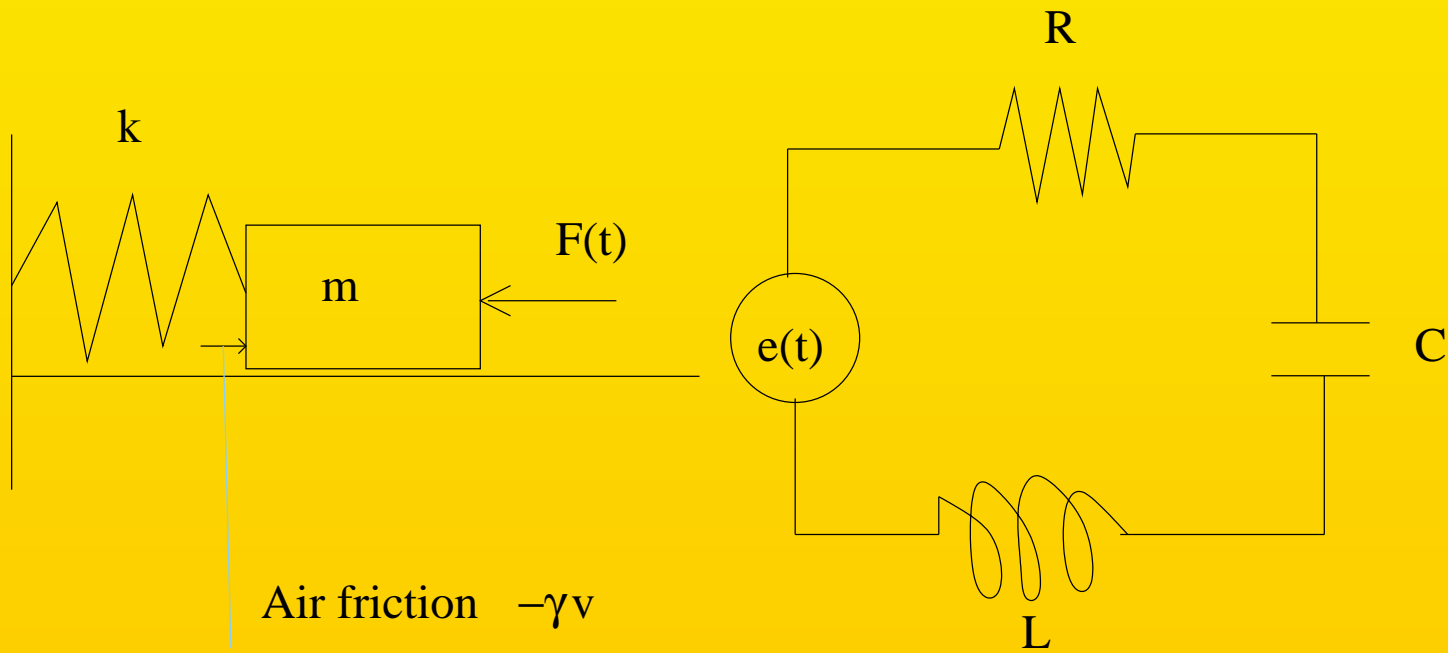
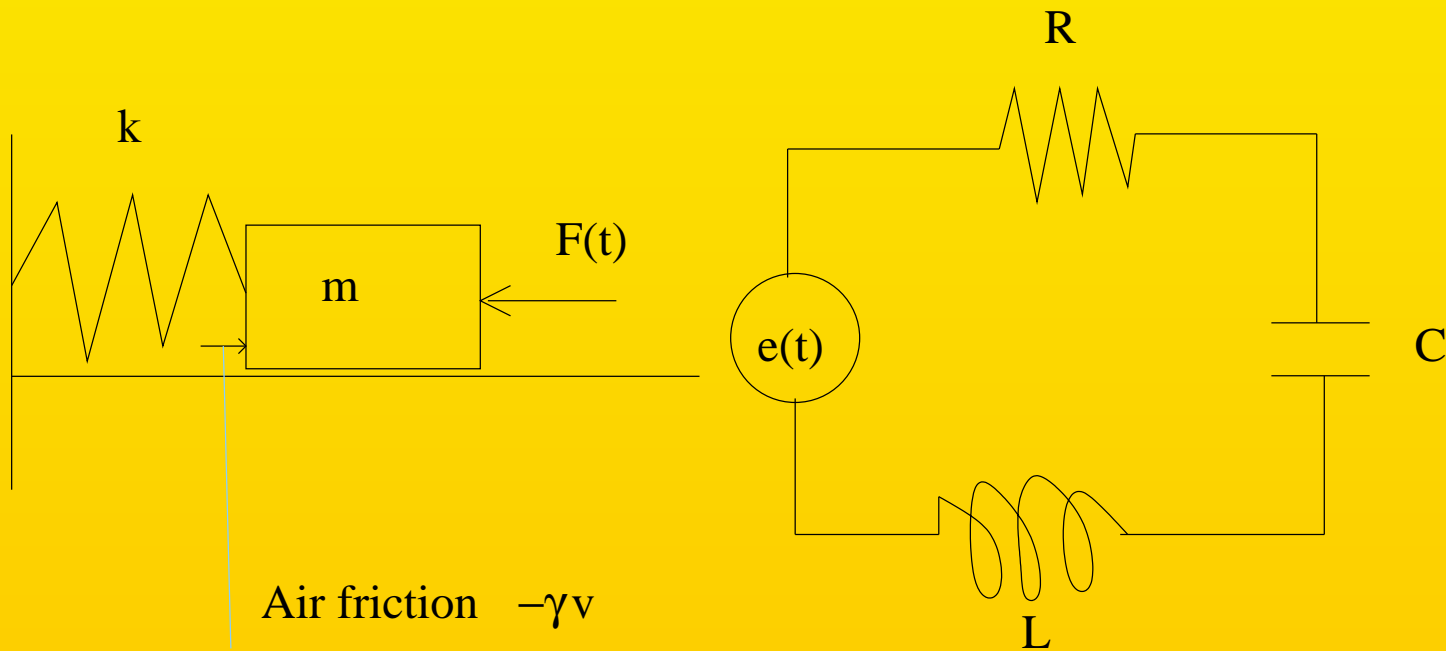


Forced oscillations. Review of power series

§3.9,5.1

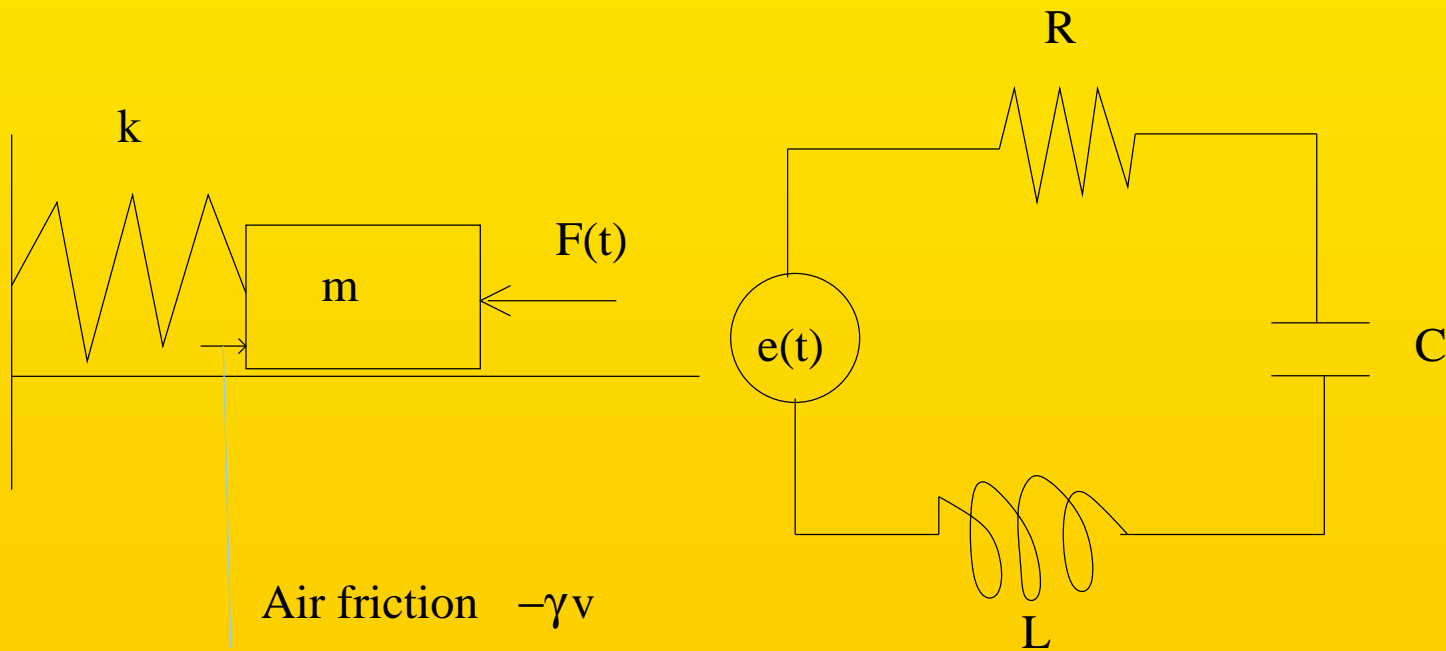


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Note that for this solution, there is oscillation with frequency ω and amplitude $A_1 = \frac{1}{\omega_0^2 - \omega^2} \frac{F_0}{m}$, which becomes unbounded as ω approaches ω_0 . The general solution of the equation is a particular solution, for example this, plus the general solution of the homogeneous equation, $A \sin(\omega_0 t + \phi)$

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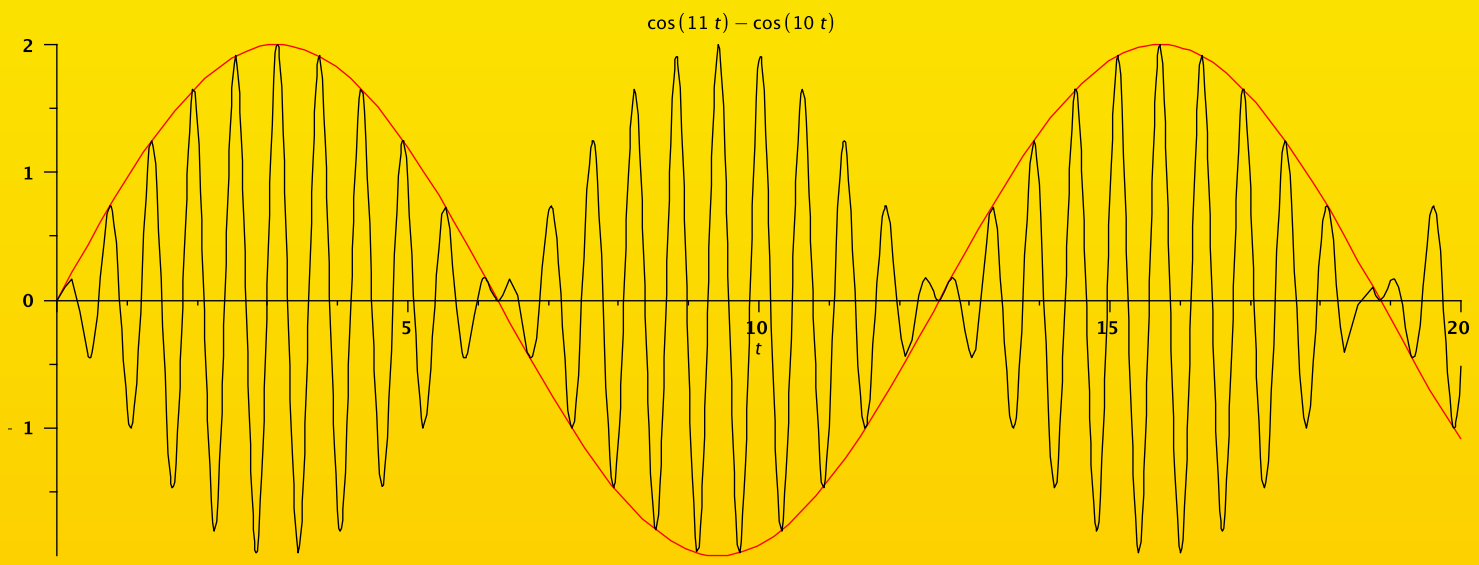
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Substituting we get

$$-Bt(\underline{m\omega_0^2 - k}) \sin \omega t + \underline{2B\omega} \cos(\omega_0 t) - \underline{A_1} \cos(\omega_0 t) = 0$$

Thus $B = A_1/(2\omega_0)$ and we have a particular solution in the form $A_1 t \sin \omega_0 t / (2\omega_0)$

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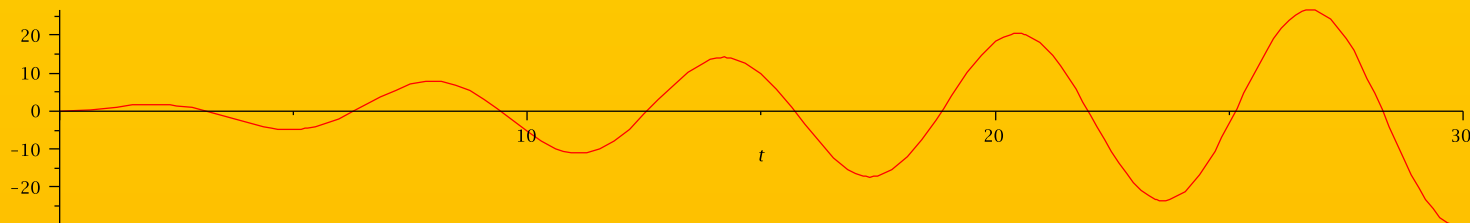
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This solution grows without bound.



Tacoma Narrows bridge, 1940

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Read the textbook for the formulas of the other constants, δ etc.

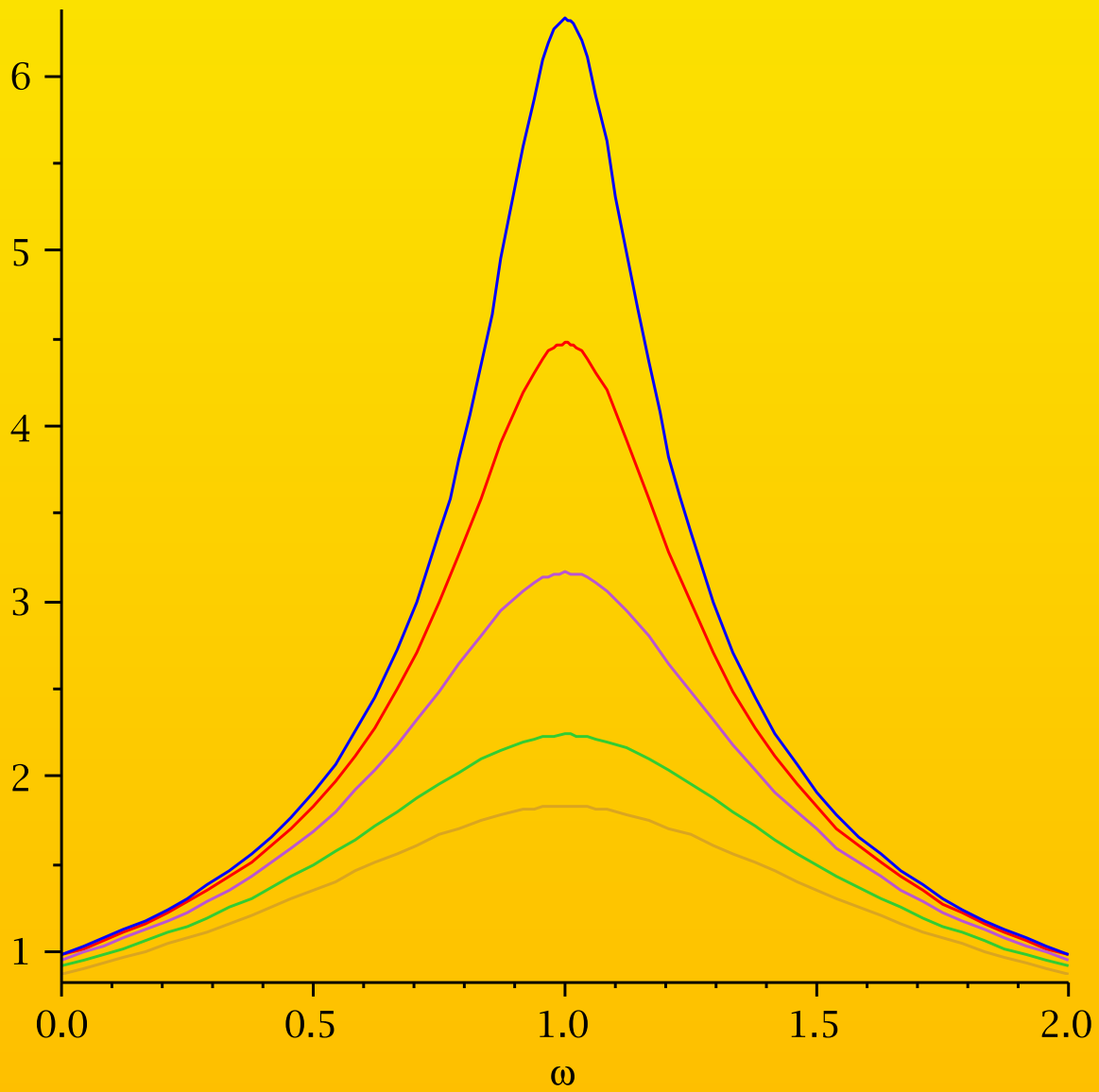


Figure 1: Response vs. frequency.

Series: short review. Please brush up

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