

Series solutions to ODEs

§5.2

- Changes of index of summation.

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- Shift in index

$$\sum_{k=1}^{\infty} c_k (x - x_0)^k = c_1 (x - x_0) + c_2 (x - x_0)^2 + \dots$$

$$= \sum_{k=0}^{\infty} c_{k+1} (x - x_0)^{k+1}$$

$$f'(x) = \sum_{k=1}^{\infty} k c_k (x - x_0)^{k-1} \stackrel{k=m+1}{=} \sum_{m=0}^{\infty} (m+1) c_{m+1} (x - x_0)^m$$

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$$y'' = xy \Rightarrow (k+1)(k+2)c_{k+2} = c_{k-1}; \text{ and } c_2 = 0$$

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Now we have all the coefficients. So

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The general solution!. y_0 and y_1 are 2 particular solutions

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- They are linearly independent! (Why?)

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What is the radius of convergence? Use ratio test, carefully since many terms are zero.

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$$\frac{\frac{x^{3k+3}}{(3k+3)(3k+2)3k(3k-1)\cdots}}{\frac{x^{3k}}{3k(3k-1)\cdots}} = \frac{x^3}{(3k+3)(3k+2)} \rightarrow 0 < 1; \quad \forall x$$

Thus the series converges for all x .

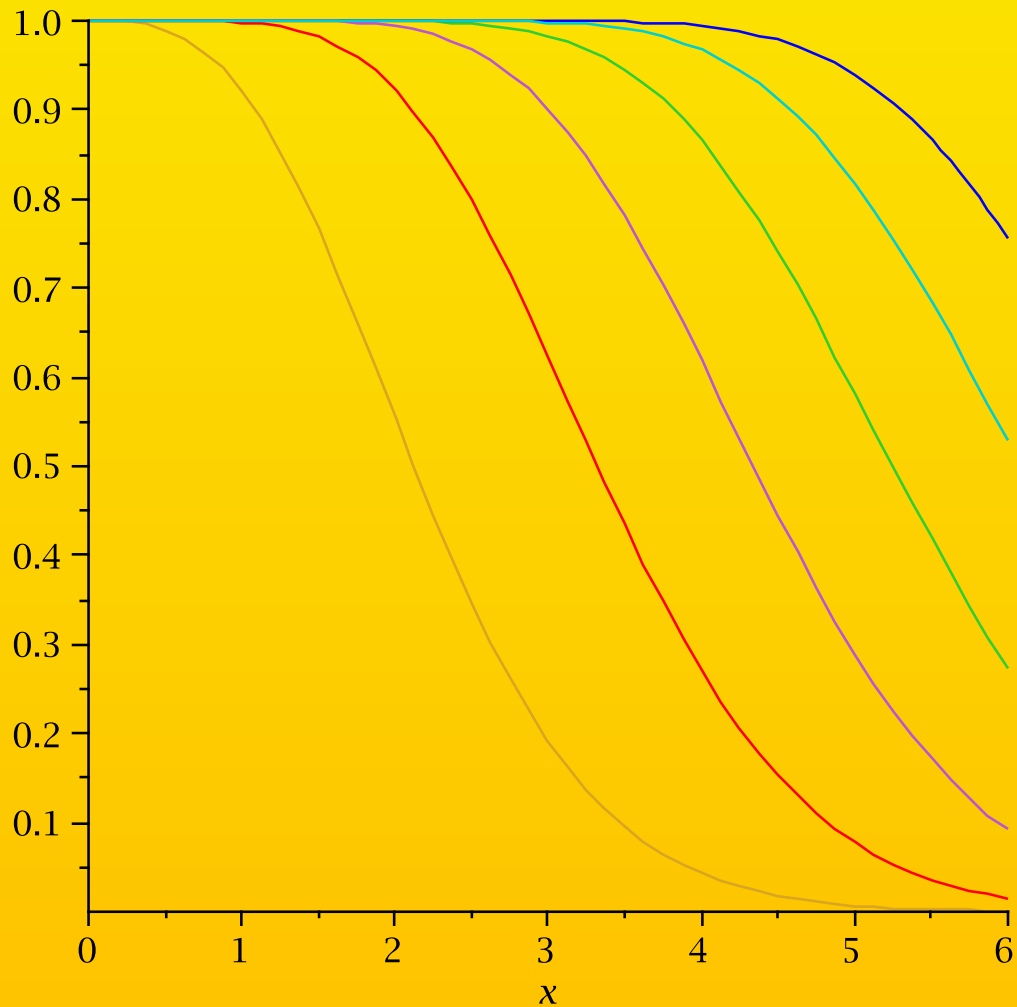
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Ratio between solution and power series with: 1 term (yellow), 2 terms (red) 3 terms (mag.) etc. The graphs which are closer to 1 mean better approximations

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The distance is 2. Thus the radius is 2.

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