GL(n,F)$ and briefly sketch how they can be shown to be equal to their arithmetic counterparts, i.e., Artin factors. The analysis involved here is different from techniques used earlier and relies on the theory of germs of Bessel functions.

Introduction

One of the ingenious ideas of Piatetski–Shapiro in the study of analytic root numbers (or ε–factors) has been the use of the theory of Bessel functions to establish their stability under highly ramified twists, as was done in a joint work with the first author in the case of $SO(2n+1)$ [**7**]. This stability was designed to overcome the lack of a local Langlands correspondence (LLC) for $SO(2n+1)$ when applying converse theorems to prove functoriality from $SO(2n+1)$ to $GL(2n)$ using the integral representations of Novodvorsky, Ginzburg, and Soudry [**27, 14, 36**]. As events unfolded, the functorial transfer from the generic spectrum of classical groups to $GL(N)$ finally took place by combining converse theorems [**7, 8**] with the analytic properties of L–functions obtained from the Langlands–Shahidi method [**25, 33, 34**]. In doing this, the necessary stability results needed to be deduced for each case within the context of the Langlands-Shahidi method [**1, 6, 10, 11, 23**].

The purpose of this note is to review the problem of stability of root numbers and discuss the progress which has been made since its use in establishing functoriality as well as a new potential application. The problem of the equality of the arithmetic factors of Artin with the analytic factors defined by representations of local groups through the local Langlands correspondence [**16, 17, 35**] has now

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been reduced, in a number of previously unavailable cases, to a proof of the stability of these local factors for supercuspidal representations [**12**]. This approach to the equality of such arithmetic and analytic factors seems to be valid in some generality [**12, 35**]. We refer to Section 3 for a discussion of this in the cases of ε–factors attached to the exterior and the symmetric square L–functions for $GL(n)$. Stability is very useful whenever one is to use global functional equations to deduce information about local factors, including their comparisons, as we shall see below.

1. What is Stability?

We refer to [**35**] for a related article on stability and reciprocity.

Let F be a non–archimedean local field and let G be a connected reductive group over F. We will need to assume G has non–trivial F–rational characters, i.e., $X(G)_F \neq \{\mathbf{1}\}$. Choose and fix a non-trivial $\delta \in X(G)_F$, $\delta \neq \mathbf{1}$. The space $X(G)_F$ of rational characters is often one dimensional, as in the case of $GL(n)$, but when it is not there is often a natural choice for δ given the problem under consideration. Then $\delta(G(F)) \subset F^\times$ is open. Let χ be a highly ramified character of $F^\times$. Then $\chi \cdot \delta$ is what we call a *highly ramified character* of $G(F)$. We will suppress any dependence on δ.

Let ${}^LG$ be the L–group of G. Fix a non–trivial character ψ_F of F. Let r be a complex analytic representation of ${}^LG$ on a finite dimensional complex vector space V. Assume we have a good theory of L–functions for r, i.e., a pair of complex functions $L(s, \pi, r)$ and $\varepsilon(s, \pi, r, \psi_F)$ for every irreducible admissible representation π of $G(F)$, satisfying a number of local and global properties [**13, 24, 26**].

Let π_1 and π_2 be two irreducible admissible representations of $G(F)$. Let ω_{π_i} denote the central character of $\pi_i, i = 1, 2$. Stability asserts:

Stability of local factors. *Assume $\omega_{\pi_1} = \omega_{\pi_2} = \omega$. Then for every sufficiently highly ramified character χ of $G(F)$, with the ramification level depending on π_1 and π_2, one has*

$$L(s, \pi_1 \otimes \chi, r) = L(s, \pi_2 \otimes \chi, r) = 1 \tag{1.1}$$

and

$$\varepsilon(s, \pi_1 \otimes \chi, r, \psi_F) = \varepsilon(s, \pi_2 \otimes \chi, r, \psi_F). \tag{1.2}$$

Set

$$\gamma(s, \pi, r, \psi_F) = \varepsilon(s, \pi, r, \psi_F) L(1 - s, \pi, \tilde{r})/L(s, \pi, r). \tag{1.3}$$

Then stability requires

$$\gamma(s, \pi_1 \otimes \chi, r, \psi_F) = \gamma(s, \pi_2 \otimes \chi, r, \psi_F). \tag{1.4}$$

This is of course to match Deligne's "arithmetic" stability results for the Artin factors [**13**]. For this reason we will call our stability "analytic" stability whenever there is a cause for confusion.

Initial cases of analytic stability in any generality were proved for Rankin product L–functions for $GL(n) \times GL(m)$ by Jacquet and Shalika using theory of conductors [**21**]. The first case of analytic stability outside the case of $GL(n) \times GL(m)$ was proved by Cogdell and Piatetski–Shapiro for the case $G = SO(2n + 1)$ as part of their program to establish functorial transfer from generic forms on $SO(2n + 1, \mathbb{A})$

to $GL(2n, \mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of a number field [**7**]. The work in [**7**] relied on the integral representation established for $L(s, \pi, St)$ in [**27, 14, 36**], where π is a generic cusp form on $SO(2n+1, \mathbb{A})$. Here St stands for the $2n$–dimensional standard representation of $Sp(2n, \mathbb{C}) = {}^L SO(2n+1)$. Analytic stability for the standard L-functions of classical groups were established, via the doubling integrals of Piatetski-Shapiro and Rallis, by Rallis and Soudry [**28**] and Brenner [**2**].

The eventual proof of functoriality for the generic spectrum of the classical groups using converse theorems [**8, 9**], as envisioned by Cogdell and Piatetski–Shapiro, employed the theory of L–functions developed through the Langlands–Shahidi method [**31, 34**]. To accomplish this we needed analytic stability in the context of the Langlands-Shahidi method. The groundwork for this in general was laid out in [**33**] and then implemented for each family of classical groups, at first on a case-by-case basis and then in the generality needed for functorialiy in [**10, 11**]. This is explained in more detail in what follows. We refer to [**32, 35**] for some definitive results concerning (1.1) and to the article [**4**] in this volume for the relevant literature on functoriality in the cases of classical or GSpin groups to $GL(n)$.

A proof of analytic stability along the lines of [**33**] for $GL(n) \times GL(n)$ was the subject matter of the third author's Ph.D. thesis [**37**]. The case of $(GL(n) \times GL(m), GL(n+m))$, which gives the Rankin product L–functions for $GL(n) \times GL(m)$ studied by Jacquet and Shalika, is not a case of a self–associate parabolic subgroup (in the sense of Section 1.2 of [**34**] and Section 2 below) when $n \neq m$ and the formula in [**33**] does not directly apply. However, analytic stability for Rankin-Selberg convolutions when $n \neq m$, which is probably the hardest of the non–self–associate cases occurring in the Langlands-Shahidi method, now seems to fit in the realm of the approach in [**33**] when the corresponding integral representation of the inverse of local coefficient is properly interpreted. This is work in progress of the second two authors.

Beside its central and important applications in establishing functoriality already alluded to, there are other applications which also require the use of stability. As we mentioned earlier, present proofs of equality of root numbers from the arithmetic side (Artin factors) to the analytic ones are based on local–global arguments for which stability is indispensable (cf. Section 3 here). In the next section we will discuss steps taken in establishing stability in the generality of the factors defined by the Langlands–Shahidi method.

2. Towards a general stability

We will now briefly explain the progress made on stability in the context of the analytic factors defined by the Langlands–Shahidi method. We start with a quick review of the definition of these factors.

Let G be a connected reductive quasisplit algebraic group over a p–adic field F. Let B be a Borel subgroup of G defined over F and write $B = TU$, where T is a maximal torus of B and U its unipotent radical. Let A_0 be the maximal split subtorus of T. Fix a standard parabolic subgroup P of G with Levi decomposition $P = MN$, with M the Levi component and N the unipotent radical, such that $T \subset M$ and $N \subset U$. This Levi decomposition is unique as we fix $M \supset T$. To define the local factors we may assume P to be maximal. Let α be the unique simple root of A_0 in the Lie algebra of U whose root subgroup lies in N.

Let $A \subset A_0$ be the split component of M, i.e., the maximal split torus in the connected component of the center of M. Let $\mathfrak{a} = \mathrm{Hom}(X(M)_F, \mathbb{R})$ be the real Lie algebra of A and $\mathfrak{a}^*_{\mathbb{C}} = X(M)_F \otimes_{\mathbb{Z}} \mathbb{C}$ its complex dual. Let ρ be a half the sum of the roots in N. Let $s \in \mathbb{C}$ and set

$$\tilde{\alpha} = \langle \rho, \alpha \rangle^{-1} \rho. \tag{2.1}$$

Then $s\tilde{\alpha} \in \mathfrak{a}^*_{\mathbb{C}}$. Let (π, V_π) be an irreducible admissible representation of $M(F)$. If

$$H_M \colon M(F) \longrightarrow \mathfrak{a}$$

is defined by

$$\exp\langle \chi, H_M(m) \rangle = |\chi(m)|$$

for all $\chi \in X(M)_F$, then

$$m \mapsto \exp\langle \nu, H_M(m) \rangle$$

gives a character of $M(F)$ for every $\nu \in \mathfrak{a}^*_{\mathbb{C}}$. We let

$$I(\nu, \pi) = \mathrm{Ind}_{M(F)N(F)}^{G(F)} (\pi \otimes \exp\langle \nu, H_M(\cdot) \rangle) \otimes \mathbf{1}$$

be the unitarily induced representation of $G(F)$. We finally let $I(s, \pi) \colon = I(s\tilde{\alpha}, \pi)$. We will use $V(\nu, \pi)$ and $V(s, \pi)$ to denote the spaces of $I(\nu, \pi)$ and $I(s, \pi)$, respectively, if confusion arises. We refer to [**34**] for details.

If P' is another standard maximal parabolic of G, write P' in its Levi decomposition $P' = M'N'$ with Levi component M' and unipotent radical N' such that $T \subset M'$ and $N' \subset U$. Assume there exists a $\tilde{w} \in W(G, A_0)$, the Weyl group of A_0 in G, such that $M' = \tilde{w}(M)$. Let w be a representative for $\tilde{w}$ and set $N_w = (wN^- w^{-1}) \cap U$, where N^- is the opposite of N. Define the intertwining operator $A(\nu, \pi, w)$ by

$$A(\nu, \pi, w) f(g) = \int_{N_w(F)} f(w^{-1} n' g) dn', \tag{2.2}$$

for $f \in V(\nu, \pi)$. Then

$$A(\nu, \pi, w) f \in V(w(\nu), w(\pi)). \tag{2.3}$$

To avoid technicalities, let us assume $\nu = s\tilde{\alpha}$, $s \in \mathbb{C}$; then it can be shown that (2.2) converges absolutely for $\mathrm{Re}(s) >> 0$ and in fact for $\mathrm{Re}(s) > 0$ if π is tempered. We refer to Chapter 4 of [**34**] for details.

If we combine our character ψ_F of F with a F–splitting of G, we then get a non–degenerate character of $U(F)$ and by restriction one of $U_M(F)$, $U_M = U \cap M$, both of which we will denote by ψ. The representation π being ψ–generic means that there exists a non–zero Whittaker functional λ in the dual space V'_π such that

$$\lambda(\pi(u) v) = \psi(u) \lambda(v) \tag{2.4}$$

for $u \in U_M(F)$ and $v \in V_\pi$. We now assume π is ψ-generic. We can then define a ψ-Whittaker functional on the space $V(\nu, \pi)$ by

$$\lambda_\psi(\nu, \pi)(f) = \int_{N'(F)} \lambda(f(w^{-1} n')) \overline{\psi(n')} dn'. \tag{2.5}$$

This is the so called canonical (induced) Whittaker functional on $V(\nu, \pi)$. The definition of the local coefficients $C_\psi(\nu, \pi)$ is through the functional equation

$$C_\psi(\nu, \pi) \lambda_\psi(w(\nu), w(\pi)) \cdot A(\nu, \pi, w) = \lambda_\psi(\nu, \pi), \tag{2.6}$$

which holds by Rodier's theorem (cf. Chapter 3 of [**34**]). Finally, we set $\nu = s\tilde{\alpha}$ and define

$$C_\psi(s,\pi)\colon = C_\psi(s\tilde{\alpha},\pi). \tag{2.7}$$

Now, let ${}^LG$ and ${}^LM$ be the L–groups of G and M, respectively. In our setting ${}^LM$ is a Levi subgroup of ${}^LG$ and one can define a unipotent subgroup ${}^LN$ of ${}^LG$ so that ${}^LM{}^LN$ is a parabolic subgroup of ${}^LG$ with unipotent radical ${}^LN$ (cf. [**31**]). Let ${}^L\mathfrak{n}\colon = \mathrm{Lie}({}^LN)$ and let r be the adjoint action of ${}^LM$ on ${}^L\mathfrak{n}$. The irreducible constituents r_i, $i = 1,\ldots,m$, of r will be the restrictions of r to the subspaces

$$V_i = \{X_{\alpha^\vee} \in {}^L\mathfrak{n} \mid \langle\tilde{\alpha},\alpha\rangle = i\}. \tag{2.8}$$

One of the results proved in [**31**] is the existence of analytic L- and ε-factors for each triple (G,M,r_i). Their definition is inductive and follows the process explained in [**31**], using the main identity for γ-factors

$$\gamma(s,\pi,r_i,\psi_F) = \varepsilon(s,\pi,r_i,\psi)L(1-s,\pi,\tilde{r}_i)/L(s,\pi,r_i) \tag{2.9}$$

and the relationship between the γ-factors and local coefficients, stated as part of Theorem 3.5 in [**31**], namely

$$C_\psi(s,\pi) = \lambda_G(\psi_F,w_0)^{-1}\prod_{i=1}^{m}\gamma(is,\pi,\tilde{r}_i,\overline{\psi}_F). \tag{2.10}$$

Here w_0 is the long element of $W(G,A_0)$ modulo that of $W(M,A_0)$, i.e., $w_0 = w_{\ell,G}\cdot w_{\ell,M}^{-1}$, where the representatives are chosen as in Remark 8.2.1 of [**34**] and the constant $\lambda_G(\psi_F,w_0)^{-1}$ is a product of Langlands λ–functions [**24, 26, 30, 31**] (Hilbert symbols). These local factors satisfy all the desired properties, which include consistency with the global functional equation whenever π occurs as a local factor of a irreducible cuspidal automorphic representation of $M(\mathbb{A})$. By an inductive argument, to prove stability for each $\gamma(s,\pi,r_i,\psi_F)$ it is enough to prove it for local coefficients. Stability in our context then becomes:

Stability for local coefficients. *Given a pair of irreducible admissible ψ–generic representations π_1 and π_2 of $M(F)$ with the same central characters,*

$$C_\psi(s,\pi_1\otimes\chi) = C_\psi(s,\pi_2\otimes\chi) \tag{2.11}$$

whenever χ is a highly ramified character of $M(F)$ with ramification level depending on π_1 and π_2.

As experience has shown, at least in a number of important cases, this can be proved by expressing $C_\psi(s,\pi)$ as a Mellin transform of a partial Bessel function on $M(F)$. This was attained by establishing an integral representation for $C_\psi(s,\pi)^{-1}$ in [**33**], Theorem 6.2. The formula given there, and in Theorem 2.1 below, is under the assumption that P is self–associate, i.e., that $\overline{N} = w_0Nw_0^{-1} = N^-$, where N^- is the unipotent subgroup opposed to N.

We first recall the partial Bessel function involved. Let ω_π be the central character of π and define $w_0(\omega_\pi)(z) = \omega_\pi(w_0^{-1}zw_0)$. Given $s\in\mathbb{C}$, set $\pi_s = \pi\otimes q^{\langle s\tilde{\alpha},H_M(\cdot)\rangle}$ and define

$$\omega_{\pi_s}(z) = \omega_\pi(z)q^{\langle s\tilde{\alpha},H_M(z)\rangle}. \tag{2.12}$$

We refer to equation (2.1) for the definition of $\tilde{\alpha}$. Fix a sufficiently large open compact subgroup $\overline{N}_0 \subset \overline{N}$ and let φ denote its characteristic function.

For an open dense subset of $n \in N(F)$, we have a decomposition

$$w_0^{-1} n = m n' \overline{n}, \tag{2.13}$$

with $m \in M(F)$, $n' \in N(F)$, and $\overline{n} \in \overline{N}(F)$. This sets up a densely defined map

$$n \mapsto (m, \overline{n})$$

from $N(F)$ into $M(F) \times \overline{N}(F)$. While $n \mapsto \overline{n}$ is a bijection for all n and $\overline{n}$ satisfying (2.13), $n \mapsto m$ may not be one; see [**33**].

For $v \in V_{\pi_s}$ let $W_v(m) = \lambda(\pi_s(m)v)$ be a Whittaker function in the Whittaker model of π_s, defined with respect to the ψ-Whittaker functional in (2.4). We choose v such that $W_v(e) = 1$. Given $z \in Z_M(F)$, the center of $M(F)$, we define the partial Bessel function

$$j_{v,\varphi}(m, \overline{n}, z) := \int_{U_{M,n}(F)\backslash U_M(F)} W_v(mu^{-1}) \varphi(zu\overline{n}u^{-1}z^{-1}) \psi(u) du, \tag{2.14}$$

where $U_{M,n}$ is the stabilizer of n in U_M.

As before, let α be the unique simple root of T in U whose root-subgroup lies in N. We may assume $H^1(F, Z_G) = 1$, which we can attain by enlarging G without changing its derived group; this will not affect our results. Lemma 5.2 of [**33**] then implies existence of a map $\alpha^\vee$ from $F^\times$ into $Z_M^0 = Z_G(F)\backslash Z_M(F)$ such that $\alpha'(\alpha^\vee(t)) = t$, $t \in F^\times$, for any root α' of T that restricts to α. Let $x_\alpha = x_\alpha(\bar{n}) \in F$ denote the α–coordinate of $w_0^{-1}\overline{n}w_0 \in N(F)$ by means of our fixed splitting.

Given $y \in F^\times$, set

$$j_{v,\varphi}(m, \overline{n}, y) := j_{v,\varphi}(m, \overline{n}, a^\vee(y^{-1} \cdot x_\alpha)), \tag{2.15}$$

whenever $x_\alpha(\bar{n}) \neq 0$. We also let $Z_M^0 U_M(F)$ act on $N(F)$ by conjugation and write $Z_M^0 U_M(F)\backslash N(F)$ for the corresponding quotient space.

Theorem 2.1. *Suppose $\omega_\pi(w_0\omega_\pi^{-1})$ is ramified. Fix $y_0 \in F$ such that $ord_F(y_0) = -d - f$, where d and f are the conductors of ψ_F and $\omega_\pi^{-1} \cdot (w_0\omega_\pi)$, respectively. Then up to the abelian Tate γ–factor attached to $\omega_\pi \cdot (w_0\omega_\pi^{-1})$ and ψ_F,*

$$C_\psi(s, \pi)^{-1} \sim \int j_{\tilde{v},\varphi}(m, \overline{n}, y_0) \omega_{\pi_s}^{-1}(x_\alpha)(w_0\omega_{\pi_s})(x_\alpha) q^{\langle s\tilde{\alpha}+\rho, H_M(m)\rangle} d\dot{n}. \tag{2.16}$$

Here the integration is over $Z_M^0 U_M(F)\backslash N(F)$, x_α is embedded in $Z_M(F)$ through $a^\vee$ and $v = \tilde{v} \otimes q^{\langle s\tilde{\alpha}, H_M(\cdot)\rangle}$, i.e., $\tilde{v}$ is the vector in V_π that corresponds to v in V_{π_s}.

This result is Theorem 6.2 of [**33**]. It is Equation (2.16) which has been the main tool in proving stability in the context of the Langlands-Shahidi method. What one has to do is to prove an asymptotic expansion for the partial Bessel function $j_{\tilde{v},\varphi}$. In the cases of classical or GSpin groups, one basically needs to deal with $M = GL(1) \times G_1$, where G_1 is one of these groups, occurring as a maximal Levi subgroup inside a larger group G of the same type. The philosophy of expressing γ–functions as a Mellin transform of a partial Bessel function goes back to Cogdell and Piatetski–Shapiro [**7**] who proved such a formula as well as the asymptotic expansion for the corresponding partial Bessel functions when $G_1 = SO(2n+1)$. Using Equation (2.16), the corresponding stability for other cases arising in the proofs of functoriality was proved in [**1, 6, 10, 11, 23**].

3. Stability and equality of factors in the case $\gamma(s,\pi,\Lambda^2,\psi_F)$

Let ρ be an n–dimensional continuous Frobenius semisimple representation of the Weil–Deligne group W'_F and let $\pi(\rho)$ be the irreducible admissible representation of $GL(n,F)$ attached to ρ through the local Langlands correspondence (LLC) [**16, 17**]. The LLC should be *robust* when it comes to forming L-, ε-, and γ-factors associated to finite dimensional representations of $GL(n,\mathbb{C})$; it should respect various parallel operations on the arithmetic and analytic sides. As examples we have the exterior and symmetric square operations. If $\rho \in Rep_n(W'_F)$, then $\Lambda^2\cdot\rho$ and $\mathrm{Sym}^2\cdot\rho$ are again Galois representations of dimension $n(n\mp 1)/2$ and as such have associated L- and ε-factors as defined in [**13**]:

$$L(s,\Lambda^2\cdot\rho),\ \varepsilon(s,\Lambda^2\cdot\rho,\psi_F)\quad\text{and}\quad L(s,\mathrm{Sym}^2\cdot\rho),\ \varepsilon(s,\mathrm{Sym}^2\cdot\rho,\psi_F).$$

On the analytic side, we have the corresponding operations for $\pi(\rho)$ as defined in [**31**], namely

$$L(s,\pi(\rho),\Lambda^2),\ \varepsilon(s,\pi(\rho),\Lambda^2,\psi_F)\quad\text{and}\quad L(s,\pi(\rho),\mathrm{Sym}^2),\ \varepsilon(s,\pi(\rho),\mathrm{Sym}^2,\psi_F).$$

In [**12**] we present an approach to the following equality of arithmetic and analytic local factors.

Equality of local factors. *With ρ and $\pi(\rho)$ as above, we have*

$$\varepsilon(s,\Lambda^2\cdot\rho,\psi_F)=\varepsilon(s,\pi(\rho),\Lambda^2,\psi_F)\quad\textit{and}\quad\varepsilon(s,\mathrm{Sym}^2\cdot\rho,\psi_F)=\varepsilon(s,\pi(\rho),\mathrm{Sym}^2,\psi_F).$$

On both the arithmetic and analytic sides, we have the associated γ-factors defined through (2.9) and its arithmetic analogue. In fact, in [**12**], using a robust deformation argument, which should apply more generally whenever LLC is available, we give an argument for the equality

$$\gamma(s,\Lambda^2\cdot\rho,\psi_F)=\gamma(s,\pi(\rho),\Lambda^2,\psi_F). \tag{3.1}$$

The proof of this is reduced to a proof of stability for the case of irreducible ρ and thus only when $\pi=\pi(\rho)$ is supercuspidal [**16, 17**]. A similar equality can then be deduced for the symmetric square

$$\gamma(s,\mathrm{Sym}^2\cdot\rho,\psi_F)=\gamma(s,\pi(\rho),\mathrm{Sym}^2,\psi_F).$$

Here we use the factorizations

$$\gamma(s,\rho\otimes\rho,\psi_F)=\gamma(s,\Lambda^2\cdot\rho,\psi_F)\gamma(s,\mathrm{Sym}^2\cdot\rho,\psi_F) \tag{3.2}$$

and

$$\gamma(s,\pi\times\pi,\psi_F)=\gamma(s,\pi(\rho),\Lambda^2,\psi_F)\gamma(s,\pi(\rho),\mathrm{Sym}^2,\psi_F). \tag{3.3}$$

The left hand sides are equal by LLC, from which the equality for the symmetric square factors is deduced from (3.1). From the equality of the γ-factors, we can deduce the equality of the L-and ε-factors. The equality of the L-factors was previously established by Henniart [**18**].

In the case of the exterior square, the defining pair (M,G) giving $\gamma(s,\pi,\Lambda^2,\psi_F)$ can be taken to be $(GL(n)\times GL(1),GSp(2n))$ and this is in fact the self–associate pair chosen in [**12**]. The corresponding local coefficient then has $\gamma(s,\pi(\rho),\Lambda^2,\psi_F)$ as one of its factors, the other being $\gamma(s,\pi,St,\psi_F)$ whose stability is well–known, either directly [**21**], or through LLC [**13**]. (We remark that the pair $(GL(n),SO(2n))$, which also gives $\gamma(s,\pi,\Lambda^2,\psi_F)$, is not self–associate whenever n is odd which was the reason to choose the pair $(GL(n)\times GL(1), GSp(2n))$ instead.) The proof given in [**12**] can be reduced to a proof of stability for supercuspidal representations:

Stability for supercuspidals. *Let π_1 and π_2 be two irreducible supercuspidal representations of $GL(n,F)$ sharing the same central characters. Then for all suitably highly ramified characters χ*

$$\gamma(s,\pi_1\otimes\chi,\Lambda^2,\psi_F)=\gamma(s,\pi_2\otimes\chi,\Lambda^2,\psi_F). \tag{3.4}$$

Assuming this for the moment, the reduction roughly follows the following steps:

Following ideas of Harris [**15**] and Henniart [**18**] one needs to prove (3.1) for a basis of the Grothendieck ring of all the finite dimensional representations of W_F, e.g., monomial representations via Brauer's theorem. This can be done using local–global arguments by comparing global functional equations [**13, 24, 26, 31, 34**] for global objects in which ρ and $\pi=\pi(\rho)$ are local restrictions.

To implement this argument one needs the stable version of equation (3.1). More precisely, one needs

Proposition 3.1 (Stable equality). *Let $n\in\mathbb{N}$ be a positive integer. Let ρ be an n–dimensional irreducible complex representation of W_F. Then for each highly ramified character χ of $F^\times$.*

$$\gamma(s,\Lambda^2(\rho\otimes\chi),\psi_F)=\gamma(s,\pi(\rho)\otimes\chi,\Lambda^2,\psi_F) \tag{3.5}$$

The proposition is proved by induction on n, using multiplicativity [**13, 24, 26, 31, 34**] for both arithmetic and analytic factors, and a robust deformation argument which can be applied to many other situations. One first uses a local–global argument to prove:

Proposition 3.2 (Base point equality). *Fix $\omega_0\in\widehat{F^\times}$. There exists a pair (ρ_0,π_0), $\pi_0=\pi(\rho_0)$, where ρ_0 is an irreducible continuous n–dimensional representation of W_F, such that $\det\rho_0=\omega_0$ and (3.5) is valid for every character χ of $F^\times$.*

By arithmetic stability [**13**] we have

$$\gamma(s,\Lambda^2(\rho_0\otimes\chi),\psi_F)=\gamma(s,\Lambda^2(\rho\otimes\chi),\psi_F) \tag{3.6}$$

for every irreducible n–dimensional representation ρ as soon as χ is sufficiently highly ramified. By stability for supercuspidals, which we are assuming,

$$\gamma(s,\pi(\rho_0)\otimes\chi,\Lambda^2,\psi_F)=\gamma(s,\pi(\rho)\otimes\chi,\Lambda^2,\psi_F) \tag{3.7}$$

as soon as χ is sufficiently highly ramified. But now (3.5) for the single pair $(\rho_0,\pi(\rho_0))$, which is established in Proposition 3.2, implies the equality of left hand sides of (3.6) and (3.7) for this pair. Then the two stability results in (3.6) and (3.7) give Proposition 3.1. Once one has this stable equality, one proves (3.1) for monomial representations using the globalization of Harris [**15**] and Henniart [**18**], and the concomitant global functional equations, plus multiplicativity of the local factors and now the stable equality to isolate a single monomial local component.

We now discuss our ideas for a proof of analytic stability for supercuspidals. The techniques for proving this are very different from the steps discussed above. Here we have to use the analysis available from the Langlands–Shahidi method [**33**]. We will use the same integral (2.16) as we used in the cases of $GL(1) \times G$ needed for functoriality, but the analysis is more subtle as we discuss below. While in the cases of $GL(1) \times G$ that occurred in the cases of functoriality there were only two relevant Bruhat cells which support Bessel functions, the case at hand requires us to deal with all such cells. The work in [**7, 10, 11**] required a complicated corrective argument, moving from one cell to the other, even those which did not support Bessel functions, back and forth. What makes things work in our current approach is a germ expansion for full Bessel functions which was established in [**22**] and gives a conceptual approach to the asymptotics of the Bessel functions. (See the appendix [**19**] to [**12**] for an updated formulation.)

The representation π being supercuspidal allows one to write the partial Bessel function $j_{\tilde{v},\varphi}(m, \overline{n}, y_0)$, used in the integral representation (2.16) given in Theorem 2.1, as an orbital integral for the matrix coefficient of our supercuspidal representation. More precisely, there is a matrix coefficient f of π, a function of compact support modulo the center, which defines the corresponding Whittaker function in (2.14) via an integral

$$W_{\tilde{v}}(g) = \int_{U_M(F)} f(ug)\psi^{-1}(u)\ du$$

and one then replace $W_{\tilde{v}}$ in (2.14) by this expression in terms of f. This only gives a partial orbital integral (Bessel function) for f in the sense of [**22, 19**] due to the appearance of the cutoff function φ in (2.14). In particular, the germ expansions of [**22, 19**] do not directly apply.

In our first approaches to supercuspidal stability, we attempted to play the compact support of f modulo the center against the (seemingly weaker) compact support of the cutoff function φ in order to be able to remove the cutoff function from the integral. Then we would have expressed our resulting full Bessel function as an orbital integral in the sense of [**22, 19**] and applied their germ expansion to obtain the asymptotics of the Bessel function as we approach other relevant Bruhat cells that could support Bessel functions. From this point on, the argument finished more or less as in the traditional proofs [**7, 5, 6**]. However, the analytic behavior of the cutoff function as we approach other relevant Bruhat cells turned out to be more subtle than we first thought and we have not been able to remove the cutoff function.

To incorporate the behavior of the cutoff function on the other relevant cells, we propose the following approach. We introduce a family of Bessel functions, one attached to each Bruhat cell. Given a Bruhat cell Y, we will first use f and the cut–off function φ to define a smooth function of compact support $f_{\varphi,Y}$ on the closure $\overline{Y}$ of Y. Using Lemma 6.1.1 of [**3**], we then extend $f_{\varphi,Y}$ to a smooth function $\tilde{f}_{\varphi,Y}$ of compact support modulo center on all of $GL(n, F)$. For each $\tilde{f}_{\varphi,Y}$, we then define the full Bessel function $j_{\tilde{f}_{\varphi,Y}}(m)$ as usual. (Although $\tilde{f}_{\varphi,Y}$ has φ built in its definition, what we will consider is the full integration of the corresponding Whittaker function on the maximal unipotent subgroup. It is for this reason that we will call $j_{\tilde{f}_{\varphi,Y}}$ a "full" Bessel function.)

The next step is truncation. Write

$$GL(n,F) = \coprod_i Y_i$$

in its disjoint Bruhat decomposition. We use the standard strict partial order between indices i given by $i > j$ if and only if Y_j lies in the boundary of Y_i. We now fix a set of open neighborhoods V_i of the Bruhat cells $Y_i, V_i \supset Y_i$. We will choose $\{V_i\}$ in such a way that $V_i \cap V_j = \emptyset$ unless $i > j$ or $j > i$. Moreover, we may assume each V_i is invariant under left and right multiplications by U_M. To wit, one can take the union of all $x.V_i'.y$ as x and y range over U_M, where V_i' is a neighborhood of Y_i. Each Y_i can be reached by letting suitable minors of elements in V_i, or equivalently V_i', go to zero, thanks to the invariance of these minors under two sided U_M-multiplications. We then set

$$U_i = V_i - \bigcup_{j<i} V_i \cap V_j = V_i - \bigcup_{j<i} V_j.$$

The sets U_i are all U_M invariant on both sides as well as disjoint. The disjoint collection $\{U_M \backslash U_i / U_M\}$ will now basically give our domains of integration of the corresponding Bessel functions $j_{\tilde{f}_{\varphi,Y_i}}$.

To prove stability, Theorem 2.1 tells us that we have to subtract the formula (2.16) for two supercuspidal representations π_1 and π_2 with equal central characters $\omega_{\pi_1} = \omega_{\pi_2}$. We will need to consider the integral of the difference of the Bessel functions $j_{\tilde{f}_{\varphi,Y_i}}$ for a pair of matrix coefficients of π_1 and π_2, respectively, whenever we are in a neighborhood of the cell Y_i. These Bessel functions $j_{\tilde{f}_{\varphi,Y_i}}$ are now full orbital integrals in the sense of Jacquet and Ye [**22, 19**] to which their germ expansion applies. Taking the difference of Bessel functions for π_1 and π_2 cancels the non–smooth contributions, which come only from the trivial cell part of the germ expansions for the Bessel functions and depend only on the central characters. The result is then sums of integrals of products of germ functions, which are smooth and independent of π_1 and π_2, times certain smooth functions of compact support which do depend on π_1 and π_2, on a pair of complementary tori (cf. [**12, 22, 19**]). Since our Bessel functions, or more appropriately their germ expansions, are in fact twisted by highly ramified characters in (2.16), the integral of each difference of Bessel functions $j_{\tilde{f}_{\varphi,Y_i}}$ will now vanish under these highly ramified twists. Passing to the limit by letting each neighborhood V_i reach their cell Y_i, we should then get (3.4), i.e., the stability for supercuspidals in this case. We are still working through the details of this approach.

One consequence of equality (3.1) would be the stability in general in this case, i.e., when π_1 and π_2 are any pair of irreducible admissible representations of $GL(n,F)$ with $\omega_{\pi_1} = \omega_{\pi_2}$.

General analytic stability. *Let π_1 and π_2 be any pair of irreducible admissible representations of $GL(n,F)$ with $\omega_{\pi_1} = \omega_{\pi_2}$. Then stability holds, i.e.,*

$$\gamma(s, \pi_1 \otimes \chi, \Lambda^2, \psi_F) = \gamma(s, \pi_2 \otimes \chi, \Lambda^2, \psi_F)$$

for every suitably ramified character χ.

This would follow from (3.1) and stability of Artin factors proved in [**13**].

4. Concluding Remarks

The technique developed in [**12**] following the formula (2.16), together with germ expansions of [**22, 19**], has opened the door to the possibility of a general approach to the problem of stability in the generality of factors defined by the Langlands–Shahidi method, at least for supercuspidal representations. The local–global arguments of [**12**], which were briefly discussed in Section 3 here, may lead to the equality of arithmetic and analytic factors from which general analytic stability would follow.

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