PROBLEMS

1 Find power series expansions for the following functions, and determine the values of x for which these expansions are valid:

(a) $\dfrac{1}{(1+x)^2}$; (b) $\dfrac{1}{(1+x)^3}$.

2 Show that

$$\sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} x^n = \frac{1}{(1-x)^4}.$$

3 Find the sum of each of the following series:

(a) $x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \cdots + \dfrac{x^{2n+1}}{2n+1} + \cdots$;

(b) $1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \cdots + \dfrac{x^{n-1}}{n!} + \cdots$;

(c) $x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots$;

(d) $x + 2x^3 + 3x^5 + \cdots + nx^{2n-1} + \cdots$.

4 Show that

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\ln(1-t)}{t}\,dt.$$

5 Show that the Bessel function

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

satisfies the differential equation $xy'' + y' + xy = 0$.

6 Obtain the series

$$\ln(x + \sqrt{1+x^2}) = x - \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots$$

by integrating another series.

7 If $\epsilon > 0$, show that the inequality $|nx^{n-1}| \le (|x| + \epsilon)^n$ is true for all sufficiently large n's. Hint: $n^{1/n}|x|^{1-(1/n)} \to |x|$.

8 On p. 41-7 in Vol. 1 of his *Lectures On Physics* (Addison-Wesley, 1964), Richard Feynman (Nobel Prize, 1965) writes:

> Thus the average energy is
>
> $$\langle E \rangle = \frac{\hbar\omega(0 + x + 2x^2 + 3x^3 + \cdots)}{1 + x + x^2 + \cdots}.$$
>
> Now the two sums which appear here we shall leave for the reader to play with and have some fun with. When we are all finished summing and substituting for x in the sum, we should get—if we make no mistakes in the sum—
>
> $$\langle E \rangle = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}.$$
>
> This, then, was the first quantum-mechanical formula ever known, or ever discussed, and it was the beautiful culmination of decades of puzzlement.

Use Problem 3(c) to have the fun that Feynman recommends. (It is not necessary to know that $\hbar$ is Planck's constant, but it *is* necessary to substitute $x = e^{-\hbar\omega/kT}$.) A few sentences on, Feynman writes: "This expression should, of course, approach kT as $\omega \to 0$. See if you can prove that it does—learn how to do the mathematics." Prove that it does.

14.4 TAYLOR SERIES AND TAYLOR'S FORMULA

We have solved the problem of determining the general nature of the function that is the sum of a convergent power series: Inside the interval of convergence, it is a continuous function with derivatives of all orders. We now investigate the converse problem of starting with a given infinitely differentiable function and expanding it in a power series. In Section 14.3 we established several such expansions for a few special functions with particularly simple derivatives. Our purpose here is to consider a method of much greater generality.

It may seem that the coefficients of a convergent power series are not connected with one another in any necessary way. In fact, however, they are bound together by an invisible chain, which we now make visible.

To this end, let us assume that a function $f(x)$ is the sum of a power series with positive radius of convergence,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots, \qquad R > 0. \tag{1}$$