

# 1 Gradient and Hamiltonian systems

## 1.1 Gradient systems

These are quite special systems of ODEs, Hamiltonian ones arising in conservative classical mechanics, and gradient systems, in some ways related to them, arise in a number of applications. They are certainly nongeneric, but in view of their origin, they are common.

A system of the form

$$X' = -\nabla V(X) \tag{1}$$

where  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is, say,  $C^\infty$ , is called, for obvious reasons, a gradient system. A critical point of  $V$  is a point where  $\nabla V = 0$ .

These systems have special properties, easy to derive.

**Theorem 1.** *For the system (1), if  $V$  is smooth, we have (i) If  $c$  is a regular point of  $V$ , then the vector field is perpendicular to the level hypersurface  $V^{-1}(c)$  along  $V^{-1}(c)$ .*

*(ii) A point is critical for  $V$  iff it is critical for (1).*

*(iii) At any equilibrium, the eigenvalues of the linearized system are real.*

*More properties, related to stability, will be discussed in that context.*

*Proof.*

□

(i) It is known that the gradient is orthogonal to level surface.

(ii) This is clear essentially by definition.

(iii) The linearization matrix elements are  $a_{ij} = -V_{x_i, x_j}$  (the subscript notation of differentiation is used). Since  $V$  is smooth, we have  $a_{ij} = a_{ji}$ , and all eigenvalues are real.

## 1.2 Hamiltonian systems

If  $\mathbf{F}$  is a conservative field, then  $\mathbf{F} = -\nabla V$  and the Newtonian equations of motion (the mass is normalized to one) are

$$q' = p \tag{2}$$

$$p' = -\nabla V \tag{3}$$

where  $q \in \mathbb{R}^n$  is the position and  $p \in \mathbb{R}^n$  is the momentum. That is

$$q' = \frac{\partial H}{\partial p} \tag{4}$$

$$p' = -\frac{\partial H}{\partial q} \tag{5}$$

where

$$H = \frac{p^2}{2} + V(q) \tag{6}$$

is the Hamiltonian. In general, the motion can take place on a manifold, and then, by coordinate changes,  $H$  becomes a more general function of  $q$  and  $p$ . The coordinates  $q$  are called generalized positions, and  $p$  are called generalized momenta; they are canonical coordinates on the phase on the cotangent manifold of the given manifold.

An equation of the form (4) is called a Hamiltonian system.

**Exercise 1.** Show that a system  $x' = F(x)$  is at the same time a Hamiltonian system and a gradient system iff the Hamiltonian  $H$  is a harmonic function.

**Proposition 1.** (i) The Hamiltonian is a constant of motion, that is, for any solution  $X(t) = (p(t), q(t))$  we have

$$H(p(t), q(t)) = \text{const} \quad (7)$$

where the constant depends on the solution.

(ii) The constant level surfaces of a smooth function  $F(p, q)$  are solutions of a Hamiltonian system

$$q' = \frac{\partial F}{\partial p} \quad (8)$$

$$p' = -\frac{\partial F}{\partial x} \quad (9)$$

*Proof.* (i) We have

$$\frac{dH}{dt} = \nabla_p H \frac{dp}{dt} + \nabla_q H \frac{dq}{dt} = -\nabla_p H \nabla_q + \nabla_q H \nabla_p = 0 \quad (10)$$

(ii) This is obtained very similarly.  $\square$

**Note 1.** It is important to mention that a system is actually called Hamiltonian if the function  $H$  is defined over a sufficiently large region, preferably the whole phase space. Otherwise, if we write a general system in the form

$$q' = F_1(p, q) \quad (11)$$

$$p' = -F_2(p, q) \quad (12)$$

and if  $(F_2, F_1)$  is irrotational, then the system is automatically Hamiltonian locally.

### 1.3 Example

As an example for both systems, we study the following problem: draw the contour plot (constant level curves) of

$$F(x, y) = y^2 + x^2(x - 1)^2 \quad (13)$$

and draw the lines of steepest descent of  $F$ .

For the first part we use Proposition 1 above and we write

$$x' = \frac{\partial F}{\partial y} = 2y \tag{14}$$

$$y' = -\frac{\partial F}{\partial x} = -2x(x-1)(2x-1) \tag{15}$$

where we can apply the nullcline method easily. The regions are depicted below. The critical points are  $(0, 0), (1/2, 0), (1, 0)$ . It is easier to analyze them using

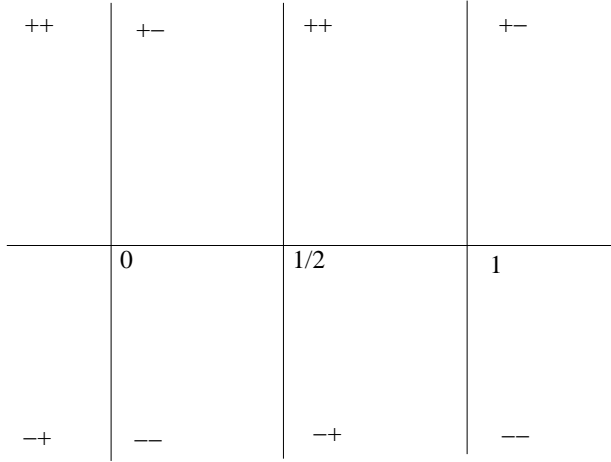


Figure 1:

the Hamiltonian. Near  $(0, 0)$   $H$  is essentially  $x^2 + y^2$ , that is the origin is a center, and the trajectories are near-circles. We can also not the symmetry  $x \rightarrow (1-x)$  so the same conclusion holds for  $x = 1$ , and the phase portrait is symmetric about  $1/2$ .

Near  $x = 1/2$  we write  $x = 1/2 + s$ ,  $H = y^2 + (1/4 - s^2)^2$  and the leading Taylor approximation gives  $H \sim y^2 - 1/2s^2$ . Then,  $1/2$  is a saddle (check). Now we can draw the phase portrait easily, noting that for large  $x$  the curves essentially become  $x^4 + y^2 = C$  “flattened circles”. Clearly, from the interpretation of the problem and the expression of  $H$  we see that *all* trajectories are closed. The perpendicular lines solve the equations

$$x' = -\frac{\partial F}{\partial x} = -2x(x-1)(2x-1) \tag{16}$$

$$y' = -\frac{\partial F}{\partial y} = -2y \tag{17}$$

We note that this equation is separated! In any case, the two equation obviously share the critical points, and the sign diagram can be found immediately from the first figure.

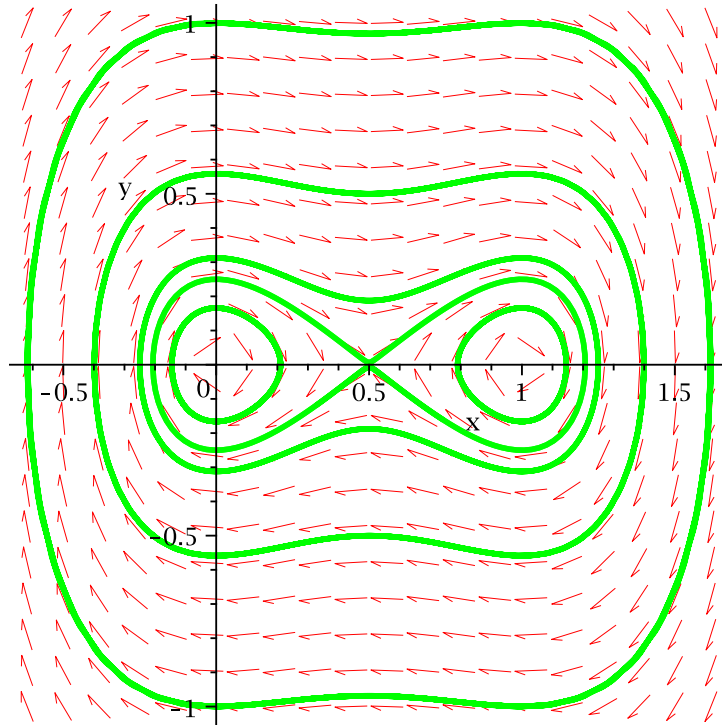


Figure 2:

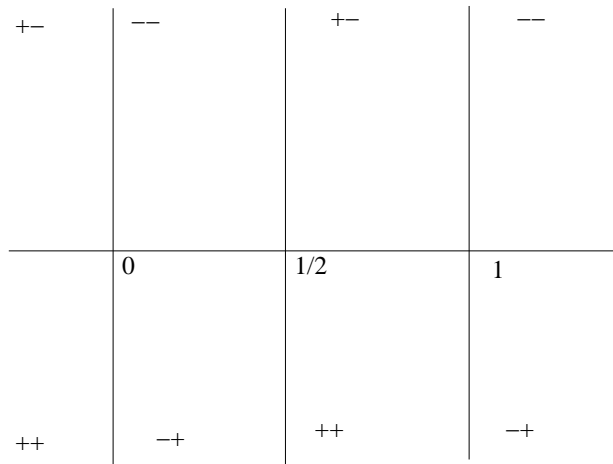


Figure 3:

**Exercise 2.** Find the phase portrait for this system, and justify rigorously its qualitative features. Find the expression of the trajectories of (16). I found

$$y = C \left( \frac{1}{(x - 1/2)^2} - 4 \right)$$

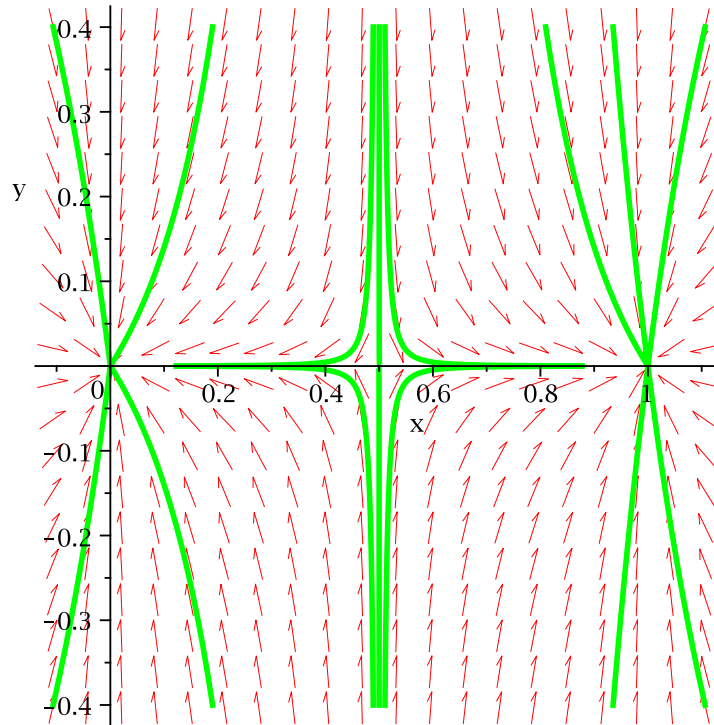


Figure 4:

## 2 Liapunov stability

Often in nonlinear systems, equilibria are of higher order (the linearization has zero eigenvalues). Clearly such points are not hyperbolic and the methods we have seen so far do not apply.

There are no general methods to deal with all cases, but an important one is based on Liapunov (or Lyapounov,...) functions.

We remember that a flow is a smooth map

$$(X, t) \rightarrow \Phi_t(X)$$

A differential system

$$\dot{x} = F(x) \tag{18}$$

generates a flow

$$(X, t) \rightarrow x(t; X)$$

where  $x(t; X)$  is the solution at time  $t$  with initial condition  $X$ .

The derivative of a function  $G$  along a vector field  $F$  is, as usual,

$$D_F(G) = \nabla F \cdot G$$

If we write the differential equation associated to  $F$ , (18), then clearly

$$D_F G = \frac{d}{dt} G(x(t))|_{t=0}$$

## 2.1 Liapunov functions

Let  $X^*$  be a fixed point of (18). A Liapunov function for (18) is a function defined in a neighborhood  $\mathcal{O}$  of  $X^*$  with the following properties

- (1)  $L$  is differentiable in  $\mathcal{O}$ .
- (2)  $L(X^*) = 0$  (this can be arranged by subtracting a constant).
- (3)  $L(x) > 0$  in  $\mathcal{O}$ .
- (4)  $D_F L \leq 0$  in  $\mathcal{O}$ .

A strict Liapunov function is a Liapunov function for which

- (4')  $D_F L < 0$  in  $\mathcal{O}$ .

Finding a Liapunov function is many times nontrivial. In systems coming from physics, often the energy is such a function. In general systems, one may try to find an exactly integrable equation which is a good approximation for the actual one in a neighborhood of  $X^*$  and look at the various constants of motion of the approximation as candidates for Liapunov functions.

**Theorem 2** (Liapunov stability). *Assume  $X^*$  is a fixed point for which there exists a Liapunov function  $L$ . Then*

- (i)  $X^*$  is stable.
- (ii) If  $L$  is a strict Liapunov function then  $X^*$  is asymptotically stable.

*Proof.* (i) Consider a small ball  $B \ni X^*$  contained in  $\mathcal{O}$ ; we denote the boundary of  $B$  (a sphere) by  $\partial B$ . Let  $\alpha$  be the minimum of  $L$  on the  $\partial B$ . By the definition of a Liapunov function, (3),  $\alpha > 0$ . Consider the following subset:

$$\mathcal{U} = \{x \in B : L(x) < \alpha\} \tag{19}$$

From the continuity of  $L$ , we see that  $\mathcal{U}$  is an open set. Clearly,  $X^* \in \mathcal{U}$ . Let  $X \in \mathcal{U}$ . Then  $x(t; X)$  is a continuous curve, and it cannot have components outside  $B$  without intersecting  $\partial B$ . But an intersection is impossible since by monotonicity,  $L(x(t)) \leq L(X) < \alpha$  for all  $t$ . Thus, trajectories starting in  $\mathcal{U}$  are confined to  $\mathcal{U}$ , proving stability.

- (ii)

1. Note first that  $X^*$  is the only critical point in  $\mathcal{O}$  since  $(d/dt)L(s(t; X_1^*)) = 0$  for any fixed point.

2. Note that trajectories  $x(t; X)$  with  $X \in \mathcal{U}$  are contained in a compact set, and thus they contain limit points. Any limit point  $x^*$  is strictly inside  $\mathcal{U}$  since  $L(x^*) < L(x(t); X) < \alpha$ .
3. Let  $x^*$  be a limit point of a trajectory  $x(t; X)$  where  $X \in \mathcal{U}$ , i.e.  $x(t_n, X) \rightarrow x^*$ . Then, by 1 and 2,  $x^* \in \mathcal{U}$  and  $x^*$  is a regular point of the field.
4. We want to show that  $x^* = X^*$ . We will do so by contradiction. Assuming  $x^* \neq X^*$  we have  $L(x^*) = \lambda > 0$ , again by (3) of the definition of  $L$ .
5. By 3 the trajectory  $\{x(t; x^*) : t \geq 0\}$  is well defined and is contained in  $\mathcal{B}$ .
6. We then have  $L(x(t; x^*)) < \lambda \forall t > 0$ .

7. The set

$$\mathcal{V} = \{X : L(x(t_{n+1} - t_n; X))\} < \lambda \quad (20)$$

is open, so

$$L(x(t_{n+1} - t_n; X_1)) < \lambda \quad (21)$$

for all  $X_1$  close enough to  $x^*$ .

8. Let  $n$  be large enough so that  $x(t_m; X) \in \mathcal{V}$  for all  $m \geq n$ .
9. Note that, by existence and uniqueness of solutions at regular points we have

$$x(t_{n+1}; X) = x(t_{n+1} - t_n; x(t_n; X)) \quad (22)$$

10. On the one hand  $L(x(t_{n+1})) \downarrow \lambda$  and on the other hand we got  $L(x(t_{n+1})) < \lambda$ . This is a contradiction.

□

## 2.2 Examples

Hamiltonian systems, in Cartesian coordinates often assume the form

$$H(q, p) = p^2/2 + V(q) \quad (23)$$

where  $p$  is the collection of spatial coordinates and  $p$  are the momenta. If this ideal system is subject to external dissipative forces, then the energy cannot increase with time.  $H$  is thus a Liapunov function for the system. If the external force is  $F(p, q)$ , the new system is generally not Hamiltonian anymore, and the equations of motion become

$$\dot{q} = p \quad (24)$$

$$\dot{p} = -\nabla V + F \quad (25)$$

and thus

$$\frac{dH}{dt} = pF(p, q) \quad (26)$$

which, in a dissipative system should be nonpositive, and typically negative. But, as we see,  $dH/dt = 0$  along the curve  $p = 0$ .

For instance, in the ideal pendulum case with Hamiltonian

$$H = \frac{1}{2}\omega^2 + (1 - \cos \theta) \quad (27)$$

The associated Hamiltonian flow is

$$\theta' = \omega \quad (28)$$

$$\omega' = -\sin \theta \quad (29)$$

Then  $H$  is a global Liapunov function at  $(0,0)$  for (30) (in fact, this is true for any system with *nonnegative* Hamiltonian). This is clear from the way Hamiltonian systems are defined.

Then  $(0,0)$  is a stable equilibrium. But, clearly, it is not asymptotically stable since  $H = \text{const} > 0$  on any trajectory not starting at  $(0,0)$ .

If we add air friction to the system (30), then the equations become

$$\theta' = \omega \quad (30)$$

$$\omega' = -\sin \theta - \kappa\omega \quad (31)$$

where  $\kappa > 0$  is the drag coefficient. Note that this time, if we take  $L = H$ , the same  $H$  defined in (27), then

$$\frac{dH}{dt} = -\kappa\omega^2 \quad (32)$$

The function  $H$  is a Liapunov function, but it is not strict, since  $H' = 0$  if  $\omega = 0$ . It is intuitively clear that the energy still decreases to zero in the limit, since  $\omega = 0$  are isolated points on any trajectory and we expect  $(0,0)$  to still be asymptotically stable. In fact, we could adjust the proof of Theorem 2 to show this. However, as we see in (26), this degeneracy is typical and then it is worth having a systematic way to deal with it. This is one application of Lasalle's invariance principle that we will prove next.

### 3 Some important concepts

We start by introducing some important concepts.

**Definition 2.** 1. An entire solution  $x(t; X)$  is a solution which is defined for all  $t \in \mathbb{R}$ .

2. A positively invariant set  $\mathcal{P}$  is a set such that  $x(t, X) \in \mathcal{P}$  for all  $t \geq 0$  and  $X \in \mathcal{P}$ . Solutions that start in  $\mathcal{P}$  stay in  $\mathcal{P}$ . Similarly one defines negatively invariant sets, and invariant sets.

3. The basin of attraction of a fixed point  $X^*$  is the set of all  $X$  such that  $x(t; X) \rightarrow X^*$  when  $t \rightarrow \infty$ .

4. Given a solution  $x(t; X)$ , the set of all points  $\xi^*$  such that solution  $x(t_n; X) \rightarrow \xi^*$  for some sequence  $t_n \rightarrow \infty$  is called the set of  $\omega$ -limit points of  $x(t; X)$ . At the opposite end, the set of all points  $\xi^*$  such that solution  $x(-t_n; X) \rightarrow \xi^*$  for some sequence  $t_n \rightarrow \infty$  is called the set of  $\alpha$ -limit points. These may of course be empty.

**Proposition 3.** Assume  $x(t; X)$  belongs to a closed, positively invariant set  $\mathcal{P}$  where the field is defined. The  $\omega$ -limit set is a closed invariant set too. A similar statement holds for the  $\alpha$ -set.

- Proof.* 1. (Closure) We show the complement is open. Let  $b$  be in the complement of the set of  $\omega$ -limit points. Then  $\liminf_{t \rightarrow \infty} d(x(t, X), b) > a > 0$  for some  $a$  and for all  $t > 0$ . If  $b'$  is close enough to  $b$ , then clearly  $\liminf_{t \rightarrow \infty} d(x(t, X), b) > a/2 > 0$  for all  $t$ .
2. (Invariance) Assume  $x(t_n, X) \rightarrow x^*$ . By assumption, the differential equation is well-defined in a neighborhood of any point in  $\mathcal{P}$ , and since  $x^* \in \mathcal{P}$ ,  $x(t; x^*)$  exists for all  $t \geq 0$ . By continuity with respect to initial conditions and of solutions, we have  $x(t; x(t_n)) \rightarrow x(t; x^*)$  as  $n \rightarrow \infty$ . Then  $x(t; x^*)$  is an omega-point to for any  $t \geq 0$ .
3. Backward invariance is proved similarly, since  $x(t_n; X) = x(t_f - t_i; x(t_i))$  for all  $t_f < t_i$ , by existence and uniqueness of solutions. Thus  $x(-T, X_1)$  is arbitrarily close to  $x(t; X)$  for some  $t$ .

□

## 4 Lasalle's invariance principle

**Theorem 3.** Let  $X^*$  be an equilibrium point for  $X' = F(X)$  and let  $L : \mathcal{U} \rightarrow \mathbb{R}$  be a Liapunov function at  $X^*$ . Let  $X^* \ni \mathcal{P} \subset \mathcal{U}$  be closed, bounded and positively invariant. Assume there is no entire trajectory in  $\mathcal{P} - \{X^*\}$  along which  $L$  is constant. Then  $X^*$  is asymptotically stable, and  $\mathcal{P}$  is contained in the basin of attraction of  $X^*$ .

*Proof.* Since  $\mathcal{P}$  is compact and positively invariant, every trajectory in  $\mathcal{P}$  has  $\omega$ -limit points. If  $X^*$  is the only limit point, the assumption follows easily (check!). So, we may assume there is an  $x^* \neq X^*$  which is also an  $\omega$ -limit point of some  $x(t; X)$ . We know that the trajectory  $x(t; x^*)$  is entire. Since  $L$  is nondecreasing along trajectories, we have  $L(x(t; X)) \rightarrow \alpha = L(x^*)$  as  $t \rightarrow \infty$ . (This is clear for the subsequence  $t_n$ , and the rest follows by inequalities: check!) On the other hand, for any  $T \in \mathbb{R}$ , positive or negative,  $x(T; x^*)$  is arbitrarily close to  $x(t, X)$  for some  $t > 0$  (check). Since  $L(x(T; x^*)) \leq \alpha$  and it is arbitrarily close to  $L(x(t, X)) \geq \alpha$ , it follows that  $L(x(T; x^*)) = \alpha$  for all  $t \geq 0$ . □

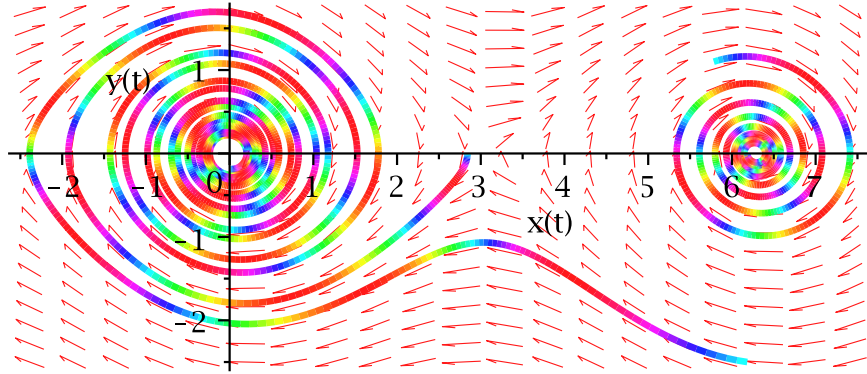


Figure 5:

#### 4.1 Example: analysis of the pendulum with drag

Intuitively, it is clear that any trajectory that starts with  $\omega = 0$  and  $\theta \in (-\pi, \pi)$  should asymptotically end up at the equilibrium point  $(0, 0)$  (other trajectories, which for the frictionless system would rotate forever, may end up in a different equilibrium,  $(2n\pi, 0)$ ). For zero initial  $\omega$ , the basin of attraction of  $(0, 0)$  should exactly be  $(-\pi, \pi)$ . In general, the energy should be exactly less than the one in this marginal case,  $E = 1 - \cos(\pi) = 2$ . Then, the region  $\theta_0 \in (-\pi, \pi)$ ,  $E < 1 - \cos(\pi) = 2$  should be the basin of attraction of  $(0, 0)$ .

So let  $c \in (0, 2)$ , and let

$$\mathcal{P}_c = \{(\theta, \omega) : \text{and } E(\theta, \omega) \leq c, \quad |\theta| \leq \arccos(1 - c) \in (-\pi, \pi)\} \quad (33)$$

In  $E, \theta$  coordinates, this is simply a closed rectangle and since  $(E, \theta)$  is a continuous map, its preimage in the  $(p, \theta)$  plane is closed too.

Now we show that  $\mathcal{P}_c$  is closed and forward invariant. We have already seen that  $\dot{H} \leq 0$  along trajectories. If a trajectory were to exit  $\mathcal{P}_c$ , it would mean, by continuity, that for some  $t$  we have  $H = c + \delta$  for a small  $\delta > 0$  or  $|\theta| = \arccos(1 - c) + \epsilon$  for a small  $\epsilon > 0$  which implies, from the formula,  $H > c$ . But this is not possible, by the monotonicity of  $H$ .

Now there is no nontrivial entire solution (that is, other than  $X^* = (0, 0)$ ) along which  $H = \text{const}$ . Indeed,  $H = \text{const}$  implies, from (32) that  $\omega = 0$  identically along the trajectory. But then, from (29) we see that  $\sin \theta = 0$  identically, which, within  $\mathcal{P}_c$  simply means  $\theta = 0$  identically. Lasalle's theorem applies, and all solutions starting in  $\mathcal{P}_c$  approach  $(0, 0)$  as  $t \rightarrow \infty$ . The phase portrait of the damped pendulum is depicted in Fig. 5

## 5 Gradient systems and Liapunov functions

Recall that a gradient system is of the form (1), that is

$$X' = -\nabla V(X) \tag{34}$$

where  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is, say,  $C^\infty$  and a critical point of  $V$  is a point where  $\nabla V = 0$ . We have the following result:

**Theorem 4.** *For the system (1): (i) If  $c$  is a regular value of  $V$ , then the vector field is orthogonal to the level set of  $V^{-1}(c)$ .*

*(ii) The equilibrium points of the system coincide with the critical points of  $V$ .*

*(iii)  $V$  is a Liapunov function for the system, and given a solution  $x$ , we have  $\frac{d}{dt}V(x(t)) = 0$  iff  $x(t) \equiv X^*$ , an equilibrium point.*

*(iv) If a critical point  $X^*$  is an isolated minimum of  $V$ ,  $V(X) - V(X^*)$  is a strict Liapunov function at  $X^*$ , and then  $X^*$  is asymptotically stable.*

*(v) Any  $\alpha$ -limit point of a solution of (1), and any  $\omega$ -limit point is an equilibrium.*

*(vi) The linearized system at any equilibrium has only real eigenvalues.*

**Note 2.** (a) By (v), any solution of a gradient system tends to a limit point or to infinity.

(b) Thus, descent lines of any smooth manifold have the same property: they link critical points, or they tend to infinity.

(c) We can use some of these properties to determine for instance that a system is not integrable. We write the associated gradient system and determine that it fails one of the properties above, for instance the linearized system at a critical point has an eigenvalue which is not real. Then there cannot exist a smooth  $H$  so that  $H(x, y(x))$  is constant along trajectories.

*Proof.* We have already shown (i) and (ii), which are in fact straightforward from the definition.

For (iii) we see that

$$\frac{d}{dt}V(x(t)) = \nabla V \frac{dx}{dt} = -|\nabla V|^2 \leq 0 \tag{35}$$

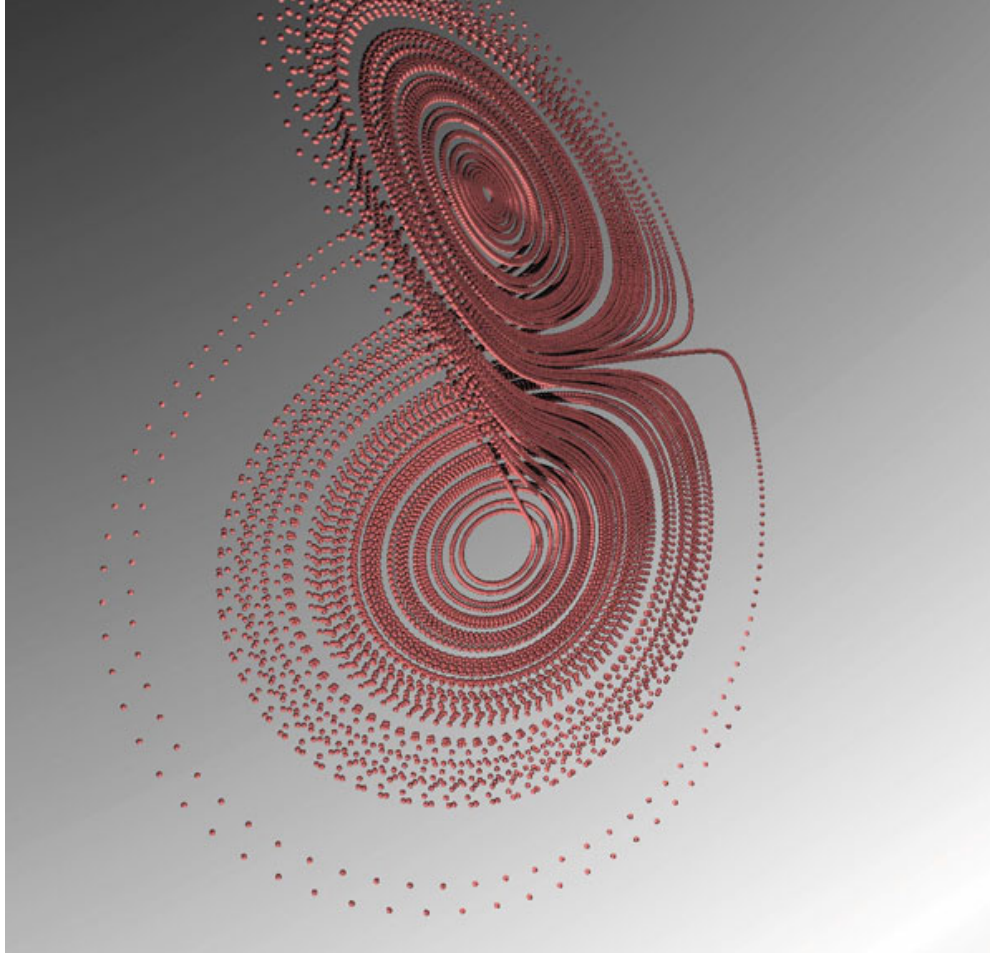


Figure 6: The Lorenz attractor

whereas, if  $\nabla V = 0$  for some point of the trajectory, then of course that point is an equilibrium, and the whole trajectory is that equilibrium.

(iv) If an equilibrium point is isolated, then  $\nabla V \neq 0$  in a set of the form  $|X - X^*| \in (0, a)$ . Then  $-\|\nabla V\|^2 < 0$  in this set. Furthermore,  $V(X) - V(X^*) > 0$  for all  $X$  with  $|X - X^*| \in (0, a)$ .

(v) Since  $V$  is a Liapunov function for (1), we have shown in the proof of Lasalle's invariance principle that  $V$  is constant along any trajectory starting at a limit point. But we see from (iii) that this implies that the trajectory reduces to a point, which is an equilibrium point.

(vi) Note that for smooth  $V$ , the linearization at  $X^*$  is simply the matrix

$$A; \quad A_{ij} = \frac{-\partial^2 V(X^*)}{\partial x_i \partial x_j} \quad (36)$$

which is *symmetric*. □

## 6 Limit sets, Poincaré maps, the Poincaré Bendixson theorem

We shall denote the  $\omega$ -limit set of a solution starting at  $X$  by  $\omega(X)$ , and likewise, its  $\alpha$ -limit set by  $\alpha(X)$ .

In two dimensions, there are typically two types of limit sets: equilibria and periodic orbits (which are thereby limit cycles). Exceptions occur when a limit set contains a number of equilibria, as we will see in examples.

Beyond two dimensions however, the possibilities are far vaster and limit sets can be quite complicated. Fig. 6 depicts a limit set for the Lorenz system, in three dimensions. Note how the trajectories seem to spiral erratically around two points. The limit set here has a fractal structure.

We begin the analysis with the two dimensional case, which plays an important role in applications.

We have already studied the system  $r' = 1/2(r-r^3)$  in Cartesian coordinates. There the circle of radius one was a periodic orbit, and a limit cycle. All trajectories, except for the trivial one  $(0,0)$  tended to it as  $t \rightarrow \infty$ .

We have also analyzed many cases of nodes, saddle points etc, where trajectories have equilibria as limit sets, or else they go to infinity.

A rather exceptional situation is that where the limit sets contain equilibria. Here is one example

## 6.1 Example: equilibria on the limit set

Consider the system

$$x' = \sin x(-\cos x - \cos y) \tag{37}$$

$$y' = \sin y(\cos x - \cos y) \tag{38}$$

The phase portrait is depicted in Fig. 7.

**Exercise 1.** *Justify the qualitative elements in Fig. 7.*

In the example above, we see that the limit set is a collection of fixed points and orbits, none of which periodic.

### 6.1.1 Closed orbits

A closed orbit is a solution whose trajectory is a closed curve (with no equilibria on it). Starting at a point  $X$ , after a finite time then, the solution returns to  $X$  since the absolute value of the velocity along the curve is bounded below. From that time on, the solution must repeat itself identically, by uniqueness of solutions. It then means that the solution is periodic, that is there is a smallest  $\tau$  so that  $\Phi_{t+\tau}(X) = \Phi(X)$ . This  $\tau$  is called the period of the orbit.

**Proposition 4.** (i) *If  $X$  and  $Z$  lie on the same solution curve, then  $\omega(X) = \omega(Z)$  and  $\alpha(X) = \alpha(Z)$ .*

(ii) *If  $D$  is a closed, positively invariant set and  $Z \in D$ , then  $\omega(Z) \subset D$ ; similarly for negatively invariant sets and  $\alpha(Z)$ .*

(iii) *A closed invariant set, and in particular a limit set, contains the  $\alpha$ -limit and the  $\omega$ -limit of every point in it.*

*Proof.* Exercise. □

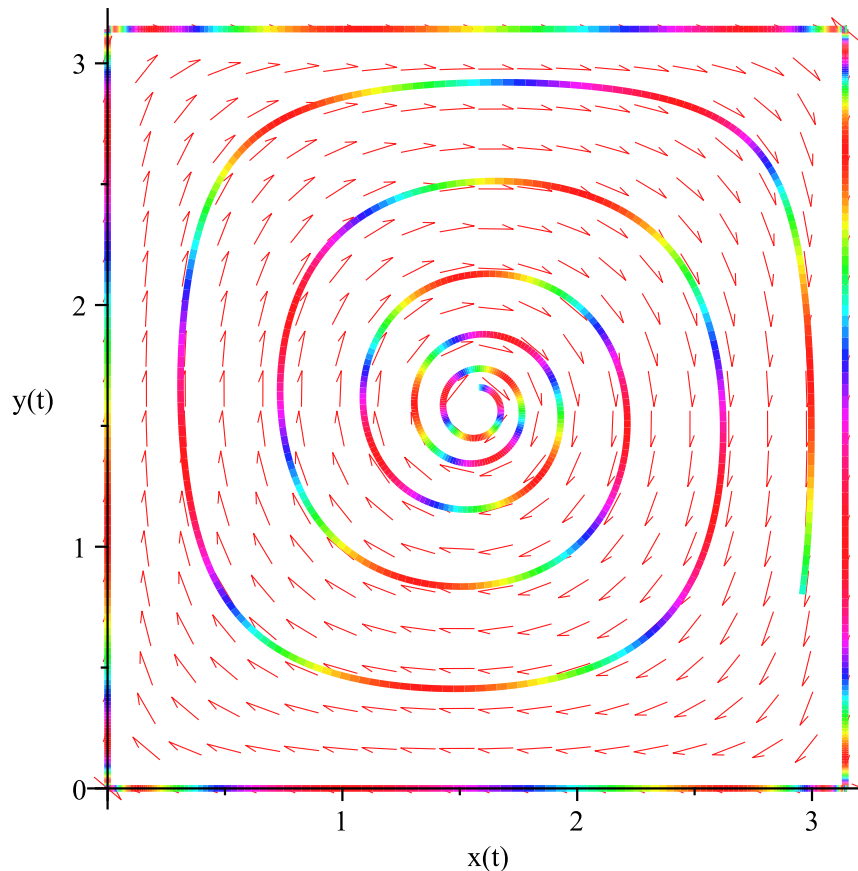


Figure 7: Phase portrait for (37).

## 7 Sections; the flowbox theorem

Consider a differential equation  $X' = F(X)$  with  $F$  smooth, and a point  $X_0$  such that  $F(X_0) \neq 0$ . Then there is a diffeomorphic change of coordinates in some neighborhood of  $X_0$ ,  $X \leftrightarrow X^-$  so that in coordinates  $X^-$  the field is simply  $\dot{X}^- = \mathbf{e}_1$  where  $\mathbf{e}_1 = (1, 0)$ .

One way to achieve this is the following. Since  $F(X_0) \neq 0$  there is a unit vector  $V_0$  at  $X_0$  which is orthogonal to  $F(X_0)$ , say  $(-F_2, F_1)/|F(X_0)|$ . we draw a line segment,  $h(u) = X_0 + uV_0$ ,  $u \in (-\epsilon, \epsilon)$ . If  $\epsilon$  is small enough, then  $F(h(u)) \neq 0$  for all  $u \in (-\epsilon, \epsilon)$  and  $F$  is not tangent to  $u$  anywhere along the segment  $\mathcal{S} = \{X_0 + uV_0 | u \in (-\epsilon, \epsilon)\}$  (that is,  $(-F_2, F_1) \cdot V_0 \neq 0$ ).

**Definition 5.** *The segment  $\mathcal{S}$  defined above is called local section at  $X_0$ .*

Note that any solution that intersects  $\mathcal{S}$  crosses it, in the same direction

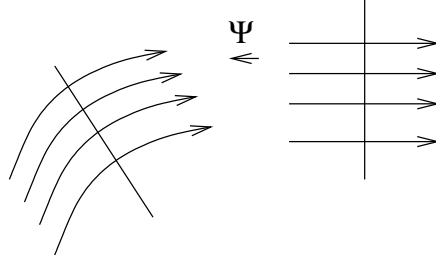


Figure 8: Flowbox and transformation

relative to  $\mathcal{S}$ . Indeed, since the field is smooth and  $(-F_2, F_1) \cdot V_0 \neq 0$ , then  $(-F_2, F_1) \cdot V_0$  has constant sign throughout  $\mathcal{S}$  and the trajectories flow towards the same side of  $\mathcal{S}$ .

To straighten the field, we construct the following map, from a neighborhood of  $X_0$  of the form  $\mathcal{N} = \{(t, u) : |t| < \delta, u \in \mathcal{S}\}$ . This is a small enough region where the flow is smooth and there are no equilibria.

We consider the function from

$$\Psi(s, u) = \Phi_s(h(u)) := x(s; h(u)) \quad (39)$$

Note that  $x(s; h(u)) \in \mathbb{R}^2$  lies in a neighborhood of  $X_0$ . Note also that  $x(0; h(u)) = h(u)$  and thus the line  $(0, u) \in \mathcal{N}$  is mapped onto the section  $\mathcal{S}$ .

Finally, we note that  $x(0, h(u_2)) - x(0, h(u_1)) = h(u_2) - h(u_1) = (u_2 - u_1)V_0$ . then  $\Psi$  is a diffeomorphism, since the Jacobian of the transformation is

$$\det(J(X_0)) = \det \begin{pmatrix} F_1 & V_1 \\ F_2 & V_2 \end{pmatrix} = |F(X_0)| \quad (40)$$

is nonzero in a neighborhood of  $X_0$ , in fact throughout  $\mathcal{S}$ , and since we have assumed that  $F$  does not become tangent to  $V_0$ .

We see that the inverse of  $\Psi$  takes a neighborhood of  $X_0$  into a neighborhood of  $X_0$ .

We saw that  $(0, u)$  is mapped onto  $\mathcal{S}$ . We also see that  $(s, u_0)$  is mapped to  $x(s; h(u_0))$  which is part of a trajectory. Thus, the inverse image of trajectories through  $\Psi$  are straight lines, as depicted. The new field is the trivial flow we have mentioned.

## 7.1 Time of arrival

We consider all solutions in the domain  $\mathcal{O}$  where the field is defined. Some of them intersect  $\mathcal{S}$ . Since the trajectories are continuous, there is a first time of arrival, the smallest  $t$  so that  $x(t, Z_0) \in \mathcal{S}$ .

This time of arrival is continuous in  $Z_0$ , as shown in the next proposition.

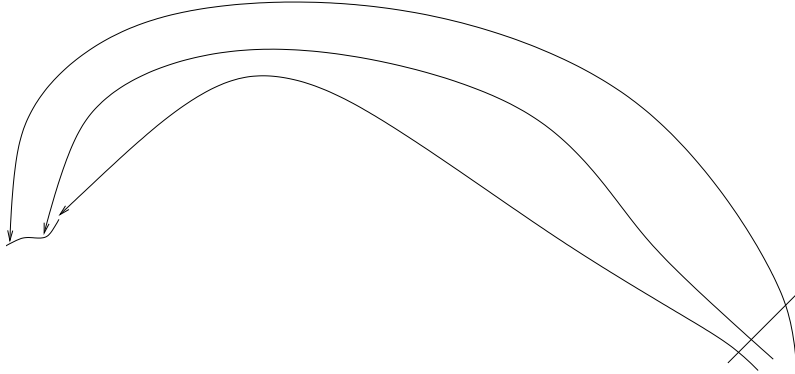


Figure 9: Time of arrival function

**Proposition 6.** *Let  $\mathcal{S}$  be a local section at  $X_0$  and assume  $\phi_{t_0}(Z_0) = X_0$ . Let  $\mathcal{W}$  be a neighborhood of  $Z_0$ . Then there is an open set  $\mathcal{U} \subset \mathcal{W}$  and a differentiable function  $\tau : \mathcal{U} \rightarrow \mathbb{R}$  such that  $\tau(Z_0) = t_0$  and*

$$\phi_{\tau(X)}(X) \in \mathcal{S} \quad (41)$$

for each  $x \in \mathcal{U}$ .

**Note 3.** *In some sense, a subsegment of the section  $\mathcal{S}$  is carried backwards smoothly through the field arbitrarily far, assuming that the flow makes sense, and that the subsegment is small enough.*

*Proof.* A point  $X_1$  belongs to the line  $\ell$  containing  $\mathcal{S}$  iff  $X = X_0 + uV_0$  for some  $u$ . Since  $V_0$  is orthogonal to  $F(X_0)$  we see that  $X \in \ell$  iff  $X \cdot F(X_0) = X_0 F(X_0)$ .

We look now at the more general function

$$G(X, t) = x(t; X) \cdot F(X_0) \quad (42)$$

We have

$$G(Z_0, t_0) = X_0 \cdot F(X_0) \quad (43)$$

by construction. We want to see whether we can apply the implicit function theorem to

$$G(X, t) - G(Z_0, t_0) = 0 \quad (44)$$

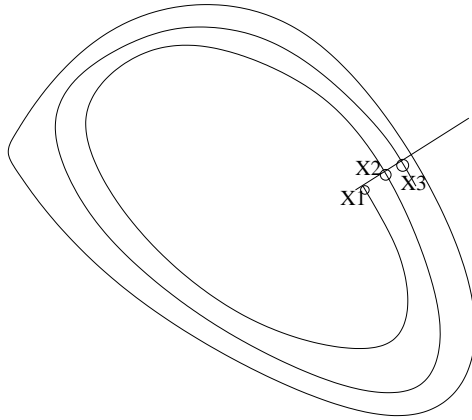
□

For this we need to check  $\frac{\partial}{\partial t} G(Z_0, t_0)$ . But this equals

$$x'(t; X) \cdot F(X_0) = |F(X_0)|^2 \neq 0 \quad (45)$$

Then, there is a neighborhood of  $t_0$  and a differentiable function  $\tau(X)$  so that

$$G(X, \tau(X)) = G(Z_0, t_0) = X_0 \cdot F(X_0) \quad (46)$$



$$X_{n+1} = P(X_n)$$

Figure 10: A Poincaré map.

## 7.2 The Poincaré map

The Poincaré map is a useful tool in determining whether closed trajectories (that is, periodic orbits) are stable or not. This means that taking an initial close enough to the periodic orbit, the trajectory thus obtained would approach the periodic orbit or not.

The basic idea is simple, we look at a section containing a point on the periodic orbit, and then follow the successive re-intersections of the perturbed orbit with the section. Now we are dealing with a discrete map  $X_{n+1} = P(X_n)$ . If  $P(X_n) \rightarrow X_0$ , the point on the closed orbit, then the orbit is asymptotically stable. See Figure 11.

It is often not easy to calculate the Poincaré map, but it is a very useful concept, and it has many theoretical applications.

Let's define the map rigorously.

Consider a periodic orbit  $\mathcal{C}$  and a point  $X_0 \in \mathcal{C}$ . We have

$$x(X_0; T) = X_0 \tag{47}$$

where  $T$  is the period of the orbit. Consider a section  $\mathcal{S}$  through  $X_0$ . Then according to Proposition 6, there is a neighborhood  $\mathcal{U}$  of  $X_0$  and a continuous function  $\tau(X)$  so that  $x(\tau(t), X) \in \mathcal{S}$  for all  $X \in \mathcal{U}$ . Then certainly  $\mathcal{S}_1 = \mathcal{U} \cap \mathcal{S}$  is an open set in  $\mathcal{S}$  in the induced topology. The return map is thus defined on  $\mathcal{S}_1$ . It means that for each point in  $X \in \mathcal{S}_1$  there is a point  $P(X) \in \mathcal{S}$ , so that  $x(\tau(X); X) = P(X)$  and  $\tau(X)$  is the smallest time with this property.

This is the Poincaré map associated to  $\mathcal{C}$  and to its section  $\mathcal{S}$ .

This can be defined for planar systems as well as for higher dimensional ones, if we now take as a section a subset of a hyperplane through a point  $X_0 \in \mathcal{C}$ . The statement and proof of Proposition 6 generalize easily to higher dimensions.

In two dimensions, we can identify the segments  $\mathcal{S}$  and  $\mathcal{S}_1$  with intervals on the real line,  $u \in (-a, a)$ , and  $u \in (-\epsilon, \epsilon)$  respectively, see also Definition 5. Then  $P$  defines an analogous transformation of the interval  $(-\epsilon, \epsilon)$ , which we still denote by  $P$  though this is technically a different function, and we have

$$P(0) = 0$$

$$P(u) \in (-a, a), \quad \forall u \in (-\epsilon, \epsilon)$$

We have the following easy result, the proof of which we leave as an exercise.

**Proposition 7.** *Assume that  $X' = F(X)$  is a planar system with a closed orbit  $\mathcal{C}$ , let  $X_0 \in \mathcal{C}$  and  $\mathcal{S}$  a section at  $X_0$ . Define the Poincaré map  $P$  on an interval  $(-\epsilon, \epsilon)$  as above, by identifying the section with a real interval centered at zero. If  $|P'(X_0)| < 1$  then the orbit  $\mathcal{C}$  is asymptotically stable.*

**Example 4.** Consider the planar system

$$r' = r(1 - r) \tag{48}$$

$$\theta' = 1 \tag{49}$$

In Cartesian coordinates it has a fixed point,  $x = y = 0$  and a closed orbit,  $x = \cos t, y = \sin t; x^2 + y^2 = 1$ . Any ray originating at  $(0, 0)$  is a section of the flow. We choose the positive real axis as  $\mathcal{S}$ . Let's construct the Poincaré map. Since  $\theta' = 1$ , for any  $X \in \mathbb{R}^+$  we have  $x(2\pi; X) = x(0, X)$ . We have  $P(1) = 1$  since 1 lies on the unit circle. In this case we can calculate explicitly the solutions, thus the Poincaré map and its derivative.

We have

$$\ln r(t) - \ln(r(t) - 1) = t + C \tag{50}$$

and thus

$$r(t) = \frac{Ce^t}{Ce^t - 1} \tag{51}$$

where we determine  $C$  by imposing the initial condition  $r(0) = x: C = x/(x-1)$ . Thus,

$$r(t) = \frac{xe^t}{1 - x + xe^t} \tag{52}$$

and therefore we get the Poincaré map by taking  $t = 2\pi$ ,

$$P(x) = \frac{xe^{2\pi}}{1 - x + xe^{2\pi}} \tag{53}$$

Direct calculation shows that  $P'(1) = e^{-2\pi}$ , and thus the closed orbit is stable. We could have seen this directly from (53) by taking  $t \rightarrow \infty$ .

Note that here we could calculate the orbits explicitly. Thus we don't quite need the Poincaré map anyway, we could just look at (52). When explicit solutions, or at least an explicit formula for the closed orbit is missing, calculating the Poincaré map can be quite a challenge.

## 8 Monotone sequences in two dimensions

There are two kinds of monotonicity that we can consider. One is monotonicity along a solution:  $X_1, \dots, X_n$  is monotone along the solution if  $X_n = x(t_n, X)$  and  $t_n$  is an increasing sequence of times. Or, we can consider monotonicity along a segment, or more generally a piece of a curve. On a piece of a smooth curve, or on an interval we also have a natural order (or two rather), by arclength parameterization of the curve:  $\gamma_2 > \gamma_1$  if  $\gamma_2$  is farther from the chosen endpoint. To avoid this rather trivial distinction (dependence on the choice of endpoint) we say that a sequence  $\{\gamma_n\}_n$  is monotone along the curve if  $\gamma_n$  is inbetween  $\gamma_{n-1}$  and  $\gamma_{n+1}$  for all  $n$ . Or we could say that a sequence is monotone if it is either increasing or else decreasing.

If we deal with a trajectory crossing a curve, then the two types of monotonicity need not coincide, in general. But for sections, they do.

**Proposition 8.** *Assume  $x(t; X); t \in [0, T]$  is a solution so that  $F$  is regular and nonzero in a neighborhood. Let  $\mathcal{S}$  be a local section for a planar system. Then monotonicity along the solution  $x(t; X)$  assumed to intersect  $\mathcal{S}$  at  $X_1, X_2, \dots$  (finitely or infinitely many intersection) and along  $\mathcal{S}$  coincide.*

Note that all intersections are supposed to be with  $\mathcal{S}$ , along which, by definition, they are always transversal.

*Proof.* We assume we have three successive distinct intersections with  $\mathcal{S}$ ,  $X_1, X_2, X_3$  (if two of them coincide, then the trajectory is a closed orbit and there is nothing to prove).

We want to show that  $X_3$  is not inside the interval  $(X_1, X_2)$  (on the section, or on its image on  $\mathbb{R}$ ). Consider the curve  $x(t; X_1)$   $t \in [0, t_2]$  where  $t_2$  is the first time of re-intersection of  $x(t; X_1)$  with  $\mathcal{S}$ . By definition  $x(t_2; X_1) = X_2$ . This is supposed to be a smooth curve, with no self-intersection (since the field is assumed regular along the curve) thus of finite length. If completed with the subsegment  $[X_1, X_2] \in \mathcal{S}$  it evidently becomes a closed curve (with a natural parameterization even)  $\mathcal{C}$ . By Jordan's lemma, we can define the inside  $\text{int } \mathcal{C}$  and the outside of the curve,  $D = \text{ext } \mathcal{C}$ . Note that the field has a definite direction along  $[X_1, X_2]$ , by the definition of a section. Note also that it points towards  $\text{ext } \mathcal{C}$ , since  $x(t; X_1)$  exits  $\text{int } \mathcal{C}$  at  $t = t_2$ .

Then, no trajectory can enter  $\text{int } \mathcal{C}$ . Indeed, it should intersect either  $x(t; X_1)$  or else  $[X_1, X_2]$ . The first option is impossible by uniqueness of solutions (solutions do not intersect at regular points). The second case is ruled out since  $[X_1, X_2]$  is an exit region, not an entry one. Now we know that  $x(t_3, X) = X_3$  where  $t_3$  is the first reintersection time. It must lie in  $\text{ext } \mathcal{C}$ , thus outside  $[X_1, X_2]$ .  $\square$

The next result shows points towards limiting points being special: parts of closed curves, or simply infinity.

**Proposition 9.** *Consider a planar system and  $Z \in \omega(X)$  (or  $Z \in \alpha(X)$ ), assumed a regular point of the field. Consider a local section  $\mathcal{S}$  through  $Z$ . Then either  $\{x(t, Z) : t > 0\} \cap \mathcal{S} = Z$  or else  $\{x(t, Z) : t > 0\} \cap \mathcal{S} = \emptyset$ .*

*Proof.* Assume there are two distinct intersection points  $x(t_1, Z) = Z_1$  and  $x(t_2, Z) = Z_2$  in  $\mathcal{S}$ . Since  $Z_1$  and  $Z_2$  are also in  $\omega(X)$ , as we have shown, then there are infinitely many points on  $x(t, X)$  arbitrarily close to  $Z_1$  and infinitely many others arbitrarily close to  $Z_2$ . These nearby points lie on the same section, since the section at  $Z_1$  contains, by definition, an open set around  $Z_1$  and similarly around  $Z_2$ . Without loss of generality (rotating and translating the figure) we can assume that  $\mathcal{S} = (-a, b) \in \mathbb{R}$  and  $[Z_1, Z_2] \subset (-a, b)$ . We know that  $x(t_j, X)$ , where  $t_j$  are the increasing times when  $x(t_j, X) \in \mathcal{S}$ , are monotone. Thus they converge. But then, by definition of convergence, they cannot be arbitrarily close to two distinct points. □

## 9 The Poincaré-Bendixson theorem

**Theorem 5** (Poincaré-Bendixson). *Let  $\Omega = \omega(X)$  be a nonempty compact limit set of a planar system of ODEs, containing no equilibria. Then  $\Omega$  is a closed orbit.*

*Proof.* First, recall that  $\Omega$  is invariant. Let  $Y \in \Omega$ . Then  $x(t; Y) \in \Omega$  for all  $t \in \mathbb{R}$ . Since  $\Omega$  is compact,  $x(t; Y)$  has (infinitely many) accumulation points in  $\Omega$ . Let  $Z$  be one of them, and let  $\mathcal{S}$  be a section through  $Z$ . Then  $x(t; Y)$  crosses  $\mathcal{S}$  infinitely many times. But there is room for only one intersection, by Proposition 9. Thus  $x(t_1; Y) = x(t_2; Y)$  for some  $t_1 \neq t_2$ , and this is enough to guarantee that  $x(t; Y)$  is closed. Then, clearly,  $\omega(Y) = \{x(t; Y) : t \in [0, T]\}$  where  $T$  is the period of the orbit.

Consider a section  $\mathcal{S}$  through  $Y$ . Since  $Y$  is in  $\omega(X)$ , there is a sequence  $t'_n$  so that  $x(t'_n, X) \rightarrow Y$ . If we use the flowbox theorem at  $Y$ , we see that  $x(t, X)$  crosses  $\mathcal{S}$  infinitely often, and arbitrarily close to  $Y$  (since this is clear for a straight flow, and initial conditions approaching the image of  $Y$ ). Denote this of successive intersections of  $\mathcal{S}$  by  $\{t_n\}_{n \in \mathbb{N}}$ .

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Consider the sequence  $x(t_n, X)$  where  $t_n$  are the successive intersection times of  $x(t, X)$  with  $\mathcal{S}$ . Since the sequence converges to  $Y$  and it is monotone one way or the other by Proposition 8, it can only be monotonic towards  $Y$ .

Since the return time  $\tau$  is continuous, we must have  $t_{n+1} - t_n \rightarrow T$ . By continuity with respect to initial conditions, we have  $x(t', X_n) - x(t', Y) \rightarrow 0$  for any  $t' \leq 2T$  (say). That is,  $x(t' + t_n, X) - x(t', Y) \rightarrow 0$  as  $n \rightarrow \infty$  for any fixed  $t' \leq 2T$ . But since  $t_{n+1} - t_n \rightarrow T$ , any sufficiently large  $t$  can be written as  $t' + t_n, t' \leq 2T$ . Let now  $Z$  be any point in  $\omega(X)$ . Then there is a sequence  $t''_n$  so that  $x(t''_n, X)$  converges  $Z$ . Now, on the one hand we have  $x(t''_n, X) - x(t''_n, Y) \rightarrow 0$  and on the other hand  $x(t''_n, X) - Z \rightarrow 0$ , and thus

$\text{dist}(Z, \omega(Y)) = 0$ , and since they are both compact sets, we have  $Z \in \omega(Y)$  completing the proof.

\*\*\*

1. Let  $\mathcal{C}_Y$  be the trajectory through  $Y$  (this is nothing else but  $\omega(Y)$ , since  $x(t; Y)$  is periodic). It is a compact set.
2. We found that  $x(t_n; X) \rightarrow Y$ . Since the return time  $\tau$  is continuous, we must have  $t_{n+1} - t_n \rightarrow T$ .

3. By continuity with respect to initial conditions,

$$\sup_{\tau \in [0, 2T]} d(x(t_n + \tau; X), x(\tau; Y)) \rightarrow 0$$

as  $n \rightarrow \infty$ .

4. Let  $t'_j$  be any increasing sequence,  $t'_j \rightarrow \infty$ . Then for every  $j$  there exists  $n(j)$  so that  $t'_j \in (t_{n(j)}, t_{n(j)+1})$ . Thus  $0 \leq t'_j - t_{n(j)} \leq 2T$  for large  $j$ .

5. By 3,

$$d(x(t'_j; X), x(t'_j - t_{n(j)}; Y)) \rightarrow 0 \text{ as } j \rightarrow \infty$$

6. Of course,  $x(t'_j - t_{n(j)}; Y) \in \mathcal{C}_Y$ . It follows thus that

$$d(x(t'_j; X), \mathcal{C}_Y) \leq d(x(t'_j; X), x(t'_j - t_{n(j)}; Y)) \rightarrow 0$$

7. By the above, any sequence  $x(t'_j)$  which converges, has the limit in  $\mathcal{C}_Y$ . By definition then,  $\omega(X) \subset \mathcal{C}_Y$ . Since  $\omega(X)$  is invariant and there is no strict subset of  $\mathcal{C}_Y$  which is invariant (why?), we have  $\omega(X) = \mathcal{C}_Y$ .

□

**Exercise 1.** *Where have we used the fact that the system is planar? Think how crucial dimensionality is for this proof.*

## 10 Applications of the Poincaré-Bendixson theorem

A limit cycle is a closed orbit  $\gamma$  which is the  $\omega$ -set of a point  $X \notin \gamma$ . There are of course closed orbits which are not limit cycles. For instance, the system  $x' = -y, y' = x$  with orbits  $x^2 + y^2 = C$  for any  $C$  clearly has no limit cycles.

But when limit cycles exist, they have at least one-sided stability.

**Corollary 10.** *Assume  $\omega(X) = \gamma$  and  $X \notin \gamma$  is a limit cycle. Then there exists a neighborhood  $\mathcal{N}$  of  $X$  so that  $\forall X' \in \mathcal{N}$  we have  $\gamma = \omega(X')$ .*

*Proof.* Let  $Y \in \gamma$  and let  $\mathcal{S}$  be a section through  $Y$ . Then, as we know, there is a sequence  $t_n \rightarrow \infty$  so that the points  $x(t_n, X) \in \mathcal{S}$  and  $x(t_n, X) \rightarrow Y$ . Take for instance a small enough open neighborhood  $\mathcal{O}$  of  $X_2 \in (x(t_2; X), x(t_3; X))$ . As we have shown, all points  $X'' \in \mathcal{O}$  have the property that  $x(t''; X'') \in \mathcal{S}$  for some  $t'' > 0$  (which, in fact, is small if  $\mathcal{O}$  is small, and, in fact,  $x(t''; X'') \in (x(t_2; X), x(t_3; X))$  as well).

We know, by continuity with respect to initial conditions, that  $x(-t_2; X'')$  exist, if  $\mathcal{O}$  is small enough. Consider the open set (by continuity)  $\mathcal{O}_1 = \{x(-t_2; X''), X'' \in \mathcal{O}\}$ . Since for any  $X_1 \in \mathcal{O}_1$  we have  $x(t, X_1) \in (x(t_2; X), x(t_3; X))$  for some  $t$ , by monotonicity, we have  $\omega(X_1) = \omega(X)$ . □

## 11 The Painlevé property

The classification of equations into integrable and nonintegrable, and in the latter case finding out whether the behavior is chaotic plays a major role in the study of dynamical systems.

As usual, for an  $n$ -th order differential equation, a constant of motion is a function  $\Phi(u_1, \dots, u_n, t)$  with a predefined degree of smoothness (analytic, meromorphic,  $C^n$  etc.) and with the property that for any solution  $y(t)$  we have

$$\frac{d}{dt}\Phi(y(t), y'(t), \dots, y^{(n-1)}(t), t) = 0$$

There are multiple precise definitions of integrability, and no one perhaps is comprehensive enough to be widely accepted. For us, let us think of a system as being integrable, relative to a certain regularity class of first integrals, if there are sufficiently many global constants of motion so that a particular solution can be found by knowledge of the values of the constants of motion.

We note once more that an integral of motion needs to be defined in a wide region. The existence of local constants along trajectories follows immediately either from the flowbox theorem, or from the implicit function theorem: indeed, if  $\mathbf{Y}' = \mathbf{F}(\mathbf{Y})$  is a system of equations near a regular point,  $\mathbf{Y}_0$ , then evidently there exists a local solution  $\mathbf{Y}(t; \mathbf{Y}_0)$ . It is easy to check that  $D_{\mathbf{Y}_0} \mathbf{Y}|_{t=0} = \mathbf{I}$ , so we can write, near  $\mathbf{Y}_0, t = 0$ ,  $\mathbf{Y}_0 = \Phi(\mathbf{Y}, t)$ . Clearly  $\Phi$  is constant along trajectories.

In general, we have an integral of motion  $\Phi(\mathbf{Y}, t) = \mathbf{Y}(-t; \mathbf{Y}(t))$ . This brings back the solution to where it started, so it must be a constant along trajectories. Not a very explicit function, admittedly, but smooth, at least locally. Given  $\mathbf{Y}(t)$  it asks, where did it start, when  $t$  was zero.  $\Phi$  is thus obtained by integrating the equation backwards in time.

Is this an integral of motion?

Not really. This cannot be defined for  $t$  which is not small enough, in general since we cannot integrate backwards from any  $t$  to zero, without running into singularities. If we think of  $t$  in the complex domain, we may think of

circumventing singularities, and define  $\Phi$  by analytic continuation around singularities. But what does that mean? If the singularities are always isolated, and in particular solutions are single valued, it does not matter which way we go. But if these are, say, square root branch points, if we avoid the singularity on one side we get  $+\sqrt{\phantom{x}}$  and on the other  $-\sqrt{\phantom{x}}$ . There is no consistency.

But we see, if we impose the condition that the equation have only isolated singularities (at least, those depending on the initial condition, or *movable*), then we have a single valued global constant of motion, take away some lower dimensional singular manifolds in  $\mathbb{C}^2$ .

Such equations are said to have the Painlevé property (PP) and are integrable, at least in the sense above. But it turns out, in those considered so far in applications, that more is true: they were all ultimately re-derived from linear equations.

## 11.1 The Painlevé equations

### 11.2 Spontaneous singularities: The Painlevé's equation $P_I$

Let us analyze local singularities of the Painlevé equation  $P_I$ ,

$$y'' = y^2 + x \tag{54}$$

In a neighborhood of a point where  $y$  is large, keeping only the largest terms in the equation (*dominant balance*) we get  $y'' = y^2$  which can be integrated explicitly in terms of elliptic functions and its solutions have double poles. Alternatively, we may search for a power-like behavior

$$y \sim A(x - x_0)^p$$

where  $p < 0$  obtaining, to leading order, the equation  $Ap(p-1)x^{p-2} = A^2(x - x_0)^2$  which gives  $p = -2$  and  $A = 6$  (the solution  $A = 0$  is inconsistent with our assumption). Let's look for a power series solution, starting with  $6(x - x_0)^{-2}$ :  $y = 6(x - x_0)^{-2} + c_{-1}(x - x_0)^{-1} + c_0 + \dots$ . We get:  $c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = -x_0/10, c_3 = -1/6$  and  $c_4$  is undetermined, thus free. Choosing a  $c_4$ , all others are uniquely determined. To show that there indeed is a convergent such power series solution we substitute  $y(x) = 6(x - x_0)^{-2} + \delta(x)$  where for consistency we should have  $\delta(x) = o((x - x_0)^{-2})$  and taking  $x = x_0 + z$  we get the equation

$$\delta'' = \frac{12}{z^2}\delta + z + x_0 + \delta^2 \tag{55}$$

Note now that our assumption  $\delta = o(z^{-2})$  makes  $\delta^2/(\delta/z^2) = z^2\delta = o(1)$  and thus the nonlinear term in (55) is *relatively* small. Thus, *to leading order*, the new equation is linear. This is a general phenomenon: taking out more and more terms out of the local expansion, the correction becomes less and less important, and the equation is better and better approximated by a linear equation. It is then natural to separate out the large terms from the small terms and write a

fixed point equation for the solution based on this separation. We write (55) in the form

$$\delta'' - \frac{12}{z^2}\delta = z + x_0 + \delta^2 \quad (56)$$

and integrate as if the right side was known. This leads to an equivalent integral equation. Since all unknown terms on the right side are chosen to be *relatively smaller*, by construction this integral equation is expected to be contractive.

Click here for Maple file of the formal calculation ( $y'' = y^2 + x$ )

The indicial equation for the Euler equation corresponding to the left side of (56) is  $r^2 - r - 12 = 0$  with solutions 4, -3. By the method of variation of parameters we thus get

$$\begin{aligned} \delta &= \frac{D}{z^3} - \frac{1}{10}x_0z^2 - \frac{1}{6}z^3 + Cz^4 - \frac{1}{7z^3} \int_0^z s^4\delta^2(s)ds + \frac{z^4}{7} \int_0^z s^{-3}\delta^2(s)ds \\ &= -\frac{1}{10}x_0z^2 - \frac{1}{6}z^3 + Cz^4 + J(\delta) \end{aligned} \quad (57)$$

the assumption that  $\delta = o(z^{-2})$  forces  $D = 0$ ;  $C$  is arbitrary. To find  $\delta$  formally, we would simply iterate (57) in the following way: We take  $r := \delta^2 = 0$  first and obtain  $\delta_0 = -\frac{1}{10}x_0z^2 - \frac{1}{6}z^3 + Cz^4$ . Then we take  $r = \delta_0^2$  and compute  $\delta_1$  from (57) and so on. This yields:

$$\delta = -\frac{1}{10}x_0z^2 - \frac{1}{6}z^3 + Cz^4 + \frac{x_0^2}{1800}z^6 + \frac{x_0}{900}z^7 + \dots \quad (58)$$

This series is actually convergent. To see that, we scale out the leading power of  $z$  in  $\delta$ ,  $z^2$  and write  $\delta = z^2u$ . The equation for  $u$  is

$$\begin{aligned} u &= -\frac{x_0}{10} - \frac{z}{6} + Cz^2 - \frac{z^{-5}}{7} \int_0^z s^8u^2(s)ds + \frac{z^2}{7} \int_0^z su^2(s)ds \\ &= -\frac{x_0}{10} - \frac{z}{6} + Cz^2 + J(u) \end{aligned} \quad (59)$$

It is straightforward to check that, given  $C_1$  large enough (compared to  $x_0/10$  etc.) there is an  $\epsilon$  such that this is a contractive equation for  $u$  in the ball  $\|u\|_\infty < C_1$  in the space of analytic functions in the disk  $|z| < \epsilon$ . We conclude that  $\delta$  is analytic and that  $y$  is meromorphic near  $x = x_0$ .

The equation  $y'' = y^2 + x^2$  does not have the Painlevé property.

Click here for Maple file of the formal calculation, for  $y'' = y^2 + x^2$

## References

- [1] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, (1955).
- [2] M.W.Hirsch, S. Smale and R.L. Devaney, *Differential Equations, Dynamical Systems & An Introduction to Chaos*, Academic Press, New York (2004).

$$\begin{aligned}
& y'' = 6y^2 \\
& y'' = 2y^3 + xy' \\
& y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 \\
& y'' = \frac{y'^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y' \\
& y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2} \left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{x} + \frac{\delta y(y-1)}{y-1} \\
& y'' = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)y'^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2}\right]
\end{aligned}$$

Figure 11: The six Painlevé equations, all equations of the form  $y'' = R(y', y, x)$ , with  $R$  rational, having the PP.

- [3] D. Ruelle, *Elements of Differentiable Dynamics and Bifurcation theory*, Academic Press, Ney York , (1989)