

Exercise 2.5 Assume f is entire, $|f(z)| \leq C_1 e^{|az|}$ in \mathbb{C} and $|f(z)| \leq C e^{-|z|}$ in a sector of opening more than π . Show that f is identically zero. (A similar statement holds under much weaker assumptions; see Exercise 2.29.)

2.2 Laplace and inverse Laplace transforms

Let $F \in L^1(\mathbb{R}^+)$ (meaning that $|F|$ is integrable on $[0, \infty)$). Then the Laplace transform

$$(\mathcal{L}F)(x) := \int_0^\infty e^{-px} F(p) dp \quad (2.6)$$

is analytic in \mathbb{H} and continuous in $\overline{\mathbb{H}}$. Note that the substitution allows us to work in space of functions with the property that $F(p)e^{-|\alpha|p}$ is in L^1 , correspondingly replacing x by $x - |\alpha|$.

Proposition 2.7 If $F \in L^1(\mathbb{R}^+)$, then

- (i) $\mathcal{L}F$ is analytic in \mathbb{H} and continuous on the imaginary axis $\partial\mathbb{H}$.
- (ii) $\mathcal{L}\{F\}(x) \rightarrow 0$ as $x \rightarrow \infty$ along any ray $\{x : \arg(x) = \theta\}$ if $|\theta| \leq \pi/2$.

Proof. (i) Continuity and analyticity are preserved by integration against a finite measure ($F(p)dp$). Equivalently, these properties follow by dominated convergence², as $\epsilon \rightarrow 0$, of $\int_0^\infty e^{-isp}(e^{-ip\epsilon} - 1)F(p)dp$ and of $\int_0^\infty e^{-xp}(e^{-p\epsilon} - 1)\epsilon^{-1}F(p)dp$, respectively, the last integral for $\operatorname{Re}(x) > 0$.

If $|\theta| < \pi/2$, (ii) follows easily from dominated convergence; for $|\theta| = \pi/2$ it follows from the Riemann-Lebesgue lemma; see Proposition 3.55. \square

Remark 2.8 Extending F on \mathbb{R}^- by zero and using the continuity in x proved in Proposition 2.7, we have $\mathcal{L}\{F\}(it) = \int_{-\infty}^\infty e^{-ipt} F(p)dp = \hat{\mathcal{F}}F(t)$. In this sense, the Laplace transform can be identified with the (analytic continuation of) the Fourier transform, restricted to functions vanishing on a half-line.

First inversion formula

Let \mathcal{H} denote the space of analytic functions in \mathbb{H} .

Proposition 2.9 (i) $\mathcal{L} : L^1(\mathbb{R}^+) \mapsto \mathcal{H}$ and $\|\mathcal{L}F\|_\infty \leq \|F\|_1$.

(ii) $\mathcal{L} : L^1 \mapsto \mathcal{L}(L^1)$ is invertible, and the inverse is given by

$$F(x) = \hat{\mathcal{F}}^{-1}\{\mathcal{L}F(it)\}(x) \quad (2.10)$$

for $x \in \mathbb{R}^+$ where $\hat{\mathcal{F}}$ is the Fourier transform (in distributions if $\mathcal{L}F \notin L^1$).

²See e.g. [52]. Essentially, if the functions $|f_n| \in L^1$ are bounded uniformly in n by $g \in L^1$ and they converge pointwise (except possibly on a set of measure zero), then $\lim f_n \in L^1$ and $\lim \int f_n = \int \lim f_n$.