

SOME CONTINUED FRACTION EXPANSIONS OF LAPLACE TRANSFORMS OF ELLIPTIC FUNCTIONS

A Thesis

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of the Ohio State University

By

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The Ohio State University
2002

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ABSTRACT

In a 1907 paper, L. Rogers used two methods to obtain continued fractions for certain Laplace transforms of Jacobi elliptic functions. His first method employed repeated integration by parts, while his second method recalled an 1889 technique of T. Stieltjes. In 1996, S. Milne used these expansions and others obtained by modular transformations to derive results about sums of squares and triangular numbers.

Working independently in the 1820's, C. Jacobi and N. Abel both introduced elliptic functions to advance the study of elliptic integrals. In 1981, D. Dumont introduced symmetric variants of the elliptic functions of Jacobi and Abel to facilitate the study of certain combinatorial problems related to coefficients in Maclaurin expansions of Jacobi elliptic functions.

In this thesis, we use Dumont's elliptic functions to rederive the continued fraction expansions of Rogers. In the classical approach used by Rogers and Milne, four families of continued fractions are obtained. In our approach, members of the same four families are derived directly by specializing parameters instead of employing modular transformations.

To these four families, we add a new set of continued fractions based on certain elliptic functions that were studied in an 1890 paper by A. Dixon. These new continued fractions were discovered in 1999 in the course of work with D. Dumont.

In memoriam patris

Glenn Lee Conrad

1921 - 1999

ACKNOWLEDGMENTS

I am deeply indebted to Dominique Dumont (Université Louis Pasteur, Strasbourg) for sharing his work with me during his visit to the Ohio State University in April 1999. During the course of that visit, he referred me to an 1890 paper of A. C. Dixon that forms a centerpiece in this thesis. I believe or at least hope that his influence is also felt here in less tangible ways. The visit was made possible in part by the enthusiasm and encouragement of Dominique Foata (also of Université Louis Pasteur). Both of them have my profoundest gratitude for all their support.

The work in chapter 2 of this thesis depends very heavily on the work of my advisor Stephen Milne. He has my gratitude both for having shared parts of that work in classes and for more mundane help such as proofreading drafts of this thesis. He also has my thanks for his general support and his patience over these past years.

I thank the members of my Examination Committee (Thomas Dowling, Surinder Sehgal, and Stephen Milne of the Department of Mathematics and Professor L. Mark Berliner of the Department of Statistics) for their comments, criticisms and suggestions.

I also thank Dijen Ray-Chauduri (Department of Mathematics, The Ohio State University) and Jerry Downhouser (Department of Evolution Ecology and Organismal

Biology, The Ohio State University) for taking time to review a preliminary draft of this thesis and offering their comments and suggestions.

This work was supported in part by National Security Agency grants MDA 904-93-H-3032, MDA 904-97-1-0019 and MDA 904-99-1-0003, and by National Science Foundation grant DMS-0100288.

This document was prepared using $\mathcal{A}\mathcal{M}\mathcal{S}$ - \LaTeX , the American Mathematical Society's \LaTeX macro system.

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PUBLICATIONS

1. G. Allison, A. Ash and E. Conrad. “Galois representations, Hecke operators, and mod- p cohomology of $GL(3, \mathbf{Z})$ with twisted coefficients”. *Experimental Mathematics*, **7:4** (1998), 361–390.

FIELDS OF STUDY

Major field: Mathematics

Specialization: Elliptic Functions

TABLE OF CONTENTS

	Abstract	ii
	Dedication	iii
	Acknowledgments	iv
	Vita	vi
	List of Tables	x
	Preface	xi
CHAPTER		
	PAGE	
1	Background	1
	1.1 Historical Background	1
	1.2 Jacobian Elliptic Functions	2
	1.3 Dixon's Elliptic Functions	8
	1.4 General Convergence	10
	1.5 The Formal Laplace Transform	13
	1.6 Continued Fractions and Hankel Determinants	15
	1.7 Basic Hypergeometric Series	20
	1.8 Relation to Recent Work	21
2	Continued Fractions for the Jacobian Functions	24
	2.1 Associated Continued Fractions	25
	2.2 Regular C-Fractions	34
	2.3 Modular Transformations	36
	2.4 Other Formulations	43

3	Continued Fractions for the Dixon Functions	46
3.1	Laplace Transform Recurrences	46
3.2	Associated Continued Fractions ($\alpha = 0$)	47
3.3	Regular C-Fractions ($\alpha = 0$)	56
3.4	Quasi C-Fractions for the Dixon Functions	60
3.5	Quasi Associated Continued Fractions	63
4	Analytical Issues	68
4.1	Laplace Transforms	68
4.2	Power Series	69
4.3	Continued fractions	70
5	Future Directions	73
APPENDICES		
A	Continued Fraction Iteration	76
A.1	Three Term Recurrences	76
A.2	C-Fractions	77
A.3	Associated Continued Fractions	78
A.4	Relation Between the Two Types	79
B	Canonical Forms	80
B.1	Regular C-Fractions	82
B.2	Associated Continued Fractions	83
	Bibliography	85

LIST OF TABLES

TABLE		PAGE
1.1	General convergence norm examples	11
B.1	Regular C-fractions	82
B.2	Associated continued fractions (Jacobi type)	83
B.3	Associated continued fractions (Dixon type)	84

PREFACE

In [25], L. J. Rogers used two methods to obtain continued fraction expansions of Laplace transforms for certain Jacobi elliptic functions. In his first method, he used integration by parts to obtain the expansion. His second method was a rediscovery of a technique introduced by T. J. Stieltjes in [28]. Stieltjes and Rogers found three families of continued fraction expansions. (A fourth family, implicit in the work of Rogers, was discovered almost a century later by Ismail and Masson [15].) In 1845, J. Heilermann showed that regular C-fraction expansions and associated continued fraction expansions could be used to evaluate certain persymmetric or Hankel determinants (see W. B. Jones and W. J. Thron [17]). S. Milne in [21, 22] obtained additional expansions and Hankel determinants using well-known modular transformations of elliptic functions (*e.g.* see [4, 7, 13, 18, 30]) which he used to help prove some results about sums of squares and triangular numbers. These results include non-trivial exact explicit infinite families of identities enumerating the number of ways to write an integer as the sum of $4N^2$ or $4N(N + 1)$ squares of integers, where N is an arbitrary positive integer.

In [26, 27], A. Schett used recurrence relations to obtain general formulas for

Taylor expansions of the Jacobi elliptic functions. D. Dumont, in [9], defined multi-modular generalizations of the Jacobi functions which he used to interpret Schett's results combinatorially.

In [8], A. Dixon conducted an extensive study of a family of elliptic functions arising from the cubic curve

$$x^3 + y^3 - 3\alpha xy = 1.$$

According to Dixon:

“The only direct references that I have come across elsewhere are certain passages in the lectures of Professor Cayley and Mr. Forsyth where the integral $\int \frac{dx}{(1-x^3)^{2/3}}$ was used to illustrate Abel's Theorem, in the treatises of Legendre, and Briot and Bouquet, and again in Professor Cayley's lectures and elsewhere where it is shewn how to turn the integral into elliptic form, and lastly at the end of Bobek's *Einleitung in die Theorie der elliptischen Functionen* where expressions are found for the coordinates of any point on the above cubic.”

Dixon's paper was brought to my attention by D. Dumont during an April 1999 visit to Ohio State University. On investigating the Dixon functions during the course of Dumont's visit, I found that they give rise to a previously unknown family of continued fraction expansions.

In Chapter 2 of this thesis, I apply the integration by parts method of Rogers to a more symmetric variant of the standard Jacobi elliptic functions to recover the three families of continued fractions of Rogers and Stieltjes and, also, a fourth family

which was implicit in the work of Rogers but eluded discovery until 1998 (Ismail and Masson [15]). These four families fall naturally into two sets of two. In Chapter 3, I develop new analogues of these continued fractions using the Dixon elliptic functions. These analogues fall into six families which are naturally grouped as two sets of three. The underlying symmetries suggest that these classifications are fundamental and complete.

The symmetries of the notation for elliptic functions used in Dumont [9] lead to elementary derivations of Landen's transformation and of Gauss's transformation. Moreover in this setting, Gauss's transformation may be written in a way which emphasizes its fundamental connection to the arithmetic-geometric mean, a connection heavily exploited by Jacobi in the *Fundamenta Nova* [16]. These results appear for the first time in Chapter 2 of this thesis.

CHAPTER 1

BACKGROUND

1.1 Historical Background

In [25], L. J. Rogers used two methods to obtain continued fraction expansions of Laplace transforms for certain Jacobi elliptic functions. In his first method, he used integration by parts to obtain the expansion. His second method was a rediscovery of a technique introduced by T. J. Stieltjes in [28]. Stieltjes and Rogers found three families of continued fraction expansions, and a fourth family, implicit in the work of Rogers, was discovered almost a century later by Ismail and Masson [15]. In 1845, J. Heilermann showed that regular C-fraction expansions and associated continued fraction expansions are intimately related to the evaluation of certain persymmetric or Hankel determinants (see [17]). S. Milne in [22] obtained additional expansions and Hankel determinants using well-known modular transformations of elliptic functions (*e.g.* see [4, 7, 13, 18, 30]) and Heilermann's correspondence. He used these expansions and determinants to help derive infinite families of non-trivial exact explicit identities enumerating sums of squares and of triangular numbers. (These identities include explicit enumerations of the number of ways to write an integer as the sum of $4N^2$

perfect squares, and as the sum of $4N(N+1)$ perfect squares, where N is an arbitrary positive integer.)

In [26, 27], A. Schett (and later, G. Viennot [29], D. Dumont [9], and Lomont and Brillhart [19],) used recurrence relations to obtain general formulas for Taylor expansions of the Jacobi elliptic functions. D. Dumont, in [9], defined variants and generalizations of the Jacobi functions which he used to interpret Schett's results combinatorially.

During a 1999 visit to the Ohio State University, Dumont directed me to an 1890 treatise [8] by A. Dixon on a family of elliptic functions arising from the cubic curve $x^3 + y^3 - 3\alpha xy = 1$. On investigating the Dixon functions, I found that they give rise to a previously unknown family of continued fraction expansions which I develop in Chapter 3 of this thesis.

1.2 Jacobian Elliptic Functions

In the 1820's, N. Abel (see [1]) and C. Jacobi (see [16, §17]) inverted elliptic integrals of the first kind to yield what, under Jacobi's influence, came to be known as elliptic functions. (Bowman [4] introduces elliptic functions in this manner.) Jacobi [16, §60–66] also showed that elliptic functions are quotients of theta functions, solutions to the heat equation. (For a modern treatment which develops elliptic functions in this manner, see Lawden [18].) For our purposes, it is convenient to develop elliptic functions directly as solutions to a coupled family of ordinary differential equations following the treatments in Abel [1] and, much more recently, Dumont [9].

Because of the symmetries, we have chosen to use notation due to Dumont [9] over the standard notation (essentially due to Jacobi, with simplifications by Gudermann and Glaisher), and over related formulations due to Abel and to Neville. This choice is aesthetic. Of these four competing formulations, the standard Jacobi-Gudermann-Glaisher is the poorest choice because it hides symmetry. The notation of Neville [24] is cumbersome because it is unnormalized. Abel's notation is closest in spirit to that used by Dumont – Abel's functions are built with a fundamental antisymmetry where Dumont's exhibit a simple symmetry. The four systems of notation are trivially equivalent – Dumont's system simply exhibits the most pleasing symmetries.

Let $a, b \in \mathbb{C}$ be given. Let $f, g, h : \mathbb{C} \rightarrow \mathbb{C}$ be the power series solution to the ordinary differential equations:

$$f'(u) = g(u) h(u) \qquad g'(u) = a^2 f(u) h(u) \qquad h'(u) = b^2 f(u) g(u)$$

subject to the following conditions:

$$f(0) = 0 \qquad g(0) = 1 \qquad h(0) = 1.$$

Dumont's bimodular Jacobi elliptic functions are the functions f , g , and h above. Specifically, Dumont in [9] defines:

$$\operatorname{sn}(u; a, b) = f(u) \qquad \operatorname{cn}(u; a, b) = g(u) \qquad \operatorname{dn}(u; a, b) = h(u).$$

Thus:

$$\frac{d}{du} \operatorname{sn}(u; a, b) = \operatorname{cn}(u; a, b) \operatorname{dn}(u; a, b) \tag{1.1}$$

$$\frac{d}{du} \operatorname{cn}(u; a, b) = -a^2 \operatorname{sn}(u; a, b) \operatorname{dn}(u; a, b) \tag{1.2}$$

$$\frac{d}{du} \operatorname{dn}(u; a, b) = b^2 \operatorname{sn}(u; a, b) \operatorname{cn}(u; a, b) \quad (1.3)$$

where:

$$\operatorname{sn}(0; a, b) = 0 \text{ and } \operatorname{cn}(0; a, b) = \operatorname{dn}(0; a, b) = 1. \quad (1.4)$$

From these definitions, one can develop a theory of elliptic functions analogous to that of the trigonometric functions. We make use of the following results in the sequel:

$$\operatorname{dn}(u; a, b) = \operatorname{cn}(u; b, a) \quad \text{symmetry} \quad (1.5)$$

$$\operatorname{cn}^2(u; a, b) = 1 + a^2 \operatorname{sn}^2(u; a, b) \quad \text{Pythagorean theorem 1} \quad (1.6)$$

$$\operatorname{dn}^2(u; a, b) = 1 + b^2 \operatorname{sn}^2(u; a, b) \quad \text{Pythagorean theorem 2} \quad (1.7)$$

In addition, by observing that the triples $(\operatorname{sn}, \operatorname{cn}, \operatorname{dn})$, $(\operatorname{sc}, \operatorname{nc}, \operatorname{dc})$ and $(\operatorname{sd}, \operatorname{cd}, \operatorname{nd})$ of the classical Jacobian elliptic functions are all solutions of the above family of differential equations, we obtain the following specializations of the Dumont bimodular elliptic functions:

$$\begin{aligned} \operatorname{sn}(u, k) &= \operatorname{sn}(u; i, ik) & \operatorname{cn}(u, k) &= \operatorname{cn}(u; i, ik) & \operatorname{dn}(u, k) &= \operatorname{dn}(u; i, ik) \\ \operatorname{sc}(u, k) &= \operatorname{sn}(u; 1, k') & \operatorname{nc}(u, k) &= \operatorname{cn}(u; 1, k') & \operatorname{dc}(u, k) &= \operatorname{dn}(u; 1, k') \\ \operatorname{sd}(u, k) &= \operatorname{sn}(u; ik', k) & \operatorname{cd}(u, k) &= \operatorname{cn}(u; ik', k) & \operatorname{nd}(u, k) &= \operatorname{dn}(u; ik', k) \end{aligned} \quad (1.8)$$

Either branch of the square root function may be taken in choosing the complementary modulus $k' := \sqrt{1 - k^2}$, though by custom the principal branch is taken.

Dumont's functions are closer to the elliptic functions introduced by N. Abel in [1]. Both Jacobi and Abel first obtained elliptic functions by inverting elliptic integrals.

Jacobi in [16] inverted Legendre's elliptic integral of the first kind. Abel modified this same integral to introduce symmetry before he inverted it. Since Abel's formulation is not as well known as that of Jacobi, we review it here. Abel started with the elliptic integral

$$u = u(x; a, b) := \int_0^x \frac{ds}{\sqrt{(1 - a^2s^2)(1 + b^2s^2)}}.$$

Inverting this integral, he obtained elliptic functions which he called ϕ , f , and F :

$$\begin{aligned}\phi(u; a, b) &:= x \\ f(u; a, b) &:= \sqrt{1 - a^2x^2} \\ F(u; a, b) &:= \sqrt{1 + b^2x^2}\end{aligned}$$

with the branch of square root taken so that $f(0) = F(0) = 1$. Using formal methods, Abel reduced this system to the differential equations

$$\begin{aligned}\frac{\partial}{\partial u} \phi(u; a, b) &= f(u; a, b) F(u; a, b) \\ \frac{\partial}{\partial u} f(u; a, b) &= -a^2 \phi(u; a, b) F(u; a, b) \\ \frac{\partial}{\partial u} F(u; a, b) &= b^2 \phi(u; a, b) f(u; a, b)\end{aligned}$$

with initial conditions of $\phi(0) = 0$ and $f(0) = F(0) = 1$. Comparing Abel's differential equations to those of Dumont, we see that

$$\phi(u; a, b) = \operatorname{sn}(u; ia, b) \quad f(u; a, b) = \operatorname{cn}(u; ia, b) \quad F(u; a, b) = \operatorname{dn}(u; ia, b).$$

Furthermore, Dumont's sn function is the inversion of the elliptic integral

$$\int_0^x \frac{ds}{\sqrt{(1+a^2s^2)(1+b^2s^2)}}.$$

The Pythagorean relations are motivated by the two factors inside the square root.

In 1944, E. Neville [24] introduced a two-parameter family of elliptic functions which can be viewed as symmetric variants of those of Jacobi. Neville's functions differ from those of Jacobi, Abel and Dumont in that they are not normalized at the origin.

The modular transformations used by Milne in [21, 22] have the form:

$$F(u, k_1) = \alpha G(\beta u, k_2),$$

where F and G are elliptic functions. In the bimodular setting, these modular transformations reduce to the following form:

$$\begin{aligned} \operatorname{sn}(cu; a, b) &= c \operatorname{sn}(u; ca, cb) \\ \operatorname{cn}(cu; a, b) &= \operatorname{cn}(u; ca, cb) \\ \operatorname{dn}(cu; a, b) &= \operatorname{dn}(u; ca, cb) \end{aligned} \tag{1.9}$$

We call these identities the “multiplication formulas”. (They are easily verified using the differential equations given above.)

Using equations (1.8) and (1.9), we can write the bimodular functions in terms of the classical Jacobian functions as follows:

$$\begin{aligned} a \operatorname{sn}(u; a, ak') &= \operatorname{sc}(au, k) \\ \operatorname{cn}(u; a, ak') &= \operatorname{nc}(au, k) \\ \operatorname{dn}(u; a, ak') &= \operatorname{dc}(au, k) \end{aligned} \tag{1.10}$$

The following symmetries are easily verified:

$$\operatorname{sn}(u; a, b) = \operatorname{sn}(u; b, a) \qquad \operatorname{cn}(u; a, b) = \operatorname{dn}(u; a, b) \qquad (1.11)$$

$$\operatorname{sn}(u; a, b) = \operatorname{sn}(u; \pm a, \pm b) \qquad \operatorname{cn}(u; a, b) = \operatorname{cn}(u; \pm a, \pm b) \qquad (1.12)$$

$$\operatorname{sn}(u; a, b) = -\operatorname{sn}(-u; a, b) \qquad \operatorname{cn}(u; a, b) = \operatorname{cn}(-u; a, b) \qquad (1.13)$$

We now characterize the bimodular functions:

Lemma 1. *Let $a, b \in \mathbb{C}^\times$.*

1. **Trivial case:** $\operatorname{sn}(u; 0, 0) = 0$ and $\operatorname{cn}(u; 0, 0) = \operatorname{dn}(u; 0, 0) = 1$;
2. **Sine case:** $\operatorname{sn}(u; a, 0) = \frac{1}{a} \sinh au$, $\operatorname{cn}(u; a, 0) = \cosh au$, and $\operatorname{dn}(u; a, 0) = 1$.
3. **Tangent case:** $\operatorname{sn}(u; a, \pm a) = \frac{1}{a} \tan au$ and $\operatorname{cn}(u; a, \pm a) = \operatorname{dn}(u; a, \pm a) = \sec au$.
4. **Elliptic case:** *If $\frac{b}{a} \neq \pm 1$, then $\operatorname{sn}(u; a, b)$, $\operatorname{cn}(u; a, b)$ and $\operatorname{dn}(u; a, b)$ are elliptic (i.e. doubly periodic, single-valued, and meromorphic) functions of u having two simple poles and two simple zeros in each period parallelogram.*

The first three cases follows immediately from the differential equations. The fourth case follows from algebraic addition formulas for the three functions which involve only the function in question and its derivative. (See Lawden [18], section 8.12 for details.) We display the addition formula for $\operatorname{sn}(u; a, b)$ as an example:

$$\operatorname{sn}(u + v; a, b) = \frac{\operatorname{sn}(u; a, b) \operatorname{cn}(v; a, b) \operatorname{dn}(v; a, b) + \operatorname{sn}(v; a, b) \operatorname{cn}(u; a, b) \operatorname{dn}(u; a, b)}{1 - a^2 b^2 \operatorname{sn}^2(u; a, b) \operatorname{sn}^2(v; a, b)}.$$

We see that this is an algebraic addition formula involving the function sn and its derivative by rewriting it as a polynomial equation:

$$P(s_1, s_2, s_3, t_2, t_3) = 0.$$

where:

$$s_1 = \text{sn}(u + v; a, b) \quad s_2 = \text{sn}(u; a, b) \quad s_3 = \text{sn}(v; a, b)$$

and the t_i are the corresponding derivatives:

$$t_2 = \text{sn}'(u; a, b) \quad t_3 = \text{sn}'(v; a, b).$$

The polynomial P can clearly take the following form:

$$P(s_1, s_2, s_3, t_2, t_3) = s_1(1 - a^2 b^2 s_2^2 s_3^2) - (s_2 t_3 + s_3 t_2).$$

1.3 Dixon's Elliptic Functions

In [8], A. Dixon. studied a family of elliptic functions arising from the cubic curve:

$$x^3 + y^3 - 3\alpha xy = 1 \quad (\alpha \neq -1)$$

(The polynomial $1 - 3\alpha xy - x^3 - y^3$ has $1 - x - y$ as a factor, so, when $\alpha = -1$, the curve “factors” into a straight line and a conic section. In the remarks that follow, we assume $\alpha \neq -1$.)

These functions may be developed like the Jacobian elliptic functions by taking an ordinary differential equations initial value problem as a starting point. Let α be a complex constant. We consider the following system of equations:

$$s'(u) = c^2(u) - \alpha s(u) \tag{1.14}$$

$$c'(u) = -s^2(u) + \alpha c(u) \quad (1.15)$$

subject to:

$$s(0) = 0 \text{ and } c(0) = 1 \quad (1.16)$$

We start with:

$$f(u) = s^3(u) + c^3(u) - 3\alpha s(u) c(u)$$

Now we differentiate and expand to obtain:

$$f'(u) = 0$$

Thus $f(u)$ is constant. But $f(0) = 1$, so it follows that $f(u)$ is identically 1. This may be taken as a cubic analogue of the Pythagorean theorem.

Using the notation introduced by Dixon in [8], we define two new analytic functions as the solutions to this initial value problem, namely:

$$\text{sm}(u, \alpha) := s(u) \quad \text{and} \quad \text{cm}(u, \alpha) := c(u) \quad (1.17)$$

Thus:

$$\frac{d}{du} \text{sm}(u, \alpha) = \text{cm}^2(u, \alpha) - \alpha \text{sm}(u, \alpha) \quad (1.18)$$

$$\frac{d}{du} \text{cm}(u, \alpha) = -\text{sm}^2(u, \alpha) + \alpha \text{cm}(u, \alpha) \quad (1.19)$$

$$(1.20)$$

where

$$\operatorname{sm}(0, \alpha) = 0 \text{ and } \operatorname{cm}(0, \alpha) = 1. \quad (1.21)$$

We have shown that $\operatorname{sm}(u, \alpha)$ and $\operatorname{cm}(u, \alpha)$ satisfy the following functional relation, an analogue of the Pythagorean theorem:

$$\operatorname{sm}^3(u, \alpha) + \operatorname{cm}^3(u, \alpha) - 3\alpha \operatorname{sm}(u, \alpha) \operatorname{cm}(u, \alpha) = 1 \quad (1.22)$$

By showing that $\operatorname{sm}(u, \alpha)$ is meromorphic and single-valued and satisfies an algebraic addition theorem involving only sm and its first derivative, Dixon verified that $\operatorname{sm}(u, \alpha)$ and $\operatorname{cm}(u, \alpha)$ are elliptic functions.

1.4 General Convergence

The principal notion of convergence that we use is general convergence of power series. Our treatment of general convergence essentially follows that of Lorentzen and Waadeland in [20, *pp.* 243*ff.*].

Let x be an indeterminate. For our purposes, a power series in x is a unilateral Laurent series of the form:

$$A(x) = \sum_{n=M}^{\infty} a_n x^n$$

where M is a finite integer and the range of the sequence $\{a_n\}_{n=M}^{\infty}$ is a set of complex-valued constants. If the indeterminate x is a variable, then we follow traditional usage and refer to the power series as a Maclaurin series or as a Taylor series centered about $x = 0$. More generally, if $x = y - a$ where y is an indeterminate and a is a complex

constant, we describe the series as a Taylor series about $y = a$. Analogously, if $x = y^{-1}$, then we classify the series as a Taylor series about $y = \infty$.

Next, given a power series $A(x) = \sum_{N=M}^{\infty} a_n x^n$, we define its (minimum) degree $\delta(A)$ as follows:

$$\delta(A) := \inf \{n \in \mathbf{Z} : n \geq M \text{ and } a_n \neq 0\}. \quad (1.23)$$

Note that this extends the notion of the *minimum* degree of a polynomial. The degree of a power series in x yields a norm on the set of power series in x by defining:

$$\|A\| = 2^{-\delta(A)}. \quad (1.24)$$

(The choice here of two as a base of the exponential is arbitrary. Any real base $b > 1$ would do.)

The following table illustrates these notions with examples:

$A(x)$	$\delta(A)$	$\ A\ $
0	∞	0
1	0	1
17	0	1
$5x^2 + 8x^3 + 16x^4$	2	1/4
$5x^{-11} + 8x^{-10} + 16x^{-9}$	-11	2^{11}
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	0	1
$\frac{x}{1-x} = \sum_{n=1}^{\infty} x^n$	1	1/2

TABLE 1.1 General convergence norm examples.

To prove that this is a norm on power series in x , we must check the following:

1. For all power series $A(x)$, $\|A\| = 0$ if and only if $A(x) = 0$.
2. For all power series $A(x)$, $\|A\| \geq 0$.
3. For all power series $A(x)$ and $B(x)$, $\|A\| \cdot \|B\| = \|A \cdot B\|$.
4. (Triangle inequality) For all power series $A(x)$ and $B(x)$,

$$\|A + B\| \leq \|A\| + \|B\|$$

The first three properties follow directly from properties of the exponential. Verification of property (2) uses the fact that the base 2 in the defining equation (1.24) is real and greater than 1.

Verification of the triangle inequality is straightforward and follows from the fact that the minimum degree of the sum of two power series is at least the minimum of their minimum degrees:

$$\delta(A + B) \geq \min\{\delta(A), \delta(B)\}.$$

Since the base 2 of the exponential in the defining equation (1.24) is a positive real number, the norm is decreasing and non-negative function of the minimum degree. Thus:

$$\begin{aligned} \|A + B\| &:= 2^{-\delta(A+B)} \leq 2^{-\min\{\delta(A), \delta(B)\}} = \min\{2^{-\delta(A)}, 2^{-\delta(B)}\} \\ &\leq 2^{-\delta(A)} + 2^{-\delta(B)} =: \|A\| + \|B\| \end{aligned}$$

We say that sequence $\{A_n(x)\}$ of power series in x converges generally to a power series $A(x)$ if and only if

$$\lim_{n \rightarrow \infty} \|A_n - A\| = 0 \tag{1.25}$$

More generally, we consider a complex-valued function $f(x)$ which may be expanded about some point x_0 into a Taylor series $A(x)$ centered about x_0 . On expanding $f(x)$ into its Taylor series (in $(x - x_0)$ if $x_0 \neq \infty$; in x^{-1} if $x_0 = \infty$), we may for convergence purposes take $\|f\| := \|A\|$. The notation is somewhat misleading as the definition does depend on the choice of center x_0 of the Taylor expansion. In most of what follows, we consider Maclaurin expansions of functions, *i.e.* Taylor series expansions centered at $x_0 = 0$. We accordingly take this as a default for all implicit Taylor expansions that follow.

Thus when we say, without further qualification, that a sequence of rational functions $a_n(x)$ converges generally to a power series $A(x)$, we really mean that the associated sequence of Maclaurin expansions $A_n(x)$ converges generally to the Maclaurin series $A(x)$.

1.5 The Formal Laplace Transform

Let $a = \{a_\nu\}_{\nu=0}^\infty$ be a sequence. The *ordinary generating function* for a is the power series $A(x)$ given by:

$$A(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu$$

The *exponential generating function* for a is the power series $\bar{A}(u)$ given by

$$\bar{A}(u) = \sum_{\nu=0}^{\infty} \frac{a_{\nu} u^{\nu}}{\nu!}$$

We can frame a useful combinatorial relationship between these two power series by using the Laplace transform. Let $f(u)$ be a given function. The usual Laplace transform of f is given by:

$$\mathcal{L}(f, s) := \int_0^{\infty} f(u) e^{-su} du$$

For our purposes, it turns out to be convenient to take $s = x^{-1}$.

Now consider the exponential generating function $\bar{A}(u)$. Using elementary calculus, it is easy to see that:

$$\mathcal{L}\left(\frac{a_{\nu} u^{\nu}}{\nu!}, x^{-1}\right) = a_{\nu} x^{\nu+1}$$

Assuming that the Laplace transform commutes with the infinite sum, we would have:

$$\mathcal{L}(\bar{A}, x^{-1}) := \int_0^{\infty} \bar{A}(u) e^{-u/x} du = x A(x)$$

There is a noteworthy relation between the Laplace transform of a function and its derivative, namely:

$$\mathcal{L}(f'(u), x^{-1}) = x^{-1} \mathcal{L}(f(u), x^{-1}) - f(0). \quad (1.26)$$

(To prove this identity, assume f and its derivative are exponentially bounded and evaluate the right hand side using integration by parts.)

If $f(u)$ is an analytic function, Taylor's formula tells us that $f(u)$ can be expanded into an exponential generating function as follows:

$$f(u) = \sum_{n=0}^{\infty} \left(\frac{df(u)}{du} \right)_{u=0} \frac{u^n}{n!}.$$

Thus, the formal Laplace transform of $f(u)$ is given by the associated normal generating function:

$$x^{-1} \mathcal{L}(f(u), x^{-1}) = \sum_{n=1}^{\infty} \left(\frac{df(u)}{du} \right)_{u=0} u^n.$$

The meaning of equation (1.26) is quite clear: differentiation in function space corresponds in Laplace transform space to dropping the constant term of the associated normal generating function and shifting the remaining terms one place to the left.

1.6 Continued Fractions and Hankel Determinants

We next review a correspondence between ordinary generating functions and certain kinds of continued fractions.

Our notation for continued fractions is similar to the notation used in L. Lorentzen and H. Waadeland in [20], page 5:

$$\mathbf{K}_{n=1}^{\infty} \frac{a_n}{b_n} := \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}} = \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}}.$$

This can be made precise given a reasonable underlying notion of convergence. Given three sequences, $a = \{a_n\}$, $b = \{b_n\}$ and $c = \{c_n\}$, we can recursively define a sequence of convergents, namely:

$$\begin{aligned} \mathbf{K}_{n=1}^1(a, b, c) &:= \frac{a_1}{b_1 + c_1} \\ \mathbf{K}_{n=1}^N(a, b, c) &:= \mathbf{K}_{n=1}^{N-1} \left(a, b, \frac{a_N}{b_N + c_N} \right) \end{aligned}$$

Assuming this sequence converges, we define:

$$\mathbf{K}_{n=1}^\infty(a, b, c) := \lim_{N \rightarrow \infty} \mathbf{K}_{n=1}^N(a, b, c) \quad (1.27)$$

If $\mathbf{0}$ denotes the zero sequence, we can further define:

$$\mathbf{K}_{n=1}^N \frac{a_n}{b_n} := \mathbf{K}_{n=1}^N(a, b, \mathbf{0}) \quad (1.28)$$

$$\mathbf{K}_{n=1}^\infty \frac{a_n}{b_n} := \mathbf{K}_{n=1}^\infty(a, b, \mathbf{0}) \quad (1.29)$$

Two kinds of continued fractions have special importance in the sequel. Let $a = \{a_n\}$ and $b = \{b_n\}$ be sequences of complex numbers and let $\alpha(n)$ be a sequence of positive integers. When z is an indeterminate, the continued fractions

$$\mathbf{K}_{n=1}^\infty \frac{a_n z}{1 + b_n z} \quad \text{and} \quad \mathbf{K}_{n=1}^\infty \frac{a_n z^{\alpha(n)}}{1}$$

are known respectively as an *associated continued fraction* and a *C-fraction*. When the sequence $\alpha(n)$ is constant, the second continued fraction is called a *regular C-fraction*.

Next let $c = \{c_\nu\}_{\nu=1}^\infty$ be a sequence in \mathbb{C} . Associated with this sequence are two sequences of determinants of square matrices, namely

$$H_m^{(n)} \equiv H_m^{(n)}(c_\nu) \equiv H_m^{(n)}(c) := \det \begin{pmatrix} c_n & c_{n+1} & \cdots & c_{m+n-2} & c_{n+m-1} \\ c_{n+1} & c_{n+2} & \cdots & c_{m+n-1} & c_{m+n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{m+n-1} & c_{m+n} & \cdots & c_{2m+n-3} & c_{2m+n-2} \end{pmatrix} \quad (1.30)$$

$$\chi_m \equiv \chi_m(c_\nu) \equiv \chi_m(c) := \det \begin{pmatrix} c_1 & c_2 & \cdots & c_{m-1} & c_{m+1} \\ c_2 & c_3 & \cdots & c_m & c_{m+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_m & c_{m+1} & \cdots & c_{2m-2} & c_{2m} \end{pmatrix} \quad (1.31)$$

The matrix for χ_m is obtained from the matrix for $H_{m+1}^{(1)}$ by deleting the last row and the next to last column. In particular, for $m = 1$ we have $H_1^{(1)} = c_1$ and $\chi_1 = c_2$.¹

A determinant (such as $H_m^{(n)}$) of a matrix whose entries are symmetric about the diagonal is known by a number of names in the literature. Muir in [23] describes them as *persymmetric determinants*, while others use names such as *Hankel determinants* or *Turánian determinants*. Here we collectively refer to determinants of type $H_m^{(n)}$ and χ_m as Hankel determinants.

In Jones and Thron [17, Theorem 7.14, pages 244-246], a combinatorial connection between an ordinary generating function and its corresponding associated continued fraction may be framed in terms of Hankel determinants. For convenience, we restate this correspondence in the form of a lemma:

¹In Jones and Thron [17, theorem 7.14], χ_1 is incorrectly given as c_1 .

Lemma 2 (Heilermann). *Suppose that an associated continued fraction converges generally to a formal power series as follows:*

$$1 + \sum_{\nu=1}^{\infty} c_{\nu} z^{\nu} = 1 + \frac{a_1 z}{1 + b_1 z} + \mathbf{K}_{n=2}^{\infty} \frac{-a_n z^2}{1 + b_n z} \quad (a_n \neq 0) \quad (1.32)$$

Let $H_m^{(n)} \equiv H_m^{(n)}(c_{\nu})$ and $\chi_m \equiv \chi_m(c_{\nu})$. It follows that:

$$H_m^{(1)} \neq 0 \quad (1.33a)$$

for $m \geq 1$. Furthermore, for all $m \geq 1$:

$$a_m = \frac{H_m^{(1)} H_{m-2}^{(1)}}{\left(H_{m-1}^{(1)}\right)^2} \quad (1.33b)$$

where $H_{-1}^{(1)} := H_0^{(1)} := 1$, and

$$b_m = \frac{\chi_{m-1}}{H_{m-1}^{(1)}} - \frac{\chi_m}{H_m^{(1)}} \quad (1.33c)$$

where $\chi_0 := 0$. Conversely, if (1.33a), (1.33b) and (1.33c) all hold, then so does (1.32). In addition, it follows that:

$$H_m^{(1)} = \prod_{r=1}^m a_r^{m+1-r} \quad (m \geq 1) \quad (1.33d)$$

$$\frac{\chi_m}{H_m^{(1)}} = - \sum_{r=1}^m b_r \quad (m \geq 1) \quad (1.33e)$$

Jones and Thron [17, Theorem 7.2, pages 223-224] state a combinatorial connection between an ordinary generating function and its corresponding regular C-fraction framed in terms of Hankel determinants. We also restate this correspondence in the form of a lemma:

Lemma 3 (Heilermann). *Suppose that a regular C-fraction converges generally to a formal power series as follows:*

$$1 + \sum_{\nu=1}^{\infty} c_{\nu} z^{\nu} = 1 + \mathbf{K}_{n=1}^{\infty} \frac{a_n z}{1} \quad (a_n \neq 0) \quad (1.34)$$

Let $H_m^{(n)} \equiv H_m^{(n)}(c_{\nu})$. It follows that:

$$H_m^{(1)} \neq 0 \quad \text{and} \quad H_m^{(2)} \neq 0 \quad (1.35a)$$

for $m \geq 1$. Furthermore, for all $m \geq 1$:

$$a_1 = H_1^{(1)} \quad (1.35b)$$

$$a_{2m} = -\frac{H_{m-1}^{(1)} H_m^{(2)}}{H_m^{(1)} H_{m-1}^{(2)}} \quad \text{and} \quad a_{2m+1} = -\frac{H_{m+1}^{(1)} H_{m-1}^{(2)}}{H_m^{(1)} H_m^{(2)}} \quad (1.35c)$$

where $H_0^{(1)} := H_0^{(2)} := 1$. Conversely, if (1.35a), (1.35b) and (1.35c) and all hold, then so does (1.34). It additionally follows that:

$$H_m^{(2)} = (-1)^m H_m^{(1)} \prod_{r=1}^m a_{2r} = (-1)^m H_{m+1}^{(1)} \prod_{r=1}^m \frac{1}{a_{2r+1}} \quad (1.35d)$$

Both of these correspondences work in two directions. First, given a continued fraction representation of a normal generating function, we can evaluate certain Hankel determinants. Conversely, if these determinants are nonvanishing, then we obtain a formal correspondence. The converses are used in Milne [22, Theorems 3.10 and 3.11]. D. Dumont in [12] uses the related qd-algorithm to establish correspondences between certain formal power series and continued fractions.

1.7 Basic Hypergeometric Series

Although we make no direct use of basic hypergeometric series in this thesis, it is necessary to emphasize that there is an intimate connection between basic hypergeometric series and elliptic functions. That connection comes in the form of a parameter q . Specialists in basic hypergeometric series generally call this parameter the *base*, while specialists in elliptic functions traditionally identify q as the *nome*. Much of the analytic number theory that arises from elliptic functions is the result of the proper choice of base or nome. We give just a brief sketch here, and refer the reader to the literature (e.g. Jacobi [16, §35-46 and §61–66] or Gasper and Rahman [14, Chapters 1 and 5]) for further information.

An elliptic function is a meromorphic function with two independent periods, that is, the periods of an elliptic function form an additive group with two generators, isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Given any two generators ω_1 and ω_2 of the group of periods, a nome q may be defined by the equation:

$$q \equiv q(\omega_1, \omega_2) := \exp(2\pi i \omega_2 / \omega_1). \quad (1.36)$$

For example, using the real and imaginary periods of $\operatorname{sn}(u, k)$ as fundamental periods, Jacobi [16] defined q by:

$$q \equiv q(k) := \exp(-\pi \mathbf{K}(k') / \mathbf{K}(k)) \quad (1.37)$$

where:

$$\mathbf{K}(k) = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} \quad (1.38)$$

$$k' = \sqrt{1 - k^2}. \quad (1.39)$$

Jacobi's number theory applications arise from q -expansions of elliptic functions into Fourier series and Lambert series [16, §39–46] and from theta function identities (especially Jacobi's Triple Product Identity) and expansions [16, §35–38 and §61–66]. The Fourier series for the Jacobi elliptic functions are special cases of Ramanujan's ${}_1\psi_1$ bilateral basic hypergeometric series summation. The fundamental connection between Jacobi's number theory and his elliptic functions is the identity:

$$\sqrt{\frac{2\mathbf{K}(k)}{\pi}} = \vartheta_3(0, q) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2}, \quad (1.40)$$

where $\vartheta_3(z, q)$ is defined as in Whittaker and Watson [30, pp. 463-464]. The right hand side of this identity is the ordinary generating function for the number of ways to write an integer as the square of an integer.

Ramanujan's identity and the Triple Product Identity are key pieces for recovering Jacobi's number theory in the basic hypergeometric series setting.

1.8 Relation to Recent Work

Al-Salam and Carlitz, 1959. In a 1959 paper [3], W. Al-Salam and L. Carlitz used continued fractions and Heilermann's correspondence (described as a theorem of H. Wall) to compute determinants of Bernoulli numbers and of coefficients in the Maclaurin expansions of the Jacobi elliptic functions.

Carlitz, 1960. In 1960, L. Carlitz [5] used continued fraction expansions and Fourier series expansions for the Jacobi elliptic functions to obtain four families of

orthogonal polynomials. He obtained generating functions and discrete weight functions for each of the families. Moreover, he found relationships between the four families and showed that they occur in multiplication formulas for the Jacobi elliptic functions.

Carlitz's families are associated with the continued fractions for $\operatorname{sn}(u, k)$, $\operatorname{cn}(u, k)$, $\operatorname{dn}(u, k)$, and $\operatorname{sn}^2(u, k)$. We note that $\operatorname{cn}(u, k)$ and $\operatorname{dn}(u, k)$ are related by a modular transformation. In addition, Carlitz was not aware of the existence of a continued fraction expansion for $\operatorname{sn}(u, k) \operatorname{cn}(u, k)$, which, though implicit in the work of Rogers and Stieltjes, remained unknown until it was reported by Ismail and Masson [15] in 1998.

Dumont, 1980 - 1985. In the 1980's, D. Dumont [9, 11] applied continued fraction expansions of the Jacobi elliptic functions to obtain combinatorial interpretations of the coefficients in the Maclaurin expansions of the Jacobi elliptic functions. The 1985 paper [11] generalizes some 1881 results of M. André [2] on alternating permutations.

Chudnovsky and Chudnovsky, 1993. In a 1993 paper [6], D. Chudnovsky and G. Chudnovsky state a number of new continued fraction expansions of special functions. The coefficients in these expansions involve transcendental functions. (For example, on page 142 of the article, the authors display the expansion of a formal Laplace transform of a Weierstrass elliptic function into a quasi associated continued fraction whose coefficients are elliptic functions instead of simple constants.)

The authors had been studying "effective methods of construction of rational approximations to classical numbers and functions". Though these constructions

were “relatively simple to formulate in terms of monodromy data”, they were “evading any closed form answer, and even the simplest new constructions required inordinate amounts of computer algebra and raw numerical power.” [6, page 126–127]

Ismail and Masson, 1998. In a 1998 paper [15], M. Ismail and D. Masson discovered a previously unknown associated continued fraction expansion for the Laplace transform of $\operatorname{sn}(u, k) \operatorname{cn}(u, k)$. In the same paper, the three previously known families and the new family of continued fraction expansions were associated with generalizations of four special cases of the continuous dual Hahn polynomials.

Milne, 1996 - 2002. In [21, 22], S. Milne generalized results from Jacobi in [16] on sums of squares. Using fundamental connections between elliptic functions and theta functions, Jacobi was able to specialize Fourier series of Jacobi elliptic functions to obtain results on the enumeration of the number of ways to write an integer as the sum of 2, 4, 6 or 8 squares. Among other results, the two papers of Milne extended Jacobi’s results for four and eight squares to infinite families of $4n^2$ and $4n(n+1)$ squares. An important step in Milne’s work was the specialization of Hankel determinants of coefficients in series expansions of certain ratios of Jacobi elliptic functions. These Hankel determinants were evaluated by deriving continued fraction expansions of Laplace transforms of these ratios, and then applying Heilermann’s correspondence.

General Remarks. The formal symmetric properties of continued fractions of special functions have an important role in all of this work.

CHAPTER 2
CONTINUED FRACTIONS FOR THE JACOBIAN
FUNCTIONS

The main results of this chapter are the four associated continued fractions in Theorems 1 through 4. Each of these four theorems gives the root of a family of continued fractions of Laplace transforms of certain elliptic functions. We chose notation for elliptic function due to Dumont [9] over traditional notation for the Jacobi elliptic functions because the traditional notation tends to obscure much of the underlying symmetry. The symmetry apparent in Theorems 1 through 4 makes it clear not only that the results yield four distinct classical families, but also that this collection of families is essentially complete.

We should also note that an analyst or an engineer might wince when encountering the Laplace transform of a function which is not exponentially bounded. For our purposes, the Laplace transform is a simple formal correspondence between an exponential generating function in function space and an ordinary generating function in transform space. We have nonetheless attempted to identify all Laplace transforms where problems occur on the real axis for real values of the parameters.

2.1 Associated Continued Fractions

We now have the machinery in place to derive our families of associated continued fractions. In handling issues of convergence, we use general convergence as defined in [20] rather than classical convergence.

Theorem 1. *If $a, b \in \mathbb{C}$, then*

$$\int_0^\infty \operatorname{sn}(u; a, b) e^{-u/x} du = \frac{x^2}{1 - (a^2 + b^2)x^2 + \mathbf{K}_{n=1}^\infty \frac{a_n x^4}{1 + b_n x^2}},$$

$$\text{where } a_n = -(2n + 1)(2n)^2(2n - 1)a^2b^2,$$

$$\text{and } b_n = -(2n + 1)^2(a^2 + b^2).$$

Proof. The claim is easily verified if either of the parameters a or b happens to be 0.

We next prove this assuming that a and b are chosen so that $\operatorname{sn}(u; a, b)$, $\operatorname{cn}(u; a, b)$ and $\operatorname{dn}(u; a, b)$ are elliptic functions in u with real periods and no singularities for $u \in \mathbb{R}$.

Under the restrictions, $\operatorname{sn}(u; a, b) e^{-u/x} \rightarrow 0$ as $u \rightarrow \infty$. To reduce clutter, let:

$$S_n := \int_0^\infty \operatorname{sn}^n(u; a, b) e^{-u/x} du, \text{ and}$$

$$T_n := \int_0^\infty \operatorname{sn}^n(u; a, b) \operatorname{cn}(u; a, b) \operatorname{dn}(u; a, b) e^{-u/x} du.$$

We integrate by parts and use the Pythagorean theorem (equations 1.6 and 1.7) to show that:

$$S_n = nxT_{n-1} \quad (n > 0)$$

$$T_0 = x + (a^2 + b^2)xS_1 + 2a^2b^2xS_3$$

$$T_n = nxS_{n-1} + (n+1)(a^2 + b^2)xS_{n+1} + (n+2)a^2b^2xS_{n+3} \quad (n > 0)$$

Solving for S_1 and S_n/S_{n-2} in just the right way, we obtain:

$$S_1 = \frac{x^2}{1 - (a^2 + b^2)x^2 - 2a^2b^2x^2S_3/S_1}$$

$$S_n/S_{n-2} = \frac{n(n-1)x^2}{1 - n^2(a^2 + b^2)x^2 - n(n+1)a^2b^2x^2S_{n+2}/S_n}$$

After iterating and simplifying, we obtain the desired continued fraction.

Now let $[\text{sn}]_\nu$ be denote the sequence of coefficients in the Maclaurin expansion for $\text{sn}(u; a, b)$:

$$\text{sn}(u; a, b) = \sum_{\nu=1}^{\infty} \frac{[\text{sn}]_\nu u^{2\nu-1}}{(2\nu-1)!}$$

We apply the Heilermann correspondence using the following formal power series in x^2 :

$$L = 1 + \mathcal{L}(\text{sn}(u; a, b), x^{-1})$$

The Hankel determinants calculations are obtained by iteration:

$$H_m^{(1)}([\text{sn}]_\nu) = (ab)^{m(m-1)} \prod_{r=2}^{2m-1} r! \quad (2.1)$$

$$\frac{\chi_m^{(1)}([\text{sn}]_\nu)}{H_m^{(1)}([\text{sn}]_\nu)} = (a^2 + b^2) \sum_{r=1}^m (2r-1)^2 \quad (2.2)$$

We note in passing that, by applying well-known summation formulas, this latter identity may be rewritten as follows:

$$\frac{\chi_m^{(1)}([\text{sn}]_\nu)}{H_m^{(1)}([\text{sn}]_\nu)} = \frac{(a^2 + b^2)}{6} (2m-1)(2m)(2m+1). \quad (2.3)$$

The first observation to make is that $H_m^{(1)}$ vanishes if and only if a or b vanishes. It follows that the condition (1.33a) holds.

The next observation is that the coefficients in the Maclaurin series for $\text{sn}(u; a, b)$ are polynomials in the parameters a and b . Conditions (1.33b) and (1.33c) are rational equations that hold for infinitely many values of each of the parameters holding the other parameter fixed. By the Fundamental Theorem of Algebra, it follows that these equations must hold for all values of the parameters.

But Heilermann's correspondence works in reverse, (*e.g.* see Milne [22, Theorem 3.10]). In particular, (1.33a), (1.33b) and (1.33c) imply (1.32) which is just what we wanted. \square

By specializing a and b in Theorem 1, we obtain the following expansions:

$$\int_0^\infty \text{sn}(u, k) e^{-u/x} du = \frac{x^2}{1 + (1 + k^2)x^2} + \mathbf{K}_{n=1}^\infty \frac{-(2n+1)(2n)^2(2n-1)k^2x^4}{1 + (2n+1)^2(1+k^2)x^2} \quad (2.4)$$

$$\int_0^\infty \text{sc}(u, k) e^{-u/x} du = \frac{x^2}{1 - (1 + k'^2)x^2} + \mathbf{K}_{n=1}^\infty \frac{-(2n+1)(2n)^2(2n-1)k'^2x^4}{1 - (2n+1)^2(1+k'^2)x^2} \quad (2.5)$$

$$\int_0^\infty \text{sd}(u, k) e^{-u/x} du = \frac{x^2}{1 - (k^2 - k'^2)x^2} + \mathbf{K}_{n=1}^\infty \frac{-(2n+1)(2n)^2(2n-1)(kk')^2x^4}{1 - (2n+1)^2(k^2 - k'^2)x^2} \quad (2.6)$$

Since $\text{sc}(u, k)$ has poles on the real u -axis, the correspondence in equation (2.5) is purely formal. Specializing $a = 1$ and $b = k'$ in Theorem 1 establishes the formal correspondence between the Laplace transform for $\text{sc}(u, k)$ and its associated continued fraction.

Both sn and sd have real periods and no singularities on the real axis so we specialize as follows:

- For $\text{sn}(u; a, b)$ in equation (2.4), let $a = i$ and $b = ik$;
- For $\text{sd}(u; a, b)$ in equation (2.6), let $a = ik'$ and $b = k$.

In equations (2.4,2.5,2.6), letting $k \rightarrow 0, 1$, we obtain the elementary Laplace transforms:

$$\int_0^\infty \sin u e^{-u/x} du = \frac{x^2}{1+x^2} \quad (2.7)$$

$$\int_0^\infty \sinh u e^{-u/x} du = \frac{x^2}{1-x^2} \quad (2.8)$$

$$\int_0^\infty \tan u e^{-u/x} du = \frac{x^2}{1-2x^2} + \mathbf{K}_{n=1}^\infty \frac{-(2n+1)(2n)^2(2n-1)x^4}{1-2(2n+1)^2x^2} \quad (2.9)$$

$$\int_0^\infty \tanh u e^{-u/x} du = \frac{x^2}{1+2x^2} + \mathbf{K}_{n=1}^\infty \frac{-(2n+1)(2n)^2(2n-1)x^4}{1+2(2n+1)^2x^2} \quad (2.10)$$

We immediately obtain a corollary using the following fact:

$$\mathcal{L}(f', s) = s \mathcal{L}(f, s) - f(0) \quad (2.11)$$

Corollary 1. *If $a, b \in \mathbb{C}$, then*

$$\int_0^\infty \text{cn}(u; a, b) \text{dn}(u; a, b) e^{-u/x} du = \frac{x}{1-(a^2+b^2)x^2} + \mathbf{K}_{n=1}^\infty \frac{a_n x^4}{1+b_n x^2},$$

$$\text{where } a_n = -(2n+1)(2n)^2(2n-1)a^2b^2,$$

$$\text{and } b_n = -(2n+1)^2(a^2+b^2).$$

Proof. Letting $f(u) = \text{sn}(u; a, b)$ in (2.11), we obtain:

$$\begin{aligned} \mathcal{L}(\text{cn}(u; a, b) \text{dn}(u; a, b), x^{-1}) &= x^{-1} \mathcal{L}(\text{sn}(u; a, b), x^{-1}) - \text{sn}(0; a, b) \\ &= x^{-1} \mathcal{L}(\text{sn}(u; a, b), x^{-1}) \end{aligned}$$

□

The machinery of the proof of Theorem 1 can also be used to prove this result. Using the notation from that demonstration and starting with T_0 we have:

$$T_0 = \frac{x}{1 - (a^2 + b^2)x^2 - 2 \cdot 3a^2b^2x^2 \frac{T_2}{T_0}} \quad (2.12)$$

$$\frac{T_n}{T_{n-2}} = \frac{n(n-1)x^2}{1 - (n+1)^2(a^2 + b^2)x^2 - (n+2)(n+3)a^2b^2x^2 \frac{T_{n+2}}{T_n}} \quad (2.13)$$

Theorem 2. *If $a, b \in \mathbb{C}$, then*

$$\int_0^\infty \operatorname{sn}^2(u; a, b) e^{-u/x} du = \frac{2x^3}{1 - 4(a^2 + b^2)x^2 + \mathbf{K}_{n=2}^\infty \frac{a_n x^4}{1 + b_n x^2}},$$

$$\text{where } a_n = -2n(2n-1)^2(2n-2)a^2b^2,$$

$$\text{and } b_n = -(2n)^2(a^2 + b^2).$$

Proof. Using machinery from the proof of theorem 1, we have

$$S_2 = \frac{2x^3}{1 - 4(a^2 + b^2)x^2 - 6a^2b^2x^2 S_4/S_2},$$

with S_n/S_{n-2} is as before. Again we iterate and simplify to obtain the desired continued fraction.

To remove restrictions on a and b , we let the symbol $[\operatorname{sn}^2]_\nu$ denote the sequence of coefficients in the Maclaurin expansion

$$\operatorname{sn}^2(u; a, b) = \sum_{\nu=1}^{\infty} \frac{[\operatorname{sn}^2]_\nu u^{2\nu}}{(2\nu)!}.$$

We accordingly consider the power series

$$L = 1 + x^{-1} \mathcal{L}(\operatorname{sn}^2(u; a, b), x^{-1}).$$

Iteration yields the required Hankel determinants:

$$H_m^{(1)}([\text{sn}^2]_\nu) = (ab)^{m(m-1)} \prod_{r=2}^{2m} r! \quad (2.14)$$

$$\frac{\chi_m^{(1)}([\text{sn}^2]_\nu)}{H_m^{(1)}([\text{sn}^2]_\nu)} = (a^2 + b^2) \sum_{r=1}^m (2r)^2 \quad (2.15)$$

$$= \frac{a^2 + b^2}{6} (2m)(2m+1)(2m+2) \quad (2.16)$$

□

From this theorem, we obtain the continued fraction expansions for the Laplace transforms of the unimodular sn^2 , sc^2 , and sd^2 :

$$\int_0^\infty \text{sn}^2(u, k) e^{-u/x} du = \frac{2x^3}{1 + 4(1 + k^2)x^2} + \mathbf{K}_{n=2}^\infty \frac{-2n(2n-1)^2(2n-2)k^2x^4}{1 + (2n)^2(1 + k^2)x^2} \quad (2.17)$$

$$\int_0^\infty \text{sc}^2(u, k) e^{-u/x} du = \frac{2x^3}{1 - 4(1 + k'^2)x^2} + \mathbf{K}_{n=2}^\infty \frac{-2n(2n-1)^2(2n-2)k'^2x^4}{1 - (2n)^2(1 + k'^2)x^2} \quad (2.18)$$

$$\int_0^\infty \text{sd}^2(u, k) e^{-u/x} du = \frac{2x^3}{1 - 4(k^2 - k'^2)x^2} + \mathbf{K}_{n=2}^\infty \frac{-2n(2n-1)^2(2n-2)(kk')^2x^4}{1 - (2n)^2(k^2 - k'^2)x^2} \quad (2.19)$$

(Since $\text{sc}(u, k)$ has poles on the real u -axis, the correspondence in 2.18 is purely formal.)

Since $\text{sn}^2 0 = 0$, we obtain the following corollary with no effort by using (2.11):

Corollary 2. *If $a, b \in \mathbb{C}$, then*

$$\int_0^\infty \text{sn}(u; a, b) \text{cn}(u; a, b) \text{dn}(u; a, b) e^{-u/x} du = \frac{x^2}{1 - 4(a^2 + b^2)x^2} + \mathbf{K}_{n=2}^\infty \frac{a_n x^4}{1 + b_n x^2},$$

$$\text{where } a_n = -2n(2n-1)^2(2n-2)a^2b^2,$$

$$\text{and } b_n = -(2n)^2(a^2 + b^2).$$

The integral of $\text{sn}^2(u; a, b)$ is an example of an elliptic integral of the second kind. Following the lead of Jacobi, we define

$$\mathbf{E}(u; a, b) := \int_0^u \text{dn}^2(t; a, b) dt \quad (2.20)$$

Using the Pythagorean relation between sn and dn , it follows that:

$$\mathbf{E}(u; a, b) - u = b^2 \int_0^u \text{sn}^2(t; a, b) dt \quad (2.21)$$

$$(2.22)$$

Using (2.11), we obtain the following expansion:

Corollary 3. *If $a, b \in \mathbb{C}$, then*

$$\int_0^\infty (\mathbf{E}(u; a, b) - u) e^{-u/x} du = \frac{2b^2x^4}{1 - 4(a^2 + b^2)x^2} \mathbf{K}_{n=2}^\infty \frac{a_n x^4}{1 + b_n x^2},$$

$$\text{where } a_n = -2n(2n - 1)^2(2n - 2)a^2b^2,$$

$$\text{and } b_n = -(2n)^2(a^2 + b^2).$$

Theorem 3. *If $a, b \in \mathbb{C}$, then*

$$\int_0^\infty \text{cn}(u; a, b) e^{-u/x} du = \frac{x}{1 - a^2x^2} \mathbf{K}_{n=1}^\infty \frac{a_n x^4}{1 + b_n x^2},$$

$$\text{where } a_n = -(2n)^2(2n - 1)^2a^2b^2,$$

$$\text{and } b_n = -((2n + 1)^2a^2 + (2n)^2b^2).$$

Proof. Let

$$C_n := \int_0^\infty \text{sn}^n(u; a, b) \text{cn}(u; a, b) e^{-u/x} du$$

$$D_n := \int_0^\infty \operatorname{sn}^n(u; a, b) \operatorname{dn}(u; a, b) e^{-u/x} du$$

Proceeding as in the proof of theorem 1, we obtain:

$$C_0 = x + a^2 x D_1$$

$$C_n = n x D_{n-1} + (n+1) a^2 x D_{n+1} \quad (n \geq 1)$$

$$D_n = n x C_{n-1} + (n+1) b^2 x C_{n+1} \quad (n \geq 1)$$

Solving we obtain:

$$C_0 = \frac{x}{1 - a^2 x^2 - 2a^2 b^2 c^2 \frac{C_2}{C_0}}$$

$$\frac{C_n}{C_{n-2}} = \frac{n(n-1)x^2}{1 - ((n+1)^2 a^2 + n^2 b^2) x^2 - (n+1)(n+2) a^2 b^2 x^2 \frac{C_{n+2}}{C_n}}$$

We then obtain the associated continued fraction for C_0 (the Laplace transform of $\operatorname{cn}(u; a, b)$) by starting with C_0 and iterating using the recurrence for C_n/C_{n-2} .

To generalize this to all a, b , let $[\operatorname{cn}]_\nu$ denote the coefficients in the Maclaurin expansion of $\operatorname{cn}(u; a, b)$, *i.e.*:

$$\operatorname{cn}(u; a, b) = \sum_{\nu=1}^{\infty} \frac{[\operatorname{cn}]_\nu u^{2\nu-2}}{(2\nu-2)!}.$$

We consider the associated continued fraction corresponding to:

$$L = 1 + x \mathcal{L}(\operatorname{cn}(cu; a, b), x^{-1})$$

The corresponding Hankel determinants are:

$$H_m^{(1)}([\operatorname{cn}]_\nu) = (ab)^{m(m-1)} \prod_{r=1}^{m-1} ((2r)!)^2 \quad (2.23)$$

$$\frac{\chi_m^{(1)}([\text{cn}]_\nu)}{H_m^{(1)}([\text{cn}]_\nu)} = \left(a^2 \sum_{r=1}^m (2r-1)^2 + b^2 \sum_{r=1}^{m-1} (2r)^2 \right) \quad (2.24)$$

$$= \frac{a^2}{6} (2m-1)(2m)(2m+1) + \frac{b^2}{6} (2m-2)(2m-1)(2m) \quad (2.25)$$

□

Taking into account the singularities on the real axis for nc (*i.e.* treating the Laplace transform of $\text{nc}(u, k)$ as purely formal) and specializing as before, we obtain the following continued fraction expansions:

$$\int_0^\infty \text{cn}(u, k) e^{-u/x} du = \frac{x}{1+x^2} + \mathbf{K}_{n=1}^\infty \frac{-(2n)^2(2n-1)^2 k^2 x^4}{1 + ((2n+1)^2 + (2n)^2 k^2) x^2} \quad (2.26)$$

$$\int_0^\infty \text{nc}(u, k) e^{-u/x} du = \frac{x}{1-x^2} + \mathbf{K}_{n=1}^\infty \frac{-(2n)^2(2n-1)^2 k'^2 x^4}{1 - ((2n+1)^2 + (2n)^2 k'^2) x^2} \quad (2.27)$$

$$\int_0^\infty \text{cd}(u, k) e^{-u/x} du = \frac{x}{1+k'^2 x^2} + \mathbf{K}_{n=1}^\infty \frac{(2n)^2(2n-1)(kk')^2 x^4}{1 + ((2n+1)^2 k'^2 - (2n)^2 k^2) x^2} \quad (2.28)$$

Using the symmetry $\text{cn}(u; a, b) = \text{dn}(u; b, a)$, we obtain three more continued fraction expansions:

$$\int_0^\infty \text{dn}(u, k) e^{-u/x} du = \frac{x}{1+k^2 x^2} + \mathbf{K}_{n=1}^\infty \frac{-(2n)^2(2n-1)^2 k^2 x^4}{1 + ((2n+1)^2 k^2 + (2n)^2) x^2} \quad (2.29)$$

$$\int_0^\infty \text{dc}(u, k) e^{-u/x} du = \frac{x}{1-k'^2 x^2} + \mathbf{K}_{n=1}^\infty \frac{-(2n)^2(2n-1)^2 k'^2 x^4}{1 - ((2n+1)^2 k'^2 + (2n)^2) x^2} \quad (2.30)$$

$$\int_0^\infty \text{nd}(u, k) e^{-u/x} du = \frac{x}{1-k^2 x^2} + \mathbf{K}_{n=1}^\infty \frac{(2n)^2(2n-1)(kk')^2 x^4}{1 + ((2n)^2 k'^2 - (2n+1)^2 k^2) x^2} \quad (2.31)$$

($\text{dc}(u, k)$ has singularities on the positive real u -axis.

A related but independent family of continued fraction expansions was obtained by Ismail and Masson [15]. The members of this family are Laplace transforms of derivatives of cn and dn . In the Dumont setting, these take the following form:

Theorem 4. If $a, b \in \mathbb{C}$, then

$$\int_0^\infty \operatorname{sn}(u; a, b) \operatorname{dn}(u; a, b) e^{-u/x} du = \frac{x^2}{1 - (a^2 + 4b^2)x^2 + \mathbf{K}_{n=1}^\infty \frac{a_n x^4}{1 + b_n x^2}},$$

$$\text{where } a_n = -(2n)^2(2n+1)^2 a^2 b^2,$$

$$\text{and } b_n = -((2n+1)^2 a^2 + (2n+2)^2 b^2).$$

Proof. We use the machinery of the proof of Theorem 3 starting with D_1 . If the symbol $[\operatorname{sn} \operatorname{dn}]_\nu$ denotes the sequence of coefficients in the Maclaurin expansion:

$$\operatorname{sn}(u; a, b) \operatorname{dn}(u; a, b) = \sum_{\nu=1}^\infty \frac{[\operatorname{sn} \operatorname{dn}]_\nu u^{2\nu-1}}{(2\nu-1)!}$$

then the required Hankel determinants are given by:

$$H_m^{(1)}([\operatorname{sn} \operatorname{dn}]_\nu) = (ab)^{m(m-1)} \prod_{r=2}^m ((2m-1)!)^2 \quad (2.32)$$

$$\frac{\chi_m^{(1)}([\operatorname{sn} \operatorname{dn}]_\nu)}{H_m^{(1)}([\operatorname{sn} \operatorname{dn}]_\nu)} = \left(a^2 \sum_{r=1}^m (2r-1)^2 + b^2 \sum_{r=1}^m (2r)^2 \right) \quad (2.33)$$

$$= \frac{a^2}{6} (2m-1)(2m)(2m+1) + \frac{b^2}{6} (2m)(2m+1)(2m+2) \quad (2.34)$$

□

2.2 Regular C-Fractions

A continued fraction of the form $\mathbf{K} \frac{a_n z^{N(n)}}{1}$ is called a C-fraction if $N(n)$ is a sequence of positive integers. Every C-fraction converges generally to some Maclaurin series. If the sequence of positive integers $N(n)$ is constant, the C-fraction is said to be regular.

The recurrences of the previous section yield one family of regular C-fractions, corresponding to Laplace transforms of $\operatorname{cn}(u; a, b)$:

Theorem 5. *Let $a, b \in \mathbb{C}$. Then:*

$$\int_0^\infty \text{cn}(u; a, b) e^{-u/x} du = \frac{x}{1 + \prod_{n=1}^\infty \frac{a_n x^2}{1}},$$

$$\text{where } a_n = \begin{cases} -n^2 a^2, & \text{if } n \text{ is odd;} \\ -n^2 b^2, & \text{if } n \text{ is even.} \end{cases}$$

We can rewrite the recurrence for C_n in the previous section as follows:

$$1 = nx \frac{D_{n-1}}{C_n} + (n+1)a^2 x \frac{D_{n+1}}{C_n}.$$

After solving for C_n/D_{n-1} , we have:

$$\frac{C_n}{D_{n-1}} = \frac{nx}{1 - (n+1)a^2 x \frac{D_{n+1}}{C_n}}.$$

Interchanging parameters a and b effectively interchanges $\text{cn}(u; a, b)$ and $\text{dn}(u; a, b)$.

This in turn interchanges the letters C and D in our recurrences, whence:

$$\frac{D_n}{C_{n-1}} = \frac{nx}{1 - (n+1)b^2 x \frac{C_{n+1}}{D_n}}.$$

From $C_0 = x + a^2 x D_1$, we obtain the initial value:

$$C_0 = \frac{x}{1 - a^2 x \frac{D_1}{C_0}}.$$

Iterating yields the continued fraction of Theorem 5. Starting with D_0 gives us nothing new as the same result may be obtained from Theorem 5 by interchanging parameters a and b .

Using Heilermann's C-fraction correspondence (Lemma 3), we obtain the following Hankel determinant evaluations:

$$H_m^{(1)}([\text{cn}]_\nu) = (ab)^{m(m-1)} \prod_{r=1}^{m-1} ((2r)!)^2 \quad (2.35)$$

$$H_m^{(2)}([\text{cn}]_\nu) = (-1)^m a^{m(m+1)} b^{m(m-1)} \prod_{r=1}^m ((2r-1)!)^2 \quad (2.36)$$

Naturally, the first of these two determinants agrees with the value found in the previous section.

2.3 Modular Transformations

Rogers[25] obtained a continued fraction expansion for the Laplace transform of $\frac{\text{sn } u \text{ cn } u}{\text{dn } u}$ using the first of the two following forms of Landen's transformation:

$$\frac{\text{sn } u \text{ cn } u}{\text{dn } u} = \frac{1}{1+k'} \text{sn} \left((1+k')u, \frac{1-k'}{1+k'} \right) \quad (2.37)$$

$$\frac{\text{sn } u}{\text{cn } u \text{ dn } u} = \frac{1}{i(1+k)} \text{sn} \left(i(1+k)u, \frac{1-k}{1+k} \right) \quad (2.38)$$

With the help of the multiplication formula (1.9), we can recast these two forms of Landen's transformation as follows:

$$\frac{\text{sn } u \text{ cn } u}{\text{dn } u} = \text{sn}(u; i(1+k'), i(1-k')) \quad (2.39)$$

$$\frac{\text{sn } u}{\text{cn } u \text{ dn } u} = \text{sn}(u; 1+k, 1-k) \quad (2.40)$$

Combining (2.39) with Jacobi's reciprocal transformation in the form:

$$\text{sn} \left(ik'u, \frac{1}{k'} \right) = ik' \text{sc}(u, k), \quad \text{cn} \left(ik'u, \frac{1}{k'} \right) = \text{dc}(u, k), \quad \text{dn} \left(ik'u, \frac{1}{k'} \right) = \text{nc}(u, k), \quad (2.41)$$

and applying the multiplication formula (1.9), we obtain the following identity:

$$\frac{\text{sn } u \text{ dn } u}{\text{cn } u} = \text{sn}(u; k' + ik, k' - ik) \quad (2.42)$$

The Laplace transforms of the Jacobian elliptic functions $\frac{\text{sn } u \text{ cn } u}{\text{dn } u}$, $\frac{\text{sn } u}{\text{cn } u \text{ dn } u}$, and $\frac{\text{sn } u \text{ dn } u}{\text{cn } u}$ can all be expanded by viewing them as special cases of bimodular sn. We observe that the second and third of these expressions have singularities on the real axis. Substituting equations (2.39–2.42) into Theorems 1 and 2 yields the following correspondences:

$$\int_0^\infty \frac{\text{sn } u \text{ cn } u}{\text{dn } u} e^{-u/x} du = \frac{x^2}{1 + 2(2 - k^2)x^2} + \mathbf{K}_{n=1}^\infty \frac{-(2n+1)(2n)^2(2n-1)k^4 x^4}{1 + 2(2n+1)^2(2 - k^2)x^2} \quad (2.43)$$

$$\int_0^\infty \frac{\text{sn } u}{\text{cn } u \text{ dn } u} e^{-u/x} du = \frac{x^2}{1 - 2(1 + k^2)x^2} + \mathbf{K}_{n=1}^\infty \frac{-(2n+1)(2n)^2(2n-1)k'^4 x^4}{1 - 2(2n+1)^2(1 + k^2)x^2} \quad (2.44)$$

$$\int_0^\infty \frac{\text{sn } u \text{ dn } u}{\text{cn } u} e^{-u/x} du = \frac{x^2}{1 - 2(1 - 2k^2)x^2} + \mathbf{K}_{n=1}^\infty \frac{-(2n+1)(2n)^2(2n-1)x^4}{1 - 2(2n+1)^2(1 - 2k^2)x^2} \quad (2.45)$$

$$\int_0^\infty \frac{\text{sn}^2 u \text{ cn}^2 u}{\text{dn}^2 u} e^{-u/x} du = \frac{2x^3}{1 + 8(2 - k^2)x^2} + \mathbf{K}_{n=2}^\infty \frac{-2n(2n-1)^2(2n-2)k^4 x^4}{1 + 2(2n)^2(2 - k^2)x^2} \quad (2.46)$$

$$\int_0^\infty \frac{\text{sn}^2 u}{\text{cn}^2 u \text{ dn}^2 u} e^{-u/x} du = \frac{2x^3}{1 - 8(1 + k^2)x^2} + \mathbf{K}_{n=2}^\infty \frac{-2n(2n-1)^2(2n-2)k'^4 x^4}{1 - 2(2n)^2(1 + k^2)x^2} \quad (2.47)$$

$$\int_0^\infty \frac{\text{sn}^2 u \text{ dn}^2 u}{\text{cn}^2 u} e^{-u/x} du = \frac{2x^3}{1 - 8(1 - 2k^2)x^2} + \mathbf{K}_{n=2}^\infty \frac{-2n(2n-1)^2(2n-2)x^4}{1 - 2(2n)^2(1 - 2k^2)x^2} \quad (2.48)$$

The underlying modular transformations have very pretty formulations in the Dumont setting. First, we introduce the following invariant:

$$b' := a\sqrt{1 - \left(\frac{b}{a}\right)^2}$$

with the square root chosen so that:

$$0 \leq \arg \sqrt{1 - \left(\frac{b}{a}\right)^2} < \pi.$$

Thus $b'^2 = a^2 - b^2$. In particular, if we specialize $a = i$ and $b = ik$, then we obtain $b' = ik'$.

Theorem 6 (Landen's Transformation).

$$\begin{aligned} \operatorname{sn}(u; a + b', a - b') &= \frac{\operatorname{sn}(u; a, b) \operatorname{cn}(u; a, b)}{\operatorname{dn}(u; a, b)} \\ &= \frac{\operatorname{sn}(u; b, a) \operatorname{dn}(u; b, a)}{\operatorname{cn}(u; a, b)}, \\ \operatorname{cn}(u; a + b', a - b') &= \frac{\operatorname{cn}^2(u; a, b) + ab' \operatorname{sn}^2(u; a, b)}{\operatorname{dn}(u; a, b)} \\ &= \frac{\operatorname{dn}^2(u; b, a) + ab' \operatorname{sn}^2(u; b, a)}{\operatorname{cn}(u; b, a)}, \quad \text{and} \\ \operatorname{dn}(u; a + b', a - b') &= \frac{\operatorname{cn}^2(u; a, b) - ab' \operatorname{sn}^2(u; a, b)}{\operatorname{dn}(u; a, b)} \\ &= \frac{\operatorname{dn}^2(u; b, a) - ab' \operatorname{sn}^2(u; b, a)}{\operatorname{cn}(u; b, a)}. \end{aligned}$$

Proof. It is not hard to derive this. We start by assuming an identity of the form

$$\operatorname{sn}(u; A, B) = \frac{\operatorname{sn}(u; a, b) \operatorname{cn}(u; a, b)}{\operatorname{dn}(u; a, b)}$$

where A and B are undetermined parameters. Differentiating and applying the Pythagorean theorem, we obtain:

$$\begin{aligned} \frac{d}{du} \operatorname{sn}(u; A, B) &= \frac{1 + 2a^2 \operatorname{sn}^2(u; a, b) + a^2 b^2 \operatorname{sn}^4(u; a, b)}{\operatorname{dn}^2(u; a, b)} \\ &= \frac{(1 + a(a + b') \operatorname{sn}^2(u; a, b)) (1 + a(a - b') \operatorname{sn}^2(u; a, b))}{\operatorname{dn}^2(u; a, b)} \end{aligned}$$

$$= \frac{(\operatorname{cn}^2(u; a, b) + ab' \operatorname{sn}^2(u; a, b)) (\operatorname{cn}^2(u; a, b) - ab' \operatorname{sn}^2(u; a, b))}{\operatorname{dn}^2(u; a, b)}.$$

This suggests we take:

$$\begin{aligned} \operatorname{cn}(u; A, B) &= \frac{(\operatorname{cn}^2(u; a, b) + ab' \operatorname{sn}^2(u; a, b))}{\operatorname{dn}(u; a, b)}, & \text{and} \\ \operatorname{dn}(u; A, B) &= \frac{(\operatorname{cn}^2(u; a, b) - ab' \operatorname{sn}^2(u; a, b))}{\operatorname{dn}(u; a, b)}. \end{aligned}$$

Upon differentiating, we find $A = a + b'$ and $B = a - b'$. (If we reverse the choices of $\operatorname{cn}(u; A, B)$ and $\operatorname{dn}(u; A, B)$, we obtain identities which are equivalent to the above set.) \square

After applying the substitutions implicit in Theorem 6 to Theorems 1, 2 and 3, we obtain the following three continued fractions:

Corollary 4. *If $a, b \in \mathbb{C}$, then*

$$\int_0^\infty \frac{\operatorname{sn}(u; a, b) \operatorname{cn}(u; a, b)}{\operatorname{dn}(u; a, b)} e^{-u/x} du = \frac{x^2}{1 - 2(2a^2 - b^2)x^2} + \mathbf{K}_{n=1}^\infty \frac{a_n x^4}{1 + b_n x^2},$$

where $a_n = -(2n + 1)(2n)^2(2n - 1)b^4$,

and $b_n = -2(2n + 1)^2(2a^2 - b^2)$.

Corollary 5. *If $a, b \in \mathbb{C}$, then*

$$\int_0^\infty \frac{\operatorname{sn}^2(u; a, b) \operatorname{cn}^2(u; a, b)}{\operatorname{dn}^2(u; a, b)} e^{-u/x} du = \frac{2x^3}{1 - 8(2a^2 - b^2)x^2} + \mathbf{K}_{n=2}^\infty \frac{a_n x^4}{1 + b_n x^2},$$

where $a_n = -2n(2n - 1)^2(2n - 2)b^4$,

and $b_n = -2(2n)^2(2a^2 - b^2)$.

Corollary 6. *If $a, b \in \mathbb{C}$, then*

$$\int_0^\infty \frac{\operatorname{cn}^2(u; a, b) + ab' \operatorname{sn}^2(u; a, b)}{\operatorname{dn}(u; a, b)} e^{-u/x} du = \frac{x}{1 - (2a^2 + 2ab' - b^2)x^2} + \mathbf{K}_{n=1}^\infty \frac{a_n x^4}{1 + b_n x^2},$$

$$\text{where } a_n = -(2n)^2(2n-1)^2 b^4,$$

$$\text{and } b_n = (8n^2 + 4n + 1)(b^2 - 2a^2) - 2(4n + 1)ab'.$$

Equations (2.43) through (2.45) can be obtained as specializations of corollary 4 and equations (2.46) through (2.48) can be obtained as specializations of corollary 5. The particular specializations are easily seen to be $(a \mapsto i, b \mapsto ik)$, $(a \mapsto 1, b \mapsto k')$, and $(a \mapsto k', b \mapsto 1)$, respectively.

We can take ratios of bimodular functions just as we do their classical counterparts. Thus $\operatorname{sc}(u; a, b) = \frac{\operatorname{sn}(u; a, b)}{\operatorname{cn}(u; a, b)}$, etc. With these definitions, a number of other modular transformations of orders 1 and 2 are easily cast in the Dumont bimodular setting. We offer the following examples:

Theorem 7 (Jacobi's Imaginary Transformation).

$$i \operatorname{sc}(u; a, b) = \operatorname{sn}(iu; a, b'), \tag{2.49}$$

$$\operatorname{nc}(u; a, b) = \operatorname{cn}(iu; a, b'), \quad \text{and} \tag{2.50}$$

$$\operatorname{dc}(u; a, b) = \operatorname{dn}(iu; a, b'). \tag{2.51}$$

Theorem 8 (Jacobi's Real Transformation).

$$\operatorname{sd}(u; a, b) = \operatorname{sn}(u; ib, b'), \tag{2.52}$$

$$\operatorname{nd}(u; a, b) = \operatorname{cn}(u; ib, b'), \quad \text{and} \quad (2.53)$$

$$\operatorname{cd}(u; a, b) = \operatorname{dn}(u; ib, b'). \quad (2.54)$$

Theorem 9 (Gauss' Transformation). *If $0 \leq \arg \frac{b}{a} < \pi$, then:*

$$\operatorname{sn}(u; a + b, 2\sqrt{ab}) = \frac{\operatorname{sn}(u; a, b)}{1 - ab \operatorname{sn}^2(u; a, b)}, \quad (2.55)$$

$$\operatorname{cn}(u; a + b, 2\sqrt{ab}) = \frac{\operatorname{cn}(u; a, b) \operatorname{dn}(u; a, b)}{1 - ab \operatorname{sn}^2(u; a, b)}, \quad \text{and} \quad (2.56)$$

$$\operatorname{dn}(u; a + b, 2\sqrt{ab}) = \frac{1 + ab \operatorname{sn}^2(u; a, b)}{1 - ab \operatorname{sn}^2(u; a, b)}; \quad (2.57)$$

and:

$$\operatorname{sn}(u; a - b, 2\sqrt{ab}) = \frac{\operatorname{sn}(u; a, b)}{1 + ab \operatorname{sn}^2(u; a, b)}, \quad (2.58)$$

$$\operatorname{cn}(u; a - b, 2\sqrt{ab}) = \frac{1 - ab \operatorname{sn}^2(u; a, b)}{1 + ab \operatorname{sn}^2(u; a, b)}, \quad \text{and} \quad (2.59)$$

$$\operatorname{dn}(u; a - b, 2\sqrt{ab}) = \frac{\operatorname{cn}(u; a, b) \operatorname{dn}(u; a, b)}{1 + ab \operatorname{sn}^2(u; a, b)}. \quad (2.60)$$

Proof. Both Jacobi transformations are easily verified using the differential equations.

We sketch the derivation of Gauss's transformation.

First consider $\operatorname{sc}(u; a + b, 2\sqrt{ab})$. From the differential equations for sc , nc , and dc , we have:

$$\operatorname{sc}(u; A, B) = \operatorname{sn}(u; iA, iB')$$

$$\operatorname{nc}(u; A, B) = \operatorname{cn}(u; iA, iB')$$

$$\operatorname{dc}(u; A, B) = \operatorname{dn}(u; iA, iB')$$

Since $\arg \frac{b}{a}$ is between 0 and π , we obtain:

$$i \operatorname{sc}(u; a + b, 2\sqrt{ab}) = i \operatorname{sn}(u; i(a + b), i(a - b))$$

Applying the multiplication formula:

$$= \operatorname{sn}(u; a + b, a - b)$$

The restriction on the argument of $\frac{b}{a}$ tells us that $b'' = b$. Applying Landen's transformation:

$$= \frac{\operatorname{sn}(u; a, b') \operatorname{cn}(u; a, b')}{\operatorname{dn}(u; a, b')}$$

By the multiplication formula:

$$= \frac{i \operatorname{sn}(u; ia, ib') \operatorname{cn}(u; ia, ib')}{\operatorname{dn}(u; ia, ib')}$$

Converting back to sc , nc , and dc :

$$= \frac{i \operatorname{sc}(u; a, b) \operatorname{nc}(u; a, b)}{\operatorname{dc}(u; a, b)}.$$

Doing the same for nc and dc , and writing the right hand sides in terms of sn , cn , and dn , we obtain Gauss's transformation for sc , nc and dc :

$$\operatorname{sc}(u; a + b, 2\sqrt{ab}) = \frac{\operatorname{sn}(u; a, b)}{\operatorname{cn}(u; a, b) \operatorname{dn}(u; a, b)} \quad (2.61)$$

$$\operatorname{nc}(u; a + b, 2\sqrt{ab}) = \frac{1 - ab \operatorname{sn}^2(u; a, b)}{\operatorname{cn}(u; a, b) \operatorname{dn}(u; a, b)} \quad (2.62)$$

$$\operatorname{dc}(u; a + b, 2\sqrt{ab}) = \frac{1 + ab \operatorname{sn}^2(u; a, b)}{\operatorname{cn}(u; a, b) \operatorname{dn}(u; a, b)} \quad (2.63)$$

Three of the six forms of Gauss's transformation for sn , cn , and dn are obtained immediately by taking quotients. To obtain the other three forms, work through the above arguments starting with

$$\operatorname{sc}(u; a - b, 2\sqrt{ab}) \quad \operatorname{dc}(u; a - b, 2\sqrt{ab}) \quad \operatorname{nc}(u; a - b, 2\sqrt{ab})$$

□

Using the multiplication formula, we can put Gauss's transformation in a slightly more suggestive form. Consider the left hand side of equation 2.55.

$$\operatorname{sn}(u; a + b, 2\sqrt{ab}) = \frac{1}{2} \operatorname{sn}\left(2u; \frac{a + b}{2}, \sqrt{ab}\right)$$

The moduli on the right are respectively the arithmetic mean and the geometric mean of the moduli a and b on the left. Jacobi used this connection between Gauss's transformation and the arithmetic-geometric mean to establish some deep connections between the theory of elliptic functions and the arithmetic-geometric mean in [16, §38].

The obvious corollaries of Theorems 7, 8 and 9 are stated explicitly in Milne [22]. In addition, the continued fraction expansions in equations (2.4-2.6), (2.9-2.10), (2.17-2.19), (2.26-2.31), and (2.43-2.48) can also be found in Milne [22].

2.4 Other Formulations

Our continued fractions (and those in Milne [22]) are slightly different from those of Rogers in [25] and Stieltjes in [28]. The principal difference is in the choice of Laplace transform. Our continued fractions use the Laplace transform at $x = 0$:

$$\mathcal{L}(f(u), x^{-1}) := \int_0^\infty f(u) e^{-u/x} du$$

instead of the more common Laplace transform at $x = \infty$:¹

$$\mathcal{L}(f(u), x) := \int_0^\infty f(u) e^{-xu} du.$$

Letting $x \mapsto x^{-1}$ in the continued fraction expansions above gives the following expansions²:

Corollary 7. *If $a, b \in \mathbb{C}$, then*

$$\int_0^\infty \operatorname{sn}(u; a, b) e^{-xu} du = \frac{1}{x^2 - (a^2 + b^2) + \mathbf{K}_{n=1}^\infty \frac{a_n}{x^2 + b_n}},$$

$$\text{where } a_n = -(2n + 1)(2n)^2(2n - 1)a^2b^2,$$

$$\text{and } b_n = -(2n + 1)^2 (a^2 + b^2).$$

Corollary 8. *If $a, b \in \mathbb{C}$, then*

$$x \int_0^\infty \operatorname{sn}^2(u; a, b) e^{-xu} du = \frac{2}{x^2 - 4(a^2 + b^2) + \mathbf{K}_{n=2}^\infty \frac{a_n}{x^2 + b_n}},$$

$$\text{where } a_n = -2n(2n - 1)^2(2n - 2)a^2b^2,$$

$$\text{and } b_n = -(2n)^2 (a^2 + b^2).$$

Corollary 9. *If $a, b \in \mathbb{C}$, then*

$$\int_0^\infty \operatorname{cn}(u; a, b) e^{-xu} du = \frac{x}{x^2 - a^2 + \mathbf{K}_{n=1}^\infty \frac{a_n}{x^2 + b_n}},$$

¹The transform at $x = 0$ transforms an exponential generating function into a Taylor series centered at $x = 0$, while the transform at $x = \infty$ yields a Taylor series at $x = \infty$.

²The factor of x in the left hand side of the expansion of sn^2 in Corollary 8 was placed on the left hand side following Carlitz [5]. This factor was inadvertently omitted in Stieltjes [28].

$$\begin{aligned} \text{where} \quad a_n &= -(2n)^2(2n-1)^2a^2b^2, \\ \text{and} \quad b_n &= -((2n+1)^2a^2 + (2n)^2b^2). \end{aligned}$$

Letting $x \mapsto x^{-1}$ in the Laplace transform of the derivative of $\text{cn}(u; a, b)$ yields what would have been the classical form of this associated continued fraction:

Corollary 10. *If $a, b \in \mathbb{C}$, then*

$$\int_0^\infty \text{sn}(u; a, b) \text{cn}(u; a, b) e^{-xu} du = \frac{1}{x^2 - (a^2 + 4b^2) + \mathbf{K}_{n=1}^\infty \frac{a_n}{x^2 + b_n}},$$

$$\begin{aligned} \text{where} \quad a_n &= -(2n)^2(2n+1)^2a^2b^2, \\ \text{and} \quad b_n &= -((2n+1)^2a^2 + (2n+2)^2b^2). \end{aligned}$$

Letting $x \mapsto x^{-1}$ in our regular C-fraction expansion of the Laplace transform of cn yields the usual form for the classical continued fractions expansions for cn and dn :

Corollary 11. *If $a, b \in \mathbb{C}$, then*

$$\int_0^\infty \text{cn}(u; a, b) e^{-xu} du = \frac{1}{x + \mathbf{K}_{n=1}^\infty \frac{a_n}{x}},$$

$$\text{where} \quad a_n = \begin{cases} -n^2a^2, & \text{if } n \text{ is odd;} \\ -n^2b^2, & \text{if } n \text{ is even.} \end{cases}$$

CHAPTER 3

CONTINUED FRACTIONS FOR THE DIXON FUNCTIONS

3.1 Laplace Transform Recurrences

When $\alpha \neq 0$, the Dixon functions have slightly more complicated derivatives and a slightly more complicated Pythagorean theorem than the classical Jacobian elliptic functions.

When working with the Jacobian functions, we used the Pythagorean theorem to rewrite a polynomial $P(s, c, d)$ canonically in the form:

$$P(s, c, d) = P_0(s) + P_1(s)c + P_2(s)d + P_3(s)cd$$

where $s = \operatorname{sn} u$, $c = \operatorname{cn} u$ and $d = \operatorname{dn} u$ and the expressions P_n are polynomials in one variable.

Using the Pythagorean Theorem for the Dixon functions, we have an analogous canonical form for polynomials in the Dixon functions:

$$P(s, c) = P_0(s) + P_1(s)c + P_2(s)c^2$$

where $s = \operatorname{sm} u$ and $c = \operatorname{cm} u$ and the expressions P_n are polynomials in one variable.

Using notation similar to that in Chapter 2 above, we define the following Laplace transforms:

$$S_n := \int_0^\infty \text{sm}^n(u, \alpha) \exp(-u/x) du \quad (3.1)$$

$$C_n := \int_0^\infty \text{sm}^n(u, \alpha) \text{cm}(u, \alpha) \exp(-u/x) du \quad (3.2)$$

$$D_n := \int_0^\infty \text{sm}^n(u, \alpha) \text{cm}^2(u, \alpha) \exp(-u/x) du \quad (3.3)$$

After integrating by parts and reducing to canonical form using the Dixon analogue of the Pythagorean Theorem, we obtain the following recurrences:

$$S_0 = x \quad (3.4)$$

$$S_n = nx D_{n-1} - n\alpha x S_n \quad (n > 0) \quad (3.5)$$

$$C_0 = x + \alpha x C_0 - x S_2 \quad (3.6)$$

$$C_n = nx S_{n-1} + (1 + 2n)\alpha x C_n - (n + 1)x S_{n+2} \quad (n > 0) \quad (3.7)$$

$$D_0 = x + 2\alpha x D_0 - 2x C_2 \quad (3.8)$$

$$D_n = nx C_{n-1} + (2 + 2n)\alpha x D_n - (n + 2)x C_{n+2} \quad (n > 0) \quad (3.9)$$

When $\alpha = 0$, we obtain simple recurrences that may be manipulated like those in Chapter 2 to obtain associated continued fractions and regular C-fractions. In the general case ($\alpha \neq 0$), we employ a trick to obtain generalizations of these results.

3.2 Associated Continued Fractions ($\alpha = 0$)

We start by setting $\alpha = 0$ in the recurrences (3.4-3.9). We then algebraically eliminate all but one of the letters S , C and D to obtain three sets of three initial value problems.

For Laplace transforms of powers of $\text{sm}(u, 0)$, we have the following three initial value problems:

$$\frac{S_n}{S_{n-3}} = \frac{n(n-1)(n-2)x^3}{1 + 2n(n^2 + 1)x^3 - n(n+1)(n+2)x^3 \frac{S_{n+3}}{S_n}} \quad (3.10)$$

subject to:

$$S_1 = \frac{x^2}{1 + 4x^3 - 6x^3 \frac{S_4}{S_1}} \quad (3.11)$$

$$S_2 = \frac{2x^3}{1 + 20x^3 - 24x^3 \frac{S_5}{S_2}} \quad (3.12)$$

$$S_3 = \frac{6x^4}{1 + 60x^3 - 60x^3 \frac{S_6}{S_3}} \quad (3.13)$$

We obtain three associated continued fractions by iteration. To use the Heilermann correspondence to obtain evaluations of the Hankel determinants, we define the Maclaurin series coefficients as follows:

$$\begin{aligned} \text{sm}(u, 0) &=: \sum_{\nu=1}^{\infty} \frac{[\text{sm}]_{\nu} u^{3\nu-2}}{(3\nu-2)!} \\ \text{sm}^2(u, 0) &=: \sum_{\nu=1}^{\infty} \frac{[\text{sm}^2]_{\nu} u^{3\nu-1}}{(3\nu-1)!} \\ \text{sm}^3(u, 0) &=: \sum_{\nu=1}^{\infty} \frac{[\text{sm}^3]_{\nu} u^{3\nu}}{(3\nu)!} \end{aligned}$$

Theorem 10.

$$\int_0^{\infty} \text{sm}(u, 0) e^{-u/x} du = \frac{x^2}{1 + 4x^3} + \mathbf{K}_{n=1}^{\infty} \frac{-a_n x^6}{1 + b_n x^3}$$

where:

$$a_n = (3n-2)(3n-1)^2(3n)^2(3n+1); \text{ and}$$

$$b_n = 2(3n + 1) ((3n + 1)^2 + 1).$$

Furthermore:

$$\begin{aligned} H_m^{(1)}([\text{sm}]_\nu) &= \prod_{r=0}^{m-1} (3r)! (3r + 1)! \\ \frac{\chi_m([\text{sm}]_\nu)}{H_m^{(1)}([\text{sm}]_\nu)} &= - \sum_{r=1}^m 2(3r - 2) ((3r - 2)^2 + 1) \\ &= -\frac{1}{6}(3m - 2)(3m - 1)(3m)(3m + 1) \end{aligned}$$

Theorem 11.

$$\int_0^\infty \text{sm}^2(u, 0) e^{-u/x} du = \frac{2x^3}{1 + 20x^3 + \mathbf{K}_{n=1}^\infty \frac{-a_n x^6}{1 + b_n x^3}}$$

where:

$$a_n = (3n - 1)(3n)^2(3n + 1)^2(3n + 2); \text{ and}$$

$$b_n = 2(3n + 2) ((3n + 2)^2 + 1).$$

Furthermore:

$$\begin{aligned} H_m^{(1)}([\text{sm}^2]_\nu) &= \prod_{r=0}^{m-1} (3r + 1)! (3r + 2)! \\ \frac{\chi_m([\text{sm}^2]_\nu)}{H_m^{(1)}([\text{sm}^2]_\nu)} &= - \sum_{r=1}^m 2(3r - 1) ((3r - 1)^2 + 1) \\ &= -\frac{1}{6}(3m - 1)(3m)(3m + 1)(3m + 2) \end{aligned}$$

Theorem 12.

$$\int_0^\infty \text{sm}^3(u, 0) e^{-u/x} du = \frac{6x^4}{1 + 60x^3 + \mathbf{K}_{n=1}^\infty \frac{-a_n x^6}{1 + b_n x^3}}$$

where:

$$a_n = (3n)(3n + 1)^2(3n + 2)^2(3n + 3); \text{ and}$$

$$b_n = 2(3n + 3) \left((3n + 3)^2 + 1 \right).$$

Furthermore:

$$H_m^{(1)}([\text{sm}^3]_\nu) = 2^{2-3m} \prod_{r=1}^m (3r - 1)! (3r)!$$

$$\frac{\chi_m([\text{sm}^3]_\nu)}{H_m^{(1)}([\text{sm}^3]_\nu)} = - \sum_{r=1}^m 2(3r) \left((3r)^2 + 1 \right)$$

$$= -\frac{1}{6}(3m)(3m + 1)(3m + 2)(3m + 3)$$

For Laplace transforms of $\text{sm}^n(u, 0)$ $\text{cm}(u, 0)$, we have the following three initial value problems:

$$\frac{C_n}{C_{n-3}} = \frac{n(n-1)(n-2)x^3}{1 + ((n-1)n^2 + (n+1)^2(n+2))x^3 - (n+1)(n+2)(n+3)x^3 \frac{C_{n+3}}{C_n}} \quad (3.14)$$

subject to:

$$C_0 = \frac{x}{1 + 2x^3 - 6x^3 \frac{C_3}{C_0}} \quad (3.15)$$

$$C_1 = \frac{x^2}{1 + 12x^3 - 24x^3 \frac{C_4}{C_1}} \quad (3.16)$$

$$C_2 = \frac{2x^3}{1 + 40x^3 - 60x^3 \frac{C_5}{C_2}} \quad (3.17)$$

We again obtain three associated continued fractions by iteration. To use the

Heilermann correspondence to obtain evaluations of the Hankel determinants, we define the Maclaurin series coefficients as follows:

$$\begin{aligned}\text{cm}(u, 0) &=: \sum_{\nu=1}^{\infty} \frac{[\text{cm}]_{\nu} u^{3\nu-3}}{(3\nu-3)!} \\ \text{sm}(u, 0) \text{cm}(u, 0) &=: \sum_{\nu=1}^{\infty} \frac{[\text{sm} \cdot \text{cm}]_{\nu} u^{3\nu-2}}{(3\nu-2)!} \\ \text{sm}^2(u, 0) \text{cm}(u, 0) &=: \sum_{\nu=1}^{\infty} \frac{[\text{sm}^2 \cdot \text{cm}]_{\nu} u^{3\nu-1}}{(3\nu-1)!}\end{aligned}$$

Theorem 13.

$$\int_0^{\infty} \text{cm}(u, 0) e^{-u/x} du = \frac{x}{1+2x^3} + \mathbf{K}_{n=1}^{\infty} \frac{-a_n x^6}{1+b_n x^3}$$

where:

$$\begin{aligned}a_n &= (3n-2)^2(3n-1)^2(3n)^2; \text{ and} \\ b_n &= ((3n-1)^2(3n) + (3n+1)^2(3n+2)).\end{aligned}$$

Furthermore:

$$\begin{aligned}H_m^{(1)}([\text{cm}]_{\nu}) &= \prod_{r=0}^{m-1} ((3r)!)^2 \\ \frac{\chi_m([\text{cm}]_{\nu})}{H_m^{(1)}([\text{cm}]_{\nu})} &= -\sum_{r=1}^m ((3r-4)^2(3r-3) + (3r-2)^2(3r-1)) \\ &= -\frac{m}{2}(27m^3 - 42m^2 + 27m - 8)\end{aligned}$$

Theorem 14.

$$\int_0^{\infty} \text{sm}(u, 0) \text{cm}(u, 0) e^{-u/x} du = \frac{x^2}{1+12x^3} + \mathbf{K}_{n=1}^{\infty} \frac{-a_n x^6}{1+b_n x^3}$$

where:

$$a_n = (3n - 1)^2(3n)^2(3n + 1)^2; \text{ and}$$

$$b_n = ((3n)^2(3n + 1) + (3n + 2)^2(3n + 3)).$$

Furthermore:

$$\begin{aligned} H_m^{(1)}([\text{sm} \cdot \text{cm}]_\nu) &= \prod_{r=0}^{m-1} ((3r + 1)!)^2 \\ \frac{\chi_m([\text{sm} \cdot \text{cm}]_\nu)}{H_m^{(1)}([\text{sm} \cdot \text{cm}]_\nu)} &= - \sum_{r=1}^m ((3r - 3)^2(3r - 2) + (3r - 1)^2(3r)) \\ &= -\frac{3m^2}{2}(9m^2 - 2m + 1) \end{aligned}$$

Theorem 15.

$$\int_0^\infty \text{sm}^2(u, 0) \text{cm}(u, 0) e^{-u/x} du = \frac{2x^3}{1 + 40x^3} \mathbf{K}_{n=1}^\infty \frac{-a_n x^6}{1 + b_n x^3}$$

where:

$$a_n = (3n)^2(3n + 1)^2(3n + 2)^2; \text{ and}$$

$$b_n = ((3n + 1)^2(3n + 2) + (3n + 3)^2(3n + 4)).$$

Furthermore:

$$\begin{aligned} H_m^{(1)}([\text{sm}^2 \cdot \text{cm}]_\nu) &= 2^{-m} \prod_{r=0}^{m-1} ((3r + 2)!)^2 \\ \frac{\chi_m([\text{sm}^2 \cdot \text{cm}]_\nu)}{H_m^{(1)}([\text{sm}^2 \cdot \text{cm}]_\nu)} &= - \sum_{r=1}^m ((3r - 2)^2(3r - 1) + (3r)^2(3r + 1)) \\ &= -\frac{m}{2}(27m^3 + 30m^2 + 15m + 4) \end{aligned}$$

For Laplace transforms of $\text{sm}^n(u, 0) \text{cm}^2(u, 0)$, we could handle things in a similar manner, but it turns out there is a simpler procedure available to us. Differentiating powers of $\text{sm}(u, 0)$ with respect to u , we have:

$$\frac{d}{du} \text{sm}^{n+1}(u, 0) = (n + 1) \text{sm}^n(u, 0) \text{cm}^2(u, 0).$$

Next we recall the connection between a Laplace transform and its derivative:

$$\mathcal{L}(f'(u), x^{-1}) = x^{-1} \mathcal{L}(f(u), x) - f(0).$$

Since $\text{sm}(0, 0) = 0$, it follows that:

$$\mathcal{L}(\text{sm}^n(u, 0) \text{cm}^2(u, 0), x^{-1}) = \frac{1}{(n + 1)x} \mathcal{L}(\text{sm}^{n+1}(u, 0), x^{-1}).$$

But, if we instead proceed in the manner above, then we would have the following three initial value problems:

$$\frac{D_n}{D_{n-3}} = \frac{n(n-1)(n-2)x^3}{1 + (n+1)(n^2 + (n+2)^2)x^3 - (n+2)(n+3)(n+4)x^3 \frac{D_{n+3}}{D_n}} \quad (3.18)$$

subject to:

$$D_0 = \frac{x}{1 + 4x^3 - 24x^3 \frac{D_3}{D_0}} \quad (3.19)$$

$$D_1 = \frac{x^2}{1 + 20x^3 - 60x^3 \frac{D_4}{D_1}} \quad (3.20)$$

$$D_2 = \frac{2x^3}{1 + 52x^3 - 120x^3 \frac{D_5}{D_2}} \quad (3.21)$$

We would then obtain the following continued fractions:

$$\int_0^{\infty} \text{cm}^2(u, 0) e^{-u/x} du = \frac{x}{1 + 4x^3} \mathbf{K}_{n=1}^{\infty} \frac{-a_n x^6}{1 + b_n x^3}$$

where:

$$a_n = (3n - 2)(3n - 1)^2(3n)^2(3n + 1); \text{ and}$$

$$b_n = (3n + 1) \left((3n)^2 + (3n + 2)^2 \right).$$

$$\int_0^{\infty} \text{sm}(u, 0) \text{cm}^2(u, 0) e^{-u/x} du = \frac{x^2}{1 + 20x^3} \mathbf{K}_{n=1}^{\infty} \frac{-a_n x^6}{1 + b_n x^3}$$

where:

$$a_n = (3n - 1)(3n)^2(3n + 1)^2(3n + 2); \text{ and}$$

$$b_n = (3n + 2) \left((3n + 1)^2 + (3n + 3)^2 \right).$$

$$\int_0^{\infty} \text{sm}^2(u, 0) \text{cm}(u, 0) e^{-u/x} du = \frac{2x^3}{1 + 52x^3} \mathbf{K}_{n=1}^{\infty} \frac{-a_n x^6}{1 + b_n x^3}$$

where:

$$a_n = (3n)(3n + 1)^2(3n + 2)^2(3n + 3); \text{ and}$$

$$b_n = (3n + 3) \left((3n + 2)^2 + (3n + 4)^2 \right).$$

Our observations about the Laplace transforms of derivatives tell us that these three continued fractions should differ from their counterparts in the first three theorems only in the entry in the first numerator. The expressions for a_n are identical

to those in their counterparts in the first three theorems of this section. Some simple algebra shows that the expressions for b_n are equal to their counterparts in the first three theorems in this section. Since the first numerator entries are as expected, the extra calculations serve as a check on the original calculations.¹

Consider the list of positive integers in ascending order with each integer listed twice:

$$1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, \dots$$

There are essentially six ways to break this up into ascending 6-tuples, allowing for the possibility of missing leading entries in the first 6-tuple. The a_n terms in the associated continued fraction for the Laplace transform of $\text{sm}(u, 0)$ correspond to the partitions:

$$\{(1, 2, 2, 3, 3, 4), (4, 5, 5, 6, 6, 7), (7, 8, 8, 9, 9, 10), \dots\}$$

The associated continued fractions for $\text{sm}^2(u, 0)$ and $\text{sm}^3(u, 0)$ correspond respectively to:

$$\{(2, 3, 3, 4, 4, 5), (5, 6, 6, 7, 7, 8), (8, 9, 9, 10, 10, 11), \dots\}$$

and to:

$$\{(3, 4, 4, 5, 5, 6), (6, 7, 7, 8, 8, 9), (9, 10, 10, 11, 11, 12), \dots\}$$

These are the three possible “odd” partitions, odd in the sense that the first integer in each 6-tuple appears exactly once. The three remaining continued fractions are associated in the same way with the three possible even partitions.

¹This process involves foresight. In actual practice, I first calculated the three continued fractions using the recurrence relations, observed that they were essentially the same, and then realized why this was the case. In any case, the extra calculations served as to check the original calculations.

3.3 Regular C-Fractions ($\alpha = 0$)

Starting with the original recurrences in this chapter (equations (3.4-3.9)), we can use the relation between S_n and D_{n-1} to eliminate either the letter S or the letter D .

In terms of general α , we obtain the following four recurrence relations:

$$\frac{S_n}{C_{n-2}} = \frac{n(n-1)x^2}{(1+n\alpha x)(1-2n\alpha x) + n(n+1)x^2 \frac{C_{n+1}}{S_n}} \quad (3.22)$$

$$\frac{C_n}{S_{n-1}} = \frac{nx}{1 - (2n+1)\alpha x + (n+1)x \frac{S_{n+2}}{C_n}} \quad (3.23)$$

$$\frac{C_n}{D_{n-2}} = \frac{\frac{n(n-1)x^2}{1-(n-1)\alpha x}}{1 - (2n+1)\alpha x + \frac{(n+1)(n+2)x^2}{1-(n+2)\alpha x} \frac{D_{n+1}}{C_n}} \quad (3.24)$$

$$\frac{D_n}{C_{n-1}} = \frac{nx}{1 - (2n+2)\alpha x + (n+2)x \frac{C_{n+2}}{D_n}} \quad (3.25)$$

The recurrences with general α will be used again in §3.4. When $\alpha = 0$, these recurrences reduce to C-fraction recurrences, which we tabulate here:

$$\frac{S_n}{C_{n-2}} = \frac{n(n-1)x^2}{1 + n(n+1)x^2 \frac{C_{n+1}}{S_n}} \quad (3.26)$$

$$\frac{C_n}{S_{n-1}} = \frac{nx}{1 + (n+1)x \frac{S_{n+2}}{C_n}} \quad (3.27)$$

$$\frac{C_n}{D_{n-2}} = \frac{n(n-1)x^2}{1 + (n+1)(n+2)x^2 \frac{D_{n+1}}{C_n}} \quad (3.28)$$

$$\frac{D_n}{C_{n-1}} = \frac{nx}{1 + (n+2)x \frac{C_{n+2}}{D_n}} \quad (3.29)$$

We need initial conditions to generate C-fractions, and the six candidate starting points give just three C-fraction initial conditions:

$$S_1 = x^2 - 2x^2C_2 = \frac{x^2}{1 + 2x\frac{C_2}{S_1}}$$

$$C_0 = x - xS_2 = \frac{x}{1 + x\frac{S_2}{C_0}}$$

$$C_1 = x^2 - 2xS_3 = \frac{x}{1 + 2x\frac{S_3}{C_1}}$$

The three that fail to give good initial conditions for a C-fraction are as follows:

$$S_2 = 2x^3 - 2x^3S_2 - 6x^2C_3 = \frac{2x^3}{1 + 2x^3 + 6x^2\frac{C_3}{S_2}}$$

$$S_3 = 6x^4 - 12x^3S_3 - 12x^2C_4 = \frac{6x^4}{1 + 12x^3 + 12x^2\frac{C_4}{S_3}}$$

$$C_2 = 2x^3 - 4x^3C_2 - 3x^2S_4 = \frac{2x^3}{1 + 4x^3 + 3x^2\frac{S_4}{C_2}}$$

On expanding the Laplace transform of $\text{sm}(u, 0)$, we have the following regular C-fraction expansion:

Theorem 16 (C-fraction for $\mathcal{L}(\text{sm}(u, 0))$).

$$\int_0^\infty \text{sm}(u, 0) e^{-u/x} du = \frac{x^2}{1+} \mathbf{K}_{n=1}^\infty \frac{a_n x^3}{1}$$

where for integers $r \geq 1$:

$$a_{2r-1} = (3r - 2)(3r - 1)^2$$

$$a_{2r} = (3r)^2(3r + 1).$$

The Heilermann correspondence yields the following new Hankel determinant evaluation:

$$H_m^{(2)}([\text{sm}]_\nu) = (-1)^m \prod_{r=0}^{m-1} ((3r + 2)!)^2.$$

The C-fraction for $\text{cm}(u, 0)$ is likewise regular:

Theorem 17 (C-fraction for $\mathcal{L}(\text{cm}(u, 0))$).

$$\int_0^\infty \text{cm}(u, 0) e^{-u/x} du = \frac{x}{1+} \mathbf{K}_{n=1}^\infty \frac{a_n x^3}{1}$$

where for integers $r \geq 1$:

$$a_{2r-1} = (3r - 2)^2(3r - 1)$$

$$a_{2r} = (3r - 1)(3r)^2.$$

Applying the C-fraction correspondence:

$$H_m^{(2)}([\text{cm}]_\nu) = (-1)^m \prod_{r=0}^{m-1} (3r + 1)!(3r + 2)!.$$

Finally, the Laplace transform of $\text{sm}(u, 0) \text{cm}(u, 0)$ yields a regular C-fraction:

Theorem 18 (C-fraction for $\mathcal{L}(\text{sm}(u, 0) \text{cm}(u, 0))$).

$$\int_0^\infty \text{sm}(u, 0) \text{cm}(u, 0) e^{-u/x} du = \frac{x^2}{1+} \mathbf{K}_{n=1}^\infty \frac{a_n x^3}{1}$$

where for integers $r \geq 1$:

$$a_{2r-1} = (3r - 1)^2(3r)$$

$$a_{2r} = (3r)(3r + 1)^2.$$

Again, applying the Heilermann C-fraction correspondence:

$$H_m^{(2)}([\text{sm} \cdot \text{cm}]_\nu) = (-1)^m \prod_{r=1}^m (3r-1)! (3r)!.$$

Now consider the set of positive integers written in increasing order with each integer written twice:

$$1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, \dots$$

There are three ways to divide this list into ascending triples if we permit ourselves to discard ones. If we discard no ones, then we have:

$$\sigma_1 := \{(1, 1, 2), (2, 3, 3), (4, 4, 5), (5, 6, 6), \dots\}.$$

If we discard the first one, we obtain:

$$\sigma_2 := \{(1, 2, 2), (3, 3, 4), (4, 5, 5), (6, 6, 7), \dots\}.$$

If we discard both ones, we obtain:

$$\sigma_3 := \{(2, 2, 3), (3, 4, 4), (5, 5, 6), (6, 7, 7), \dots\}.$$

The sequence of products of the triples in sequence σ_2 yields the sequence of numerator coefficients in the C-fraction for $\mathcal{L}(\text{sm}(u, 0))$. The sequence of triples σ_1 is similarly associated with $\text{cm}(u, 0)$. The remaining sequence of triples σ_3 is associated with the Laplace transform of the product of the two families. This correspondence suggests that these three continued fractions form a complete set.

3.4 Quasi C-Fractions for the Dixon Functions

If we remove the restriction that $\alpha = 0$, then we no longer obtain C-fractions or associated continued fractions from the recurrences. We do, however, obtain continued fractions that converge generally to the Maclaurin series for the Laplace transforms. Furthermore, these continued fractions are α -generalizations of the continued fractions above in the sense that if we formally let $\alpha = 0$, then we obtain the C-fractions and associated continued fractions above as special cases. The three continued fractions in this section have the rough form of regular C-fractions in x^3 whose denominator terms are alternately replaced by linear and quadratic functions of αx .

In this section we will carry out this program for the regular C-fractions of the previous section. The recurrences (3.22) and (3.23) are well-suited to the task at hand.

We do ask the reader's forbearance if the derivations seem tedious or the continuous fractions seem inelegant. The Chudnovskys [6, page 127] explain their difficulties in finding effective, explicit and practical constructions of rational approximations to classical numbers and functions, saying, "As we understand the cause of the difficulties now, there is simply no closed form expression for the best rational approximations or continued fractions of classical functions (even more so for numbers)." Apparently such constructions are quite rare.

For $\text{sm}(u, \alpha)$, we note that:

$$S_1 = xD_0 - \alpha xS_1.$$

Now the expansion D_0 can be solved for D_0 as follows:

$$D_0 = \frac{1}{1 - 2\alpha x}(x - 2xC_2).$$

Once we have eliminated D_0 , we can solve and simplify to obtain:‘

$$S_1 = \frac{x^2}{(1 + \alpha x)(1 - 2\alpha x) + 2x^2 \frac{C_2}{S_1}}.$$

On iterating, we obtain:

Theorem 19.

$$\int_0^\infty \text{sm}(u, \alpha) e^{-u/x} du = \frac{x^2}{(1 + \alpha x)(1 - 2\alpha x) + \prod_{n=0}^\infty \frac{a_n x^3}{b_n(x)}}$$

where, for positive integers r :

$$a_{2r-1} = (3r - 2)(3r - 1)^2$$

$$b_{2r-1}(x) = 1 + (6r - 1)\alpha x$$

and:

$$a_{2r} = (3r)^2(3r + 1)$$

$$b_{2r}(x) = (1 + (3r + 1)\alpha x)(1 - 2(3r + 1)\alpha x).$$

For $\mathcal{L}(\text{cm}(u, \alpha))$, we have:

$$C_0 = x + \alpha x C_0 - x S_2 = \frac{x}{1 - \alpha x + x \frac{S_2}{C_0}}.$$

On iterating:

Theorem 20.

$$\int_0^{\infty} \text{cm}(u, \alpha) e^{-u/x} du = \frac{x}{1 - \alpha x} \mathbf{K}_{n=0}^{\infty} \frac{a_n x^3}{b_n(x)}$$

where, for positive integers r :

$$a_{2r-1} = (3r - 2)^2(3r - 1)$$

$$b_{2r-1}(x) = (1 + (3r - 1)\alpha x)(1 - 2(3r - 1)\alpha x)$$

and:

$$a_{2r} = (3r - 1)(3r)^2$$

$$b_{2r}(x) = 1 + (6r + 1)\alpha x.$$

For the Laplace transform of the product of the two functions, we have:

$$C_1 = x^2 + 3\alpha x C_1 - 2x S_3 = \frac{x^2}{1 - 3\alpha x + 2x \frac{S_3}{C_1}}.$$

Thus, on iterating:

Theorem 21.

$$\int_0^{\infty} \text{sm}(u, 0) \text{cm}(u, \alpha) e^{-u/x} du = \frac{x}{1 - 3\alpha x} \mathbf{K}_{n=0}^{\infty} \frac{a_n x^3}{b_n(x)}$$

where, for positive integers r :

$$a_{2r-1} = (3r - 1)^2(3r)$$

$$b_{2r-1}(x) = (1 + (3r)\alpha x)(1 - 2(3r)\alpha x)$$

and:

$$a_{2r} = (3r)(3r + 1)^2$$
$$b_{2r}(x) = 1 + (6r + 3)\alpha x.$$

We note that the numerator sequences a_n in the three continued fractions of this section are the same as the numerator sequences for the C-fractions of the previous section. The remarks at the end of the previous section suggest the existence of just these three families.

3.5 Quasi Associated Continued Fractions

Just as the C-fractions for $\alpha = 0$ do not generalize to C-fractions for the Dixon functions, so also the six associated continued fractions fail to generalize to associated continued fractions. However, all six associated continued fractions do generalize to continued fractions having the form of an associated continued fraction in x^3 whose coefficients have been replaced by polynomials of degree at most 5 in αx . Though more complicated than any of the continued fractions that appear so far in this thesis, they are still much simpler in form than the results stated in Chudnovsky and Chudnovsky [6]. The polynomials themselves exhibit a considerable amount of symmetry. Moreover, the underlying cubic nature of these continued fractions suggests connections to cubic transformations of hypergeometric series.

Iterating the general cases of the recurrences of section 3.3, we obtain the following

recurrence for Laplace transforms of powers of $\text{sm}(u, \alpha)$:

$$E_n(x, \alpha)x^3 \frac{S_{n-3}}{S_n} = F_n(x, \alpha) + G_n(\alpha x)x^3 - H_n(x, \alpha)x^3 \frac{S_{n+3}}{S_n} \quad (3.30)$$

where:

$$E_n \equiv E_n(x, \alpha) := (1 - (2n + 3)\alpha x)n(n - 1)(n - 2) \quad (3.31)$$

$$F_n \equiv F_n(x, \alpha) := (1 - (2n - 3)\alpha x)(1 - 2n\alpha x)(1 - (2n + 3)\alpha x)(1 + n\alpha x) \quad (3.32)$$

$$G_n \equiv G_n(x, \alpha) := (1 - (2n + 3)\alpha x)n(n - 1)^2 + (1 - (2n - 3)\alpha x)n(n + 1)^2 \quad (3.33)$$

$$H_n \equiv H_n(\alpha, x) := (1 - (2n - 3)\alpha x)n(n + 1)(n + 2) \quad (3.34)$$

Thus:

$$\frac{S_n}{S_{n-3}} = \frac{E_n x^3}{F_n + 4x^3 + H_n x^3 \frac{S_{n+3}}{S_n}}.$$

For initial conditions:

$$S_1 = \frac{(1 - 5\alpha x)x^2}{F_1^* + 4x^3 - 6x^3 \frac{S_4}{S_1}} \quad (3.35)$$

$$S_2 = \frac{E_2 x^2}{F_2 + G_2 x^3 - H_2 x^3 \frac{S_5}{S_2}} \quad (3.36)$$

$$S_3 = \frac{E_3 x^2}{F_3 + G_3 x^3 - H_3 x^3 \frac{S_6}{S_3}} \quad (3.37)$$

where:

$$F_1^* := (1 - 2\alpha x)(1 - 5\alpha x)(1 + \alpha x) \quad (3.38)$$

Iterating, we obtain as theorems the expansions for the three Laplace transforms in terms of the polynomials E_n , F_n , G_n and H_n defined above:

Theorem 22.

$$\begin{aligned} & \int_0^\infty \text{sm}(u, \alpha) e^{-u/x} du \\ &= \frac{(1 - 5\alpha x)x^2}{(1 - 2\alpha x)(1 - 5\alpha x)(1 + \alpha x) + 4x^3 +} \\ & \quad \frac{-(1 - 11\alpha x)2^2 3^2 4x^6}{F_4 + G_4 x^3 +} \mathbf{K}_{n=2}^\infty \frac{-H_{3n-2} E_{3n+1} x^6}{F_{3n+1} + G_{3n+1} x^3}. \end{aligned}$$

Theorem 23.

$$\int_0^\infty \text{sm}^2(u, \alpha) e^{-u/x} du = \frac{E_2 x^6}{F_2 + G_2 x^3 +} \mathbf{K}_{n=1}^\infty \frac{-H_{3n-1} E_{3n+2} x^6}{F_{3n+2} + G_{3n+2} x^3}.$$

Theorem 24.

$$\int_0^\infty \text{sm}^3(u, \alpha) e^{-u/x} du = \frac{E_3 x^6}{F_3 + G_3 x^3 +} \mathbf{K}_{n=1}^\infty \frac{-H_{3n} E_{3n+3} x^6}{F_{3n+3} + G_{3n+3} x^3}.$$

For the Laplace transforms related to $\text{cm}(u, \alpha)$, we solve for C_n in terms of C_{n-3} and C_{n+3} . Arranging the result in suitable form for computing the continued fraction, we obtain:

$$\frac{C_n}{C_{n-3}} = \frac{M_n x^3}{P_n + Q_n x^3 - R_n x^3 \frac{C_{n+3}}{C_n}} \quad (3.39)$$

where:

$$M_n \equiv M_n(x, \alpha) := ((1 - (2n + 4)\alpha x)(1 + (n + 2)\alpha x)n(n - 1)(n - 2)) \quad (3.40)$$

$$P_n \equiv P_n(x, \alpha) := ((1 - (2n - 2)\alpha x)((1 - (2n + 1)\alpha x)((1 - (2n + 4)\alpha x)$$

$$\times ((1 + (n - 1)\alpha x)(1 + (n + 2)\alpha x) \quad (3.41)$$

$$\begin{aligned} Q_n \equiv Q_n(x, \alpha) := & ((1 - (2n + 4)\alpha x)((1 + (n + 2)\alpha x)n^2(n - 1) \\ & + ((1 - (2n - 2)\alpha x)((1 + (n + 2)\alpha x)(n + 1)^2(n + 2) \end{aligned} \quad (3.42)$$

$$R_n \equiv R_n(x, \alpha) := ((1 - (2n - 2)\alpha x)(1 + (n - 1)\alpha x)(n + 1)(n + 2)(n + 3) \quad (3.43)$$

For initial conditions:

$$C_0 = \frac{M_0^* x}{P_0^* + 2x^3 - 6x^3 \frac{C_3}{C_0}} \quad (3.44)$$

$$C_1 = \frac{M_1^* x^2}{P_1^* + 12x^3 - R_1 x^3 \frac{C_4}{C_1}} \quad (3.45)$$

$$C_2 = \frac{2M_2^* x^3}{P_2 + Q_2 x^3 - R_2 x^3 \frac{C_5}{C_2}} \quad (3.46)$$

where:

$$M_0^* := (1 - 4\alpha x)(1 + 2\alpha x) \quad M_1^* := (1 - 6\alpha x)(1 + 3\alpha x)$$

$$M_2^* := (1 - 8\alpha x)(1 + 4\alpha x)$$

$$P_0^* := (1 - \alpha x)(1 - 4\alpha x)(1 + 2\alpha x) \quad P_1^* := (1 - 3\alpha x)(1 - 6\alpha x)(1 + 3\alpha x).$$

Theorem 25.

$$\begin{aligned} & \int_0^\infty \text{cm}(u, \alpha) e^{-u/x} du \\ &= \frac{(1 - 4\alpha x)(1 + 2\alpha x)x}{(1 - \alpha x)(1 - 4\alpha x)(1 + 2\alpha x) + 2x^3 +} \\ & \quad \frac{-6M_3 x^6}{P_3 + Q_3 x^3 +} \mathbf{K}_{n=2}^\infty \frac{-R_{3n-3} M_{3n} x^6}{P_{3n} + Q_{3n} x^3}. \end{aligned}$$

Theorem 26.

$$\int_0^\infty \text{sm}(u, \alpha) \text{cm}(u, \alpha) e^{-u/x} du = \frac{(1 - 6\alpha x)(1 + 3\alpha x)x^2}{(1 - 3\alpha x)(1 - 6\alpha x)(1 + 3\alpha x) + 12x^3} \mathbf{K}_{n=1}^\infty \frac{-R_{3n-2}M_{3n+1}x^6}{P_{3n+1} + Q_{3n+1}x^3}.$$

Theorem 27.

$$\int_0^\infty \text{sm}^2(u, \alpha) \text{cm}(u, \alpha) e^{-u/x} du = \frac{2(1 - 8\alpha x)(1 + 4\alpha x)x^3}{P_2 + Q_2x^3} \mathbf{K}_{n=1}^\infty \frac{-R_{3n-1}M_{3n+2}x^6}{P_{3n+2} + Q_{3n+2}x^3}.$$

The remarks at the end of Section 3.2 on symmetries and completeness carry over to this section.

CHAPTER 4

ANALYTICAL ISSUES

4.1 Laplace Transforms

Since our underlying notion of convergence is general convergence of power series, a sufficient condition for the existence of a (formal) Laplace transform of a function is that the function be analytic at the origin. (This condition may be weakened, but we have no need here to do so.) The result in this case is a power series which might not represent an analytic function. We offer the following example as an illustration.

$$\int_0^{\infty} \frac{1}{1-x} e^{-u/x} dx = x \sum_{n=0}^{\infty} n! x^n. \quad (4.1)$$

Using classical notions of convergence, neither side makes sense. The integrand on the left is a function that has a pole of order 1 at $x = 1$ on the real axis. Thus the integral diverges classically. On the right hand side, the ratio of the absolute values of the consecutive terms is $(n+1)|x|$ and thus, by the ratio test, the right hand side converges if and only if $x = 0$. (Note that the left hand side is undefined when $x = 0$.)

We can, however, make sense of both sides of the equation using the notion of a formal Laplace transform of a series, defined in chapter 1. Formal properties of

the Laplace transform are preserved. These include linearity and the relation of the Laplace transform of an analytic function to the transform of its derivative.

Using the formal Laplace transform, the example follows almost trivially from the geometric series expansion:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

4.2 Power Series

A sequence of power series f_n converges generally to a power series if

$$\|f_n - f\| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

where $\|\cdot\|$ denotes the power series norm. A few examples should help to illustrate some differences between general convergence and some classical notions of convergence.

Our first example is the sequence $f_n(x) = e^x - x/n$. The terms of this sequence may be expanded into power series as follows:

$$f_n(x) = 1 + (1 - 1/n)x + \sum_{r=2}^{\infty} \frac{x^r}{r!}.$$

Since $\lim_{n \rightarrow \infty} |f_n(x) - e^x| = 0$, it follows that $f_n(x)$ converges classically to

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

But $\|f_n(x) - f_{n+1}(x)\| = \|x/n(n+1)\| = 1/2$, so $f_n(x)$ does not converge generally. Note however that it converges both classically and termwise to e^x .

As a second example, consider: $g_n(x) = e^x - 1/n!$. For this sequence, classical convergence to e^x is uniform and computationally very fast, but from the standpoint of general convergence, this sequence is computationally worse than the sequence $f_n(x)$:

$$\|g_n(x) - g_{n+1}(x)\| = \|(1/n! - 1/(n+1)!)x^0\| = 1.$$

From a functional standpoint, a sequence of analytic functions may converge either pointwise or uniformly, but fail to converge generally. Although general convergence does guarantee termwise convergence, the two examples above show that the converse is sometimes false.

Finally, a simple example shows that general convergence does not guarantee convergence. Consider the sequence of polynomials defined by:

$$h_n(x) = \sum_{r=0}^n r!x^r.$$

The sequence h_n converges generally to the (nonanalytic) power series:

$$h(x) = \sum_{r=0}^{\infty} r!x^r.$$

The power series is easily seen to diverge for all $x \neq 0$ by the ratio test, or by observing that the sequence of terms does not converge to zero.

4.3 Continued fractions

Following an example in D. Dumont in [12], we use the quotient-difference algorithm as described in Jones and Thron [17, Theorem 7.7] to expand the divergent series:

$$f(x) := \sum_{r=0}^{\infty} r!x^r$$

into its corresponding regular C-fraction:

$$f(x) = \frac{1}{1 + \mathbf{K}_{n=1}^{\infty} \frac{a_n x}{1}}$$

where, for $m = 1, 2, 3, \dots$:

$$a_{2m-1} = a_{2m} = -m.$$

As noted above, the series expansion for $f(x)$ diverges (*e.g.* by D'Alembert's ratio test) for all $z \neq 0$.

The following argument shows that the continued fraction on the left converges generally to some power series. More importantly, the argument shows that each of the continued fractions of Chapter 2 and Chapter 3 likewise converge generally.

Like most of the continued fractions considered in this thesis, the continued fraction in this example has tails of the form:

$$\frac{Az^\alpha}{1 + O(z)} = Az^\alpha + O(z^{\alpha+1})$$

where $\alpha \geq 1$ and where $O(z^\beta)$ denotes the set of Laurent series of minimum degree β , or, in the language of general convergence:

$$O(z^\beta) := \{f(z) \in \mathbf{L}(z) : \|f(z)\| \leq 2^{-\beta}\}.$$

(The notation $\mathbf{L}(z)$ denotes the field of unilateral formal Laurent series in indeterminate z).

Working backwards from the tail, we have the form:

$$\frac{Bz^\alpha}{1 + \frac{Az^\alpha}{1 + O(z)}} = \frac{Bz^\alpha}{1 + Az^\alpha + O(z^{\alpha+1})}$$

Applying the division algorithm for power series, can reduce this further:

$$= Bz^\alpha - ABz^{2\alpha} + O(z^{2\alpha+1})$$

Iterating up to the top of the continued fraction and comparing with other tails, we have:

$$\mathbf{K}_{n=1}^N(a_n z^\alpha, 1, O(z)) - \mathbf{K}_{n=1}^{N+m}(a_n z^\alpha, 1, O(z)) = O(z^{N\alpha+1}) \quad (4.2)$$

This simply means that the power series expansions of the two convergents are in exact agreement, term for term, up to terms of degree less than or equal to $N\alpha$. In the example, to compute the first N terms of the power series directly from the continued fraction using the division algorithm, we need to expand the N th convergent of the continued fraction.

CHAPTER 5

FUTURE DIRECTIONS

- Is there an analogue of Hankel determinants which would enable us to do number theory on the Dixon functions analogous to Milne's work on the Jacobi elliptic functions?

An important ingredient in the work of Milne [21, 22] is a set of Fourier series expansions for elliptic functions. Most of these were obtained by C. Jacobi in [16, §38] and most are consequences of S. Ramanujan's ${}_1\psi_1$ summation theorem (see Gasper and Rahman [14, Chapter 5]). Although Dixon did not obtain Fourier series expansions for his functions, we do note that the Dixon functions are ${}_1\psi_1$ summable and yield Fourier series. A much deeper issue is the connection between elliptic function invariants and the combinatorics or number theory. In particular, it would be necessary to find analogues in Dixon's theory to Jacobi's one-square theorem, *i.e.*:

$$\sqrt{\frac{2\mathbf{K}(k)}{\pi}} = \vartheta_3(0, q),$$

which relates the real period $4\mathbf{K}(k)$ of $\operatorname{sn}(u, k)$ to the ordinary generating function $\vartheta_3(0, q)$ for the number of ways to write an integer as a perfect square. If, in addition, the continued fractions of sections 3.4 and 3.5 can be used to

compute some reasonable analogue of Hankel determinants of Maclaurin coefficients, then there may well be some Dixon analogues of the infinite families of sums of squares results in Milne [22].

- Are the Dixon($\alpha = 0$) functions part of some family of functions which generalize the continued fractions of Sections 3.2 and 3.3?

Here we ask whether there are regular C-fraction generalizations of the C-fractions of Section 3.3 which correspond to reasonable generalizations of the Dixon ($\alpha = 0$) functions. Do the associated continued fractions of Section 3.2 generalize in a similar manner? Perhaps there is a different way of extending these functions by adding a free parameter which yields better generalizations of the continued fractions for $\alpha = 0$. In this case, if the generalizations have usable Fourier expansions, then there is a chance of finding analogues of Milne's results in another way.

- What families of orthogonal polynomials arise from the three-term recurrences for the Dixon functions?

Since Ramanujan's ${}_1\psi_1$ summation theorem can be used to yield Fourier series for the Dixon functions, there is a reasonable chance that the recurrences in Chapter 3 yield interesting families of orthogonal polynomials, perhaps yielding analogues of the specialized families of Carlitz [5], or perhaps analogues of more classical families (*e.g.* see Ismail and Masson [15]). There may also be additional families of orthogonal polynomials mentioned in [19] with analogues in the Dixon setting.

- Has the system of orthogonal polynomials in Carlitz [5] been completely characterized?

Carlitz was unaware of the existence of an associated continued fraction expansion for $\operatorname{sn}(u, k) \operatorname{cn}(u, k)$.

Appendix A

CONTINUED FRACTION ITERATION

The purpose of the first section of this appendix is to demystify the process of obtaining the continued fractions from the Laplace transform recurrences in Chapters 2 and 3 of this thesis. We do this both by eliminating the clutter of coefficients and by showing more of the details of the underlying computations.

The remaining sections in this appendix briefly introduce C-fractions and associated continued fractions, and show a simple relationship between them.

A.1 Three Term Recurrences

Consider the following three-term recurrence relation:

$$x_n = a_n x_{n-1} + b_n x_n + c_n x_{n+1} \quad (n \geq 0). \quad (\text{A.1})$$

We assume that $x_n \neq 0$ for all $n \geq -1$. To avoid trivialities, we also assume that $a_n \neq 0$, $b_n \neq 1$, and $c_n \neq 0$ for all n . Under these assumptions, we next divide the recurrence by x_n to obtain:

$$1 = a_n \frac{x_{n-1}}{x_n} + b_n + c_n \frac{x_{n+1}}{x_n}.$$

We rewrite this as follows:

$$a_n \frac{x_{n-1}}{x_n} = 1 - b_n - c_n \frac{x_{n+1}}{x_n}.$$

Now solve for the fraction on the left:

$$\frac{x_{n-1}}{x_n} = \frac{1 - b_n - c_n \frac{x_{n+1}}{x_n}}{a_n}.$$

Finally, take reciprocals to obtain a recurrence satisfied by consecutive quotients:

$$\frac{x_n}{x_{n-1}} = \frac{a_n}{1 - b_n - c_n \frac{x_{n+1}}{x_n}}. \quad (\text{A.2})$$

This recurrence (A.2) is the key step.

If we start with $n = 0$, then we may use weak mathematical induction to show that the following sequence of statements holds:

$$\begin{aligned} \frac{x_0}{x_{-1}} &= \frac{a_0}{1 - b_0 - c_0 \frac{x_1}{x_0}} \\ &= \frac{a_0}{1 - b_0 - \frac{c_0 a_1}{1 - b_1 - c_1 \frac{x_2}{x_1}}} \\ &= \dots \end{aligned}$$

A.2 C-Fractions

If we additionally assume that $b_n = 0$ for all n , recurrence (A.1) reduces to:

$$x_n = a_n x_{n-1} + c_n x_{n+1} \quad (n \geq 0)$$

and recurrence (A.2) reduces to:

$$\frac{x_n}{x_{n-1}} = \frac{a_n}{1 - c_n \frac{x_{n+1}}{x_n}}$$

This can be simplified by making the substitutions:

$$y_n = (-1)^n x_n \prod_{m=0}^{n-1} c_m$$

$$A_n = -a_n c_{n-1}$$

On applying the substitutions, the two recurrences reduce to:

$$y_n = A_n y_{n-1} - y_{n+1} \quad (n \geq 0) \quad (\text{A.3})$$

$$\frac{y_n}{y_{n-1}} = \frac{a_n}{1 + \frac{y_{n+1}}{y_n}} \quad (\text{A.4})$$

The resulting continued fraction $\mathbf{K} \frac{A_n}{1}$ is traditionally called a C-fraction. The C-fractions in this thesis are all special cases $\mathbf{K} \frac{A_n z^{\alpha(n)}}{1}$ where z is a power series variable and $\alpha(n)$ is a sequence of positive integers. When the sequence of exponents $\alpha(n)$ is constant, the resulting C-fractions are said to be regular.

A.3 Associated Continued Fractions

Returning to the recurrences at the beginning of this appendix, we make the following substitutions:

$$y_n = (-1)^n x_n \prod_{m=0}^{n-1} c_m$$

$$A_n = -a_n c_{n-1}$$

$$B_n = -b_n$$

On applying the substitutions, we have:

$$y_n = A_n y_{n-1} - B_n y_n - y_{n+1} \quad (n \geq 0) \quad (\text{A.5})$$

$$\frac{y_n}{y_{n-1}} = \frac{A_n}{1 + B_n + \frac{y_{n+1}}{y_n}} \quad (\text{A.6})$$

The resulting continued fraction $\mathbf{K} \frac{A_n}{1+B_n}$ is a more general form than the C-fractions of the previous section. An associated continued fraction has the form $\mathbf{K} \frac{A_n z^2}{1+B_n z}$ where z is a power series variable.

A.4 Relation Between the Two Types

Certain associated continued fractions may be obtained from C-fractions by considering only the “even” recurrences associated with recurrence relation (A.3). Specifically, we iterate the recurrence in both directions to eliminate the “odd” terms:

$$y_n = A_{n-1} A_n y_{n-2} - (A_n + A_{n+1}) y_n + y_{n+2} \quad (\text{A.7})$$

whence:

$$\frac{y_n}{y_{n-2}} = \frac{A_{n-1} A_n}{1 + (A_n + A_{n+1}) - \frac{y_{n+2}}{y_n}} \quad (\text{A.8})$$

This is reasonable provided that $A_n + A_{n+1} \neq -1$.

Two good starting points for the continued fraction iteration are y_1/y_{-1} and y_2/y_0 .

Appendix B

CANONICAL FORMS

In this appendix we tabulate the regular C-fractions of sections 2.2 and 3.3 and the associated continued fractions of sections 2.1 and 3.2. The following example will serve as a guide to reading the tables:

Problem: Expand $F := \mathcal{L}(\text{dc}(u, k), x^{-1})$ into a regular C-fraction.

Solution Use Table B.1. From §1.2, we have:

$$\text{dc}(u, k) = \text{dn}(u; 1, k') = \text{cn}(u; k', 1).$$

Thus $a^2 = (k')^2 = 1 - k^2$ and $b^2 = 1$. Reading the entries in the table,

$$\begin{aligned} z = x^2 \quad a_{2r} &= -(2r - 1)^2(1 - k^2) \\ a_1 = 1 \quad a_{2r+1} &= -4r^2 \end{aligned}$$

From the second column, the correspondence has the form $1 + xF$. In other words:

$$1 + x\mathcal{L}(\text{dc}(u, k), x^{-1}) = 1 + \frac{x^2}{1 - \frac{(1 - k^2)x^2}{1 - \frac{4x^2}{1 - \frac{9(1 - k^2)x^2}{1 - \frac{16x^2}{1 - \dots}}}}}$$

Solving for the Laplace transform is straightforward:

$$\mathcal{L}(\text{dc}(u, k), x^{-1}) = \frac{x}{1 - \frac{(1 - k^2)x^2}{1 - \frac{4x^2}{1 - \frac{9(1 - k^2)x^2}{1 - \frac{16x^2}{1 - \dots}}}}$$

Since $\text{dc}(u, k)$ has poles on the real line, this is a formal Laplace transform.

B.1 Regular C-Fractions

Heilermann's C-fraction correspondence (Lemma 3) in Chapter 1 establishes a correspondence between a normal generating function and a regular C-fraction as follows:

$$1 + \sum_{\nu=1}^{\infty} c_{\nu} z^{\nu} = 1 + \mathbf{K}_{n=1}^{\infty} \frac{a_n z}{1}.$$

The following table summarizes the data for the four regular C-fraction correspondences:

Elliptic function	Power series	z	a_{2r} ($r \geq 1$)
$f(u)$	$(F = \mathcal{L}(f, x^{-1}))$	a_1	a_{2r+1} ($r \geq 1$)
$\text{cn}(u; a, b)$	$1 + xF$	x^2	$-(2r - 1)^2 a^2$
		1	$-(2r)^2 b^2$
$\text{sm}(u, 0)$	$1 + xF$	x^3	$(3r - 2)(3r - 1)^2$
		1	$(3r)^2(3r + 1)^2$
$\text{cm}(u, 0)$	$1 + x^2 F$	x^3	$(3r - 2)^2(3r - 1)$
		1	$(3r - 1)(3r)^2$
$\text{sm}(u, 0) \text{ cm}(u, 0)$	$1 + xF$	x^3	$(3r - 1)^2(3r)$
		1	$(3r)(3r + 1)^2$

TABLE B.1 Regular C-fractions.

B.2 Associated Continued Fractions

For the associated continued fractions, in Lemma 2, the stated Heilermann correspondence takes the form:

$$1 + \sum_{\nu=1}^{\infty} c_{\nu} z^{\nu} = 1 + \frac{a_1 z}{1 + b_1 z + \mathbf{K}_{n=2}^{\infty} \frac{-a_n z^2}{1 + b_n z}}.$$

We first summarize the correspondences for the four families of Laplace transforms of Jacobi elliptic functions:

Elliptic function $f(u)$	Power series ($F = \mathcal{L}(f, x^{-1})$)	z a_1	a_n ($n \geq 2$) b_n ($n \geq 1$)
$\text{sn}(u; a, b)$	$1 + F$	x^2 1	$(2n - 3)(2n - 2)^2(2n - 1)a^2b^2$ $-(2n - 1)^2(a^2 + b^2)$
$\text{sn}^2(u; a, b)$	$1 + x^{-1}F$	x^2 2	$(2n - 2)(2n - 1)^2(2n)a^2b^2$ $-(2n)^2(a^2 + b^2)$
$\text{cn}(u; a, b)$	$1 + xF$	x^2 1	$(2n - 3)^2(2n - 2)^2a^2b^2$ $-((2n - 1)^2a^2 + (2n - 2)^2b^2)$
$\text{sn}(u; a, b) \text{ dn}(u; a, b)$	$1 + F$	x^2 1	$(2n - 1)^2(2n - 2)^2a^2b^2$ $-((2n - 1)^2a^2 + (2n)^2b^2)$

TABLE B.2 Associated continued fractions (Jacobi type).

The table that follows is a similar summary for the Dixon ($\alpha = 0$) elliptic functions:

Elliptic function	Power series	z	a_n ($n \geq 2$)
$f(u)$	$(F = \mathcal{L}(f, x^{-1}))$	a_1	b_n ($n \geq 1$)
$\text{sm}(u, 0)$	$1 + xF$	x^3	$(3n - 5)(3n - 4)^2(3n - 3)^2(3n - 2)$
		1	$2(3n - 2)((3n - 2)^2 + 1)$
$\text{sm}^2(u, 0)$	$1 + F$	x^3	$(3n - 4)(3n - 3)^2(3n - 2)^2(3n - 1)$
		2	$2(3n - 1)((3n - 1)^2 + 1)$
$\text{sm}^3(u, 0)$	$1 + x^{-1}F$	x^3	$(3n - 3)(3n - 2)^2(3n - 1)^2(3n)$
		6	$2(3n)((3n)^2 + 1)$
$\text{cm}(u, 0)$	$1 + x^2F$	x^3	$(3n - 5)^2(3n - 4)^2(3n - 3)^2$
		1	$(3n - 4)^2(3n - 3) + (3n - 2)^2(3n - 1)$
$\text{sm}(u, 0) \text{ cm}(u, 0)$	$1 + xF$	x^3	$(3n - 4)^2(3n - 3)^2(3n - 2)^2$
		1	$(3n - 3)^2(3n - 2) + (3n - 1)^2(3n)$
$\text{sm}^2(u, 0) \text{ cm}(u, 0)$	$1 + F$	x^3	$(3n - 3)^2(3n - 2)^2(3n - 1)^2$
		2	$(3n - 2)^2(3n - 1) + (3n)^2(3n + 1)$

TABLE B.3 Associated continued fractions (Dixon type).

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