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★**The geometry and topology of Coxeter groups.**

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Books on Coxeter groups come in many flavours. When trying to get a feeling for the direction this book is heading, one should consider the tessellation of the hyperbolic plane by right-angled pentagons depicted on the front page and look for its re-occurrence in the book. It is impossible to describe the contents of this impressive book in one sentence; maybe it is fair to say that this book presents Coxeter groups in their natural environment of metric geometry.

This book is one of those that grows with the reader: A graduate student can learn many properties, details and examples of Coxeter groups, while an expert can read about aspects of recent results in the theory of Coxeter groups and use the book as a guide to the literature. I strongly recommend this book to anybody who has any interest in geometric group theory. Anybody who reads (parts of) this book with an open mind will get a lot out of it.

When the reader finally arrives at figure 6.2 on page 87 (the re-occurrence of the front picture in the book), he or she (1) has seen the wonderful introduction: it reads like the menu of our favourite restaurant; in this book we are going to read about geometric and abstract reflection groups, geometrisation of abstract reflection groups, the Davis-Moussong complex, many examples, the reflection group trick, buildings, and homology and cohomology; moreover, we learn that apparently the author is particularly interested in right-angled Coxeter groups; (2) has learned some basics of geometric group theory: Cayley graphs; word metrics; the relation between the Cayley 2-complex and presentations by generators and relations; aspherical spaces; in particular, we encounter the notions of nonpositively curved polyhedra and word hyperbolic groups; (3) has studied Coxeter groups: dihedral groups as starting examples; the equivalence between a reflection system (i.e., a connected graph together with a family of involutory automorphisms that stabilise at least one edge such that the graph without the stabilised edges falls into two connected components) and a Coxeter system (i.e., a group with a distinguished set of generators such that relations only occur between at most two generators); a solution of the word problem in Coxeter systems; Coxeter diagrams; (4) has seen many features of Coxeter groups: special subgroups; parabolic subgroups; shortest elements in special cosets; normalisers of special subgroups; subgroups generated by reflections; (5) has learned about mirror structures and the basic construction (see also pages 381–387 of [M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer, Berlin, 1999; MR1744486 (2000k:53038)]).

Altogether, the first chapters of this book provide a good foundation for studying Coxeter groups. The book has not yet committed itself to a certain cause; any good book on Coxeter groups could start like this, whence anybody interested in learning the basics of Coxeter groups is well-advised to read the first few chapters of this book.

In Chapter 6, finally, one of the main themes occurs: curvature. The three types of model spaces of constant curvature are introduced: Euclidean space, n-spheres, and hyperbolic space, motivated by the classification of simply connected complete Riemannian manifolds of constant sectional curvature. This chapter then treats geometric reflection groups: spherical, Euclidean, and hyperbolic. Notably, it contains a characterisation of the representability of a Coxeter system as a spherical or Euclidean or hyperbolic geometric reflection group by properties of the cosine matrix, which is obtained from the Coxeter matrix (Theorem 6.8.12). Finite reflection groups are studied and we learn why finite Coxeter groups are also called spherical Coxeter groups (Theorem 6.12.9). Moreover, this chapter contains a wealth of information about hyperbolic reflection groups such as Andreev's Theorem (6.10.2; see [E. M. Andreev, Mat. Sb. (N.S.) **81 (123)** (1970), 445–478; MR0259734 (41 #4367)]) and Vinberg's Theorem (6.11.8; see [E. B. Vinberg, Uspekhi Mat. Nauk **40** (1985), no. 1(241), 29–66, 255; MR0783604 (86m:53059)]). A special class of hyperbolic reflection groups are the hyperbolic polygon groups (remember the picture on the front page) introduced in Example 6.5.3.

Chapter 7 introduces "one of the fundamental objects of study in this book" (page 126), the complex Σ: Let (W, S) be a Coxeter system. A subset T of S is called spherical if the Coxeter group W_T generated by the elements of T is finite (cf. Theorem 6.12.9 mentioned above); the group W_T is called a spherical special subgroup. A spherical coset is a coset of a spherical special subgroup in W, and the set of all spherical cosets is denoted by $W\mathcal{S}$, i.e., $W\mathcal{S} = \bigcup_{T \text{ spherical}} W/W_T$; the set $W\mathcal{S}$ is partially ordered by inclusion. Then the simplicial complex Σ is defined as the geometric realisation of this poset $W\mathcal{S}$. A lot of attention is dedicated to finding a suitable cell structure on Σ. It turns out (Proposition 7.3.4) that there exists a natural one so that its vertex set is the Coxeter group W, its 1-skeleton is the Cayley graph $\mathrm{Cay}(W, S)$, its 2-skeleton is a Cayley 2-complex, each cell is a Coxeter polytope, the link of each vertex is isomorphic to the nerve of (W, S), a subset of W is the vertex set of a cell if and only if it is a spherical coset, and the poset of cells in Σ is $W\mathcal{S}$. It is an easy but very instructive exercise in geometric group theory (Lemma 7.3.5) that Σ is simply connected. In the notes the author modestly admits that the complex Σ is known as both the Davis-Moussong complex and the Davis complex. It has been introduced in [M. W. Davis, Ann. of Math. (2) **117** (1983), no. 2, 293–324; MR0690848 (86d:57025)]; the cell structure has been studied in G. Moussong's Ph.D. thesis ["Hyperbolic Coxeter groups", Ohio State Univ., Columbus, OH, 1988], where it was used to prove that the Davis-Moussong complex is CAT(0) (see Chapter 12 of this book).

Chapter 8 deals with the algebraic topology of the Davis complex and the complex obtained from the basic construction in Chapter 5. In particular, it is proved that the Davis-Moussong complex is contractible (Theorem 8.2.13). Moreover, we learn about ends of Coxeter groups. Here, the ends of a finitely generated group are defined as the ends of the corresponding Cayley graph. Since switching to another finite set of generators defines a quasi-isometry of the underlying Cayley graph, this gives a well-defined concept of ends of the group which is independent of the chosen finite set of generators. Chapter 15 returns to questions from algebraic topology. The main theme is the homology and cohomology of Coxeter groups.

Chapter 9 continues to deal with the algebraic topology of the Davis-Moussong complex and the complex obtained from the basic construction: we learn when the former is simply connected at infinity and when the latter is simply connected.

One of the highlights of Chapter 10, which deals with manifolds, is Example 10.5.3 (see also Remark 5b.2 on page 384 of [M. W. Davis and T. Januszkiewicz, J. Differential Geom. **34** (1991), no. 2, 347–388; MR1131435 (92h:57036)]). It establishes that for each $n \geq 5$ there is a closed n-manifold with a non-positively curved piecewise Euclidean metric such that its universal cover is not homeomorphic to Euclidean space, which answers a question of M. Gromov (page 187 of [in *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, 183–213, Ann. of Math. Stud., 97, Princeton Univ. Press, Princeton, N.J., 1981; MR0624814 (82m:53035)]). It should not come as a big surprise that this example is provided by the Davis-Moussong complexes of Coxeter systems with certain properties. Section 10.4 presents the Local Smith Theorem (10.4.3; see [P. A. Smith, Ann. of Math. (2) **40** (1939), 690–711; MR0000177 (1,30c)]), Newman's Theorem (10.4.4), and Edwards' Polyhedral Manifold Characterisation Theorem (10.4.10; see [R. D. Edwards, in *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, 111–127, Acad. Sci. Fennica, Helsinki, 1980; MR0562601 (81g:57010)]), while Section 10.6 deals with the question of when the Davis-Moussong complex is a manifold. Other features of this chapter are reflection groups on homology manifolds (10.7) and a characterisation of when a Coxeter group is a virtual Poincaré duality group (10.9), i.e., when a Coxeter group is of type FP and its classifying space satisfies Poincaré duality with arbitrary local coefficients; by Theorem 10.9.2 [see M. W. Davis, Duke Math. J. **91** (1998), no. 2, 297–314; MR1600586 (99b:20067)] this happens if and only if the corresponding Coxeter system decomposes as a product of a spherical Coxeter system and a Coxeter system of type HM^n.

The topic of Chapter 11 is the reflection group trick. This term "refers to a method for converting a finite aspherical CW complex into a closed aspherical manifold which retracts back onto it. The upshot is that closed aspherical manifolds are at least as complicated as finite aspherical complexes" (page 212). It lies in the nature of such a machinery that it produces all kinds of counter-examples, i.e., examples of aspherical manifolds whose fundamental group does *not* have a certain property: for each $n \geq 4$ there exists a closed aspherical n-manifold whose fundamental group is not residually finite (Example 11.2.1), there is a closed aspherical n-manifold whose fundamental group contains an infinitely divisible abelian group (Example 11.2.2), and there exists a closed aspherical n-manifold whose fundamental group has unsolvable word problem (Example 11.2.3). A very prominent class of examples treated in this chapter are the groups constructed by M. Bestvina and N. Brady [see Invent. Math. **129** (1997), no. 3, 445–470; MR1465330 (98i:20039)], which are of type FL but are not finitely presented (and hence not of type F). These groups are subgroups of right-angled Artin groups. Mathematicians who are very fond of conjectures, and their respective implications and exclusions, can make their day by reading Section 11.4: the book deals with the Reduced Projective Class Group Conjecture, the Whitehead Group Conjecture, the Assembly Map Conjecture in L-Theory, and various incarnations of these. Later, in Section 12.8, these conjectures are revisited, now armed with the knowledge that the Davis-Moussong complex is CAT(0).

Chapter 12 establishes a fact that has already been used in Example 10.5.3 mentioned above: the Davis-Moussong complex, endowed with its natural piecewise Euclidean structure, is CAT(0)

(Theorem 12.3.3). The fact that Coxeter groups are CAT(0) groups is a spectacular result. First, this means that the Bruhat-Tits fixed point lemma can be applied, so that any finite subgroup of a Coxeter group is conjugate to a subgroup of a spherical special subgroup of the Coxeter group; in fact, the conjugacy problem is solvable (Theorem 12.3.4). Secondly, standard arguments from building theory (see Chapter 18 below) show that this implies that buildings are CAT(0). Other spectacular things are happening in Chapter 12: For example, we learn exactly when a Coxeter group is word hyperbolic (Corollary 12.6.3), namely if and only if the Davis-Moussong complex admits a piecewise hyperbolic $\mathrm{CAT}(-1)$ metric. One of the features of word hyperbolic groups is that the word problem is solvable. In Section 12.7 we learn about one of the major achievements of D. Krammer's Ph.D. thesis ["The conjugacy problem for Coxeter groups", Univ. Utrecht, Utrecht, 1994], another spectacular result: Any free abelian subgroup of a Coxeter group has a subgroup of finite index which is a subgroup of a standard free abelian subgroup (Theorem 12.7.4); for a recent addition to this theme the reader may also want to consult the article by P.-E. Caprace and F. Haglund ["On geometric flats in the CAT(0) realisation of Coxeter groups and Tits buildings", Canad. J. Math., to appear].

The main theme of Chapter 13 is the strong (reflection) rigidity of Coxeter groups/systems. A Coxeter group is called strongly rigid if any two fundamental sets of generators turning the Coxeter group into a Coxeter system, are conjugate. A Coxeter system (W, S) is called strongly reflection rigid if each fundamental system S' of the Coxeter group W contained in the set R_S of reflections of the Coxeter system (W, S), is conjugate to S. There are examples of Coxeter systems which are not rigid, e.g., the dihedral groups D_{2n} for n even (Examples 13.1.1 and 13.1.6) and Mühlherr's irreducible nonspherical example (Figure 13.2 on page 259; see also [B. Mühlherr, Des. Codes Cryptogr. **21** (2000), no. 1-3, 189; MR1801199 (2001j:20059)]). The key technique distilled from Mühlherr's example is the twisting of Coxeter graphs (Proposition 13.1.7; also [N. Brady et al., Geom. Dedicata **94** (2002), 91–109; MR1950875 (2004b:20052)]). Positive rigidity results are available in the following situations: irreducible spherical Coxeter systems of rank at least two distinct from H_3 or H_4 are strongly reflection rigid (see page 260; also page 330 of [W. N. Franzsen and R. B. Howlett, Adv. Geom. **3** (2003), no. 3, 301–338; MR1997410 (2004d:20041)]), two-spherical Coxeter systems without spherical component are strongly reflection rigid (Proposition 13.1.9; also [P.-E. Caprace and B. Mühlherr, Proc. Lond. Math. Soc. (3) **94** (2007), no. 2, 520–542; MR2311001 (2008c:20072)]), Coxeter groups admitting a Coxeter system of type PM^n, $n > 1$, are strongly rigid (Theorem 13.4.1); a Coxeter system is of type PM^n if its nerve is an orientable gallery-connected pseudomanifold of dimension $n-1$. Additional information on the rigidity of Coxeter groups can be found in [P. Bahls, *The isomorphism problem in Coxeter groups*, Imp. Coll. Press, London, 2005; MR2162146 (2006e:20069)].

Chapter 14 presents a number of virtual properties of Coxeter groups, i.e., properties that Coxeter groups have up to passing to a subgroup of finite index. Theorem 14.1.2 [see G. A. Margulis and E. B. Vinberg, J. Lie Theory **10** (2000), no. 1, 171–180; MR1748082 (2001h:22016)] states that a Coxeter group is either virtually free abelian or contains a finite index subgroup that surjects onto a nonabelian free group. Note also that, if a Coxeter group is virtually abelian, then it is a product of a finite group and a cocompact Euclidean reflection group (Theorem 12.3.5). Moreover, a Coxeter group is virtually free if and only if it does not contain a surface subgroup (Theorem 14.2.1; see also [C. McA. Gordon, D. D. Long and A. W. Reid, J. Pure Appl. Algebra **189** (2004), no. 1-3, 135–148; MR2038569 (2004k:20077)]), i.e., the fundamental group of a closed orientable surface of genus greater than zero.

Chapter 16 deals with the Euler characteristic. Section 16.1 contains background information, while Section 16.2 presents the Euler Characteristic Conjecture (16.2.1): The Euler characteristic of a $2k$-dimensional closed aspherical manifold is conjectured to have the sign $(-1)^k$. One should note that the Euler characteristic of an odd-dimensional closed manifold is known to be zero by Poincaré duality. Also, the Euler characteristic of a closed even-dimensional Riemannian manifold is equal to the integral over the manifold of the Euler form by the Chern-Gauss-Bonnet Theorem. The chapter concludes with the Flag Complex Conjecture. Proposition 16.3.2 states that the Euler Characteristic Conjecture for homology manifolds with non-positively curved piecewise Euclidean metrics coming from cubical structures is equivalent to the Flag Complex Conjecture. Theorem 16.3.3 gives some information on cases in which the Flag Complex Conjecture has been known to hold at the time of print.

In Chapter 17 the growth series of Coxeter groups is investigated. Besides rationality and reciprocity results (Sections 17.1 and 17.3), the relation to the topics of Chapter 14 is explored: For an infinite Coxeter system (W, S) the properties that W is amenable, that W is virtually abelian, and that W has subexponential growth, are equivalent (Proposition 17.2.1).

Buildings are introduced in Chapter 18. The combinatorial definition is given in Section 18.1 and the geometric realisation is defined in Section 18.2. In this book the geometric realisation of a building is such that its apartments are isomorphic to the Davis-Moussong complex. The advantage of the Davis-Moussong realisation in comparison with the “usual definition” (Example 18.2.2) is that it makes buildings CAT(0) (Theorem 18.3.1; also [M. W. Davis, in *Geometry and cohomology in group theory (Durham, 1994)*, 108–123, Cambridge Univ. Press, Cambridge, 1998; MR1709955 (2000i:20068)]). The chapter concludes with a section on the Euler-Poincaré measure.

Chapter 19 introduces the Hecke algebra associated to a Coxeter system and explains how it can be completed to a von Neumann algebra. The main references are [M. W. Davis et al., Geom. Topol. **11** (2007), 47–138; MR2287919 (2008g:20084); J. Dymara, Geom. Topol. **10** (2006), 667–694 (electronic); MR2240901 (2007h:20027)]. Chapter 20 continues in the direction of weighted L^2-(co)homology. In Section 20.2 the relationship between weighted L^2-Betti numbers and the Euler characteristic is explored. Section 20.3 explains why (co)homology is concentrated in dimension zero [see also J. Dymara, op. cit.]. More highlights are the decomposition theorems (Section 20.6) and the L^2-cohomology of buildings (Section 20.8). A reader interested in the latter should also consult [J. Dymara and T. Januszkiewicz, Invent. Math. **150** (2002), no. 3, 579–627; MR1946553 (2003j:20052)].

Reviewed by *Ralf Gramlich*