

§10.3: Fourier Series, Cont.

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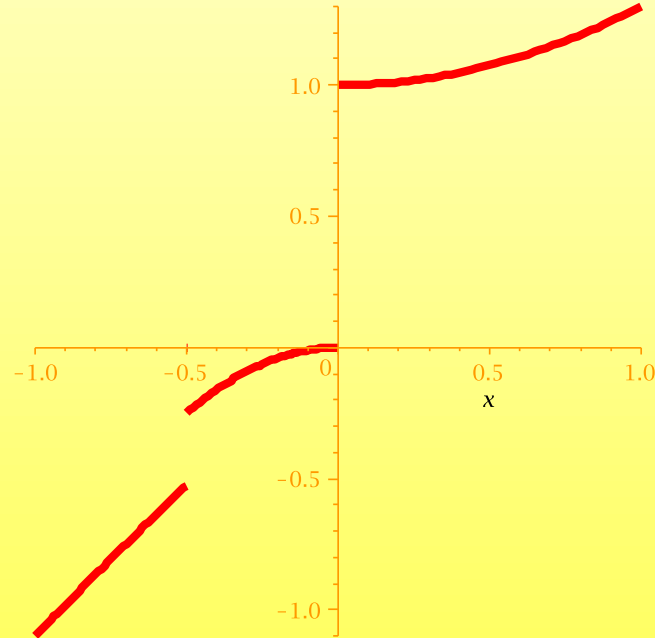
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When is this decomposition possible?

Piecewise differentiable functions

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Then

$$\frac{1}{2}(f(x_+) + f(x_-)) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \frac{m\pi x}{L} + \sum_{m=1}^{\infty} b_m \sin \frac{m\pi x}{L}$$

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at **all points**. Note again that $\frac{1}{2}(f(x_+) + f(x_-))$ is simply $f(x)$ at all ordinary points. Furthermore, a_m, b_m are given by (2).

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$$f = \frac{2}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \frac{1}{7} \sin 7x + \dots \right)$$

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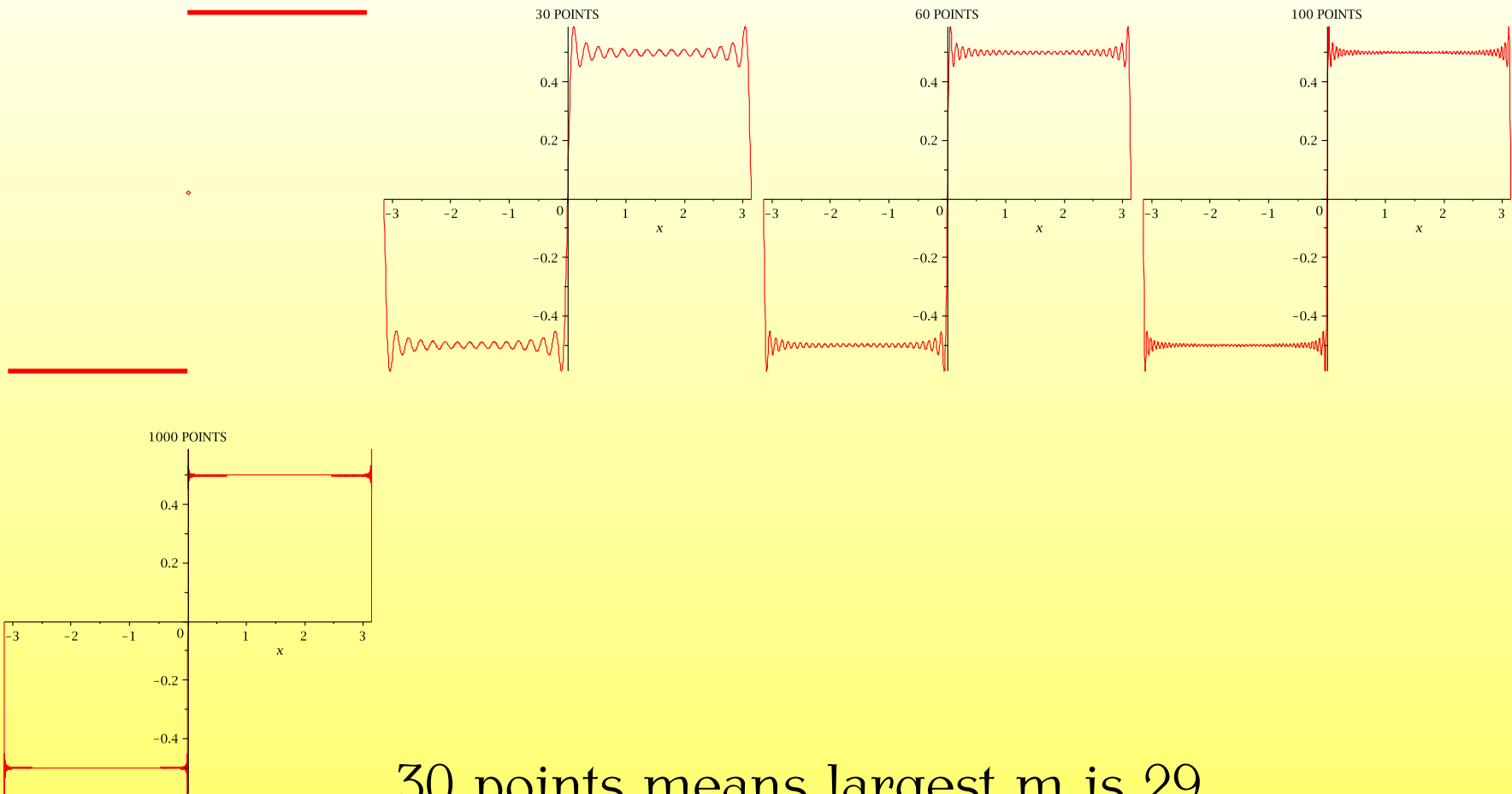
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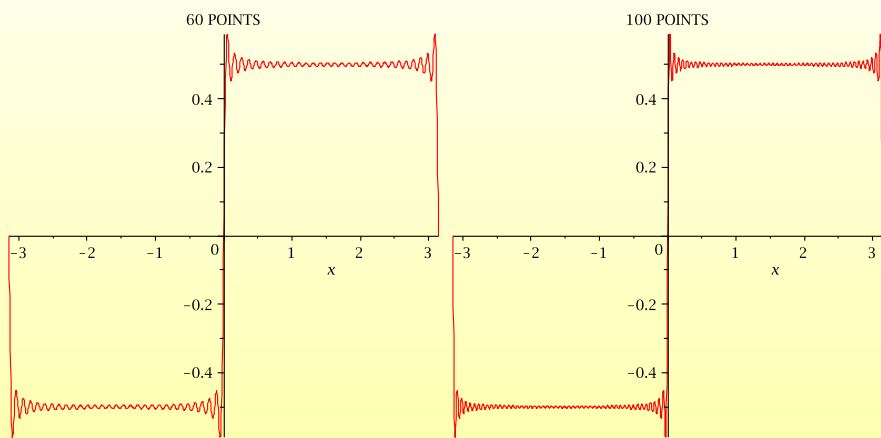
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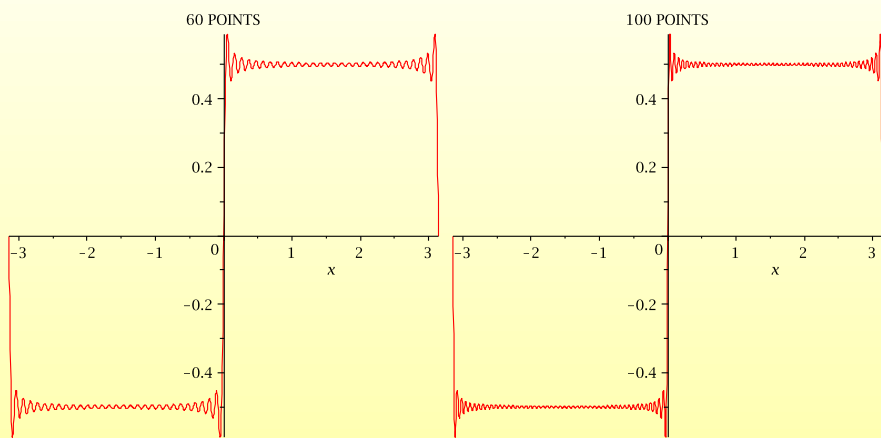
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30 points means largest m is 29.

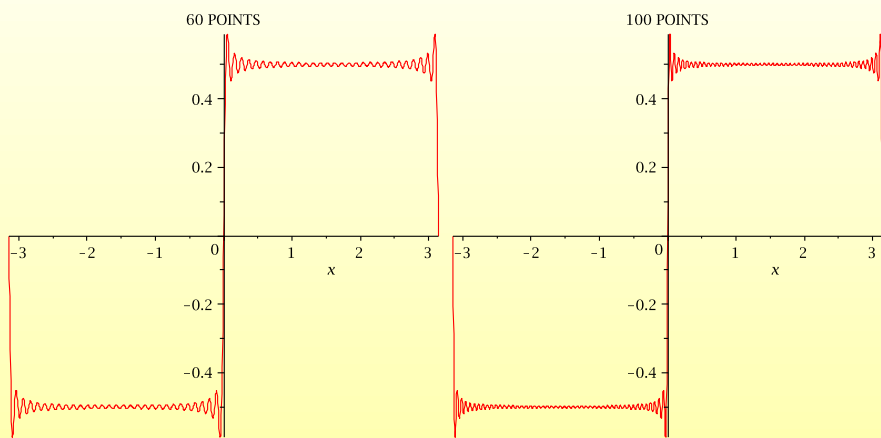


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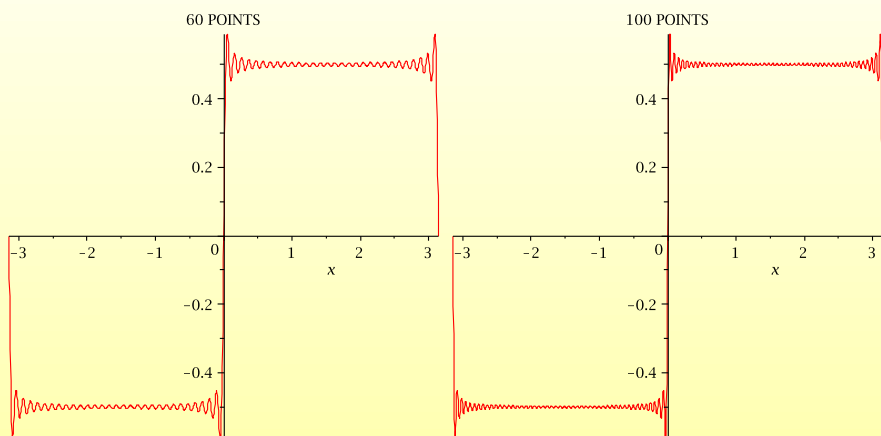
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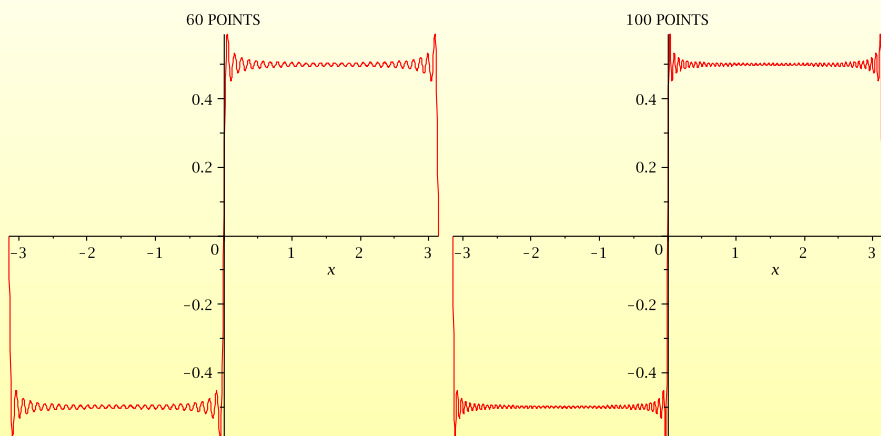
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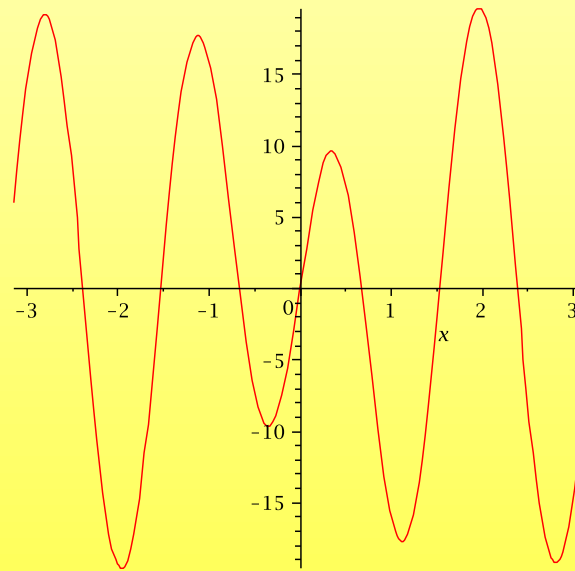
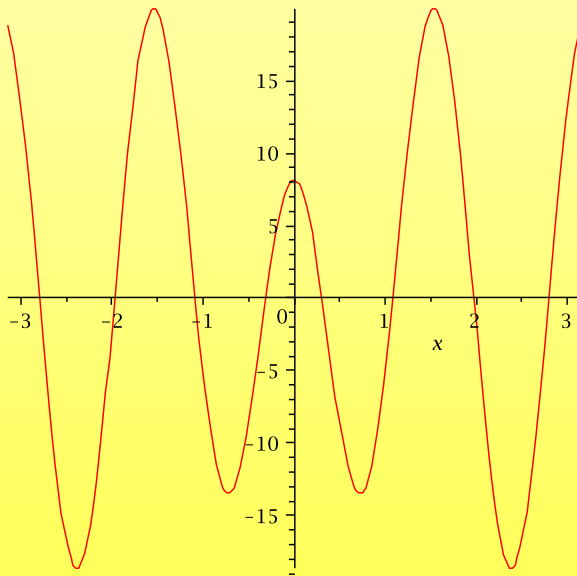
Odd and even functions The formulas can be substantially simplified if the functions are even, or if they are odd.

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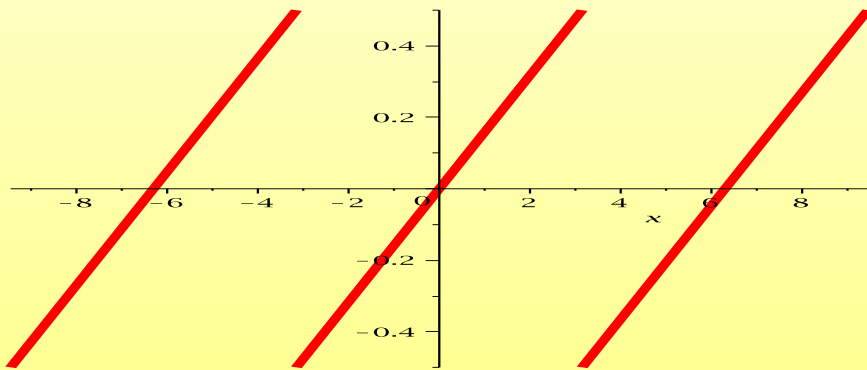
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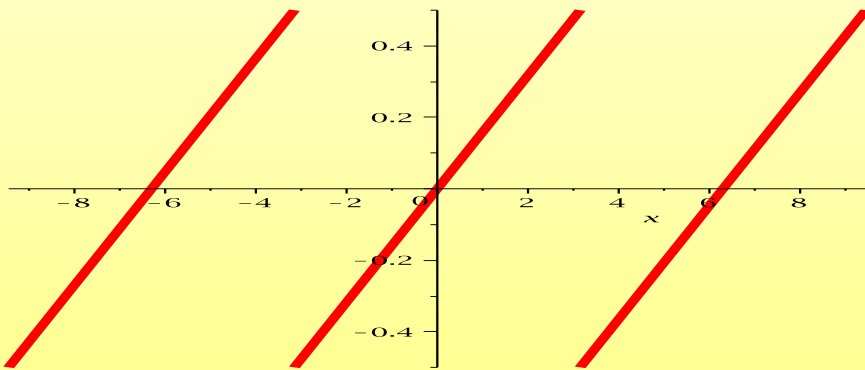
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We have: f is odd.

Thus

$$a_m = 0; \quad b_m = \frac{2}{\pi} \int_0^\pi x \sin(m\pi x) dx = \frac{2}{m} (-1)^{m+1} \quad (8)$$

```
> assume(m, integer);
```

```
> F:=(x+Pi)/Pi/2-floor((x+Pi)/Pi/2)-1/2;
```

$$F := \frac{1}{2} \frac{x + \pi}{\pi} - \text{floor}\left(\frac{1}{2} \frac{x + \pi}{\pi}\right) - \frac{1}{2} \quad (1)$$

```
> plot(F, x=-3*Pi..3*Pi, discontinuous=true);
```

```
> cm:=2/Pi*int(F*sin(m*x), x=0..Pi);
```

$$cm := \frac{(-1)^{1+m}}{\pi m} \quad (2)$$

```
> S:=sum(cm*sin(m*x), m=1..10);
```

$$S := \frac{\sin(x)}{\pi} - \frac{1}{2} \frac{\sin(2x)}{\pi} + \frac{1}{3} \frac{\sin(3x)}{\pi} - \frac{1}{4} \frac{\sin(4x)}{\pi} + \frac{1}{5} \frac{\sin(5x)}{\pi} - \frac{1}{6} \frac{\sin(6x)}{\pi} + \frac{1}{7} \frac{\sin(7x)}{\pi} - \frac{1}{8} \frac{\sin(8x)}{\pi} + \frac{1}{9} \frac{\sin(9x)}{\pi} - \frac{1}{10} \frac{\sin(10x)}{\pi} \quad (3)$$

```
> plot(S, x=0..Pi);
```

```
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3. Extend it in many other number of ways.
4. Then, the Fourier series, calculated on $[-L, L]$ will converge to f

on $[0, L]$ by the general theorem (and to whatever we extended it with elsewhere).

