Weighted L^2–cohomology of Coxeter groups

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Given a Coxeter system (W, S) and a positive real multiparameter $\mathbf{q}$, we study the "weighted L^2–cohomology groups," of a certain simplicial complex Σ associated to (W, S). These cohomology groups are Hilbert spaces, as well as modules over the Hecke algebra associated to (W, S) and the multiparameter $\mathbf{q}$. They have a "von Neumann dimension" with respect to the associated "Hecke–von Neumann algebra" $\mathcal{N}_{\mathbf{q}}$. The dimension of the i–th cohomology group is denoted $b^i_{\mathbf{q}}(\Sigma)$. It is a nonnegative real number which varies continuously with $\mathbf{q}$. When $\mathbf{q}$ is integral, the $b^i_{\mathbf{q}}(\Sigma)$ are the usual L^2–Betti numbers of buildings of type (W, S) and thickness $\mathbf{q}$. For a certain range of $\mathbf{q}$, we calculate these cohomology groups as modules over $\mathcal{N}_{\mathbf{q}}$ and obtain explicit formulas for the $b^i_{\mathbf{q}}(\Sigma)$. The range of $\mathbf{q}$ for which our calculations are valid depends on the region of convergence of the growth series of W. Within this range, we also prove a Decomposition Theorem for $\mathcal{N}_{\mathbf{q}}$, analogous to a theorem of L Solomon on the decomposition of the group algebra of a finite Coxeter group.

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1 Introduction

Suppose (W, S) is a Coxeter system. (The precise definition will be given in Section 2. For now, it suffices to say that W is a group and S is a set of involutions which generate W.) Associated to (W, S) there is a certain contractible simplicial complex Σ on which W acts properly and cocompactly. (The definition of Σ can be found in Davis [12; 15; 16; 17], Davis and Meier [20] and Moussong [36], as well as in Section 6.) Let $i\colon S \to I$ be a function to some index set I so that $i(s) = i(s')$ whenever s and s' are conjugate. Given an I–tuple $\mathbf{q} = (q_i)_{i\in I}$ of positive real numbers, the second author [27] defined certain "weighted L^2–cohomology spaces, " here denoted $L^2_{\mathbf{q}}\mathcal{H}^i(\Sigma)$. The weighted L^2–cochain complex, $L^2_{\mathbf{q}}C^*(\Sigma)$, is a subcomplex of the complex $C^*(\Sigma; \mathbf{R})$ of ordinary cellular cochains. It consists of those cochains which are square summable with respect to an inner product defined via a weight function

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depending on the multiparameter $\mathbf{q}$. As we explain in Section 5 and Section 7, to each of the Hilbert spaces $L^2_{\mathbf{q}}\mathcal{H}^i(\Sigma)$ one can attach a "von Neumann dimension." It is a nonnegative real number, denoted $b^i_{\mathbf{q}}(\Sigma)$ and called the *i–th $L^2_{\mathbf{q}}$–Betti number of* Σ.

Our principal interest in the weighted L^2–cohomology of Σ lies in the fact that it computes the L^2–cohomology of buildings of type (W,S). Here $\mathbf{q}$ is an I–tuple of positive integers called the "thickness vector" of the building. (So, for buildings, only $\mathbf{q}$ with integral components can occur.)

The theory of the weighted L^2–cohomology of Σ is closely tied to several other topics, for example, growth series of Coxeter groups, decompositions of "Hecke–von Neumann algebras" and the Singer Conjecture. Moreover, as $\mathbf{q}$ goes from 0 to ∞, $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$ interpolates between ordinary cohomology and cohomology with compact supports. For these reasons, we believe that the study of weighted L^2–cohomology of Coxeter groups has intrinsic interest, independent of its connection to buildings.

Let $\mathbf{t} := (t_i)_{i\in I}$ be an I–tuple of indeterminates. Write t_s instead of $t_{i(s)}$. If $s_1\cdots s_k$ is a reduced expression for an element $w\in W$, then the monomial $t_w := t_{s_1}\cdots t_{s_k}$ is independent of the choice of reduced expression for w. The *growth series* for W is the power series in $\mathbf{t}$ defined by

$$W(\mathbf{t}) := \sum_{w\in W} t_w.$$

It is a rational function of $\mathbf{t}$; see Bourbaki [4] and Serre [38]. We give several explicit formulas for it in Lemma 3.3 of Section 3. (In the case where I is a singleton, so that $\mathbf{t}$ is a single indeterminate t, we have $t_w = t^{l(w)}$, where $l(w)$ denotes the word length of w. So, in the case of a single indeterminate, $W(t) = \sum t^{l(w)}$ is the usual growth series.)

Let $\mathbf{R}^{(W)}$ denote the vector space of finitely supported, real-valued functions on W and let $(e_w)_{w\in W}$ be its standard basis.

As we explain in Section 4, associated to each multiparameter $\mathbf{q}$, there is a deformation of the group algebra of W called the "Hecke algebra" (or sometimes the "Iwahori–Hecke algebra") of W. We denote it by $\mathbf{R}_{\mathbf{q}}[W]$. When $\mathbf{q} = \mathbf{1}$ (the I–tuple with all components equal to 1), $\mathbf{R}_{\mathbf{q}}[W]$ is the group algebra of W. (No matter what $\mathbf{q}$ is, the underlying vector space of $\mathbf{R}_{\mathbf{q}}[W]$ is always $\mathbf{R}^{(W)}$.)

Also associated to $\mathbf{q}$, there is an inner product $\langle\ ,\ \rangle_{\mathbf{q}}$ on $\mathbf{R}^{(W)}$ defined by $\langle e_w, e_{w'}\rangle_{\mathbf{q}} = q_w\delta_{ww'}$, where $\delta_{ww'}$ is the Kronecker delta. The completion of $\mathbf{R}^{(W)}$ with respect to this inner product is denoted $L^2_{\mathbf{q}}(W)$ or simply $L^2_{\mathbf{q}}$ when W is understood. $L^2_{\mathbf{q}}$ is an $\mathbf{R}_{\mathbf{q}}[W]$–bimodule. There is an anti-involution on $\mathbf{R}_{\mathbf{q}}[W]$, denoted by $x\to x^*$ and defined by $(e_w)^* := e_{w^{-1}}$. Moreover, $\langle yx, z\rangle_{\mathbf{q}} = \langle y, zx^*\rangle_{\mathbf{q}}$, ie right translation by x^* is the adjoint of right translation by x. As is explained in [27] and Proposition 5.1, this makes $\mathbf{R}_{\mathbf{q}}[W]$ into a "Hilbert algebra" in the sense of Dixmier [24]. It follows that there is an associated von Neumann algebra $\mathcal{N}_{\mathbf{q}}$ acting on $L^2_{\mathbf{q}}$ from the right. It can be defined as the algebra of bounded linear operators on $L^2_{\mathbf{q}}$ which commute with the left $\mathbf{R}_{\mathbf{q}}[W]$–action. $\mathcal{N}_{\mathbf{q}}$ is the *Hecke–von Neumann algebra* associated to $\mathbf{q}$. ($\mathcal{N}_{\mathbf{q}}$ is a completion of $\mathbf{R}_{\mathbf{q}}[W]$ acting from the right on $L^2_{\mathbf{q}}$.) As in the case of a von Neumann algebra associated to a group algebra, $\mathcal{N}_{\mathbf{q}}$ is equipped with a trace which one can use to define the "dimension" of any $\mathbf{R}_{\mathbf{q}}[W]$–stable closed subspace V of a finite direct sum of copies of $L^2_{\mathbf{q}}$.

Suppose W acts as a reflection group on a CW complex $\mathcal{U}$ with a strict fundamental domain Z. Assume further that for each $s \in S$ there is a subcomplex $Z_s \subseteq Z$, called a "mirror" of Z, so that s acts on $\mathcal{U}$ as a reflection across Z_s. Then $\mathcal{U}$ is formed by gluing together copies of Z, one for each element of W. In other words, $\mathcal{U} \cong (W \times Z)/\sim$, where the equivalence relation $\sim$ is defined in an obvious fashion. (See Section 6.) The complex Σ can be described in this manner: the fundamental chamber for W on Σ is denoted by K instead of Z.

$C^i_c(\mathcal{U})$ is the space of finitely supported, real-valued, cellular i–cochains on $\mathcal{U}$. For each oriented i–cell σ of $\mathcal{U}$, let e_σ be its characteristic function. So, $\{e_\sigma\}_{\sigma\in\{i\text{–cells}\}}$ is a basis for $C^i_c(\mathcal{U})$. As in [27], there is a definition of an inner product on $C^i_c(\mathcal{U})$ similar to the definition of $\langle\ ,\ \rangle_{\mathbf{q}}$ on $\mathbf{R}^{(W)}$. The e_σ form an orthogonal basis; however, the norm of e_σ need not be 1. Instead, one uses $\mathbf{q}$ to weight the inner product so that $\langle e_\sigma, e_\sigma\rangle_{\mathbf{q}} = q_w$, where w is the shortest element of W such that $\sigma \subseteq wZ$. Let $L^2_{\mathbf{q}}C^i(\mathcal{U})$ denote the completion of $C^i_c(\mathcal{U})$ with respect to this inner product.

As explained in [27], as well as in Section 7, $L^2_{\mathbf{q}}C^i(\mathcal{U})$ can be identified with a $\mathbf{R}_{\mathbf{q}}[W]$–stable subspace of $\oplus L^2_{\mathbf{q}}$. The coboundary maps are $\mathbf{R}_{\mathbf{q}}[W]$–equivariant. So, the (reduced) cohomology group $L^2_{\mathbf{q}}\mathcal{H}^i(\mathcal{U})$ is a closed $\mathbf{R}_{\mathbf{q}}[W]$–stable subspace of $\oplus L^2_{\mathbf{q}}$ and therefore, has a well-defined von Neumann dimension, $b^i_{\mathbf{q}}(\mathcal{U})$. The alternating sum of the $b^i_{\mathbf{q}}(\mathcal{U})$ is denoted $\chi_{\mathbf{q}}(\mathcal{U})$ and called the $L^2_{\mathbf{q}}$*–Euler characteristic* of $\mathcal{U}$. It is proved in [27] (and in Proposition 7.4) that $\chi_{\mathbf{q}}(\Sigma) = 1/W(\mathbf{q})$. (Recall $W(\mathbf{t})$ is a rational function.) Moreover, the Betti numbers $b^i_{\mathbf{q}}(\mathcal{U})$ are continuous functions of $\mathbf{q}$ (Theorem 7.7).

Let $\mathcal{R}$ denote the region of convergence of $W(\mathbf{t})$ and let

$$\mathcal{R}^{-1} := \{\mathbf{q} \mid \mathbf{q}^{-1} \in \mathcal{R}\},$$

where $\mathbf{q}^{-1} := (q_i^{-1})_{i\in I}$. The closures of these regions are denoted $\overline{\mathcal{R}}$ and $\overline{\mathcal{R}^{-1}}$, respectively. (When I is a singleton, we write q instead of $\mathbf{q}$ and t instead of $\mathbf{t}$. In this case, $W(t)$ is a power series in one variable. As such, it has a radius of convergence ρ and $\mathcal{R} = (0, \rho)$.)

The main result of this paper, Theorem 10.3, is a calculation of $L^2_{\mathbf{q}}\mathcal{H}^i(\mathcal{U})$ (as a $\mathcal{N}_{\mathbf{q}}$–module) for $\mathbf{q} \in \overline{\mathcal{R}} \cup \overline{\mathcal{R}^{-1}}$. It also gives a formula for the $b^i_{\mathbf{q}}(\mathcal{U})$ in this range of $\mathbf{q}$. Roughly speaking, the answer is that for $\mathbf{q} \in \overline{\mathcal{R}}$, $L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{U})$ looks like ordinary cohomology while for $\mathbf{q} \in \overline{\mathcal{R}^{-1}}$, it looks like cohomology with compact supports. Before stating the result precisely, we need to set up some notation and recall some background.

Given $T \subseteq S$, the subgroup W_T generated by T is called a *special subgroup*. It is also a Coxeter group. The subset T is *spherical* if W_T is finite. Let $\mathcal{S}$ denote the poset of spherical subsets of S. Given an element $w \in W$, set $\mathrm{In}(w) := \{s \in S \mid l(ws) < l(w)\}$, ie $\mathrm{In}(w)$ is the set of letters in S with which a reduced expression for w can end. It turns out that for any $w \in W$, $\mathrm{In}(w)$ is always a spherical subset of S. For each $T \in \mathcal{S}$, let $W^T := \{w \in W \mid \mathrm{In}(w) = T\}$ and let $\mathbf{Z}(W^T)$ denote the free abelian group of finitely supported functions on W^T. For any $U \subseteq S$, Z^U denotes the union of those mirrors Z_s, with $s \in U$.

(a) The homology of $\mathcal{U}$ is computed in Davis [13]. The answer is
$$H_*(\mathcal{U}) \cong \bigoplus_{T \in \mathcal{S}} H_*(Z, Z^T) \otimes \mathbf{Z}(W^T).$$
(This implies, in particular, that Σ is acyclic.) The answer for cohomology is similar, except that it is necessary to replace $\mathbf{Z}(W^T)$ by the abelian group of all functions $W^T \to \mathbf{Z}$.

(b) The cohomology with compact supports of $\mathcal{U}$ can be computed as in Davis [15]. The answer is
$$H^*_c(\mathcal{U}) \cong \bigoplus_{T \in \mathcal{S}} H^*(Z, Z^{S-T}) \otimes \mathbf{Z}(W^T).$$

Another proof of the formulas in (a) and (b) is given in our paper [18] or in Davis [17].

Given $U \subseteq S$, in Section 5, we define idempotents a_U and h_U in $\mathcal{N}_{\mathbf{q}}$ by
$$a_U := \frac{1}{W_U(\mathbf{q})} \sum_{w \in W_U} e_w,$$
$$h_U := \frac{1}{W_U(\mathbf{q}^{-1})} \sum_{w \in W_U} \varepsilon_w q_w^{-1} e_w,$$

where $\varepsilon_w := (-1)^{l(w)}$. These idempotents are defined provided $\mathbf{q} \in \mathcal{R}_U$ in the case of a_U and provided $\mathbf{q} \in \mathcal{R}_U^{-1}$ in the case of h_U. ($\mathcal{R}_U$ denotes the region of convergence for $W_U(\mathbf{t})$.) Let $A_U \subseteq L^2_{\mathbf{q}}$ stand for $\operatorname{Im} a_U$ if $\mathbf{q} \in \mathcal{R}_U$ and for the 0–space, otherwise. A_U is a closed $\mathbf{R}_{\mathbf{q}}[W]$–stable subspace of $L^2_{\mathbf{q}}$. Another closed $\mathbf{R}_{\mathbf{q}}[W]$–stable subspace is defined by

$$D_U := A_{S-U} \cap \Big(\sum_{V \subset U} A_{S-V} \Big)^{\perp}.$$

(Throughout this paper we will denote inclusion of a subset by $\subseteq$ and use $\subset$ for inclusion of a proper subset.)

Here is the precise statement of our calculation of $L^2_{\mathbf{q}}$–cohomology. (Compare it with statements (a) and (b) above.)

The Main Theorem (Theorem 10.3 in Section 10)

(a) *If* $\mathbf{q} \in \overline{\mathcal{R}}$, *then*

$$L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{U}) \cong \bigoplus_{T \in \mathcal{S}} H^*(Z, Z^T) \otimes D_T.$$

(b) *If* $\mathbf{q} \in \overline{\mathcal{R}^{-1}}$, *then*

$$L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{U}) \cong \bigoplus_{T \in \mathcal{S}} H^*(Z, Z^{S-T}) \otimes D_{S-T}.$$

(To compare this with the previous answers for ordinary cohomology and cohomology with compact supports, we note that, by Theorem 11.2, for $\mathbf{q} \in \mathcal{R}$, $\{e_w h_T a_{S-T}\}_{w \in W^T}$ spans a dense subspace of D_T; while for $\mathbf{q} \in \mathcal{R}^{-1}$, $\{e_w h_{S-T} a_T)\}_{w \in W^T}$ spans a dense subspace of D_{S-T}.)

The proof of the Main Theorem depends on the following result.

The Decomposition Theorem (Theorem 9.11 in Section 9)

(a) *If* $\mathbf{q} \in \overline{\mathcal{R}}$, *then*

$$\sum_{T \in \mathcal{S}} D_T$$

is a direct sum decomposition and a dense subspace of $L^2_{\mathbf{q}}$.

(b) *If* $\mathbf{q} \in \overline{\mathcal{R}^{-1}}$, *then*

$$\sum_{T \in \mathcal{S}} D_{S-T}$$

is a direct sum decomposition and a dense subspace of $L^2_{\mathbf{q}}$.

In the case when W is finite and $\mathbf{q} = \mathbf{1}$ (ie when the Hecke algebra is the group algebra) a similar result was proved by Solomon [39] in 1968. In Section 11 we give a version of the Decomposition Theorem (namely, Theorem 11.1) which is more transparently a generalization of Solomon's Theorem than the version stated above. The Decomposition Theorem is also compatible with the theory of representations of Hecke algebras developed by Kazhdan–Lusztig [19] (cf Remark 11.4).

Although the Main Theorem is a consequence of the Decomposition Theorem, our proof of the Decomposition Theorem ultimately is based on a special case of the Main Theorem from [27]. The result of [27] states that, for $\mathbf{q} \in \mathcal{R}$, the $L^2_{\mathbf{q}}$–homology of Σ vanishes except in dimension 0. (NB To calculate homology, $L^2_{\mathbf{q}}\mathcal{H}_*(\Sigma)$, from $L^2_{\mathbf{q}}C_*(\Sigma)$ one does not use the usual boundary map but rather, the adjoint of the usual coboundary map.) In Section 8 we apply this vanishing result to show that, for $\mathbf{q} \in \mathcal{R}$, the relative $L^2_{\mathbf{q}}$–homology of certain pairs of subcomplexes of Σ vanishes except in the bottom dimension. (These pairs of subcomplexes are dubbed "ruins" in Section 6.) For $\mathbf{q} \in \mathcal{R}$, these vanishing results are essentially an equivalent version of the Decomposition Theorem. One then uses a certain isomorphism $j\colon \mathcal{N}_{\mathbf{q}} \to \mathcal{N}_{\mathbf{q}^{-1}}$ to convert the statement of Decomposition Theorem for $\mathbf{q} \in \overline{\mathcal{R}}$ into its statement for $\mathbf{q} \in \overline{\mathcal{R}^{-1}}$.

The key role played by the case $\mathbf{q} \in \mathcal{R}$ in this sketch of the proof is probably the most compelling reason for studying weighted L^2–cohomology with $\mathbf{q}$ an I–tuple of arbitrary positive real numbers. When W is infinite, the vector $\mathbf{q} \in \mathcal{R}$ never has all its components equal to positive integers. So, on the face of it, the case $\mathbf{q} \in \mathcal{R}$ of the Main Theorem would never seem to be applicable to nonspherical buildings. However, because of various dualities (such as the j–isomorphism) which switch $\mathbf{q}$ with $\mathbf{q}^{-1}$, the results for $\mathbf{q} \in \mathcal{R}$ are equivalent to results for $\mathbf{q} \in \mathcal{R}^{-1}$ and these are applicable to buildings.

For $\mathbf{q} \in \overline{\mathcal{R}} \cup \overline{\mathcal{R}^{-1}}$, the Main Theorem (in particular, its version as Theorem 10.4) gives a complete calculation of $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$. On the other hand, our knowledge about what happens for $\mathbf{q} \notin \overline{\mathcal{R}} \cup \overline{\mathcal{R}^{-1}}$ is fragmentary. For example, suppose Σ is an n–manifold. Then by the Main Theorem, $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$ is concentrated in dimension 0 for $\mathbf{q} \in \overline{\mathcal{R}}$ and in dimension n for $\mathbf{q} \in \overline{\mathcal{R}^{-1}}$. We note that when Σ is a manifold (without boundary), W is infinite and so, $\mathbf{1} \notin \mathcal{R} \cup \mathcal{R}^{-1}$. When $\mathbf{q} = \mathbf{1}$, a version of the Singer Conjecture asserts that the weighted L^2–cohomology of Σ vanishes except in dimension $\frac{n}{2}$. (When n is odd, this is to be interpreted as meaning that $L^2_{\mathbf{1}}\mathcal{H}^*(\Sigma)$ vanishes in all dimensions.) In [21] the first and fourth authors explained some evidence for this conjecture. For a general $\mathbf{q}$, in the case where Σ is a n–manifold, there is a version of Poincaré duality which exchanges $\mathbf{q}$ with $\mathbf{q}^{-1}$ (as well as dimension k with dimension $n-k$); see Dymara [26] or Proposition 14.1. So, when Σ is a manifold, knowledge of

$L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$ for $\mathbf{q} \le \mathbf{1}$ also determines it for $\mathbf{q} > \mathbf{1}$. In Section 14 we explain that the right generalization of this version of the Singer Conjecture for $\mathbf{q} = \mathbf{1}$ is the following.

Conjecture (Conjecture 14.7) *Suppose Σ is an n–manifold. Then*

$$L^2_{\mathbf{q}}\mathcal{H}^k(\Sigma) = 0 \text{ for } k > \frac{n}{2} \quad \text{and} \quad \mathbf{q} \le \mathbf{1}.$$

In Section 16, by modifying the arguments of the first and fourth authors [21], we prove it in the case where W is right-angled and $n \le 4$. In the same section, we give examples where Σ is a 4–manifold and where for certain $\mathbf{q} \notin \overline{\mathcal{R}} \cup \overline{\mathcal{R}^{-1}}$, the $L^2_{\mathbf{q}}$–cohomology fails to be concentrated in a single dimension (it is nonzero in both dimension 2 and 3.)

Next, we make a few remarks concerning buildings. Buildings come in different types, where the "type" of a building is a Coxeter system (W, S). In the case of a classical building associated to an algebraic group, its type is always a spherical or Euclidean reflection group. The simplest example of a Euclidean reflection group is when W is the infinite dihedral group acting on the real line and S consists of the two reflections about the endpoints of a fundamental interval. A building of this type is a tree. (See Example 13.1 for more details.) Many other types for buildings are possible. Most of our interest in this paper lies with these nonclassical types.

Roughly speaking, a building of type (W, S) consists of a set Φ of "chambers" and a family, indexed by S, of "adjacency relations" on Φ. An example of a building is W itself – the adjacency relation corresponding to $s \in S$ is defined by calling two distinct elements of W *s–adjacent* if they form to the same coset of $W_{\{s\}}$.

To define the "geometric realization" X of a building, one first declares the geometric realization of any chamber to be isomorphic to the fundamental chamber K of Σ. One then amalgamates copies of K, one for each element of Φ, by gluing together chambers corresponding to s–adjacent elements of Φ along the mirror corresponding to s. Details of this construction can be found in [14], as well as in Section 13. (NB When W is an irreducible Euclidean reflection group, K is a simplex and X has the structure of a simplicial complex in which the top-dimensional simplices are the chambers; however, in the general case, this is not the correct picture of the geometric realization of a building.)

A group G of automorphisms of a building is *chamber transitive* if it acts transitively on Φ. If the building admits a chamber transitive automorphism group, then, for any given $\varphi_0 \in \Phi$, the number of chambers which are s–adjacent to φ_0 is independent of the choice of φ_0. We denote this number by q_s, ie

$$q_s = \mathrm{Card}\{\varphi \in \Phi \mid \varphi \text{ is } s\text{–adjacent to } \varphi_0 \text{ and } \varphi \neq \varphi_0\}.$$

Moreover, if s and s' are conjugate in W, then $q_s = q_{s'}$. We assume throughout that the building has finite thickness, ie that each q_s is finite. We then get a well-defined I–tuple of integers $\mathbf{q} := (q_i)_{i \in I}$, called the *thickness vector* of the building, where I is the set of conjugacy classes in S and where $q_i := q_s$ for any representative s for i. For example, the thickness vector of W (considered as a building) is $\mathbf{1}$.

How do Hecke algebras arise in the theory of buildings? Let Φ be a building of finite thickness with a chamber transitive automorphism group G and with thickness vector $\mathbf{q}$. Fix a base chamber $\varphi_0 \in \Phi$. Using the "W–distance" from φ_0, one gets a retraction $r\colon \Phi \to W$. Let $C_c(\Phi)$ denote the space of finitely supported, real-valued functions on Φ. It is an algebra with product given by convolution. Consider the subspace $J \subseteq C_c(\Phi)$ consisting of those functions which are constant on the fibers of r. It is a subalgebra. As a vector space, J can be identified with $\mathbf{R}^{(W)}$; however, the product is not the usual one for the group algebra. As the reader has probably guessed, J is identified with the Hecke algebra $\mathbf{R}_{\mathbf{q}}[W]$, where the multiparameter $\mathbf{q}$ is the thickness of Φ.

Let X denote the geometric realization of the building Φ. The retraction $r\colon \Phi \to W$ induces a topological retraction $X \to \Sigma$, which we continue to denote by r. This induces an inclusion $r^*\colon C_c^*(\Sigma) \to C_c^*(X)$ of (finitely supported) cellular cochains. The standard inner product on $C_c^*(X)$ restricts to the inner product $\langle\ ,\ \rangle_{\mathbf{q}}$ on $C_c^*(\Sigma)$. In this way, $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$ is identified with a subspace of $\mathcal{H}^*(X)$, the ordinary reduced L^2–cohomology of X.

Since Φ has finite thickness, G is locally compact and hence, has a Haar measure μ, which we normalize by the condition, $\mu(B) = 1$, where B denotes the stabilizer of a chamber. Given μ, we have the Hilbert space $L^2(G, \mu)$ of square integrable functions on G and a von Neumann algebra $\mathcal{N}(G)$. Since $\mathcal{H}^i(X)$ is an $\mathcal{N}(G)$–module, it has a "dimension" with respect to $\mathcal{N}(G)$. This number is called the *i–th L^2–Betti number* and denoted $b^i(X; G)$. It is proved in [27] (under slightly stronger hypotheses), as well as in Theorem 13.8 of Section 13, that $b^i(X; G) = b^i_{\mathbf{q}}(\Sigma)$.

In [28] the second and third authors calculated $\mathcal{H}^*(X)$ under the assumption that the thickness vector $\mathbf{q}$ is very large. The result of [28] is similar to statement (b) of our Main Theorem: it says that for $\mathbf{q} \gg \mathbf{1}$,

$$\mathcal{H}^*(X) \cong \bigoplus_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes \widehat{D}_{S-T}.$$

where $\widehat{D}_{S-T}$ is a specific subrepresentation of $L^2(G/B)$ analogous to the subspace $D_{S-T} \subset L^2_{\mathbf{q}}$. (The notation in [28] is different.) In Corollary 13.11 of Section 13 we use our Main Theorem to show that, in fact, this formula is valid for all $\mathbf{q} \in \overline{\mathcal{R}^{-1}}$.

If W is a Euclidean reflection group, then the radius of convergence of $W(t)$ is 1 (cf Proposition 3.10). It follows that $\mathbf{q} \in \overline{\mathcal{R}^{-1}}$ whenever $\mathbf{q} \geq \mathbf{1}$. From this we deduce the following (known) result.

Theorem (Corollary 14.6 in Section 14) *Suppose X is an affine building (ie of Euclidean type) and that its automorphism group is chamber transitive. Then $\mathcal{H}^*(X)$ is concentrated in the top dimension.*

We want to emphasize three points:

(i) There are many interesting classes of buildings which are neither spherical nor affine.

(ii) The rational function $W(\mathbf{q})$ is completely explicit (cf Lemma 3.3) and is easily calculated in any given case.

(iii) In practice, calculation shows few I–tuples of positive integers lie outside $\mathcal{R}^{-1}$.

With regard to (i), when W is right-angled, there is a building Φ of type (W,S) and thickness $\mathbf{q}$ for any I–tuple of positive integers $\mathbf{q}$ (cf Example 13.3). When (W,S) satisfies the "crystallographic condition" (that all $m_{ij} = 2, 3, 4, 6$ or ∞), Tits proved the existence of "Kac–Moody groups" over finite fields. These give locally finite buildings with chamber transitive automorphism groups. (In this Kac–Moody case, all q_s are restricted to be the order of a finite field.)

With regard to (iii), consider the following concrete example. Suppose W is the group generated by the reflections across the faces of a right-angled dodecahedron in hyperbolic 3–space. Let I be a singleton and $\mathbf{q} = q$, a positive integer. Suppose X_q is the (dodecahedral) building of type W and thickness q. If $q \geq 7$, our results give that $\mathcal{H}^*(X_q)$ is concentrated in the top dimension ($= 3$). Previously, this was only known for $q > 10^{60}$ (cf [28]). On the other hand, by Theorem 16.12, for $2 \leq q \leq 6$, $\mathcal{H}^*(X_q)$ is concentrated in dimension 2 (and for $q = 1$ it vanishes identically).

The results of this paper raise more questions than they answer. Here are two such:

- Is there a version of this theory for weighted differential forms?
- Is there a version of this for groups other than Coxeter groups?

The short answer to both is "yes." In both cases a good deal of foundational work remains to be done.

As for the first question, there exists a literature on weighted L^2 de Rham cohomology on a Riemannian manifold M, for example, Bueler [8]. The inner product on the vector space of compactly supported, smooth forms on M is modified via a weight function of the form $x \to q^{d(x)}$, where $q \in (0,\infty)$, $x \in M$ and $d(x)$ is the distance from a basepoint. As one would expect, when $M = \mathbf{R}^n$, the weighted L^2–cohomology is concentrated in dimension 0 if $q < 1$ and in dimension n if $q > 1$. In Section 16, we use a version of this weighted de Rham theory on hyperbolic space equipped with an isometric action of a group W generated by reflections across the faces of a fundamental polytope K. This time the weight function is a step function of the form $x \to q^{l(w)}$, where $w \in W$ is such that $x \in wK$. In this case, the de Rham version and the cellular version of weighted L^2–cohomology are canonically isomorphic.

As for the second question, given a discrete group Γ, a CW complex X equipped with a cellular Γ–action and a positive real number q, one can deform the standard inner product on $C^*_c(X)$ via a weight function of the form $\gamma \to q^{l(\gamma)}$ and then define the weighted L^2 (cellular) cohomology groups of X. As before, as q varies from 0 to ∞, these groups interpolate between ordinary cohomology and cohomology with compact supports. The missing feature for a general group Γ (as opposed to a Coxeter group) is that we do not have a deformation of the group algebra analogous to the Hecke algebra. We will say more about this question in Section 18. We believe that this topic also has an intrinsic interest and we hope to write more about it in the future.

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2 Coxeter systems

A *Coxeter matrix* on a set S is an $S \times S$ symmetric matrix $M = (m_{st})$ with entries in $\mathbb{N} \cup \{\infty\}$ such that each diagonal entry is 1 and each off-diagonal entry is ≥ 2. The matrix M gives a presentation for an associated *Coxeter group* W: the set of generators is S and there is a relation

$$(st)^{m_{st}} = 1,$$

for each pair (s,t) of elements in S with $m_{st} \neq \infty$. The purpose of this section is to recall some standard facts about such groups. Proofs of most of these facts can be found in Bourbaki [4].

The natural map $S \to W$ is injective and henceforth, we identify S with its image in W. Moreover, each element of S has order 2 in W and the order of st in W is m_{st}. The pair (W,S) is a *Coxeter system*.

Given an element $w \in W$, $l(w)$ denotes its word length. An expression for w as a word in S, $w = s_1 \cdots s_l$, is a *reduced expression* if $l = l(w)$.

Given $T \subseteq S$, W_T denotes the subgroup generated by T. Such a W_T is a *special subgroup* of W. The pair (W_T, T) is the Coxeter system whose Coxeter matrix is given by the restriction of M to T [4, Theorem 2 (i), p 12]. The subset T is *spherical* if W_T is finite.

For $T \subseteq S$ and $w \in W$, the coset wW_T contains a unique element of minimum length. This element is said to be $(\varnothing, T)$*–reduced*. Moreover, an element w is $(\varnothing, T)$–reduced if and only if $l(wt) > l(w)$ for all $t \in T$. (See [4, Example 3, pp 31–32].) Let X_T denote the set of $(\varnothing, T)$–reduced elements of W.

If W_T is finite, then it contains a unique element w_T of maximum length, called the *element of longest length*. It is characterized by the property that $l(w_T t) < l(w_T)$ for all $t \in T$ [4, Example 22, p 40].

Given $w \in W$, set $\mathrm{In}\, w := \{s \in S \mid l(ws) < l(w)\}$. It follows from the "Exchange Condition" (cf [4, p 7]) that $s \in \mathrm{In}\, w$ if and only if w has a reduced expression with final letter s. Thus, $\mathrm{In}\, w$ is the set of letters with which a reduced expression for w can end. A key fact [12, Lemma 7.12] is that $\mathrm{In}\, w$ is always a spherical subset of S.

For any spherical subset $T \subseteq S$, define

$$W^T := \{w \in W \mid \mathrm{In}\, w = T\}. \tag{2–1}$$

The simplicial complex Σ Given a poset $\mathcal{P}$ and an element $p \in \mathcal{P}$, define the subposet $\mathcal{P}_{>p} := \{x \in \mathcal{P} \mid x > p\}$. Subposets $\mathcal{P}_{<p}$, $\mathcal{P}_{\geq p}$ and $\mathcal{P}_{\leq p}$ are defined similarly. Associated to any poset $\mathcal{P}$ there is a simplicial complex $|\mathcal{P}|$, called its *geometric realization*; its vertex set is $\mathcal{P}$ and a nonempty finite subset of $\mathcal{P}$ spans a simplex if and only if it is totally ordered.

Let $\mathcal{S}$ denote the set of spherical subsets of S, partially ordered by inclusion and let

$$\mathcal{S}^{(i)} = \{T \in \mathcal{S} \mid \mathrm{Card}(T) = i\}. \tag{2–2}$$

$\mathcal{S}$ has a minimum element, namely, $\varnothing$. $\mathcal{S}_{>\varnothing}$ is the poset of simplices of a simplicial complex denoted by $L(W, S)$ (or L for short) and called the *nerve* of (W, S). (In other words, the vertex set of L is S and a nonempty subset $T \subseteq S$ spans a simplex if and only if it is spherical.) $\mathcal{S}^{(i)}$ is the set of $(i-1)$–simplices in L.

We are also interested in $W\mathcal{S}$, the *poset of spherical cosets*. It is defined as the disjoint union of the sets W/W_T, $T \in \mathcal{S}$. Thus, a typical element of $W\mathcal{S}$ is a coset wW_T for some $T \in \mathcal{S}$. The partial order is inclusion.

The geometric realization of $\mathcal{S}$ is denoted K and the geometric realization of $W\mathcal{S}$ by Σ. The group W acts properly on the simplicial complex Σ; the orbit space is the finite complex K. The most important property of Σ is that it is contractible [12, Theorem 10.3 and Section 14].

3 Growth series

Suppose we are given a Coxeter system (W, S), an index set I and a function $i\colon S \to I$ so that $i(s) = i(s')$ whenever s and s' are conjugate in W. (The largest possible choice for $\operatorname{Im} i$ is the set of conjugacy classes of elements in S and the smallest possible choice is a singleton.) Let $\mathbf{t} = (t_i)_{i\in I}$ stand for an I–tuple of indeterminates. Write t_s for $t_{i(s)}$. If $s_1 \cdots s_l$ is a reduced expression for w, then define t_w to be the monomial $t_w := t_{s_1} \cdots t_{s_l}$. It follows from Tits' solution of the word problem for Coxeter groups (see Tits [41] or Brown [7]) that t_w is independent of the choice of reduced expression for w.

For any subset X of W, define a power series in $\mathbf{t}$

$$X(\mathbf{t}) := \sum_{w\in X} t_w. \tag{3–1}$$

$W(\mathbf{t})$ is the *growth series* of W and, for any subset T of S, $W_T(\mathbf{t})$ is the growth series of the special subgroup W_T.

Notation The region of convergence of $W_T(\mathbf{t})$ in $(0, +\infty)^I$ is denoted $\mathcal{R}_T$. Write $\mathcal{R}$ instead of $\mathcal{R}_S$. Put $\mathcal{R}_T^{-1} := \{\mathbf{z} \in (0, +\infty)^I \mid \mathbf{z}^{-1} \in \mathcal{R}_T\}$. Denote the closure of the region of convergence by $\overline{\mathcal{R}}$ and put $\partial\mathcal{R} := \overline{\mathcal{R}} - \mathcal{R}$. Define $\overline{\mathcal{R}^{-1}}$ and $\partial\mathcal{R}^{-1}$ similarly.

From the fact that all the coefficients in $W(\mathbf{t})$ are nonnegative real numbers, we immediately get the following lemmas.

Lemma 3.1 *If $U \subseteq T \subseteq S$, then $\mathcal{R} \subseteq \mathcal{R}_T \subseteq \mathcal{R}_U$.*

Lemma 3.2 *Suppose $\mathbf{q} \in (0, \infty)^I$. Then the following two conditions are equivalent:*

(a) $\mathbf{q} \in \partial\mathcal{R}$.
(b) $1/W(\mathbf{q}) = 0$ *and* $1/W(\lambda\mathbf{q}) > 0$ *for all* $\lambda \in (0, 1)$.

Note that if T is spherical, then $W_T(\mathbf{t})$ is a polynomial in $\mathbf{t}$ and so $\mathcal{R}_T = (0, +\infty)^I$. If, for each $i \in I$, $t_i \geq 1$, then $\mathbf{t} \notin \mathcal{R}_T$ whenever W_T is infinite.

Define $\varepsilon(T) := (-1)^{\operatorname{Card}(T)}$.

Lemma 3.3 (Bourbaki [4, Example 26, pp 42–43], Serre [38, Proposition 26] and Steinberg [40])

(i) *Suppose W $(= W_S)$ is finite and let $t_S = t_{w_S}$ be the monomial corresponding to the element of longest length in W. Then*

(a) $$W(\mathbf{t}) = t_S W(\mathbf{t}^{-1}).$$

(b) $$\frac{t_S}{W(\mathbf{t})} = \sum_{T \subseteq S} \frac{\varepsilon(T)}{W_T(\mathbf{t})}.$$

(ii) *As in Section 2, for each $T \subseteq S$, suppose X_T denotes the set of $(\varnothing, T)$–reduced elements in W. Then*

$$W(\mathbf{t}) = X_T(\mathbf{t}) W_T(\mathbf{t}).$$

(iii) *As in* (2–1)*, for each spherical subset T of S, suppose W^T denotes the set of $w \in W$ with $\mathrm{In}(w) = T$. Then*

(a) $$\frac{W^T(\mathbf{t})}{W(\mathbf{t})} = \sum_{T' \subseteq T} \frac{\varepsilon(T - T')}{W_{S-T'}(\mathbf{t})}.$$

(b) $$\frac{W^T(\mathbf{t})}{W(\mathbf{t})} = \sum_{U \in \mathcal{S}_{\geq T}} \frac{\varepsilon(U - T)}{W_U(\mathbf{t}^{-1})}.$$

(iv) $$\frac{1}{W(\mathbf{t}^{-1})} = \sum_{T \in \mathcal{S}} \frac{\varepsilon(T)}{W_T(\mathbf{t})}.$$

Corollary 3.4 *For polynomials $f, g \in \mathbf{Z}[\mathbf{t}]$ with integral coefficients,*

$$W(\mathbf{t}) = \frac{f(\mathbf{t})}{g(\mathbf{t})}.$$

The next lemma follows immediately from the definitions.

Lemma 3.5 *Suppose (W, S) decomposes as a product $(W_1 \times W_2, S_1 \cup S_2)$. Then $W(\mathbf{t}_1, \mathbf{t}_2) = W_1(\mathbf{t}_1) W_2(\mathbf{t}_2)$. Moreover, $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$, where $\mathcal{R}$, $\mathcal{R}_1$ and $\mathcal{R}_2$ are the regions of convergence for $W(\mathbf{t}_1, \mathbf{t}_2)$, $W_1(\mathbf{t}_1)$ and $W_2(\mathbf{t}_2)$, respectively.*

Example 3.6 (The infinite dihedral group) Suppose $S = \{s_1, s_2\}$ and $m_{s_1 s_2} = \infty$, so that W is the infinite dihedral group D_∞. Its nerve is the 0–sphere. Also, suppose $I = \{1, 2\}$ and that $S \to I$ sends s_j to j. Using Lemma 3.3(iv), we compute:

$$\frac{1}{W(\mathbf{t})} = \frac{1 - t_1 t_2}{(1+t_1)(1+t_2)}.$$

So, $\mathcal{R} = \{(z_1, z_2) \mid |z_1||z_2| < 1\}$. In particular, $(0,1)^2 \subset \mathcal{R}$.

Example 3.7 Suppose $W = (D_\infty)^n$, the n–fold product of infinite dihedral groups. Its nerve L is then the n–fold join of copies of S^0, ie it is the boundary complex of an n–octahedron. By Lemma 3.5 and Example 3.6, $(0,1)^I \subset \mathcal{R}$.

The case of a single indeterminate Suppose I is a singleton. Then $\mathbf{t}$ is a single indeterminate, call it t, the monomial t_w is just $t^{l(w)}$ and $W(t)$ is the usual growth series. Let ρ denote its radius of convergence. An immediate corollary to Lemma 3.2 is the following.

Corollary 3.8 *$1/W(\rho) = 0$ and $\rho = \min\{|t| \mid t \in \mathbf{C}$ and $1/W(t) = 0\}$.*

A corollary to Lemma 3.5 is the following.

Corollary 3.9 *Suppose (W, S) decomposes as a product $(W_1 \times W_2, S_1 \cup S_2)$. Then $W(t) = W_1(t)W_2(t)$ and $\rho = \min(\rho_1, \rho_2)$, where ρ, ρ_1 and ρ_2 are the radii of convergence for $W(t)$, $W_1(t)$ and $W_2(t)$, respectively.*

In the next proposition we list six other conditions which are equivalent to the condition that the radius of convergence of $W(t)$ be 1.

Proposition 3.10 *The following conditions on a Coxeter system (W, S) are equivalent.*

(i) *W is amenable.*

(ii) *W does not contain a free group on two generators.*

(iii) *W does not virtually map onto the free group on two generators F_2 (ie W does not have a finite index subgroup Γ which maps onto F_2).*

(iv) *W is virtually abelian.*

(v) *(W, S) decomposes as $(W_0 \times W_1, S_0 \cup S_1)$ where W_1 is finite and W_0 is a cocompact Euclidean reflection group.*

(vi) $\rho = 1$.

(vii) W has subexponential growth.

Proof (i) $\Longrightarrow$ (ii) This is a standard fact.

(ii) $\Longrightarrow$ (iii) Suppose for some subgroup Γ of W we have a surjection $f\colon \Gamma \to F_2$ where F_2 is the free group on $\{x_1, x_2\}$. Choose $\gamma_1 \in f^{-1}(x_1)$, $\gamma_2 \in f^{-1}(x_2)$. Then $\langle \gamma_1, \gamma_2 \rangle$ is a free subgroup of W.

(iii) $\Longrightarrow$ (iv) It is proved by Margulis and Vinberg [35] (and independently by Gonciulea [31]) that when W is not virtually abelian there is a subgroup Γ of finite index in W which maps onto a nonabelian free group.

(iv) $\Longrightarrow$ (v) Moussong [36] proved Σ has a $CAT(0)$ metric (so W is a "$CAT(0)$ group"). This implies that any abelian subgroup of W is finitely generated. So, if W is virtually abelian, then it is virtually free abelian. We suppose that has W has a rank n free abelian subgroup of finite index. Then W is a virtual PD^n–group. By [13, Theorem B], W decomposes as in (v), where the complex Σ_0 for (W_0, S_0) is a $CAT(0)$ homology n–manifold. By the Flat Torus Theorem [5], the "min set" of the free abelian subgroup on Σ_0 is isometric to $\mathbf{R}^n$. Hence, $\Sigma_0 = \mathbf{R}^n$ and W_0 acts as an isometric reflection group on it.

(v) $\Longrightarrow$ (vi) Since a Euclidean reflection group is virtually free abelian, it has polynomial growth and therefore, the radius of convergence of its growth series is 1. (In fact, the poles of its growth series are all roots of unity; see Remark 3.11 below.)

(vi) $\Longrightarrow$ (vii) This is obvious.

(vii) $\Longrightarrow$ (i) This follows by the Følner condition for amenability. □

Remark 3.11 Suppose W is a (cocompact) Euclidean reflection group. First consider the case where (W, S) is irreducible. Let W' be the finite linear reflection group obtained by quotienting out the translation subgroup of W and let $m_1, \dots, m_n$ be the exponents of W'. According to [4, Example 10, p 245], the growth series of W is given by the following formula of Bott:

$$W(t) = \prod_{i=1}^{n} \frac{1 + t + \cdots + t^{m_i}}{1 - t^{m_i}}.$$

In particular, all the poles of $W(t)$ are roots of unity. We can reach the same conclusion without the assumption of irreducibility, since the growth series of (W, S) is the product of the growth series of its irreducible factors.

Note In the case where $\mathbf{t}$ is a single indeterminate, most of the results of this section come from [4, Exercise 26, pp 42–43]. The idea of extending the results from this exercise to an I–tuple of indeterminates comes from [38]. Lemma 3.3(iii)(a) is from [4, Exercise 26 d), p 43], while (iii)(b) is due to Steinberg [40].

4 Hecke algebras

Let A be a commutative ring with unit. Denote by $A^{(W)}$ the free A–module on W consisting of all finitely supported functions $W \to A$ and denote by $A[W]$ this A–module equipped with its structure as the group ring of W. Let $(e_w)_{w \in W}$ be the standard basis for $A^{(W)}$. We are primarily interested in the case where $A = \mathbf{R}$, the field of real numbers.

As in the previous section, $i\colon S \to I$ is a function such that $i(s) = i(s')$ whenever s and s' are conjugate. Let $\mathbf{q} = (q_i)_{i \in I} \in A^I$ be a fixed I–tuple. As before, write q_s for $q_{i(s)}$. By [4, Exercise 23, p 57], there is a unique ring structure on $A^{(W)}$ such that

$$e_s e_w = \begin{cases} e_{sw}, & \text{if } l(sw) > l(w), \\ q_s e_{sw} + (q_s - 1)e_w, & \text{if } l(sw) < l(w), \end{cases} \tag{4–1}$$

for all $w \in W$. We will use the notation $A_{\mathbf{q}}[W]$ to denote $A^{(W)}$ with this ring structure. Note that if $\mathbf{q}$ is the constant I–tuple $\mathbf{1} := (1, \dots, 1)$, then $A_{\mathbf{q}}[W] = A[W]$. So, $A_{\mathbf{q}}[W]$ is a deformation of the group ring. It is called the *Hecke algebra* of W associated to the *multiparameter* $\mathbf{q}$.

From (4–1) it follows that

$$e_u e_v = e_{uv}, \qquad \text{for all } u, v \in W \text{ with } l(uv) = l(u) + l(v),$$

and

$$e_s^2 = (q_s - 1)e_s + q_s.$$

The function $e_w \to e_{w^{-1}}$ induces a linear involution $*$ of $A^{(W)}$, ie

$$\Big(\sum a_w e_w\Big)^* := \sum a_{w^{-1}} e_w. \tag{4–2}$$

Lemma 4.1 *Formula* (4–2) *defines an anti-involution of the ring* $A_{\mathbf{q}}[W]$*. In other words, for all* $x, y \in A_{\mathbf{q}}[W]$, $(xy)^* = y^* x^*$.

Proof For each $w \in W$, let L_w (resp. R_w) denote left (resp. right) translation by e_w defined by $L_w(x) = e_w x$ (resp. $R_w(x) = x e_w$). A quick calculation using (4–1) gives: $R_s = *L_s*$, for all $s \in S$. If $s_1 \cdots s_l$ is a reduced expression for w, then $R_w = R_{s_l} \cdots R_{s_1} = *L_{s_l} \cdots L_{s_1} * = *L_{w^{-1}}*$. Therefore, $xe_w = R_w(x) = *L_{w^{-1}} * (x) = (e_{w^{-1}} x^*)^*$. Hence, $R_w = *L_{w^{-1}}*$, for all $w \in W$. So, $(xe_w)^* = (e_{w^{-1}} x^*)^{**} = e_{w^{-1}} x^* = e_w^* x^*$. The lemma follows. □

Using the involution $*$, we deduce the following right hand version of (4–1):

$$e_w e_s = \begin{cases} e_{ws}, & \text{if } l(ws) > l(w), \\ q_s e_{ws} + (q_s - 1) e_w, & \text{if } l(ws) < l(w). \end{cases} \tag{4–3}$$

Proof of (4–3) Apply $*$, to get

$$\begin{aligned} (e_w e_s)^* &= e_s e_{w^{-1}} \\ &= \begin{cases} e_{sw^{-1}}, & \text{if } l(sw^{-1}) > l(w^{-1}), \\ q_s e_{sw^{-1}} + (q_s - 1) e_{w^{-1}}, & \text{if } l(sw^{-1}) < l(w^{-1}). \end{cases} \end{aligned}$$

Hence,
$$\begin{aligned} e_w e_s &= (e_w e_s)^{**} \\ &= \begin{cases} e_{ws}, & \text{if } l(sw) > l(w), \\ q_s e_{ws} + (q_s - 1) e_w, & \text{if } l(sw) < l(w). \end{cases} \end{aligned}$$
□

For each $w \in W$, define q_w by the same formula used to define t_w, ie if $s_1 \cdots s_l$ is a reduced expression for w, then

$$q_w := q_{s_1} \cdots q_{s_l}. \tag{4–4}$$

Also, set

$$\varepsilon_w := (-1)^{l(w)}. \tag{4–5}$$

The maps $e_w \to q_w$ and $e_w \to \varepsilon_w$ extend linearly to ring homomorphisms $A_{\mathbf{q}}[W] \to A$.

Following Kazhdan–Lusztig [32], define an isomorphism $j_{\mathbf{q}}\colon A_{\mathbf{q}}[W] \to A_{\mathbf{q}^{-1}}[W]$ by the formula:

$$j_{\mathbf{q}}(e_w) := \varepsilon_w q_w e_w. \tag{4–6}$$

It is easily checked that $j_{\mathbf{q}}$ is an algebra homomorphism and that $(j_{\mathbf{q}})^{-1} = j_{\mathbf{q}^{-1}}$. Hence, $j_{\mathbf{q}}$ is an isomorphism of Hecke algebras. It is called the *j–isomorphism* and denoted simply by j when there is no ambiguity.

Note Most of the material in this section is taken from [4, Exercise 23, p 57].

5 Hecke–von Neumann algebras

From now on A is the field of real numbers $\mathbf{R}$ and $\mathbf{q} = (q_i)_{i \in I}$ is an I–tuple of positive reals. Define an inner product $\langle\ ,\ \rangle_{\mathbf{q}}$ on $\mathbf{R}^{(W)}$ $(= \mathbf{R}_{\mathbf{q}}[W])$ by

$$\Big\langle \sum a_w e_w, \sum b_w e_w \Big\rangle_{\mathbf{q}} := \sum a_w b_w q_w, \tag{5–1}$$

where q_w was defined by (4–4). As in [34], sometimes it is convenient to normalize $(e_w)_{w \in W}$ to an orthonormal basis for $\mathbf{R}_{\mathbf{q}}[W]$ by setting

$$\widetilde{e}_w := q_w^{-1/2} e_w. \tag{5–2}$$

The completion of $\mathbf{R}^{(W)}$ with respect to the inner product $\langle\ ,\ \rangle_{\mathbf{q}}$ is denoted $L^2_{\mathbf{q}}(W)$, or simply $L^2_{\mathbf{q}}$, when there is no ambiguity.

Proposition 5.1 [27, Proposition 2.1] *The inner product defined by* (5–1)*, multiplication defined by equations* (4–1) *and the anti-involution* $*$ *defined by* (4–2)*, give* $\mathbf{R}_{\mathbf{q}}[W]$ *a Hilbert algebra structure in the sense of [23, A.54]. This means, in particular, that*

(i) $(xy)^* = y^* x^*$,

(ii) $\langle x, y\rangle_{\mathbf{q}} = \langle y^*, x^*\rangle_{\mathbf{q}}$,

(iii) $\langle xy, z\rangle_{\mathbf{q}} = \langle y, x^* z\rangle_{\mathbf{q}}$,

(iv) *for any* $x \in \mathbf{R}_{\mathbf{q}}[W]$*, left translation by* x*,* $L_x\colon \mathbf{R}_{\mathbf{q}}[W] \to \mathbf{R}_{\mathbf{q}}[W]$*, defined by* $L_x(y) = xy$*, is continuous,*

(v) *the products* xy *over all* $x, y \in \mathbf{R}_{\mathbf{q}}[W]$ *are dense in* $\mathbf{R}_{\mathbf{q}}[W]$.

Since the action of $\mathbf{R}_{\mathbf{q}}[W]$ on itself by multiplication is continuous, $L^2_{\mathbf{q}}$ is a $\mathbf{R}_{\mathbf{q}}[W]$–bimodule.

An element $x \in L^2_{\mathbf{q}}$ is *bounded* if right multiplication by x is bounded on $\mathbf{R}_{\mathbf{q}}[W]$ (or equivalently, if left multiplication by x is bounded). Let $\mathbf{R}^b_{\mathbf{q}}[W]$ be the set of all bounded elements.

As in [23] there are two von Neumann algebras associated with this situation. They are denoted by $\mathcal{N}_{\mathbf{q}}[W]$ and $\mathcal{N}'_{\mathbf{q}}[W]$ or simply by $\mathcal{N}_{\mathbf{q}}$ and $\mathcal{N}'_{\mathbf{q}}$ when there is no ambiguity. $\mathcal{N}_{\mathbf{q}}$ acts from the right on $L^2_{\mathbf{q}}$ and $\mathcal{N}'_{\mathbf{q}}$ from the left. Here are two equivalent definitions of $\mathcal{N}_{\mathbf{q}}$:

(i) $\mathcal{N}_{\mathbf{q}}$ is the algebra of all bounded linear endomorphisms of $L^2_{\mathbf{q}}$ which commute with the left $\mathbf{R}_{\mathbf{q}}[W]$–action.

(ii) $\mathcal{N}_{\mathbf{q}}$ is the weak closure of $\mathbf{R}^b_{\mathbf{q}}[W]$ acting from the right on $L^2_{\mathbf{q}}$.

If we interchange the roles of left and right in the above, we get the two equivalent definitions of $\mathcal{N}'_{\mathbf{q}}$.

Lemma 5.2 *If $T \subset S$, then the inclusion $\mathbf{R}_{\mathbf{q}}[W_T] \hookrightarrow \mathbf{R}_{\mathbf{q}}[W]$ induces inclusions $\mathbf{R}^b_{\mathbf{q}}[W_T] \hookrightarrow \mathbf{R}^b_{\mathbf{q}}[W]$ and $\mathcal{N}_{\mathbf{q}}[W_T] \hookrightarrow \mathcal{N}_{\mathbf{q}}$.*

Proof Let $L^2_{\mathbf{q}}(wW_T) \subset L^2_{\mathbf{q}}(W)$ denote the subspace of functions which are supported on the coset wW_T. Then $L^2_{\mathbf{q}}(W)$ decomposes as an orthogonal direct sum of spaces of the form $L^2_{\mathbf{q}}(wW_T)$. Suppose $\lambda \in \mathcal{N}_{\mathbf{q}}[W_T)$. Right multiplication by λ preserves the summands, and acts on each summand in the same way. The norm in the space $L^2_{\mathbf{q}}(wW_T)$ is the norm in $L^2_{\mathbf{q}}(W_T)$ rescaled by a factor of $q_w^{-1/2}$, so that the operator norms of right multiplication by λ on each of these subspaces is bounded hence, $\lambda \in \mathcal{N}_{\mathbf{q}}$. □

The j–isomorphism From the definitions, the isomorphism $j\colon \mathbf{R}_{\mathbf{q}}[W] \to \mathbf{R}_{\mathbf{q}^{-1}}[W]$ defined by (4–6) takes the orthonormal basis $(\widetilde{e}_w)$ for $L^2_{\mathbf{q}}$, defined by (5–2), to the orthonormal basis $(\widetilde{e}_w)$ for $L^2_{\mathbf{q}^{-1}}$. So, it is an isometry. Therefore, it extends to an isometry of Hilbert spaces $j\colon L^2_{\mathbf{q}} \to L^2_{\mathbf{q}^{-1}}$. From this, it is obvious that j takes a bounded element of $L^2_{\mathbf{q}}$ to a bounded element of $L^2_{\mathbf{q}^{-1}}$. Hence, it extends to an isomorphism of von Neumann algebras $j\colon \mathcal{N}_{\mathbf{q}} \to \mathcal{N}_{\mathbf{q}^{-1}}$.

The von Neumann trace Define the *trace* of an element $\varphi \in \mathcal{N}_{\mathbf{q}}$ by

$$\mathrm{tr}_{\mathcal{N}_{\mathbf{q}}}(\varphi) := \langle e_1\varphi, e_1\rangle_{\mathbf{q}},$$

where e_1 denotes the basis element of $L^2_{\mathbf{q}}$ corresponding to the identity element of W. If $\Phi\colon \bigoplus_{i=1}^n L^2_{\mathbf{q}} \to \bigoplus_{i=1}^n L^2_{\mathbf{q}}$ is a bounded linear map of left $\mathbf{R}_{\mathbf{q}}[W]$–modules, then we can represent Φ as right multiplication by an $n \times n$ matrix (φ_{ij}) with entries in $\mathcal{N}_{\mathbf{q}}$. Define

$$\mathrm{tr}_{\mathcal{N}_{\mathbf{q}}}(\Phi) := \sum_{i=1}^n \mathrm{tr}_{\mathcal{N}_{\mathbf{q}}}(\varphi_{ii}).$$

Hilbert $\mathcal{N}_{\mathbf{q}}$–modules and von Neumann dimension

Definition 5.3 A subspace V of a finite orthogonal direct sum of copies of $L^2_{\mathbf{q}}$ is called a *Hilbert $\mathcal{N}_{\mathbf{q}}$–module* if it is a closed subspace and if it is stable under the diagonal left action of $\mathbf{R_q}[W]$.

A *map* of Hilbert $\mathcal{N}_{\mathbf{q}}$–modules means a bounded linear map of left $\mathbf{R_q}[W]$–modules. A map is *weakly surjective* if it has dense image; it is a *weak isomorphism* if it is injective and weakly surjective.

Let $V \subseteq \bigoplus_{i=1}^n L^2_{\mathbf{q}}$ be a Hilbert $\mathcal{N}_{\mathbf{q}}$–module and let $p_V\colon \bigoplus_{i=1}^n L^2_{\mathbf{q}} \to \bigoplus_{i=1}^n L^2_{\mathbf{q}}$ be the orthogonal projection onto V. The *von Neumann dimension* of V is the nonnegative real number defined by

$$\dim_{\mathcal{N}_{\mathbf{q}}} V = \operatorname{tr}_{\mathcal{N}_{\mathbf{q}}}(p_V). \tag{5–3}$$

As usual, one shows that $\dim_{\mathcal{N}_{\mathbf{q}}} V$ doesn't depend on the choice of embedding of V into a finite direct sum of copies of $L^2_{\mathbf{q}}$. If a subspace $V \subseteq \bigoplus L^2_{\mathbf{q}}$ is $\mathbf{R_q}[W]$–stable but not necessarily closed, one defines $\dim_{\mathcal{N}_{\mathbf{q}}} V := \dim_{\mathcal{N}_{\mathbf{q}}} \overline{V}$. This dimension function satisfies the usual list of properties:

(i) $\dim_{\mathcal{N}_{\mathbf{q}}} V = 0$ if and only if $V = 0$.

(ii) For any two Hilbert $\mathcal{N}_{\mathbf{q}}$–modules V and V',
$$\dim_{\mathcal{N}_{\mathbf{q}}}(V \oplus V') = \dim_{\mathcal{N}_{\mathbf{q}}} V + \dim_{\mathcal{N}_{\mathbf{q}}} V'.$$

(iii) $\dim_{\mathcal{N}_{\mathbf{q}}} L^2_{\mathbf{q}} = 1$.

(iv) If $f\colon V \to V'$ is a weak isomorphism of Hilbert $\mathcal{N}_{\mathbf{q}}$–modules, then $\dim_{\mathcal{N}_{\mathbf{q}}} V = \dim_{\mathcal{N}_{\mathbf{q}}} V'$.

(v) Suppose that (W', S') and (W'', S'') are Coxeter systems, that $S' \to I'$ and $S'' \to I''$ are indexing functions, that $\mathbf{q}'$ and $\mathbf{q}''$ are multiparameters, $S = S' \cup S''$ and $I = I' \cup I''$ are disjoint unions that $(W, S) = (W' \times W'', S' \cup S'')$ and that $\mathbf{q}$ is the multiparameter for (W, S) formed by combining $\mathbf{q}'$ and $\mathbf{q}''$. Let V' (resp. V'') be a Hilbert $\mathcal{N}_{\mathbf{q}'}[W']$–module (resp. $\mathcal{N}_{\mathbf{q}''}[W'']$–module). Then the completed tensor product $V := V' \otimes V''$ is naturally a Hilbert $\mathcal{N}_{\mathbf{q}}$–module and
$$\dim_{\mathcal{N}_{\mathbf{q}}[W]}(V' \otimes V'') = (\dim_{\mathcal{N}_{\mathbf{q}'}[W']} V')(\dim_{\mathcal{N}_{\mathbf{q}''}[W'']} V'').$$

(vi) Suppose that $T \subset S$ and that V_T is a Hilbert $\mathcal{N}_{\mathbf{q}}[W_T]$–module. The *induced* Hilbert $\mathcal{N}_{\mathbf{q}}$–module V is defined to be the completed tensor product
$$V := L^2_{\mathbf{q}}(W) \otimes_{\mathbf{R_q}[W_T]} V_T.$$

Its dimension is given by

$$\dim_{\mathcal{N}_{\mathbf{q}}} V = \dim_{\mathcal{N}_{\mathbf{q}}[W_T]} V_T .$$

Idempotents in $\mathcal{N}_{\mathbf{q}}$ and growth series Given a subset T of S, recall $\mathcal{R}_T$ denotes the region of convergence for $W_T(\mathbf{t})$.

Lemma 5.4 *Given $T \subseteq S$ and $\mathbf{q} \in \mathcal{R}_T$, there is an idempotent $a_T \in \mathcal{N}_{\mathbf{q}}$ defined by*

$$a_T := \frac{1}{W_T(\mathbf{q})} \sum_{w \in W_T} e_w .$$

Proof Define

$$\widetilde{a}_T = \sum_{w \in W_T} e_w . \tag{5–4}$$

Then $\langle \widetilde{a}_T, \widetilde{a}_T \rangle_{\mathbf{q}} = \sum q_w = W_T(\mathbf{q})$, so $\widetilde{a}_T \in L^2_{\mathbf{q}}(W_T)$ if and only if $\mathbf{q} \in \mathcal{R}_T$. Assume this. Recall that for each $s \in S$, X_s denotes the set of $(\varnothing, \{s\})$–reduced elements in W. Using (4–3), we calculate that for each $s \in T$,

$$\begin{aligned} \widetilde{a}_T e_s &= \sum_{w \in X_s \cap W_T} e_w e_s + e_{ws} e_s \\ &= \sum e_{ws} + q_s e_w + (q_s - 1) e_{ws} \\ &= q_s \widetilde{a}_T . \end{aligned}$$

Hence, for $w \in W_T$,

$$\widetilde{a}_T e_w = q_w \widetilde{a}_T \quad \text{and} \quad \widetilde{a}_T \widetilde{e}_w = q_w^{1/2} \widetilde{a}_T . \tag{5–5}$$

Therefore,

$$(\widetilde{a}_T)^2 = W_T(\mathbf{q}) \widetilde{a}_T . \tag{5–6}$$

We claim $\widetilde{a}_T$ is a bounded element of $L^2_{\mathbf{q}}(W_T)$ (hence, by Lemma 5.2, it lies in $\mathcal{N}_{\mathbf{q}}$). To see this, note that if $x = \sum x_w \widetilde{e}_w \in \mathbf{R}_{\mathbf{q}}[W_T]$, then (5–5) can be rewritten as

$$\widetilde{a}_T \sum x_w \widetilde{e}_w = \left(\sum x_w q_w^{1/2} \right) \widetilde{a}_T$$

and hence, $\| \widetilde{a}_T x \|_{\mathbf{q}} \le \| \widetilde{a}_T \|_{\mathbf{q}} \| x \|_{\mathbf{q}}$. So, we get an idempotent defined by

$$a_T = \frac{\widetilde{a}_T}{W_T(\mathbf{q})} . \tag{5–7}$$

□

Lemma 5.5 *Given a subset T of S and an I–tuple $\mathbf{q} \in \mathcal{R}_T^{-1}$, there is an idempotent $h_T \in \mathcal{N}_{\mathbf{q}}$ defined by*

$$h_T := \frac{1}{W_T(\mathbf{q}^{-1})} \sum_{w \in W_T} \varepsilon_w q_w^{-1} e_w,$$

where q_w and ε_w are defined by (4–4) *and* (4–5)*, respectively.*

Proof The proof is similar to the previous one. Define

$$\widetilde{h}_T := \sum_{w \in W_T} \varepsilon_w q_w^{-1} e_w. \tag{5–8}$$

Then $\langle \widetilde{h}_T, \widetilde{h}_T \rangle_{\mathbf{q}} = \sum q_w^{-1} = W_T(\mathbf{q}^{-1})$, so $\widetilde{h}_T \in L^2_{\mathbf{q}}(W_T)$ if and only if $\mathbf{q}^{-1} \in \mathcal{R}_T$. Assume this. For $s \in T$, we calculate

$$\begin{aligned}
\widetilde{h}_T e_s &= \sum_{w \in X_s \cap W_T} \varepsilon_w q_w^{-1} e_w e_s + \varepsilon_{ws} q_{ws}^{-1} e_{ws} e_s \\
&= \sum \varepsilon_w q_w^{-1} e_{ws} + \varepsilon_{ws} q_w^{-1} q_s^{-1} (q_s e_w + (q_s - 1) e_{ws}) \\
&= -\sum \varepsilon_w q_w^{-1} e_w + \varepsilon_{ws} q_w^{-1} q_s^{-1} e_{ws} \\
&= -\widetilde{h}_T.
\end{aligned}$$

Therefore, for $w \in W_T$,

$$\widetilde{h}_T e_w = \varepsilon_w \widetilde{h}_T, \tag{5–9}$$

$$(\widetilde{h}_T)^2 = \sum_{w \in W_T} \varepsilon_w q_w^{-1} \widetilde{h}_T e_w = W_T(\mathbf{q}^{-1}) \widetilde{h}_T. \tag{5–10}$$

As before, it follows that $\widetilde{h}_T \in \mathbf{R}^b_{\mathbf{q}}[W_T]$ and hence, that $\widetilde{h}_T \in \mathcal{N}_{\mathbf{q}}$. So, by (5–10), we get an idempotent defined by

$$h_T := \frac{\widetilde{h}_T}{W_T(\mathbf{q}^{-1})}. \tag{5–11}$$

□

Using (4–3) we get the following right hand versions of (5–5) and (5–9) for $T \subseteq S$ and $w \in W_T$:

$$e_w a_T = q_w a_T, \tag{5–12}$$

$$e_w h_T = \varepsilon_w h_T. \tag{5–13}$$

What is the effect of the j–isomorphism on these idempotents? It follows immediately from definitions (4–6), (5–4) and (5–8) that

$$j(\widetilde{a}_T) = \widetilde{h}_T \quad \text{and} \quad j(\widetilde{a}_T) = \widetilde{h}_T.$$

Hence, by the definitions in Lemma 5.4 and Lemma 5.5,

$$j(a_T) = h_T \quad \text{and} \quad j(h_T) = a_T. \tag{5–14}$$

Using (5–5), (5–9), (5–12) and (5–13), we easily calculate that for any $U \subseteq T \subseteq S$:

$$a_U a_T = a_T = a_T a_U \quad \text{whenever } \mathbf{q} \in \mathcal{R}_U, \tag{5–15}$$

$$h_U h_T = h_T = h_T h_U \quad \text{whenever } \mathbf{q} \in \mathcal{R}_U^{-1}. \tag{5–16}$$

If $s_1 \cdots s_l$ is a reduced expression for w, then $s_l \cdots s_1$ is a reduced expression for w^{-1}. It follows that

$$q_{w^{-1}} = q_w \quad \text{and} \quad \varepsilon_{w^{-1}} = \varepsilon_w.$$

So,
$$a_T^* = a_T \quad \text{and} \quad h_T^* = h_T,$$

whenever the idempotents a_T and h_T make sense. In other words, the maps $x \to x a_T$ and $x \to x h_T$ are orthogonal projections from $L^2_{\mathbf{q}}$ onto Hilbert submodules.

Remark The "a" in a_T is for "average," while the "h" in h_T is for "harmonic."

Definition 5.6 For each $T \subseteq S$, let $\alpha_T \colon \mathbf{R}[W_T] \to \mathbf{R}$ and $\beta_T \colon \mathbf{R}_{\mathbf{q}}[W_T] \to \mathbf{R}$ be the algebra homomorphisms defined by $e_w \to q_w$ and $e_w \to \varepsilon_w$, respectively. α_T is the *symmetric character* and β_T is the *alternating character*.

The next lemma follows immediately from equations (5–5) and (5–9).

Lemma 5.7

(i) *Supposing $\mathbf{q} \in \mathcal{R}_T$, the action of $\mathbf{R}_{\mathbf{q}}[W_T]$ on $L^2_{\mathbf{q}} a_T$ by right multiplication is via the character α_T.*

(ii) *Supposing $\mathbf{q}^{-1} \in \mathcal{R}_T$, the action of $\mathbf{R}_{\mathbf{q}}[W_T]$ on $L^2_{\mathbf{q}} h_T$ by right multiplication is via the character β_T.*

Some Hilbert $\mathcal{N}_{\mathbf{q}}$–submodules of $L^2_{\mathbf{q}}$ To simplify notation, for each $s \in S$, write a_s and h_s for the idempotents $a_{\{s\}}$ and $h_{\{s\}}$. Let $A_s = L^2_{\mathbf{q}} a_s$ and $H_s = L^2_{\mathbf{q}} h_s$ be the corresponding Hilbert $\mathcal{N}_{\mathbf{q}}$–submodules of $L^2_{\mathbf{q}}$.

Lemma 5.8 *For each $s \in S$, the subspaces A_s and H_s are the orthogonal complements of each other in $L^2_{\mathbf{q}}$.*

Proof

$$\begin{aligned} a_s + h_s &= \frac{1}{1+q_s}(1+e_s) + \frac{1}{1+q_s^{-1}}(1-q_s^{-1}e_s) \\ &= 1. \end{aligned}$$

So, a_s and h_s are orthogonal projections onto complementary subspaces. □

For each $T \subseteq S$, set

$$A_T := \bigcap_{s\in T} A_s \quad \text{and} \quad H_T := \bigcap_{s \in T} H_s. \tag{5–17}$$

For any subspace $E \subset L^2_{\mathbf{q}}$, let $E^{\perp}$ denote its orthogonal complement. Since $\perp$ takes sums to intersections and intersections to closures of sums:

$$\Big(\sum_{s\in T} A_s\Big)^{\perp} = H_T, \qquad \Big(\sum_{s\in T} H_s\Big)^{\perp} = A_T, \tag{5–18}$$

$$\overline{\sum_{s\in T} A_s} = (H_T)^{\perp}, \qquad \overline{\sum_{s\in T} H_s} = (A_T)^{\perp}. \tag{5–19}$$

Lemma 5.9 *Let A_S be the subspace of $L^2_{\mathbf{q}}$ defined in* (5–17).

(i) *For all $x \in A_S$ and $w \in W$, $xe_w = q_w x$.*

(ii) *If $\mathbf{q} \notin \mathcal{R}$, then $A_S = 0$.*

(iii) *If $\mathbf{q} \in \mathcal{R}$, then A_S is the line spanned by a_S and $L^2_{\mathbf{q}} a_S = A_S$. Hence,*

$$\dim_{\mathcal{N}_{\mathbf{q}}} A_S = \frac{1}{W(\mathbf{q})}.$$

There is also a version of this lemma for H_S.

Lemma 5.10 *Let H_S be the subspace of $L^2_{\mathbf{q}}$ defined in* (5–17).

(i) *For all $x \in H_S$ and $w \in W$, $xe_w = \varepsilon_w x$.*

(ii) *If $\mathbf{q}^{-1} \notin \mathcal{R}$, then $H_S = 0$.*

(iii) *If $\mathbf{q}^{-1} \in \mathcal{R}$, then H_S is the line spanned by h_S and $L^2_{\mathbf{q}} h_S = H_S$. Hence,*

$$\dim_{\mathcal{N}_{\mathbf{q}}} H_S = \frac{1}{W(\mathbf{q}^{-1})}.$$

We prove only the first version, the proof of the second version being entirely similar.

Proof of Lemma 5.9 (i) As in Definition 5.6, $\alpha_T \colon \mathbf{R}[W_T] \to \mathbf{R}$ denotes the symmetric character. The $\alpha_{\{s\}}$–eigenspace of $\mathbf{R}_{\mathbf{q}}[W_{\{s\}}]$ on $L^2_{\mathbf{q}}$ is $\operatorname{Ker}(q_s - e_s) = \operatorname{Ker} h_s = L^2_{\mathbf{q}} a_s = A_s$. Since the subalgebras $\mathbf{R}_{\mathbf{q}}[W_{\{s\}}]$ generate $\mathbf{R}_{\mathbf{q}}[W]$, the intersection of the A_s, $s \in S$, is the α_S–eigenspace for $\mathbf{R}_{\mathbf{q}}[W]$.

(ii) If $x = \sum x_w e_w \in A_S$, then

$$q_w x_w = \langle e_w, x\rangle_{\mathbf{q}} = \langle 1, xe_w^*\rangle_{\mathbf{q}} = \langle 1, xe_{w^{-1}}\rangle_{\mathbf{q}} = \langle 1, q_w x\rangle_{\mathbf{q}} = q_w x_1.$$

In other words, the coefficients x_w are all equal. Hence, $\langle x, x\rangle_{\mathbf{q}} = x_1^2 W(\mathbf{q})$. So, if $\mathbf{q} \notin \mathcal{R}$, $x \notin L^2_{\mathbf{q}}$ unless $x = 0$ and if $\mathbf{q} \in \mathcal{R}$, x must be a scalar multiple of a_S.

(iii) By Lemma 5.7, if $\mathbf{q} \in \mathcal{R}$, then $L^2_{\mathbf{q}} a_S \subseteq A_S$. Since $a_S \neq 0$ and $\dim_{\mathbf{R}} A_S = 1$, the inclusion is an equality. Hence,

$$\dim_{\mathcal{N}_{\mathbf{q}}} A_S = \dim_{\mathcal{N}_{\mathbf{q}}} L^2_{\mathbf{q}} a_S = \operatorname{tr}_{\mathcal{N}_{\mathbf{q}}} a_S = \frac{1}{W(\mathbf{q})}. \qquad \square$$

Corollary 5.11 *For any $T \subseteq S$:*

(i) *$A_T = L^2_{\mathbf{q}} a_T$ if $\mathbf{q} \in \mathcal{R}_T$ and $A_T = 0$ if $\mathbf{q} \notin \mathcal{R}_T$.*

(ii) *$H_T = L^2_{\mathbf{q}} h_T$ if $\mathbf{q}^{-1} \in \mathcal{R}_T$ and $H_T = 0$ if $\mathbf{q}^{-1} \notin \mathcal{R}_T$.*

Proof Since A_T and H_T are induced from Hilbert $\mathbf{R}_{\mathbf{q}}[W_T]$–modules, this follows from Lemma 5.9 and Lemma 5.10. $\square$

6 Some cell complexes

The basic construction Suppose we are given the following data:

- a Coxeter system (W,S),
- a CW complex Z and
- a family of subcomplexes $(Z_s)_{s\in S}$.

The Z_s are called the *mirrors* of Z. Given these data there is a classical construction of a CW complex $\mathcal{U}=\mathcal{U}(W,Z)$ with a W–action so that Z is a strict fundamental domain. We recall the construction.

For each subset T of S, set

$$Z_T := Z \cap \bigcap_{s\in T} Z_s,$$
$$Z^T := \bigcup_{s\in T} Z_s. \tag{6–1}$$

For each cell c of Z and each point $z\in Z$, set

$$S(z) := \{s\in S \mid z\in Z_s\}, \tag{6–2}$$
$$S(c) := \{s\in S \mid c\subseteq Z_s\}. \tag{6–3}$$

Define $\mathcal{U}(W,Z) := (W\times Z)/\sim$ where $\sim$ is the equivalence relation defined by: $(w,z)\sim(w',z')$ if and only if $z=z'$ and the cosets $wW_{S(z)}$ and $w'W_{S(z)}$ are equal. Write $[w,z]$ for the image of (w,z) in $\mathcal{U}$. The group W acts on $\mathcal{U}$ via $w\cdot[w',z]=[ww',z]$. The orbit space is Z. Identifying Z with the image of $1\times Z$ in $\mathcal{U}$, we see that Z is a strict fundamental domain. wZ, the translate of Z by w, is identified with the image of $w\times Z$. The CW structure on $\mathcal{U}$ is defined by declaring the family (wc), with $w\in W$ and c a cell of Z, to be the set of cells in $\mathcal{U}$. (Note that wc is the image of $w\times c$ in $\mathcal{U}$.)

The setwise stabilizer of a cell c of Z is the special subgroup $W_{S(c)}$. Moreover, $W_{S(c)}$ fixes each point of c.

The family $(Z_s)_{s\in S}$ is *W–finite* if $Z_T=\varnothing$ whenever W_T is infinite. This condition insures that each isotropy subgroup is finite. It is equivalent to the condition that W act properly on $\mathcal{U}$. We shall assume it throughout.

The complex Σ The complex Σ can be described in terms of the basic construction. As in Section 2, denote the geometric realization of the poset $\mathcal{S}$ by K and the geometric realization of $W\mathcal{S}$ by Σ. For each $s \in S$, let K_s be the geometric realization of the subposet $\mathcal{S}_{\geq\{s\}}$. It is a subcomplex of K. The space $\mathcal{U}(W,K)$ is naturally a simplicial complex. The natural map $W \times \mathcal{S} \to W\mathcal{S}$, defined by $(w,T) \to wW_T$, induces a map of geometric realizations $W \times K \to \Sigma$ and this descends to W-equivariant map $\mathcal{U}(W,K) \to \Sigma$. As in [12], it is easily seen that this map is a simplicial isomorphism, ie

$$\Sigma \cong \mathcal{U}(W,K). \tag{6–4}$$

Cellulation of Σ by Coxeter cells As explained in Moussong [36] and Davis [14; 16] and below, Σ has another cell structure: its cellulation by "Coxeter cells."

Suppose, for the moment, that W is finite and $\mathrm{Card}(S) = n$. Associated to (W,S) there is a n–dimensional convex polytope P called the *Coxeter cell* of type W. P is defined as the convex hull of a generic W–orbit in the canonical representation of W on $\mathbf{R}^n$. W acts simply transitively on the vertex set of P; moreover, a subset of vertices spans a face if and only if it has the form wW_Tv_0 for some special coset wW_T and for a given choice of base vertex v_0 in the interior of the fundamental simplicial cone. This identifies the face poset of P with $W\mathcal{S}$. In other words, it gives a simplicial isomorphism between Σ and the barycentric subdivision of P.

Returning to the case where (W,S) is arbitrary, note that for any element $wW_T \in W\mathcal{S}$, the poset $W\mathcal{S}_{\leq wW_T}$ is identified with the face poset of P_T, the Coxeter cell of type W_T. So, the subcomplex $|W\mathcal{S}_{\leq wW_T}|$ of Σ is identified with the barycentric subdivision of P_T. This defines the cell structure on Σ: each simplicial subcomplex $|W\mathcal{S}_{\leq wW_T}|$ is identified with a Coxeter cell of type W_T. So, the vertex set of Σ is W and a subset of W is the vertex set of a cell if and only if it is a coset wW_T for some $w \in W$ and $T \in \mathcal{S}$. We shall use the notation Σ_{cc} to denote Σ equipped with this cell structure, where the subscript cc stands for "Coxeter cell." (In [27] this cell structure is denoted Σ_d, where the subscript d stood for "dual cell.") The poset of cells of Σ_{cc} is $W\mathcal{S}$.

Suppose $U \subseteq S$. Let $\mathcal{S}(U) := \{T \in \mathcal{S} \mid T \subseteq U\}$. Define $\Sigma(U)$ to be the subcomplex of Σ_{cc} consisting of all Coxeter cells of type T, with $T \in \mathcal{S}(U)$. If $K(U) := \Sigma(U) \cap K$, then it is not difficult to see that

$$\Sigma(U) = \mathcal{U}(W, K(U)) = W \times_{W_U} \mathcal{U}(W_U, K(U)). \tag{6–5}$$

Moreover, $\mathcal{U}(W_U, K(U))$ is the complex Σ_{W_U} associated to (W_U, U).

Ruins Given $U \subseteq S$ and $T \in \mathcal{S}(U)$, define three subcomplexes of $\Sigma(U)$:

$$\begin{aligned}\Omega(U,T) &: \text{the union of closed cells of type } T', \text{ with } T' \in \mathcal{S}(U)_{\geq T},\\ \widehat{\Omega}(U,T) &: \text{the union of closed cells of type } T'',\ T'' \in \mathcal{S}(U),\ T'' \notin \mathcal{S}(U)_{\geq T},\\ \partial\Omega(U,T) &: \text{the cells of } \Omega(U,T) \text{ of type } T'', \text{ with } T'' \notin \mathcal{S}(U)_{\geq T}.\end{aligned}$$

The subcomplex $\Omega(U,T)$ is the union of all cells of type T'', where $T'' \leq T'$ for some $T' \in \mathcal{S}(U)_{\geq T}$. So,

$$\partial\Omega(U,T) = \Omega(U,T) \cap \widehat{\Omega}(U,T)$$

and

$$\Sigma(U) = \Omega(U,T) \cup \widehat{\Omega}(U,T).$$

The pair $(\Omega(U,T), \partial\Omega(U,T))$ is called the (U,T)*–ruin*. For example, for $T = \varnothing$, we have $\Omega(U,\varnothing) = \Sigma(U)$ and $\partial\Omega(U,\varnothing) = \varnothing$. The key step in our proofs of the main results in Section 9 and Section 10 is the computation of certain homology groups of (U,T)–ruins.

Similarly, define

$$\begin{aligned}K(U,T) &:= \Omega(U,T) \cap K,\\ \partial K(U,T) &:= \partial\Omega(U,T) \cap K,\\ \widehat{K}(U,T) &:= \widehat{\Omega}(U,T) \cap K.\end{aligned}$$

so that

$$\begin{aligned}\Omega(U,T) &= \mathcal{U}(W, K(U,T)),\\ \partial\Omega(U,T) &= \mathcal{U}(W, \partial K(U,T)),\\ \widehat{\Omega}(U,T) &= \mathcal{U}(W, \widehat{K}(U,T)).\end{aligned}$$

7 Weighted L^2–(co)homology

Notation is as in the previous section: Z is a CW complex, $(Z_s)_{s\in S}$ is a W–finite family of subcomplexes and $\mathcal{U} = \mathcal{U}(W,Z)$.

We begin by defining a chain complex of Hilbert $\mathcal{N}_{\mathbf{q}}$–modules for the CW complex $\mathcal{U}$. In this case, each orbit of cells contributes an $\mathcal{N}_{\mathbf{q}}$–module of the form A_T for some $T \in \mathcal{S}$. Next we define chain complexes of $\mathcal{N}_{\mathbf{q}}$–modules in the cases of the cellulation of Σ (and its subcomplexes of ruins) by Coxeter cells. In these cases each orbit of cells contributes a $\mathcal{N}_{\mathbf{q}}$–module of the form H_T, $T \in \mathcal{S}$.

Weighted (co)chain complexes for $\mathcal{U}(W, Z)$ Orient the cells of Z arbitrarily and then orient the remaining cells of $\mathcal{U}$ so that for each positively oriented cell c of Z and each $w \in W$, wc is positively oriented.

As usual, $\mathbf{q}$ is an I–tuple of positive real numbers. Given a cell c of Z define a measure $\mu_{\mathbf{q}}$ on its orbit Wc by

$$\mu_{\mathbf{q}}(wc) := q_u, \tag{7–1}$$

where u is the shortest element in the coset $wW_{S(c)}$ (ie u is the $(\varnothing, S(c))$–reduced element in this coset). This extends in a natural way to a measure, also denoted by $\mu_{\mathbf{q}}$, on $\mathcal{U}^{(i)}$ (where $\mathcal{U}^{(i)}$ denotes the entire set of i–cells in $\mathcal{U}$). As in [27], define the $\mathbf{q}$*–weighted L^2-(co)chains* on $\mathcal{U}$ (in dimension i) to be the Hilbert space

$$L^2_{\mathbf{q}}C_i(\mathcal{U}) = L^2_{\mathbf{q}}C^i(\mathcal{U}) := L^2(\mathcal{U}^{(i)}, \mu_{\mathbf{q}}). \tag{7–2}$$

We have coboundary and boundary maps,

$$\delta^i\colon L^2_{\mathbf{q}}C^i(\mathcal{U}) \to L^2_{\mathbf{q}}C^{i+1}(\mathcal{U}) \quad \text{and} \quad \partial_i\colon L^2_{\mathbf{q}}C_i(\mathcal{U}) \to L^2_{\mathbf{q}}C_{i-1}(\mathcal{U})$$

defined by the usual formulas:

$$\delta^i(f)(\gamma) := \sum [\beta : \gamma] f(\beta), \tag{7–3}$$

$$\partial_i(f)(\alpha) := \sum [\alpha : \beta] f(\beta), \tag{7–4}$$

where the first sum is over all i–cells β incident to the $(i+1)$–cell γ while the second is over all β whose boundary contains the $(i-1)$–cell α. In contrast to the standard situation (where $\mathbf{q} = \mathbf{1}$), the maps δ^i and ∂_{i+1} are not adjoint to one another. Define $\partial^{\mathbf{q}}_i\colon L^2_{\mathbf{q}}C_i(\mathcal{U}) \to L^2_{\mathbf{q}}C_{i-1}(\mathcal{U})$ by

$$\partial^{\mathbf{q}}_i(f)(\alpha) := \sum [\alpha : \beta]\mu_{\mathbf{q}}(\beta)\mu_{\mathbf{q}}(\alpha)^{-1} f(\beta). \tag{7–5}$$

A quick calculation (cf [27, Section 1]) then shows that $\delta^* = \partial^{\mathbf{q}}$. Since $\delta^2 = 0$, taking adjoints, we get $(\partial^{\mathbf{q}})^2 = 0$. Hence, $(L^2_{\mathbf{q}}C_*(\mathcal{U}), \partial^{\mathbf{q}})$ is also a chain complex.

One defines the $\mathbf{q}$*–weighted L^2-(co)homology* of $\mathcal{U}$ in dimension i by

$$L^2_{\mathbf{q}}H^i(\mathcal{U}) := H^i((L^2_{\mathbf{q}}C^*(\mathcal{U}), \delta)) = \operatorname{Ker}\delta^i / \operatorname{Im}\delta^{i-1},$$
$$L^2_{\mathbf{q}}H_i(\mathcal{U}) := H_i((L^2_{\mathbf{q}}C_*(\mathcal{U}), \partial^{\mathbf{q}})) = \operatorname{Ker}\partial^{\mathbf{q}}_i / \operatorname{Im}\partial^{\mathbf{q}}_{i+1}.$$

Notice that while we are using the ordinary coboundary map δ, the boundary map $\partial^{\mathbf{q}}$ is not the usual one: it is modified by coefficients depending on $\mathbf{q}$. There is a standard problem with these (co)homology groups: the quotients need not be Hilbert spaces. To remedy this, define *reduced* weighted L^2-(co)homology by

$$L^2_{\mathbf{q}}\mathcal{H}^i(\mathcal{U}) := \operatorname{Ker}\delta^i/\overline{\operatorname{Im}\delta^{i-1}},$$
$$L^2_{\mathbf{q}}\mathcal{H}_i(\mathcal{U}) := \operatorname{Ker}\partial^{\mathbf{q}}_i/\overline{\operatorname{Im}\partial^{\mathbf{q}}_i}.$$

Since $\delta^* = \partial^{\mathbf{q}}$ and $(\partial^{\mathbf{q}})^* = \delta$, we have the Hodge decomposition:

$$L^2_{\mathbf{q}}C^i(\mathcal{U}) = (\operatorname{Ker}\delta^i \cap \operatorname{Ker}\partial^{\mathbf{q}}_i) \oplus \overline{\operatorname{Im}\delta^{i-1}} \oplus \overline{\operatorname{Im}\partial^{\mathbf{q}}_{i+1}}.$$

Thus both $L^2_{\mathbf{q}}\mathcal{H}^i(\mathcal{U})$ and $L^2_{\mathbf{q}}\mathcal{H}_i(\mathcal{U})$ can be identified with the space $\operatorname{Ker}\delta^i \cap \operatorname{Ker}\partial^{\mathbf{q}}_i$ of harmonic cochains. In particular, $L^2_{\mathbf{q}}\mathcal{H}^i(\mathcal{U}) \cong L^2_{\mathbf{q}}\mathcal{H}_i(\mathcal{U})$.

Lemma 7.1 *The chain complexes* $(L^2_{\mathbf{q}}C_*(\mathcal{U}), \partial^{\mathbf{q}})$ *and* $(L^2_{\mathbf{q}^{-1}}C_*(\mathcal{U}), \partial)$ *are isomorphic.*

Proof For a chain f on $\mathcal{U}$, define another chain $\theta(f)$ by $\theta(f)(\beta) := \mu_{\mathbf{q}}(\beta)f(\beta)$ and note that $\theta(f)$ is $\mathbf{q}$–square summable if and only if f is $\mathbf{q}^{-1}$–square summable. Hence, it defines a linear isomorphism $\theta = \theta_{\mathbf{q}}\colon L^2_{\mathbf{q}^{-1}}C_*(\mathcal{U}) \to L^2_{\mathbf{q}}C_*(\mathcal{U})$. Using (7–4) and (7–5), computation shows that $\theta \circ \partial = \partial^{\mathbf{q}} \circ \theta$. So, θ is a chain isomorphism. □

Remark 7.2 We have canonical inclusions of chain complexes:

$$C_*(\mathcal{U};\mathbf{R}) \hookrightarrow (L^2_{\mathbf{q}}C_*(\mathcal{U}), \partial) \hookrightarrow C^{lf}_*(\mathcal{U};\mathbf{R}). \tag{7–6}$$

So, using the isomorphism $\theta_{\mathbf{q}^{-1}}$ of Lemma 7.1 we get inclusions:

$$C_*(\mathcal{U};\mathbf{R}) \hookrightarrow L^2_{\mathbf{q}^{-1}}C_*(\mathcal{U}) \hookrightarrow C^{lf}_*(\mathcal{U};\mathbf{R}). \tag{7–7}$$

Similarly, we have inclusions of cochain complexes:

$$C^*_c(\mathcal{U};\mathbf{R}) \hookrightarrow L^2_{\mathbf{q}}C^*(\mathcal{U}) \hookrightarrow C^*(\mathcal{U};\mathbf{R}). \tag{7–8}$$

(Here $C^{lf}_*(\ )$ and $C^*_c(\ )$ stand for, respectively, infinite cellular chains and finitely supported cellular cochains.) The second map in (7–6) (or the second map in (7–7)) is obtained by dualizing the first map in (7–8). Similarly, the second map in (7–8) is obtained by dualizing the first map in (7–6).

As was indicated in the Introduction and as will be explained further in Section 12, for $\mathbf{q} \in \mathcal{R}$, the first maps in (7–6) and (7–7) induce monomorphisms with dense image

$$H_i(\mathcal{U};\mathbf{R}) \hookrightarrow \mathcal{H}_i(L^2_{\mathbf{q}^{-1}}C_*(\mathcal{U}), \partial),$$
$$H_i(\mathcal{U};\mathbf{R}) \hookrightarrow L^2_{\mathbf{q}}\mathcal{H}_i(\mathcal{U}).$$

(The first monomorphism agrees with one's intuition.) Similarly, for $\mathbf{q} \in \mathcal{R}^{-1}$, the first map in (7–8) induces a monomorphism with dense image

$$\text{can}\colon H^i_c(\mathcal{U};\mathbf{R}) \hookrightarrow L^2_{\mathbf{q}}\mathcal{H}^i(\mathcal{U}).$$

Dualizing we get isomorphisms:

$$L^2_{\mathbf{q}}\mathcal{H}^i(\mathcal{U}) \xrightarrow{\cong} H^i(\mathcal{U};\mathbf{R}) \text{ for } \mathbf{q} \in \mathcal{R}$$

and

$$L^2_{\mathbf{q}}\mathcal{H}_i(\mathcal{U}) \xrightarrow{\cong} H^{lf}_i(\mathcal{U};\mathbf{R}) \text{ for } \mathbf{q} \in \mathcal{R}^{-1}.$$

All this is reminiscent of a well-known result of Cheeger–Gromov [11] that if a discrete amenable group A acts properly on a CW complex X, then the canonical map $L^2H^*(X) \to H^*(X;\mathbf{R})$ is injective. So, for $\mathbf{q} \in \mathcal{R}$, weighted L^2–cohomology behaves as if W were amenable.

The Hilbert $\mathcal{N}_{\mathbf{q}}$–module structure on $L^2_{\mathbf{q}}C^*(\mathcal{U})$ Following [27], realize $L^2_{\mathbf{q}}$ as $L^2(W,\nu_{\mathbf{q}})$, where $\nu_{\mathbf{q}}$ is the measure on W defined by $\nu_{\mathbf{q}}(w) = q_w$. For each subset T of S, the Hilbert $\mathcal{N}_{\mathbf{q}}$–submodule $A_T \subset L^2_{\mathbf{q}}$, defined by (5–17), is then identified with $L^2(W,\nu_{\mathbf{q}})^{W_T}$, the subspace of L^2 functions which are constant on each right coset wW_T.

Since each cell of $\mathcal{U}$ has the form wc for some cell c of Z and some $w \in W$, we have

$$L^2_{\mathbf{q}}C^i(\mathcal{U}) = \bigoplus_{c \in Z^{(i)}} L^2(Wc,\mu_{\mathbf{q}}),$$

where the sum ranges over all i–cells c of Z. Moreover, $L^2(Wc,\mu_{\mathbf{q}})$ can be identified with $A_{S(c)}$ via the isometry $\phi_c\colon L^2(Wc,\mu_{\mathbf{q}}) \to A_{S(c)}$ defined by

$$\phi_c(f) = \sqrt{W_{S(c)}(q)}\Big(\sum_{u \in X_{S(c)}} f(uc)e_u a_{S(c)}\Big),$$

where the summation is over all $(\varnothing, S(c))$–reduced elements u and where $a_{S(c)} \in \mathcal{N}_{\mathbf{q}}$ is the idempotent defined in Lemma 5.4. So, we get an isometry

$$L^2_{\mathbf{q}}C^i(\mathcal{U}) = \bigoplus_{c \in Z^{(i)}} L^2(Wc,\mu_{\mathbf{q}}) \xrightarrow{\cong} \bigoplus_{c \in Z^{(i)}} A_{S(c)}.$$

Since each $A_{S(c)}$ is a left $\mathbf{R}_{\mathbf{q}}[W]$–submodule of $L^2_{\mathbf{q}}$, this gives $L^2_{\mathbf{q}}C^i(\mathcal{U})$ the structure of a Hilbert $\mathcal{N}_{\mathbf{q}}$–module as in Definition 5.3 (provided we assume, as we shall, that Z is a finite complex). It also gives an isometric embedding

(7–9)
$$\Phi\colon L^2_{\mathbf{q}}C^i(\mathcal{U}) \hookrightarrow \bigoplus_{c \in Z^{(i)}} L^2_{\mathbf{q}} = C^i(Z) \otimes L^2_{\mathbf{q}}.$$

It is shown in [27, Lemma 3.2] that δ and $\partial^{\mathbf{q}}$ are maps of Hilbert $\mathcal{N}_{\mathbf{q}}$–modules. (It is *not* true that $\delta_{\mathbf{q}}$ and ∂ are maps of Hilbert $\mathcal{N}_{\mathbf{q}}$–modules; however, it is possible to give $L^2_{\mathbf{q}}C^*(\mathcal{U})$ and $L^2_{\mathbf{q}}C_*(\mathcal{U})$ the structure of Hilbert $\mathbf{R}_{\mathbf{q}^{-1}}[W]$–modules so that they are maps of Hilbert $\mathbf{R}_{\mathbf{q}^{-1}}[W]$–modules. To do this, one transports the $\mathbf{R}_{\mathbf{q}^{-1}}[W]$–module structure from $L^2_{\mathbf{q}^{-1}}C_*(\mathcal{U})$ via the isomorphism θ of Lemma 7.1.) It follows that $\operatorname{Ker}\delta$, $\operatorname{Ker}\partial^{\mathbf{q}}$, $\overline{\operatorname{Im}\delta}$ and $\overline{\operatorname{Im}\partial^{\mathbf{q}}}$ are Hilbert $\mathcal{N}_{\mathbf{q}}$–modules. Hence, $L^2_{\mathbf{q}}\mathcal{H}^i(\mathcal{U})$ (or $L^2_{\mathbf{q}}\mathcal{H}_i(\mathcal{U})$) is also a Hilbert $\mathcal{N}_{\mathbf{q}}$–module.

Weighted (co)chain complexes for cellulations by Coxeter cells Let $\langle T\rangle$ denote the Coxeter cell in Σ corresponding to $W_T \in W\mathcal{S}$ (the W_T–coset of the identity element). Then $W\langle T\rangle$, the W–orbit of $\langle T\rangle$, is the set of all Coxeter cells in Σ of type W_T. Define a measure $\mu_{\mathbf{q}}$ on $\Sigma^{(i)}_{cc}$ by $\mu_{\mathbf{q}}(w\langle T\rangle) = q_u$, where $u = p_T(w)$ is the shortest element in wW_T. Define the $\mathbf{q}$*–weighted* L^2*-(co)chains* on Σ (in dimension i) to be the Hilbert space

$$L^2_{\mathbf{q}}C_i(\Sigma_{cc}) = L^2_{\mathbf{q}}C^i(\Sigma_{cc}) := L^2(\Sigma^{(i)}_{cc}, \mu_{\mathbf{q}}).$$

We have
$$L^2_{\mathbf{q}}C_i(\Sigma_{cc}) = \bigoplus_{T\in\mathcal{S}^{(i)}} L^2(W\langle T\rangle, \mu_{\mathbf{q}}).$$

Choose arbitrary orientations for cells of the form $\langle T\rangle$, $T \in \mathcal{S}$. We use the following orientation convention for the remaining cells in $W\langle T\rangle$: if $u \in X_T$ (ie if u is $(\varnothing, T)$–reduced as defined in Section 2), then orient $u\langle T\rangle$ so that left translation by u is an orientation-preserving map $\langle T\rangle \to u\langle T\rangle$.

As in (7–3), δ: $L^2_{\mathbf{q}}C^i(\Sigma_{cc}) \to L^2_{\mathbf{q}}C^{i+1}(\Sigma_{cc})$ is the usual coboundary map. Its adjoint $\partial^{\mathbf{q}}$: $L^2_{\mathbf{q}}C_{i+1}(\Sigma_{cc}) \to L^2_{\mathbf{q}}C_i(\Sigma_{cc})$ is defined similarly to (7–5).

Next, we determine the formula for the restriction of $\partial^{\mathbf{q}}$ to the summand $L^2(W\langle U\rangle, \mu_{\mathbf{q}})$, where $U \in \mathcal{S}^{(i+1)}$. Any $w \in W$ can be uniquely decomposed as $w = uv$ with $u \in X_U$ and $v \in W_U$. Suppose $T \in \mathcal{S}^{(i)}$ is obtained by deleting one element of U and $w \in X_T$. If $w \in X_T$, then $v \in W_U \cap X_T$. For any $f \in L^2(W\langle U\rangle, \mu_{\mathbf{q}})$, we have the following formula for $\partial^{\mathbf{q}}$:

$$\partial^{\mathbf{q}} f(w\langle T\rangle) = \varepsilon_v q_v^{-1} f(u\langle U\rangle), \tag{7–10}$$

where $w = uv$ as above.

The group W_T acts nontrivially on the cell $\langle T\rangle$. In fact, $v \in W_T$ is ε_v orientation-preserving. Hence, the right $\mathbf{R}_{\mathbf{q}}[W_T]$–action on $L^2(W\langle T\rangle, \mu_{\mathbf{q}})$ is via the alternating character β_T of Definition 5.6. Therefore, $L^2(W\langle T\rangle, \mu_{\mathbf{q}})$ can be identified with H_T.

A specific isometry $\psi\colon L^2(W\langle T\rangle, \mu_{\mathbf{q}}) \to H_T$ can be defined by

$$\psi_T(f) = \sqrt{W_T(q^{-1})}\Big(\sum_{u \in X_T} f(u\langle T\rangle) e_u\Big) h_T, \tag{7–11}$$

where h_T is the idempotent of $\mathcal{N}_{\mathbf{q}}$ defined in Lemma 5.5. So, we have an isometry:

$$L^2_{\mathbf{q}} C_i(\Sigma_{cc}) = \bigoplus_{T \in \mathcal{S}^{(i)}} L^2(W\langle T\rangle, \mu_{\mathbf{q}}) \xrightarrow{\cong} \bigoplus_{T \in \mathcal{S}^{(i)}} H_T. \tag{7–12}$$

Since each H_T is a left $\mathbf{R}_{\mathbf{q}}[W]$–submodule of $L^2_{\mathbf{q}}$, this gives $L^2_{\mathbf{q}} C^i(\Sigma_{cc})$ the structure of a Hilbert $\mathcal{N}_{\mathbf{q}}$–module. It also gives an isometric embedding

$$\Psi\colon L^2_{\mathbf{q}} C_i(\Sigma_{cc}) \hookrightarrow \bigoplus_{c \in \mathcal{S}^{(i)}} L^2_{\mathbf{q}} = C_i(K) \otimes L^2_{\mathbf{q}}.$$

We use the isomorphism in (7–12) to transport the Hilbert $\mathcal{N}_{\mathbf{q}}$–module structure from the right hand side of (7–12) to $L^2_{\mathbf{q}} C_i(\Sigma_{cc})$. It is proved in [27, Lemma 4.3] that δ and $\partial^{\mathbf{q}}$ are maps of Hilbert $\mathcal{N}_{\mathbf{q}}$–modules. We shall give the argument in Lemma 8.1 below. Hence, we get reduced L^2-(co)homology groups:

$$L^2_{\mathbf{q}} \mathcal{H}^i(\Sigma_{cc}) = \operatorname{Ker} \delta^i / \overline{\operatorname{Im} \delta^{i-1}} \quad \text{and} \quad L^2_{\mathbf{q}} \mathcal{H}_i(\Sigma_{cc}) = \operatorname{Ker} \partial^{\mathbf{q}}_i / \overline{\operatorname{Im} \partial^{\mathbf{q}}_i},$$

which are also Hilbert $\mathcal{N}_{\mathbf{q}}$–modules. It is proved in [27, Section 5] that the (co)homology groups of $L^2_{\mathbf{q}} C_*(\Sigma_{cc})$ are the same as those of $L^2_{\mathbf{q}} C_*(\Sigma)$, ie $L^2_{\mathbf{q}} H_*(\Sigma_{cc}) \cong L^2_{\mathbf{q}} H_*(\Sigma)$, $L^2_{\mathbf{q}} H^*(\Sigma_{cc}) \cong L^2_{\mathbf{q}} H^*(\Sigma)$ and $L^2_{\mathbf{q}} \mathcal{H}^*(\Sigma_{cc}) \cong L^2_{\mathbf{q}} \mathcal{H}^*(\Sigma)$. (The point is that the simplicial structure on Σ is a subdivision of Σ_{cc}.)

The chain complex $(L^2_{\mathbf{q}} C_*(\Sigma_{cc}), \partial^{\mathbf{q}})$ looks like this:

$$L^2_{\mathbf{q}} \longleftarrow \bigoplus_{s \in S} H_s \longleftarrow \bigoplus_{T \in \mathcal{S}^{(2)}} H_T \longleftarrow \cdots$$

(We shall describe the boundary maps explicitly in Lemma 8.1 in the next section.)

$L^2_{\mathbf{q}}$–Betti numbers and the $L^2_{\mathbf{q}}$–Euler characteristic Define

$$c^i_{\mathbf{q}}(\mathcal{U}) := \dim_{\mathcal{N}_{\mathbf{q}}} L^2_{\mathbf{q}} C^i(\mathcal{U}),$$

where $\dim_{\mathcal{N}_{\mathbf{q}}}$ denotes the von Neumann dimension defined by (5–3). For any cell $\sigma \subset Z$, its stabilizer is the special subgroup $W_{S(\sigma)}$, where as before $S(\sigma) = \{s \in S \mid \sigma \subseteq Z_s\}$. So, the summand of $L^2_{\mathbf{q}} C^i(\mathcal{U})$ corresponding to the orbit of an i–cell σ is isomorphic to $A_{S(\sigma)}$. Its dimension is $1/W_{S(\sigma)}(\mathbf{q})$. Hence,

$$c^i_{\mathbf{q}}(\mathcal{U}) = \sum_{\sigma \in Z^{(i)}} \frac{1}{W_{S(\sigma)}(\mathbf{q})}. \tag{7–13}$$

The i–*th* $L^2_{\mathbf{q}}$–*Betti number* of $\mathcal{U}$ is defined by

$$b^i_{\mathbf{q}}(\mathcal{U}) := \dim_{\mathcal{N}_{\mathbf{q}}} L^2_{\mathbf{q}}\mathcal{H}^i(\mathcal{U}). \tag{7–14}$$

A standard argument (cf [29, Theorem 3.6.1, p 205]) gives

$$\sum(-1)^i b^i_{\mathbf{q}}(\mathcal{U}) = \sum(-1)^i c^i_{\mathbf{q}}(\mathcal{U}). \tag{7–15}$$

(This is a version of Atiyah's Formula.) We denote either side of (7–15) by $\chi_{\mathbf{q}}(\mathcal{U})$ and call it the $L^2_{\mathbf{q}}$–*Euler characteristic* of $\mathcal{U}$.

Proposition 7.3 (Rationality of Euler characteristics) *$\chi_{\mathbf{q}}(\mathcal{U}) = f(\mathbf{q})/g(\mathbf{q})$ where f and g are polynomials in $\mathbf{q}$ with integral coefficients.*

Proof For each $T \in \mathcal{S}$, we have the subcomplex Z_T (resp. ∂Z_T) defined as the union of those cells σ such that $T \subseteq S(\sigma)$ (resp. $T \subset S(\sigma)$). By (7–13) and (7–15),

$$\chi_{\mathbf{q}}(\mathcal{U}) = \sum_{T \in \mathcal{S}} \frac{\chi(Z_T) - \chi(\partial Z_T)}{W_T(\mathbf{q})}.$$

□

Proposition 7.4 [27, Corollary 3.4]

$$\chi_{\mathbf{q}}(\Sigma) = \frac{1}{W(\mathbf{q})}.$$

Proof We use the cellulation of Σ by Coxeter cells. If $T \in \mathcal{S}$, then

$$\dim_{\mathcal{N}_{\mathbf{q}}} L^2(W\langle T\rangle, \mu_{\mathbf{q}}) = \dim_{\mathcal{N}_{\mathbf{q}}} H_T = \frac{1}{W_T(\mathbf{q}^{-1})}.$$

Hence,

$$c^i_{\mathbf{q}}(\Sigma_{cc}) = \sum_{T \in \mathcal{S}^{(i)}} \frac{1}{W_T(\mathbf{q}^{-1})}$$

and

$$\chi_{\mathbf{q}}(\Sigma) = \sum_{T \in \mathcal{S}} \frac{\varepsilon(T)}{W_T(\mathbf{q}^{-1})} = \frac{1}{W(\mathbf{q})},$$

where the last equality is by Lemma 3.3(iv). □

Remark The relationship between Euler characteristics (of groups acting on buildings) and growth series of Coxeter groups was first pointed out by Serre [38] (in the case where with fundamental chamber K is a simplex). Serre showed that the "Euler–Poincaré" measure on the automorphism group of the building is (suitably normalized) Haar measure multiplied by $1/W(\mathbf{q})$, where, as in Section 13, $\mathbf{q}$ is the "thickness vector" of the building.

Cohomology in dimension 0 The vertex set of Σ, with its cellulation by Coxeter cells, can be identified with W. So, $L^2_{\mathbf{q}}C^0(\Sigma_{cc}) \cong L^2_{\mathbf{q}}$. A 0–cochain is a cocycle if and only if it is the constant function on W. If c denotes the constant, then its norm, with respect to the inner product $\langle\ ,\ \rangle_{\mathbf{q}}$ is $|c|\sum q_w$ and this is $<\infty$ if and only if $\mathbf{q}\in\mathcal{R}$ or $c=0$. This proves the following result of Dymara [27].

Proposition 7.5 [27] *$L^2_{\mathbf{q}}H^0(\Sigma)$ is nonzero if and only if $\mathbf{q}\in\mathcal{R}$. Moreover, when $\mathbf{q}\in\mathcal{R}$, $b^0_{\mathbf{q}}(\Sigma)=1/W(\mathbf{q})$.*

Remark 7.6 It is easy to see that the space $\mathcal{U}$ $(=\mathcal{U}(W,Z))$ is connected if and only if Z is connected and $Z_s\neq\varnothing$ for each $s\in S$. (This also follows from [13, Theorem A] or [12, Theorem 10.1].) Suppose these conditions hold. An argument similar to the one in the previous paragraph then shows that $L^2_{\mathbf{q}}H^0(\mathcal{U})$ is nonzero if and only if $\mathbf{q}\in\mathcal{R}$ and when this is the case, $b^0_{\mathbf{q}}(\mathcal{U})=1/W(\mathbf{q})$.

The continuity of Betti numbers

Theorem 7.7 *Suppose (W,S) is a Coxeter system and that $\mathcal{U}=\mathcal{U}(W,Z)$ is as above. Then for each integer i, the function $\mathbf{q}\to b^i_{\mathbf{q}}(\mathcal{U})$ is continuous.*

For the proof we will need the next two lemmas.

Lemma 7.8 *Let Y be a Hilbert space, X a closed subspace of Y, P_X the orthogonal projection onto X and $y\in Y$ a unit vector. Set*

$$A(y):=\inf\{\|x\| \mid x\in X,\ \langle x,y\rangle=1\}.$$

Then $\langle P_X(y),y\rangle=A(y)^{-2}$. (By convention, $(+\infty)^{-2}=0$.)

Proof Put $a:=\langle P_X(y),y\rangle$. Since $\langle P_X(y),y\rangle=\|P_X(y)\|^2$, we see that $a\geq 0$ with equality if and only if $X\perp y$. Suppose first that $a=0$. Then the left hand side of the formula in the lemma is 0. Since $X\perp y$, $\{\|x\| \mid x\in X,\ \langle x,y\rangle=1\}=\varnothing$, so $A(y)=+\infty$ and hence, the right hand side is also 0.

Suppose $a>0$. Every $x\in X$ can be written as $bP_X(y)+x'$, where $x'\perp P_X(y)$. Then $\langle x,y\rangle=b\langle P_X(y),y\rangle+\langle x',y\rangle=ba$. (Notice that for $x'\in X$, $x'\perp P_X(y)$ if and only if $x'\perp y$.) So, $\langle x,y\rangle=1$ implies $b=1/a$. Therefore,

$$\begin{aligned}A(y)&=\inf\bigl\{\|x\| \mid x\in X,\ \langle x,y\rangle=1\bigr\}\\&=\inf\bigl\{\|(1/a)\,P_X(y)+x'\| \mid x'\in X,\ x'\perp P_X(y)\bigr\}\\&=\|(1/a)\,P_X(y)\|=(1/a)\,\|P_X(y)\|=\sqrt{a}/a=1/\sqrt{a}.\end{aligned}$$

So, $A(y)^{-2}=a$. □

Given I–tuples $\mathbf{q}$ and $\mathbf{q}'$ of real numbers, write $\mathbf{q} \le \mathbf{q}'$ if $q_i \le q_i'$ for each $i \in I$.

Definition 7.9 A function $f\colon \mathbf{R}^I \to \mathbf{R}$ is *increasing* (resp. *decreasing*) if $\mathbf{q} \le \mathbf{q}'$ implies $f(\mathbf{q}) \le f(\mathbf{q}')$ (resp. $f(\mathbf{q}) \ge f(\mathbf{q}')$).

Let $|\ |$ denote the "maximum norm" on $\mathbf{R}^I$ defined by $|\mathbf{q}| := \max\{|q_i|\}$.

Definition 7.10 A function $f\colon \mathbf{R}^I \to \mathbf{R}$ is *left continuous* at $\mathbf{q}_0$ if for any positive number ε, there is a positive number δ so that if $\mathbf{q} \le \mathbf{q}_0$ and $|\mathbf{q}_0 - \mathbf{q}| < \delta$, then $|f(\mathbf{q}_0) - f(\mathbf{q})| < \varepsilon$. *Right continuity* is similarly defined.

Lemma 7.11 *If a decreasing function $f\colon \mathbf{R}^I \to \mathbf{R}$ is both left and right continuous, then it is continuous (and similarly, if f is increasing).*

Proof Given a point $\mathbf{q}_0 \in \mathbf{R}^I$ and a number $\varepsilon > 0$, choose δ small enough to work in the definitions of both left and right continuous at $\mathbf{q}_0$. Let $\mathbf{d}$ denote the I–tuple with each component equal to δ. Assuming f is decreasing, for any $\mathbf{q}$ in a δ–neighborhood of $\mathbf{q}_0$, we have

$$f(\mathbf{q}_0) - \varepsilon \le f(\mathbf{q}_0 + \mathbf{d}) \le f(\mathbf{q}) \le f(\mathbf{q}_0 - \mathbf{d}) \le f(\mathbf{q}_0) + \varepsilon. \qquad \square$$

Proof of Theorem 7.7 We have spaces of cochains, cocycles and coboundaries

$$C^i_{\mathbf{q}} := L^2_{\mathbf{q}} C^i(\mathcal{U}), \qquad Z^i_{\mathbf{q}} := L^2_{\mathbf{q}} Z^i(\mathcal{U}), \qquad B^i_{\mathbf{q}} := L^2_{\mathbf{q}} B^i(\mathcal{U}),$$

as well as, spaces of chains, cycles and boundaries

$$C_i^{\mathbf{q}} := C^i_{\mathbf{q}}, \qquad Z_i^{\mathbf{q}} := L^2_{\mathbf{q}} Z_i(\mathcal{U}), \qquad B_i^{\mathbf{q}} := L^2_{\mathbf{q}} B_i(\mathcal{U}).$$

($Z^i_{\mathbf{q}}$ and $B^i_{\mathbf{q}}$ are defined using the coboundary map δ, while $Z_i^{\mathbf{q}}$ and $B_i^{\mathbf{q}}$ are defined using its adjoint $\partial^{\mathbf{q}}$.) We also have their von Neumann dimensions:

$$\begin{aligned} c^i_{\mathbf{q}} &:= \dim C^i_{\mathbf{q}}, & z^i_{\mathbf{q}} &:= \dim Z^i_{\mathbf{q}}, & a^i_{\mathbf{q}} &:= \dim B^i_{\mathbf{q}}, \\ c_i^{\mathbf{q}} &:= c^i_{\mathbf{q}}, & z_i^{\mathbf{q}} &:= \dim Z_i^{\mathbf{q}}, & a_i^{\mathbf{q}} &:= \dim B_i^{\mathbf{q}}, \end{aligned}$$

where, to simplify notation, we are writing $\dim(\ )$ instead of $\dim_{\mathcal{N}_{\mathbf{q}}}(\ )$.

We note that by formula (7–13), $\mathbf{q} \to c^i_{\mathbf{q}}$ is a continuous decreasing function (since each $W_{S(\sigma)}(\mathbf{q})$ is a polynomial with nonnegative coefficients).

Claim 1 The function $\mathbf{q} \to z^i_{\mathbf{q}}$ is left continuous and decreasing.

Proof of Claim 1 In (7–9) we defined an isometric embedding Φ of $C^i_{\mathbf{q}}$ into the sum $\bigoplus_{\sigma\in Z^{(i)}} L^2_{\mathbf{q}}$. Let e_1^σ be the element of $\bigoplus_{\sigma\in Z^{(i)}} L^2_{\mathbf{q}}$ with σ–component equal to e_1 and all other components equal 0. Then

$$z^i_{\mathbf{q}} = \sum_{\sigma\in Z^{(i)}} \langle P_{\Phi(Z^i_{\mathbf{q}})}(e_1^\sigma), e_1^\sigma\rangle_{\mathbf{q}}.$$

Since $\Phi(Z^i_{\mathbf{q}}) \subseteq \Phi(C^i_{\mathbf{q}})$, we have

$$\langle P_{\Phi(Z^i_{\mathbf{q}})}(e_1^\sigma), e_1^\sigma\rangle_{\mathbf{q}} = \langle P_{\Phi(Z^i_{\mathbf{q}})} P_{\Phi(C^i_{\mathbf{q}})}(e_1^\sigma), P_{\Phi(C^i_{\mathbf{q}})}(e_1^\sigma)\rangle_{\mathbf{q}}.$$

All components of the vector $P_{\Phi(C^i_{\mathbf{q}})}(e_1^\sigma)$ are 0, except the σ–component, which is equal to $P_{A_{S(\sigma)}}(e_1) = a_{S(\sigma)} = \Phi((W_{S(\sigma)}(\mathbf{q}))^{-1/2}\delta_\sigma)$, where $\delta_\sigma \in C^i_{\mathbf{q}}$ is the function which is 1 on σ and 0 on all other cells. Thus,

$$z^i_{\mathbf{q}} = \sum_{\sigma\in Z^{(i)}} \frac{1}{W_{S(\sigma)}(\mathbf{q})} \langle P_{Z^i_{\mathbf{q}}}(\delta_\sigma), \delta_\sigma\rangle_{\mathbf{q}}.$$

Since $1/W_{S(\sigma)}(\mathbf{q})$ is continuous, we need to concentrate on $\langle P_{Z^i_{\mathbf{q}}}(\delta_\sigma), \delta_\sigma\rangle_{\mathbf{q}}$. Set $z^\sigma(\mathbf{q}) := \inf\{\|u\|_{\mathbf{q}} \mid u \in Z^i_{\mathbf{q}},\ \langle u, \delta_\sigma\rangle_{\mathbf{q}} = 1\}$. By Lemma 7.8, it suffices to prove that each of the functions $z^\sigma(\mathbf{q})$ is left continuous and increasing. Notice that if $\mathbf{q} \le \mathbf{q}'$, then $Z^i_{\mathbf{q}} \supseteq Z^i_{\mathbf{q}'}$. Moreover, if $u \in Z^i_{\mathbf{q}'}$, then $\|u\|_{\mathbf{q}} \le \|u\|_{\mathbf{q}'}$, while $\langle u, \delta_\sigma\rangle_{\mathbf{q}} = \langle u, \delta_\sigma\rangle_{\mathbf{q}'}$ (because both are equal to $u(\sigma)$). It follows that z^σ is an increasing function.

Now suppose that $(\mathbf{q}_n)$ is a sequence in $\mathbf{R}^I$ converging to $\mathbf{q}$ from below (that is, each $\mathbf{q}_n \le \mathbf{q}$) and that $u_n \in Z^i_{\mathbf{q}_n}$ is a sequence such that, $\langle u_n, \delta_\sigma\rangle_{\mathbf{q}_n} = 1$ (ie $u_n(\sigma) = 1$). Further suppose that $\lim \|u_n\|_{\mathbf{q}_n} = \xi$. We will show that $\xi \ge z^\sigma(\mathbf{q})$. (This implies that z^σ is left continuous at $\mathbf{q}$.) Write $\mathcal{U}$ as $\mathcal{U} = \bigcup_{k=1}^\infty K_k$, where the K_k are finite subcomplexes. Assume that $\xi < +\infty$ (otherwise there is nothing to prove). This implies that for every k, the restrictions $u_n|_{K_k}$ are uniformly bounded. Hence, by a diagonal argument, one can choose a subsequence (u_m) such that the $u_m|_{K_k}$ converge pointwise for each k. Let u be the pointwise limit of u_m. Then u is a cocycle and $u(\sigma) = 1$ (because all u_m satisfy these conditions). Also, for each k we have $\|u|_{K_k}\|_{\mathbf{q}} = \lim \|u_m|_{K_k}\|_{\mathbf{q}_m} \le \xi$. Therefore, $\|u\|_{\mathbf{q}} \le \xi$. So, $z^\sigma(\mathbf{q}) \le \|u\|_{\mathbf{q}} \le \xi$. □

Claim 2 $\mathbf{q} \to a^i_{\mathbf{q}}$ is left continuous and so, $\mathbf{q} \to z_i^{\mathbf{q}}$ is left continuous.

Proof of Claim 2 Since $c^{i-1}_{\mathbf{q}}$ is continuous and since $a^i_{\mathbf{q}} = c^{i-1}_{\mathbf{q}} - z^{i-1}_{\mathbf{q}}$, Claim 1 implies that $a^i_{\mathbf{q}}$ is left continuous. We have the Hodge decomposition: $C^i_{\mathbf{q}} = Z_i^{\mathbf{q}} \oplus B^i_{\mathbf{q}}$. So, $z_i^{\mathbf{q}} = c^i_{\mathbf{q}} - a^i_{\mathbf{q}}$, which is left continuous. □

Claim 3 $\mathbf{q} \to z_i^{\mathbf{q}}$ is right continuous and decreasing.

Proof of Claim 3 This is a version of Claim 1 using cycles instead of cocycles. Basically, the argument in Claim 1 works provided we use the usual boundary map ∂ instead of $\partial^{\mathbf{q}}$. To transfer this back into information about $Z_i^{\mathbf{q}}$, we need to use the isometry θ from the proof of Lemma 7.1. Set

$$\widehat{Z}_i^{\mathbf{q}} := \mathrm{Ker}(\partial_i\colon C_i^{\mathbf{q}} \to C_{i-1}^{\mathbf{q}}).$$

As before, $C_i^{\mathbf{q}}$, $Z_i^{\mathbf{q}}$ and $\widehat{Z}_i^{\mathbf{q}}$ can be embedded into $\bigoplus_{\sigma\in Z^{(i)}} L^2_{\mathbf{q}}(W)$, and

$$z_i^{\mathbf{q}} = \sum_{\sigma\in Z^{(i)}} \langle P_{\Phi(Z_i^{\mathbf{q}})}(e_1^\sigma), e_1^\sigma\rangle_{\mathbf{q}}.$$

Consider the isometry $\theta\colon \bigoplus L^2_{\mathbf{q}}(W) \to \bigoplus L^2_{\mathbf{q}^{-1}}(W)$, given by $\theta(f)(w) = \mathbf{q}_w f(w)$ on each component. By Lemma 7.1, θ restricts to a map $C_*^{\mathbf{q}} \to C_*^{\mathbf{q}^{-1}}$, which intertwines $\partial^{\mathbf{q}}$ and ∂. Therefore,

$$\theta(Z_i^{\mathbf{q}}) = \widehat{Z}_i^{\mathbf{q}^{-1}}.$$

Also, $\theta(e_1^\sigma) = e_1^\sigma$, so

$$\theta(P_{\Phi(Z_i^{\mathbf{q}})}(e_1^\sigma)) = P_{\Phi(\widehat{Z}_i^{\mathbf{q}^{-1}})}(e_1^\sigma),$$

and

$$\langle P_{\Phi(Z_i^{\mathbf{q}})}(e_1^\sigma), e_1^\sigma\rangle_{\mathbf{q}} = \langle P_{\Phi(\widehat{Z}_i^{\mathbf{q}^{-1}})}(e_1^\sigma), e_1^\sigma\rangle_{\mathbf{q}^{-1}}.$$

(Note that the map Φ depends on $\mathbf{q}$; thus, the maps on the left hand sides correspond to $\mathbf{q}$, while those on the right hand sides correspond to $\mathbf{q}^{-1}$.) Now, the argument from Claim 1 can be repeated. We get that $\langle P_{\Phi(\widehat{Z}_i^{\mathbf{q}^{-1}})}(e_1^\sigma), e_1^\sigma\rangle_{\mathbf{q}^{-1}}$ is left continuous and increasing in $\mathbf{q}^{-1}$. This implies that the function $\mathbf{q} \to \langle P_{\Phi(Z_i^{\mathbf{q}})}(e_1^\sigma), e_1^\sigma\rangle_{\mathbf{q}}$ is right continuous and decreasing. □

Claim 4 $\mathbf{q} \to a_i^{\mathbf{q}}$ is right continuous and so, $\mathbf{q} \to z_{\mathbf{q}}^i$ is right continuous.

Proof of Claim 4 This follows from Claim 3 in the same way Claim 2 followed from Claim 1. □

Claim 5 $z_{\mathbf{q}}^i$, $z_i^{\mathbf{q}}$, $a_{\mathbf{q}}^i$ and $a_i^{\mathbf{q}}$ are continuous in $\mathbf{q}$

Proof of Claim 5 The functions $z_{\mathbf{q}}^i$ and $z_i^{\mathbf{q}}$ are decreasing and left and right continuous; hence, by Lemma 7.11, continuous. Since $c_{\mathbf{q}}^i$ is continuous, $a_{\mathbf{q}}^i$ and $a_i^{\mathbf{q}}$ are also continuous. □

To finish the proof of Theorem 7.7 simply note that $b_{\mathbf{q}}^i(\mathcal{U}) = z_{\mathbf{q}}^i - a_{\mathbf{q}}^i$, which, by Claim 5, is continuous. □

In view of Proposition 7.3 and Atiyah's Conjecture (cf Eckmann [29, Section 3.10] or Lück[33, Chapter 10]), it is natural to ask the following.

Question *Is* $\mathbf{q} \to b^i_{\mathbf{q}}(\mathcal{U})$ *a piecewise rational function?*

8 Weighted L^2–homology of ruins

Cosheaves Suppose Λ is a simplicial complex with vertex set V and that $\mathcal{S}(\Lambda)$ is its face poset (including the empty face). We regard the poset $\mathcal{S}(\Lambda)$ as a category in the usual way: if τ is a face of σ, then there is a unique morphism ι^τ_σ from τ to σ (which we can think of as being the inclusion of vertex sets).

A *cosheaf* on Λ with values in a category $\mathcal{C}$ is a contravariant functor F from $\mathcal{S}(\Lambda)$ to $\mathcal{C}$. In the case of interest to us, $\mathcal{C}$ will be the category of Hilbert $\mathcal{N}_{\mathbf{q}}$–modules.

Now suppose that the simplicial complex Λ is ordered (in other words, suppose that its vertex set is totally ordered). Then for any $n \geq 0$, the vertices of an n–simplex form an ordered set isomorphic to $\{0, 1, \dots, n\}$ with its usual order. For $0 \leq i \leq n$ and any n–simplex σ, the i–th face of σ is defined to be the $(n-1)$–simplex spanned by all vertices of σ except the i–th. If one writes ∂_i for ι^τ_σ, where τ is the i–th face of σ, then the relations between the morphisms become the familiar "simplicial identities" as in [42, 8.1].

A cosheaf F of abelian groups on an ordered simplicial complex Λ gives rise to a chain complex $C_*(\mathcal{S}(\Lambda); F)$ defined as follows: $C_n = 0$ for $n < 0$, and for $n \geq 0$,

$$C_n = \bigoplus_{\sigma \in \mathcal{S}^{(n)}(\Lambda)} F(\sigma),$$

where the indexing set is the set of $(n-1)$–simplices of Λ. (The indices on C_* have been shifted up by one from the conventions in [19].) Under the natural isomorphism

$$\mathrm{Hom}(C_n, C_{n-1}) \cong \bigoplus_{\substack{\sigma \in \mathcal{S}^{(n)}(\Lambda), \\ \tau \in \mathcal{S}^{(n-1)}(\Lambda)}} \mathrm{Hom}(F(\sigma), F(\tau)),$$

the boundary map $\partial\colon C_n \to C_{n-1}$ corresponds to the matrix $(\partial_{\sigma\tau})$, where $\partial_{\sigma\tau} = 0$ unless τ is a face of σ, and is equal to $(-1)^i F(\iota^\tau_\sigma)$ if τ is the ith face of σ.

Ruined chain complexes We return to the situation where (W, S) is a Coxeter system, L is its nerve and $\mathcal{S}$ is the poset of spherical subsets of S. Let $T \in \mathcal{S}$ and let $\mathrm{Lk}(T, L)$ denote the link of T in L. (If $T = \varnothing$, $\mathrm{Lk}(\varnothing, L) := L$.) We note that the face poset of $\mathrm{Lk}(T, L)$ is isomorphic to $\mathcal{S}_{\geq T}$.

Define a cosheaf H_T of Hilbert $\mathcal{N}_{\mathbf{q}}$–modules on $\mathrm{Lk}(T, L)$ as the contravariant functor on $\mathcal{S}_{\geq T}$ defined on objects by $U \to H_U$ where H_U is defined by (5–17). For objects $U \leq V \in \mathcal{S}_{\geq T}$, the morphism $H(\iota_V^U)\colon H_V \to H_U$ is the natural inclusion. Define the *ruined chain complex* $L^2_{\mathbf{q}} C_*(H_T)$ by

$$L^2_{\mathbf{q}} C_*(H_T) := C_*(\mathcal{S}_{\geq T}; H).$$

It looks like this:

$$0 \longleftarrow H_T \longleftarrow \bigoplus_{\substack{(T\cup\{s\})\\ \in(\mathcal{S}_{\geq T})^{(k+1)}}} H_{T\cup\{s\}} \longleftarrow \cdots, \tag{8–1}$$

where $k = \mathrm{Card}(T)$. Similarly, by using the family $(A_U)_{U\in\mathcal{S}_{\geq T}}$ of Hilbert $\mathcal{N}_{\mathbf{q}}$–modules, we get a cosheaf A on $\mathrm{Lk}(T, L)$ and a chain complex $L^2_{\mathbf{q}} C_*(A_T)$.

Denote the homology of $L^2_{\mathbf{q}} C_*(H_T)$ and $L^2_{\mathbf{q}} C_*(A_T)$ by $L^2_{\mathbf{q}} H_*(H_T)$ and $L^2_{\mathbf{q}} H_*(A_T)$, respectively.

The relationship between ruins and ruined chain complexes Recall that for any $U \subseteq S$ and $T \in \mathcal{S}(U)$, $\Omega(U, T)$ is the subcomplex of Σ_{cc} consisting of all closed Coxeter cells of type T', with $T' \in \mathcal{S}(U)_{\geq T}$.

To simplify notation, the chain complex $L^2_{\mathbf{q}} C_*(\Omega(U, T), \partial(\Omega(U, T))$ will be denoted $L^2_{\mathbf{q}} C_*(\Omega(U, T), \partial)$ and similarly for its homology.

Since the cell structure always will be given by Coxeter cells, we will omit the subscript cc from our notation. We say a Coxeter cell is *type* T, $T \in \mathcal{S}$, as a shorthand for type W_T.

It follows from (6–5) and the fact that $\mathcal{U}(W_U, K(U))$ deformation retracts onto Σ_U that the $\mathcal{N}_{\mathbf{q}}[W]$–modules $L^2_{\mathbf{q}} C_*(\Sigma(U))$ and $L^2_{\mathbf{q}} \mathcal{H}_*(\Sigma(U))$ are induced from the $\mathcal{N}_{\mathbf{q}}[W_U]$–modules $L^2_{\mathbf{q}} C_*(\Sigma_{W_U})$ and $L^2_{\mathbf{q}} \mathcal{H}_*(\Sigma_{W_U})$, respectively. So, we can calculate von Neumann dimensions over $\mathcal{N}_{\mathbf{q}}[W]$ by calculating with respect to $\mathcal{N}_{\mathbf{q}}[W_U]$.

Lemma 8.1 (i) *There is a isomorphism of chain complexes of $\mathcal{N}_{\mathbf{q}}$–modules:*

$$\Psi'\colon L^2_{\mathbf{q}} C_*(\Sigma_{cc}) \to L^2_{\mathbf{q}} C_*(H_\varnothing),$$

for $L^2_{\mathbf{q}} C_*(H_\varnothing)$ the ruined chain complex associated to the cosheaf $H_\varnothing$ on L.

(ii) *Suppose $T \in \mathcal{S}^{(k)}$. Then Ψ' induces an isomorphism of chain complexes of $\mathcal{N}_{\mathbf{q}}$–modules:*

$$L^2_{\mathbf{q}}C_*(H_T) \xrightarrow{\cong} L^2_{\mathbf{q}}C_{*+k}(\Omega(S,T),\partial).$$

In particular, $L^2_{\mathbf{q}}C_m(\Omega(S,T),\partial) = 0$ for $m < k$.

Proof (i) For each $T \in \mathcal{S}$ modify the isometry ψ_T of (7–11) to another Hilbert $\mathcal{N}_{\mathbf{q}}$–module isomorphism, $\psi'_T\colon L^2(W\langle T\rangle, \mu_{\mathbf{q}}) \to H_T$ as follows:

$$\psi'_T(f) := \sqrt{W_T(\mathbf{q}^{-1})}\,\psi_T(f) = W_T(\mathbf{q}^{-1})\Big(\sum_{u\in X_T} f(u\langle T\rangle)e_u\Big)h_T. \tag{8–2}$$

Ψ' is defined to be the direct sum of the ψ'_T. Suppose $U \in \mathcal{S}_{>\varnothing}$ and $T \subset U$ is obtained by deleting one element from U. Statement (i) follows immediately from the next claim.

Claim The following diagram commutes:

$$\begin{array}{ccc} L^2(W\langle U\rangle, \mu_{\mathbf{q}}) & \xrightarrow{\psi'_U} & H_U \\ {\scriptstyle \partial^{\mathbf{q}}_T}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ L^2(W\langle T\rangle, \mu_{\mathbf{q}}) & \xrightarrow{\psi'_T} & H_T \end{array}$$

where $\partial^{\mathbf{q}}_T$ denotes the $L^2(W\langle T\rangle, \mu_{\mathbf{q}})$–component of $\partial^{\mathbf{q}}$ and i is the natural inclusion.

Proof of Claim Using (8–2) and (7–10), we get

$$\begin{aligned}
\psi'_T(\partial^{\mathbf{q}}_T f) &= W_T(\mathbf{q}^{-1})\Big(\sum_{w\in X_T} (\partial^{\mathbf{q}}_T f)(w\langle T\rangle)e_w\Big)h_T \\
&= W_T(\mathbf{q}^{-1})\Big(\sum_{u\in X_U}\ \sum_{v\in W_U\cap X_T} \varepsilon_v q_v^{-1} f(u\langle U\rangle)e_u e_v\Big)h_T \\
&= W_T(\mathbf{q}^{-1})\Big(\sum_{u\in X_U} f(u\langle U\rangle)e_u\Big)\Big(\sum_{v\in W_U\cap X_T} \varepsilon_v q_v^{-1} e_v\Big)h_T \\
&= W_U(\mathbf{q}^{-1})\Big(\sum_{u\in X_U} f(u\langle U\rangle)e_u\Big)h_U \\
&= i(\psi'_U(f)),
\end{aligned}$$

where the next to last equality is from the following formula for h_U, valid whenever $T \subseteq U$ and $\mathbf{q} \in \mathcal{R}_U^{-1}$:

$$h_U = \Big(\sum_{v \in W_U \cap X_T} \varepsilon_v q_v^{-1} e_v \Big) h_T.$$

(This formula holds since $W_U \cap X_T$ is a set of coset representatives for W_U / W_T and since for any $v \in W_U \cap X_T$ and $w \in W_T$, we have $e_v e_w = e_{vw}$ and $q_v q_w = q_{vw}$.) □

(ii) Part (ii) of the lemma essentially follows from part (i). Write Ω for $\Omega(S,T)$. The point is that the cells of the $W\langle T' \rangle$, $T' \in (\mathcal{S}_{\geq T})^{(i+1)}$, are a basis for $L^2_{\mathbf{q}} C^i(\Omega, \partial\Omega)$. Hence,

$$L^2_{\mathbf{q}} C_i(\Omega, \partial\Omega) = \bigoplus_{T' \in (\mathcal{S}_{\geq T})^{(i+1)}} L^2(W\langle T' \rangle, \mu_{\mathbf{q}}) \cong \bigoplus H_{T'}$$

and (i) shows that the $\partial^{\mathbf{q}}$ maps are induced by the inclusions $H_{T''} \hookrightarrow H_{T'}$, with $T' \subset T''$. □

Remark The cochain complex $L^2_{\mathbf{q}} C^i(\Omega(S,T), \partial)$ is obtained by dualizing (8–1):

$$0 \longrightarrow H_T \longrightarrow \bigoplus_{\substack{(T \cup \{s\}) \\ \in (\mathcal{S}_{\geq T})^{(k+1)}}} H_{T \cup \{s\}} \longrightarrow \cdots$$

where the coboundary maps are induced by the orthogonal projections $H_{T'} \to H_{T''}$, with $T' \subset T''$, and $k = \mathrm{Card}(T)$.

The main result of this section as well as the results of Section 9–Section 12 ultimately are based on the following key theorem from [27].

Theorem 8.2 [27, Theorem 10.3] *If $\mathbf{q} \in \mathcal{R}$, then $L^2_{\mathbf{q}} H_*(\Sigma)$ is concentrated in dimension 0.*

While the proof of this in [27] is straightforward, some technical estimates are involved. In outline the argument goes as follows.

(a) Using the CAT(0)–metric of [36] it is proved, in [27, Theorem 9.1], that there is a chain contraction $H\colon C_*(\Sigma) \to C_{*+1}(\Sigma)$ and constants C and R such that for any simplex $\sigma \subset \Sigma$, (i) the L^∞–norm of $H(\sigma)$ is $< C$ and (ii) $H(\sigma)$ is supported in an R–neighborhood of the geodesic connecting the central vertex of K with σ.

(b) It follows [27, Theorem 10.1] that for $\mathbf{q} \in \mathcal{R}^{-1}$, H extends to a bounded linear map $H\colon L^2_{\mathbf{q}}C_*(\Sigma) \to L^2_{\mathbf{q}}C_{*+1}(\Sigma)$. (Actually, in [27], this is only proved for a single parameter, but the proof goes through without change in the case of a multiparameter $\mathbf{q}$.) Hence, for $\mathbf{q} \in \mathcal{R}^{-1}$, H is a chain contraction of $L^2_{\mathbf{q}}C_*(\sigma)$ with respect to the usual boundary map ∂.

(c) Finally, one uses the isometry θ of Lemma 7.1 to transport H to a chain contraction of $(L^2_{\mathbf{q}}C_*(\sigma), \partial^{\mathbf{q}})$ for $\mathbf{q} \in \mathcal{R}$.

The main result of this section is the following generalization of Theorem 8.2.

Theorem 8.3 *Suppose $T \in \mathcal{S}^{(k)}$. If $\mathbf{q} \in \mathcal{R}$, then $L^2_{\mathbf{q}}H_*(\Omega(S,T),\partial)$ is concentrated in dimension k. If $\mathbf{q} \in \partial\mathcal{R}$, the same holds for $L^2_{\mathbf{q}}\mathcal{H}_*(\Omega(S,T),\partial)$.*

Note that the third sentence of the theorem follows from the second one and the continuity of the $b^i_{\mathbf{q}}$ (Theorem 7.7).

In the special case $T = \varnothing$, we have $\Omega(S,T) = \Sigma$ and so Theorem 8.3 is Theorem 8.2. We shall use Theorem 8.2 as the first step in an inductive proof.

Before beginning the proof, note that we have an excision isomorphism:

$$L^2_{\mathbf{q}}C_*(\Omega(U,T),\partial) \cong L^2_{\mathbf{q}}C_*(\Sigma(U), \widehat{\Omega}(U,T)). \tag{8–3}$$

Also, for any $s \in T$ and $T' := T - s$, we have an excision isomorphism:

$$L^2_{\mathbf{q}}C_*(\Sigma(U-s), \widehat{\Omega}(U-s,T')) \cong L^2_{\mathbf{q}}C_*(\widehat{\Omega}(U,T), \widehat{\Omega}(U,T')). \tag{8–4}$$

Proof of Theorem 8.3 Suppose $U \subseteq S$ and $T \in \mathcal{S}^{(k)}(U)$. We shall prove, by induction on k $(= \mathrm{Card}(T))$, that $L^2_{\mathbf{q}}H_*(\Omega(U,T),\partial)$ is concentrated in dimension k. When $k = 0$ this holds by Theorem 8.2 (and the fact that $L^2_{\mathbf{q}}C_*(\Sigma(U))$ is induced from $L^2_{\mathbf{q}}C_*(\Sigma_{W_U})$). Assume by induction, that our assertion holds for $k-1$, with $k-1 \geq 0$. By (8–3), the assertion is equivalent to showing that $L^2_{\mathbf{q}}H_*(\Sigma(U), \widehat{\Omega}(U,T))$ is concentrated in dimension k. Choose an element $s \in T$ and set $T' := T - s$, $\widehat{\Omega} := \widehat{\Omega}(U,T)$, $\widehat{\Omega}' := \widehat{\Omega}(U,T')$. Consider the long exact sequence of the triple $(\Sigma(U), \widehat{\Omega}, \widehat{\Omega}')$:

$$L^2_{\mathbf{q}}H_*(\Sigma(U), \widehat{\Omega}') \to L^2_{\mathbf{q}}H_*(\Sigma(U), \widehat{\Omega}) \to L^2_{\mathbf{q}}H_{*-1}(\widehat{\Omega}, \widehat{\Omega}')$$

By (8–4), the right hand term excises to the homology of the $(U-s, T')$–ruin, while the middle term is that of the (U,T)–ruin and the left hand term is that of the (U,T')–ruin. By induction, the left hand and right hand terms are concentrated in dimension $k-1$. So, the middle term can only be nonzero in dimensions $k-1$ and k. On the other hand, by Lemma 8.1(ii), the middle term vanishes in dimensions $< k$. □

Combining this theorem with Lemma 8.1, we get the following.

Corollary 8.4 *For any $\mathbf{q} \in \mathcal{R}$ and any spherical subset T, $L^2_{\mathbf{q}} H_*(H_T)$ is concentrated in dimension 0. Therefore, for any $\mathbf{q} \in \overline{\mathcal{R}}$, the reduced homology $L^2_{\mathbf{q}}\mathcal{H}_*(H_T)$ is also concentrated in dimension 0.*

The meaning of this corollary is that, for $\mathbf{q} \in \overline{\mathcal{R}}$, the family of subspaces $(H_T)_{T \in \mathcal{S}}$ is "in general position" in $L^2_{\mathbf{q}}$.

9 The Decomposition Theorem

Lemma 9.1 (Compare Lemma 1 in [39].) *Suppose we are given subsets U, V of S and an I–tuple $\mathbf{q} \in \mathcal{R}_V \cap \mathcal{R}_U^{-1}$ (so that h_U and a_V are both defined). If $V \cap U \neq \varnothing$, then $h_U a_V = 0$.*

Proof Let $s \in V \cap U$. Then $h_U a_V = h_U h_s a_s a_V = 0$. □

We define some more subspaces of $L^2_{\mathbf{q}}$:

$$D_V := A_{S-V} \cap \Big(\sum_{U \subset V} A_{S-U} \Big)^{\perp}$$

$$G_V := H_V \cap \Big(\sum_{U \supset V} H_U \Big)^{\perp}$$

Lemma 9.2 *The following holds:*

$$\overline{\sum_{U \supseteq V} G_U} = H_V$$

$$\overline{\sum_{V \subseteq U} D_V} = A_{S-U}$$

Proof By definition of G_V, we have

$$H_V = G_V + \overline{\sum_{U \supset V} H_U},$$

and the first formula follows by induction on the size of $S - V$. Similarly,

$$A_{S-V} = D_V + \overline{\sum_{U \subset V} A_{S-U}},$$

and the second formula follows by induction on the size of V. □

Lemma 9.3 *Suppose* $\mathbf{q} \in \mathcal{R}$ *and* $U \nsubseteq V$. *Then*

$$G_U a_{S-V} = 0.$$

Proof Since $G_T \subseteq H_T$ the assertion follows from Lemma 9.1. □

If $\mathbf{q} \in \mathcal{R}$, then $H_V = 0$ for all nonspherical V (because V is spherical whenever $\mathcal{R}_V \cap \mathcal{R}_V^{-1} \neq \varnothing$). So, for $V \notin \mathcal{S}$, $G_V = 0$, and for $V = T \in \mathcal{S}$, G_T is the orthogonal complement of the image of ∂: $L^2_{\mathbf{q}} C_1(H_T) \to L^2_{\mathbf{q}} C_0(H_T) = H_T$; hence, $L^2_{\mathbf{q}} \mathcal{H}_0(H_T) = G_T$.

Denote by $R(\mathcal{N}_{\mathbf{q}})$ the Grothendieck group of Hilbert $\mathcal{N}_{\mathbf{q}}$–modules. If F is such a Hilbert module, $[F]$ denotes its class in $R(\mathcal{N}_{\mathbf{q}})$. It follows from additivity of dimension that the function $F \to \dim_{\mathcal{N}_{\mathbf{q}}} F$ induces a homomorphism $\dim_{\mathcal{N}_{\mathbf{q}}}\colon R(\mathcal{N}_{\mathbf{q}}) \to \mathbf{R}$.

Corollary 9.4 *For* $\mathbf{q} \in \overline{\mathcal{R}}$ *and* $T \in \mathcal{S}$, *the following formulas hold in the representation group* $R(\mathcal{N}_{\mathbf{q}})$:

$$[G_T] = \sum_{U \in \mathcal{S}_{\geq T}} \varepsilon(U-T)[H_U]$$

$$[H_T] = \sum_{U \in \mathcal{S}_{\geq T}} [G_U]$$

Proof Note that in $L^2_{\mathbf{q}} C_*(H_T)$ the boundary maps are maps of Hilbert $\mathcal{N}_{\mathbf{q}}$–modules. Hence, the first formula follows from Corollary 8.4 by taking the Euler characteristics. The second formula follows from this and the Möbius Inversion Formula. □

Corollary 9.5 *Suppose* $\mathbf{q} \in \overline{\mathcal{R}}$ *and* $T \in \mathcal{S}$. *Then* $\dim_{\mathcal{N}_{\mathbf{q}}} G_T = W^T(\mathbf{q})/W(\mathbf{q})$.

Proof By Lemma 5.10(iii), $\dim_{\mathcal{N}_{\mathbf{q}}} H_U = 1/W_U(\mathbf{q}^{-1})$. So,

$$\dim_{\mathcal{N}_{\mathbf{q}}} G_T = \sum_{U \in \mathcal{S}_{\geq T}} \frac{\varepsilon(U-T)}{W_U(\mathbf{q}^{-1})} = \frac{W^T(\mathbf{q})}{W(\mathbf{q})},$$

where the first equality is by Corollary 9.4 and the second by Lemma 3.3(iii)(b). □

Lemma 9.6 *If* $\mathbf{q} \in \overline{\mathcal{R}}$ *and* $U \subseteq S$, *then*

$$\sum_{\substack{T \in \mathcal{S} \\ T \subseteq U}} G_T a_{S-U}$$

is a dense subspace of A_{S-U} *and a direct sum decomposition. Moreover, if* $T \in \mathcal{S}$, *then right multiplication by* a_{S-T} *induces a weak isomorphism* $G_T \to G_T a_{S-T}$.

Proof As in Section 3, X_{S-U} denotes the set of $(\emptyset, S-U)$–reduced elements. As in [4, Example 26], X_{S-U} is the disjoint union of the W^T, $T \subseteq U$. Hence, $X_{S-U}(\mathbf{q}) = \sum_{T\subseteq U} W^T(\mathbf{q})$. Dividing this by $W(\mathbf{q})$ and using Lemma 3.3(ii), we get

$$\frac{1}{W_{S-U}(\mathbf{q})} = \sum_{T\subseteq U} \frac{W^T(\mathbf{q})}{W(\mathbf{q})}.$$

By Lemma 9.2, $$L^2_{\mathbf{q}} = \overline{\sum_{T\in\mathcal{S}} G_T}.$$

Multiplying on the right by a_{S-U} and using Lemma 9.3 we obtain:

$$A_{S-U} = \overline{\sum_{T\in\mathcal{S}} G_T a_{S-U}} = \overline{\sum_{\substack{T\in\mathcal{S}\\ T\subset U}} G_T a_{S-U}}.$$

So, $$\begin{aligned}\dim_{\mathcal{N}_{\mathbf{q}}} A_{S-U} &\le \sum_{\substack{T\in\mathcal{S}\\ T\subset U}} \dim_{\mathcal{N}_{\mathbf{q}}} G_T a_{S-U} \le \sum_{\substack{T\in\mathcal{S}\\ T\subset U}} \dim_{\mathcal{N}_{\mathbf{q}}} G_T = \sum_{T\subset U} \frac{W^T(\mathbf{q})}{W(\mathbf{q})}\\ &= \frac{1}{W_{S-U}(\mathbf{q})} = \dim_{\mathcal{N}_{\mathbf{q}}} A_{S-U},\end{aligned}$$

where the last equality is from Lemma 5.9(ii). Hence, both inequalities are equalities and

$$\dim_{\mathcal{N}_{\mathbf{q}}} \overline{G_T a_{S-T}} = \dim_{\mathcal{N}_{\mathbf{q}}} G_T.$$

It follows that right multiplication by a_{S-T} is a weak isomorphism from G_T to $G_T a_{S-T}$ and that the sum is direct. □

Remark In what follows we will use the symbol $\biguplus$ to denote the sum of submodules of $L^2_{\mathbf{q}}$, once we have proved that the sum is direct.

Since $G_V = 0$ for nonspherical V and $\mathbf{q} \in \overline{\mathcal{R}}$, we can restate Lemma 9.6 as follows:

$$A_{S-U} = \overline{\biguplus_{V\subseteq U} G_V a_{S-U}}.$$

Letting $U = S$, we get the following corollary.

Corollary 9.7 *If $\mathbf{q} \in \overline{\mathcal{R}}$, then*

$$\sum_{V\subset S} G_V$$

is a dense subspace of $L^2_{\mathbf{q}}$ and a direct sum decomposition.

The fact that the sum of G_V is direct has the following two corollaries.

Corollary 9.8 *Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of S. If $\mathbf{q} \in \overline{\mathcal{R}}$, then*

$$\overline{\biguplus_{U\in\mathcal{A}} G_U} \cap \overline{\biguplus_{U\in\mathcal{B}} G_U} = \overline{\biguplus_{U\in\mathcal{A}\cap\mathcal{B}} G_U}.$$

Corollary 9.9 *If $\mathbf{q} \in \overline{\mathcal{R}}$ and $V \subseteq S$, then*

$$H_V = \overline{\biguplus_{U\supseteq V} G_U}.$$

Lemma 9.10 *If $\mathbf{q} \in \overline{\mathcal{R}}$ and $V \subseteq S$, then*

$$D_V = \overline{G_V a_{S-V}}.$$

In particular, $D_V = 0$ if $V \notin \mathcal{S}$.

Proof Since, by definition, $D_V \subseteq A_{S-V}$, the equality $D_V = D_V a_{S-V}$ holds and since $D_V \subseteq \big(\sum_{U\subset V} A_{S-U}\big)^{\perp}$, we have

$$D_V \subseteq \Big(\sum_{U\subset V} A_{S-U}\Big)^{\perp} a_{S-V}.$$

Using equations (5–19), we compute

$$\Big(\sum_{U\subset V} A_{S-U}\Big)^{\perp} = \bigcap_{U\subset V} A_{S-U}^{\perp} = \bigcap_{U\subset V} \overline{\sum_{s\in S-U} H_s}.$$

By Corollary 9.9, $H_s = \overline{\biguplus_{X\ni s} G_X}$. Therefore

$$\Big(\sum_{U\subset V} A_{S-U}\Big)^{\perp} = \bigcap_{U\subset V} \overline{\sum_{s\in(S-U)} \biguplus_{X\ni s} G_X} = \bigcap_{U\subset V} \overline{\biguplus_{X\not\subseteq U} G_X}.$$

Using Corollary 9.8 we obtain

$$\Big(\sum_{U\subset V} A_{S-U}\Big)^{\perp} = \overline{\biguplus_{X\not\subseteq U\ \forall U\subset V} G_X} = \overline{\biguplus_{X\not\subset V} G_X}.$$

Thus, we have
$$D_V \subseteq \left(\overline{\biguplus_{X\not\subset V} G_X}\right) a_{S-V} = \overline{\sum_{X\not\subset V} G_X a_{S-V}}.$$

By Lemma 9.3, the only nonzero term in the last sum is when $X = V$. Therefore, $D_V \subseteq \overline{G_V a_{S-V}}$.

To prove the opposite inclusion, note that, by Lemma 9.3, for all $U \subset V$, we have $G_V a_{S-V} a_{S-U} = G_V a_{S-U} = 0$. Therefore, since $\operatorname{Ker} a_{S-U} = A^{\perp}_{S-U}$, we have $G_V a_{S-V} \subseteq A^{\perp}_{S-U}$ for all $U \subset V$. Since $G_V a_{S-V} \subseteq A_{S-V}$, it follows from the definition of D_V that $G_V a_{S-V} \subseteq D_V$. □

We shall need the following Decomposition Theorem. (Of course, there is also a corresponding version with the D_V replaced by G_V.)

Theorem 9.11 (The Decomposition Theorem) *If $\mathbf{q} \in \overline{\mathcal{R}} \cup \overline{\mathcal{R}^{-1}}$, then*

$$\sum_{V \subseteq S} D_V$$

is direct and a dense subspace of $L^2_{\mathbf{q}}$. Moreover, if $\mathbf{q} \in \overline{\mathcal{R}}$, then the only nonzero terms in the sum are those with $V \in \mathcal{S}$, and if $\mathbf{q} \in \overline{\mathcal{R}^{-1}}$, then the only nonzero terms in the sum are those with $S - V \in \mathcal{S}$.

Proof If $\mathbf{q} \in \overline{\mathcal{R}}$, then we let $U = S$ in Lemma 9.2 to obtain:

$$L^2_{\mathbf{q}} = \overline{\sum_{V \subseteq S} D_V}.$$

The assertion follows, since by Lemma 9.10, all nonspherical V have 0 contributions, and by Lemma 9.10, Lemma 9.6 and Corollary 9.7, the dimensions of the nontrivial terms add up to 1.

If $\mathbf{q} \in \overline{\mathcal{R}^{-1}}$, the result follows from Corollary 9.7 by applying the j–homomorphism. □

Corollary 9.12 *Let $\mathcal{A}$ be a collection of subsets of S and let $U \subseteq S$.*

If $\mathbf{q} \in \overline{\mathcal{R}} \cup \overline{\mathcal{R}^{-1}}$, then

$$D_U \cap \overline{\biguplus_{U \in \mathcal{A}} D_V} = \begin{cases} 0 & \text{if } U \notin \mathcal{A}, \\ D_U & \text{if } U \in \mathcal{A}. \end{cases}$$

Corollary 9.13 *If $\mathbf{q} \in \overline{\mathcal{R}} \cup \overline{\mathcal{R}^{-1}}$ and $U \subseteq S$, then*

$$A_U = \overline{\biguplus_{V \subseteq S-U} D_V}.$$

Corollary 9.14 (Compare Lemma 3.3 and Corollary 9.5.) *Suppose $T \in \mathcal{S}$.*

(i) *For $\mathbf{q} \in \mathcal{R}$, $\dim_{\mathcal{N}_\mathbf{q}} D_T = W^T(\mathbf{q})/W(\mathbf{q})$.*

(ii) *For $\mathbf{q} \in \mathcal{R}^{-1}$, $\dim_{\mathcal{N}_\mathbf{q}} D_{S-T} = W^T(\mathbf{q}^{-1})/W(\mathbf{q}^{-1})$.*

Proof (i) By Lemma 9.10, a_{S-T} maps G_T monomorphically onto a dense subspace of D_T. So, $\dim_{\mathcal{N}_\mathbf{q}} D_T = \dim_{\mathcal{N}_\mathbf{q}} G_T = W^T(\mathbf{q})/W(\mathbf{q})$, where the second equality is by Corollary 9.5.

(ii) For $\mathbf{q} \in \mathcal{R}^{-1}$, the following formulas hold in the representation ring $R(\mathcal{N}_\mathbf{q})$:

$$[A_T] = \sum_{U \in \mathcal{S}_{\geq T}} [D_{S-U}],$$
$$[D_{S-T}] = \sum_{U \in \mathcal{S}_{\geq T}} \varepsilon(U-T)[A_U],$$

where the first formula is from Corollary 9.13 and the second follows from the first by the Möbius Inversion Formula. So, as in Corollary 9.5,

$$\dim_{\mathcal{N}_\mathbf{q}} D_{S-T} = \sum_{U \in \mathcal{S}_{\geq T}} \frac{\varepsilon(U-T)}{W_U(\mathbf{q})} = \frac{W^T(\mathbf{q}^{-1})}{W(\mathbf{q}^{-1})},$$

where the second equality is Lemma 3.3(iii)(b). □

In Section 11 we will need the following version of Lemma 9.6 and Lemma 9.10. Its proof is essentially the same as the proofs of these lemmas, except that we use Theorem 9.11 and its corollaries instead of the corresponding statements involving the G_U.

Lemma 9.15 (Compare Lemma 9.6 and Lemma 9.10.) *Suppose $\mathbf{q} \in \overline{\mathcal{R}}$ and $U \subseteq S$. Then*

(i) $$\sum_{\substack{T \in \mathcal{S} \\ T \subseteq U}} D_T h_U$$

is a dense subspace of H_U and a direct sum decomposition. Moreover, if $T \in \mathcal{S}$, then the right multiplication by h_T induces a weak isomorphism $D_T \to D_T h_T$.

(ii) $G_U = \overline{D_U h_U}$.

10 Decoupling cohomology

We retain notation from Section 6 and Section 7, eg Z is a finite CW complex, $(Z_s)_{s\in S}$ is a family of subcomplexes and $\mathcal{U} = (W \times Z)/\sim$. As in (6–1), given $U \subseteq S$, Z^U denotes the union of mirrors Z_s, $s \in U$.

For any Hilbert $\mathcal{N}_{\mathbf{q}}$–submodule E of $L^2_{\mathbf{q}}$, define

$$L^2_{\mathbf{q}} C^i(\mathcal{U}; E) := \Phi^{-1}(C^i(Z) \otimes E),$$

where Φ: $L^2_{\mathbf{q}} C^i(\mathcal{U}) \hookrightarrow C^i(Z) \otimes L^2_{\mathbf{q}}$ is the monomorphism defined in (7–9). In other words,

$$L^2_{\mathbf{q}} C^i(\mathcal{U}; E) = \bigoplus_{c\in Z^{(i)}} (\phi_c)^{-1}(A_{S(c)} \cap E), \tag{10–1}$$

where $S(c)$ is the subset of S defined in (6–3) and ϕ_c: $L^2(Wc, \mu_{\mathbf{q}}) \to A_{S(c)}$ is the isomorphism defined in 7.

Proposition 10.1 *Suppose $\mathbf{q} \in \mathcal{R} \cup \mathcal{R}^{-1}$. Then the map Φ restricts to an isomorphism of cochain complexes:*

$$L^2_{\mathbf{q}} C^*(\mathcal{U}; D_U) \xrightarrow{\cong} C^*(Z, Z^U) \otimes D_U.$$

Proof (Compare the proof of Theorem B in [13].) Let $c \in Z$ be an i–cell. By Corollary 9.13,

$$A_{S(c)} = \overline{\biguplus_{V\subseteq S-S(c)} D_V}.$$

If $c \not\subseteq Z^U$, then $S(c) \subseteq S - U$ and therefore, by Corollary 9.12, $A_{S(c)} \cap D_U = D_U$ and so, by (10–1), ϕ_c: $L^2_{\mathbf{q}} C^i(Wc; D_U) \to D_U$ is an isomorphism. If $c \subseteq Z^U$, then $S(c) \not\subseteq S - U$ and therefore, $A_{S(c)} \cap D_U = 0$ and so, by (10–1), $L^2_{\mathbf{q}} C^i(Wc; D_U) = 0$. Hence, a cochain in $C^i(Z) \otimes D_U$ is in the image of the restriction of Φ if and only if it evaluates to 0 on the orbit of every i–cell $c \subseteq Z^U$. □

Suppose $\mathbf{q} \in \mathcal{R} \cup \mathcal{R}^{-1}$. Let Θ_U: $C^*(Z, Z^U) \otimes D_U \to L^2_{\mathbf{q}} C^*(\mathcal{U}; D_U)$ be the inverse of the isomorphism of Proposition 10.1. Define

$$\Theta: \bigoplus_{U\subseteq S} C^*(Z, Z^U) \otimes D_U \longrightarrow L^2_{\mathbf{q}} C^*(\mathcal{U})$$

to be the sum of the Θ_U.

Proposition 10.2 *If $\mathbf{q} \in \mathcal{R} \cup \mathcal{R}^{-1}$, then*

$$\Theta\colon \bigoplus_{U \subseteq S} C^*(Z, Z^U) \otimes D_U \longrightarrow L^2_{\mathbf{q}} C^*(\mathcal{U})$$

is a weak isomorphism of cochain complexes of Hilbert $\mathcal{N}_{\mathbf{q}}$–modules.

Proof We have:

$$\bigoplus_{U \subseteq S} L^2_{\mathbf{q}} C^*(\mathcal{U}; D_U) = L^2_{\mathbf{q}} C^*(\mathcal{U}; \bigoplus_{U \subseteq S} D_U).$$

By the Decomposition Theorem (Theorem 9.11), we have a weak isomorphism,

$$L^2_{\mathbf{q}} C^*(\mathcal{U}; \bigoplus D_U) \to L^2_{\mathbf{q}} C^*(\mathcal{U}).$$

Combining this with the isomorphism of Proposition 10.1, the proposition follows. □

A weak isomorphism of chain complexes of Hilbert modules induces a weak isomorphism on the level of reduced cohomology [19, Lemma 5]. Furthermore, if two Hilbert $\mathcal{N}_{\mathbf{q}}$–modules are weakly isomorphic, then they are isometric [29, Lemma 2.5.3]. So, we have the following corollary to Proposition 10.2.

Theorem 10.3 (Compare [13] and [15, Theorem A].)

(i) *If $\mathbf{q} \in \mathcal{R}$, then*

$$L^2_{\mathbf{q}} \mathcal{H}^*(\mathcal{U}) \cong \bigoplus_{T \in \mathcal{S}} H^*(Z, Z^T) \otimes D_T.$$

(ii) *If $\mathbf{q} \in \mathcal{R}^{-1}$, then*

$$L^2_{\mathbf{q}} \mathcal{H}^*(\mathcal{U}) \cong \bigoplus_{T \in \mathcal{S}} H^*(Z, Z^{S-T}) \otimes D_{S-T}.$$

The special case $\mathcal{U} = \Sigma$ is the following.

Theorem 10.4

(i) (Theorem 8.2 or [27, Corollary 10.4]) *If $\mathbf{q} \in \mathcal{R}$, then $L^2_{\mathbf{q}} H^*(\Sigma)$ is concentrated in dimension 0 and*

$$L^2_{\mathbf{q}} H^0(\Sigma) = L^2_{\mathbf{q}} \mathcal{H}^0(\Sigma) \cong A_S.$$

So, $b^0_{\mathbf{q}}(\Sigma) = \chi_{\mathbf{q}}(\Sigma) = 1/W(\mathbf{q}) = \dim_{\mathcal{N}_{\mathbf{q}}} A_S$.

(ii) *If* $\mathbf{q} \in \mathcal{R}^{-1}$, *then*

$$L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma) \cong \bigoplus_{T\in\mathcal{S}} H^*(K, K^{S-T}) \otimes D_{S-T}.$$

So
$$b^i_{\mathbf{q}}(\Sigma) = \sum_{T\in\mathcal{S}} \frac{W^T(\mathbf{q}^{-1})}{W(\mathbf{q}^{-1})} b^i(K, K^{S-T}),$$

where $b^i(K, K^{S-T}) = \dim_{\mathbf{R}} H^i(K, K^{S-T}; \mathbf{R})$.

(In the formula for $b^i_{\mathbf{q}}(\Sigma)$ in Theorem 10.4(ii) we used the formula for $\dim_{\mathcal{N}_{\mathbf{q}}} D_{S-T}$ from Corollary 9.14.)

11 A generalization of a theorem of Solomon

When W is finite and $\mathbf{q} = \mathbf{1}$, L. Solomon [39] proved some results very similar to the Decomposition Theorem (Theorem 9.11). In this special case, formulas (5–7) and (5–11) for the idempotents a_T and h_T become

$$a_T := \frac{1}{\mathrm{Card}(W_T)} \sum_{w\in W_T} e_w$$

and

$$h_T := \frac{1}{\mathrm{Card}(W_T)} \sum_{w\in W_T} \varepsilon_w e_w$$

and we recognize a_T and h_T as the familiar elements of "symmetrization" and "alternation" in the group algebra $\mathbf{R}[W_T]$.

Solomon's Theorem [39] *Suppose W is finite. Then there are direct sum decompositions of the regular representation:*

$$L^2(W) = \sum_{T\subseteq S} L^2(W) a_T h_{S-T},$$
$$L^2(W) = \sum_{T\subseteq S} L^2(W) h_{S-T} a_T.$$

Our generalization of Solomon's Theorem is the following.

Theorem 11.1 (i) *If* $\mathbf{q} \in \mathcal{R}$, *then*

$$\sum_{T\in\mathcal{S}} \overline{L^2_{\mathbf{q}} h_T a_{S-T}} \quad \text{and} \quad \sum_{T\in\mathcal{S}} \overline{L^2_{\mathbf{q}} a_{S-T} h_T}$$

are direct sum decompositions and dense subspaces of $L^2_{\mathbf{q}}$.

(ii) *If* $\mathbf{q} \in \mathcal{R}^{-1}$, *then*

$$\sum_{T\in\mathcal{S}} \overline{L^2_{\mathbf{q}} h_{S-T} a_T} \quad \text{and} \quad \sum_{T\in\mathcal{S}} \overline{L^2_{\mathbf{q}} a_T h_{S-T}}$$

are direct sum decompositions and dense subspaces of $L^2_{\mathbf{q}}$.

This is an immediate consequence of Corollary 9.7, Theorem 9.11 and the following theorem.

Theorem 11.2 *Suppose* $T \in \mathcal{S}$.

(i) *If* $\mathbf{q} \in \mathcal{R}$, *then* $\overline{L^2_{\mathbf{q}} a_{S-T} h_T} = G_T$ *and* $\overline{L^2_{\mathbf{q}} h_T a_{S-T}} = D_T$.

(ii) *If* $\mathbf{q} \in \mathcal{R}^{-1}$, *then* $\overline{L^2_{\mathbf{q}} a_T h_{S-T}} = G_{S-T}$ *and* $\overline{L^2_{\mathbf{q}} h_{S-T} a_T} = D_{S-T}$.

Proof (i) Suppose $\mathbf{q} \in \mathcal{R}$. By Lemma 9.6, right multiplication by a_{S-T} is a weak isomorphism from G_T to $G_T a_{S-T}$. So, by Lemma 9.10, $\overline{L^2_{\mathbf{q}} h_T a_{S-T}} = \overline{G_T a_{S-T}} = D_T$. Similarly, by Lemma 9.15, $\overline{L^2_{\mathbf{q}} a_{S-T} h_T} = G_T$.

(ii) Applying the j–isomorphism to the two equations in (i), we get the two equations in (ii). □

Remark 11.3 It seems probable that $\overline{L^2_{\mathbf{q}} a_{S-U} h_U} = G_U$ and $\overline{L^2_{\mathbf{q}} h_U a_{S-U}} = D_U$ whenever $\mathbf{q} \in \mathcal{R}_{S-U} \cap \mathcal{R}^{-1}_U$ (so that h_U and a_{S-U} are both defined).

Remark 11.4 In [32] Kazhdan and Lusztig study the regular representation of the Hecke algebra $\mathbf{R}_{\mathbf{q}}[W]$ on itself and they generalize Solomon's Theorem in a slightly different direction. W can be infinite. First, they define a basis $\{C_w\}_{w\in W}$ for $\mathbf{R}_{\mathbf{q}}[W]$, called the "Kazhdan–Lusztig basis." It has many good properties. Next they partition of W into "left cells." This partition is strictly finer than the partition of W into the W^T, $T \in \mathcal{S}$. Given a left cell Z, they define a certain subquotient $\mathcal{I}_Z/\mathcal{I}'_Z$ of $\mathbf{R}_{\mathbf{q}}[W]$ such that $\{C_w\}_{w\in Z}$ projects to a basis for the subquotient. If we sum these representations over all $Z \subseteq W^T$, we obtain a representation analogous to our D_{S-T}. (Compare [18].) So, our Decomposition Theorem (Theorem 9.11) is a partial generalization of Kazhdan–Lusztig theory to the Hecke–von Neumann algebra $\mathcal{N}_{\mathbf{q}}$. It seems likely that left cells can be used to get a further direct sum decomposition of the D_{S-T}, although we do not yet know how to prove this.

12 Relationship with ordinary homology and cohomology with compact supports

As in Section 6, Z is a CW complex which is a strict fundamental domain for a W–action on $\mathcal{U}$ $(=\mathcal{U}(W,Z))$.

Theorem 12.1 (i) *For* $\mathbf{q}\in\mathcal{R}$, *the canonical map,* can: $H_*(\mathcal{U};\mathbf{R})\to L^2_{\mathbf{q}}\mathcal{H}_*(\mathcal{U})$, *is an injection with dense image.*

(ii) *For* $\mathbf{q}\in\mathcal{R}^{-1}$, *the canonical map,* can: $H^*_c(\mathcal{U};\mathbf{R})\to L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{U})$, *is an injection with dense image.*

For $T\in\mathcal{S}$, put $\mathcal{A}_T:=\mathbf{R}_{\mathbf{q}}[W]\widetilde{a}_T$, where $\widetilde{a}_T$ is defined by (5–4) and let $\widehat{\mathcal{A}}_T$ be the subspace spanned by $\{e_w\widetilde{a}_T\mid w\in W^T\}$. Similarly, for any $T\in\mathcal{S}$, $\mathfrak{H}_T:=\mathbf{R}_{\mathbf{q}}[W]\widetilde{h}_T$, where $\widetilde{h}_T$ is defined by (5–8), and $\widehat{\mathfrak{H}}_T$ is the subspace spanned by $\{e_w\widetilde{h}_T\mid w\in W^T\}$. As usual, $A_T=L^2_{\mathbf{q}}a_T$. Let $i\colon\mathcal{A}_T\to A_T$ and $k\colon\mathfrak{H}_T\to H_T$ be the inclusions. Let $p_T\colon A_T\to D_{S-T}$ and $q_T\colon H_T\to G_T$ be the orthogonal projections.

Lemma 12.2 *Suppose* $T\in\mathcal{S}$. *For* $\mathbf{q}\in\mathcal{R}^{-1}$, *the map* $p_T\circ i\colon\widehat{\mathcal{A}}_T\to D_{S-T}$ *is injective with dense image. Similarly, for* $\mathbf{q}\in\mathcal{R}$, $q_T\circ k\colon\widehat{\mathfrak{H}}_T\to G_T$ *is injective with dense image.*

Proof First, $i\colon\mathcal{A}_T\to A_T$ is injective with dense image. It is fairly easy to prove the following version of the Decomposition Theorem for $\mathcal{A}_T$:

$$\mathcal{A}_T=\biguplus_{U\in\mathcal{S}_{\geq T}}\widehat{\mathcal{A}}_U.$$

(See [18].) By Corollary 9.13, the kernel of p_T is $\sum_{U\in\mathcal{S}_{>T}}A_U$. So, i takes $\biguplus_{U\in\mathcal{S}_{>T}}\widehat{\mathcal{A}}_U$ onto a dense subspace of $\operatorname{Ker}p_T$. It follows that i takes $\widehat{\mathcal{A}}_T$ injectively to a subspace whose closure is complementary subspace for $\operatorname{Ker}p_T$. This proves the statement concerning $\mathbf{q}\in\mathcal{R}^{-1}$. The proof of the last sentence of the lemma is similar. □

Proof of Theorem 12.1 We only prove (ii), the proof of (i) being similar. Suppose $\mathbf{q}\in\mathcal{R}^{-1}$. By [15] there is an isomorphism $\bigoplus H^*(Z,Z^{S-T})\otimes\widehat{\mathcal{A}}_T\to H^*_c(\mathcal{U};\mathbf{R})$. (Throughout this proof sums are over all $T\in\mathcal{S}$.) The isomorphism is defined as follows. For each $T\in\mathcal{S}$ and $w\in W^T$, put $b_w:=e_w\widetilde{a}_T$ and define

$$\iota_T\colon C^*(Z,Z^{S-T})\otimes\mathbf{R}^{(W^T)}\to C^*_c(\mathcal{U};\mathbf{R})$$

by $\iota_T(c \otimes e_w) = e_w \widetilde{a}_T(c)$. The isomorphism of [15] is induced by $\oplus\iota_T^*$. (In [15] we only showed this was an isomorphism for $\mathbf{q} = \mathbf{1}$; however, the argument is valid for any $\mathbf{q}$.) We have a commutative diagram:

$$\begin{array}{ccc}
\bigoplus H^*(Z, Z^{S-T}) \otimes \widehat{\mathcal{A}}_T & \xrightarrow{\oplus\iota_T^*} & H_c^*(\mathcal{U}; \mathbf{R}) \\
 & & \Big\downarrow \text{can} \\
\Big\downarrow d & & L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{U}) \\
 & & \Big\downarrow \oplus\pi_T \\
\bigoplus H^*(Z, Z^{S-T}) \otimes D_T & \longleftarrow & \bigoplus L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{U}; D_T)
\end{array}$$

The map $\pi_T\colon \mathcal{H}^*(\mathcal{U}; L^2_{\mathbf{q}}) \to H^*(\mathcal{U}; D_T))$ is the coefficient homomorphism induced by orthogonal projection $L^2_{\mathbf{q}} \to D_T$ and $d := \oplus(p_T \circ i)$ is the coefficient homomorphism induced from the maps $p_T \circ i\colon \widehat{\mathcal{A}}_T \to D_T$ of Lemma 12.2. The bottom horizontal map is given by Proposition 10.1. By Theorem 10.3, the map $\oplus\pi_T$ is a weak isomorphism. In other words, up to a weak isomorphism of Hilbert $\mathcal{N}_{\mathbf{q}}$–modules, the canonical map $H_c^*(\mathcal{U}; \mathbf{R}) \to L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{U})$ is identified with d. By Lemma 12.2, each $p_T \circ i$ is injective with dense image and therefore, so is d. This proves (ii). □

13 L^2–cohomology of buildings

As in [37], a *building* consists of the following data:

- a set Φ,
- a Coxeter system (W, S),
- a collection of equivalence relations on Φ indexed by S,
- a function $\delta\colon \Phi \times \Phi \to W$.

This data must satisfy certain additional conditions which we will explain below. One condition is that for $s \in S$, each s–equivalence class contains at least two elements.

The elements of Φ are called *chambers*. Given $s \in S$, two chambers φ and φ' are s*–equivalent* if they are equivalent via the equivalence relation corresponding to s. If, in addition, $\varphi \neq \varphi'$, they are s*–adjacent*. A *gallery* is a sequence $(\varphi_0, \dots, \varphi_n)$ of adjacent chambers; its *type* is the word $(s_1, \dots, s_n)$ in the letters of S, where φ_{i-1} and φ_i are s_i–adjacent. Given $T \subseteq S$, $(\varphi_0, \dots, \varphi_n)$ is a T*–gallery* if each $s_i \in T$. The gallery is *reduced* if $w = s_1 \cdots s_n$ is a reduced expression.

Another condition for Φ to be a building is that there exist a W*–valued distance function* $\delta\colon \Phi\times\Phi\to W$. This means that there is a reduced gallery of type $(s_1,\dots,s_n)$ from φ to φ' if and only if $s_1\cdots s_n$ is a reduced expression for $\delta(\varphi,\varphi')$.

The s*–mirror* (or "s–panel") of a chamber φ is the s–equivalence class containing φ. More generally, given a subset $T\subseteq S$, the T*–residue* of φ is the T–gallery connected component containing φ. Each such T–residue is naturally a building with associated Coxeter system (W_T,T). The residue is *spherical* if T is a spherical.

Example 13.1 (Trees) Suppose W is the infinite dihedral group (so that $\mathrm{Card}(S)=2$). Any tree is bipartite, ie its vertices can be labeled by the two elements of S so that the vertices of any edge have distinct labels. Suppose T is a tree with such a labeling and suppose no vertex of T is of valence 1. Let Φ be its set of edges. Given $s\in S$, call two edges s*–equivalent* if they meet at a vertex of type s. An $\{s\}$–residue is the set of edges in the star of a vertex of type s. A gallery in Φ corresponds to an edge path in T. The type of the gallery is the word obtained by taking the types of the vertices crossed by the corresponding edge path. This word is reduced if and only if the edge path does not backtrack. Given two edges φ,φ' of T, there is a (unique) minimal gallery connecting them. The corresponding word represents an element of $w\in W$ and $\delta(\varphi,\varphi'):=w$. Thus, every such tree T defines a building of type (W,S). Not surprisingly, we will define the "geometric realization of a building" so that for the building Φ corresponding to T, its geometric realization will be T.

A building Φ of type (W,S) has *finite thickness* if for each $s\in S$, each s–equivalence class is finite. If Φ has finite thickness, then it follows from the existence of a W–distance function that each of its spherical residues is finite.

Let us say that Φ is *regular* if for each $s\in S$, the s–equivalence classes have constant cardinality. When finite, we denote this number by q_s+1. It is known [37] that if s and s' are conjugate in W, then $q_s=q_{s'}$. Let I be the set of conjugacy classes of elements in S. Then for any regular building Φ, the integers q_s define an I–tuple $\mathbf{q}$ called the *thickness vector* of Φ.

A group G of automorphisms of a building is *chamber transitive* if it acts transitively on Φ. When this is the case, we have $\Phi\cong G/B$, where B denotes the stabilizer of some given chamber ζ. If G_s denotes the stabilizer of the s–mirror containing ζ, then the chambers $g\zeta$ and $g'\zeta$ are s–equivalent if and only if g and g' belong to the same coset of G_s. Obviously, if G is chamber transitive, then the building is regular. For the remainder of this section, we suppose that Φ has finite thickness and that G is chamber transitive.

Given a subset T of S, denote the stabilizer of the T–residue containing ζ by G_T. Thus, $G_\varnothing = B$ and $G_{\{s\}} = G_s$. If Φ has finite thickness and $T \in \mathcal{S}$, then the number of elements in a T–residue is $\mathrm{Card}(G_T/B)$. (This number is known to be $W_T(\mathbf{q})$.)

Fix a chamber $\zeta \in \Phi$ and let r (or r_ζ) denote the function $\Phi \to W$ defined by $\varphi \to \delta(\zeta, \varphi)$. Since $\Phi \cong G/B$, we can regard r as a function from G/B to W. Since B fixes ζ, $r\colon G/B \to W$ is B–invariant. In other words, r induces a map $\overline{r}\colon B\backslash G/B \to W$.

A *Tits system* is a quadruple (G, B, N, S), where G is a group, B and N are subgroups of G, $W := N/N \cap B$, S is a subset of W and where the conditions listed in [4, pp 15–26] are satisfied. Given $w \in W$, put $C(w) := BwB$. The conditions imply the following:

- For each $s \in S$, $G_s := B \cup C(s)$ is a subgroup of G.
- (W, S) is a Coxeter system.
- There is a building with set of chambers G/B such that two chambers gB and $g'B$ are s–equivalent if and only if $gG_s = g'G_s$.
- Suppose $r\colon G/B \to W$ is defined by $gB \to \delta(B, gB)$ where δ is W–distance in the building. Then the induced map $\overline{r}\colon B\backslash G/B \to W$ is a bijection.

One says that the building *comes from a BN–pair*.

Definition 13.2 The Coxeter system (W, S) is *right-angled* if $m_{st} = 2$ or ∞ for each pair s, t of distinct elements in S.

Example 13.3 (Regular right-angled buildings [14, pp 112–113]) For any right-angled Coxeter system (W, S) (cf Definition 13.2) and any S–tuple $\mathbf{q} = (q_s)_{s\in S}$ of positive integers, there is a regular building Φ of type (W, S) with thickness vector $\mathbf{q}$. In the case where W is the infinite dihedral group this is well-known: as in Example 13.1, the building is a (bipartite) tree with edge set Φ. It is "regular" in the sense that for each $s \in S$ there are exactly $q_s + 1$ edges meeting at each vertex of type s.

In the general case, the construction goes as follows. For each $s \in S$, choose a finite group Γ_s with $\mathrm{Card}(\Gamma_s) = q_s + 1$ and let Γ be the "graph product" of the $(\Gamma_s)_{s\in S}$ where the graph is the 1–skeleton of L. In other words, Γ is the quotient of the free product of the $(\Gamma_s)_{s\in S}$ by the normal subgroup generated by all commutators $[g_s, g_t]$ with $g_s \in \Gamma_s$, $g_t \in \Gamma_t$ and $m_{st} = 2$. As in [14], we get a building with $\Phi = \Gamma$ and with two elements $g, g' \in \Gamma$ in an s–equivalence class if and only if they determine same coset in Γ/Γ_s. We leave the following two facts as exercises for the reader:

- Two regular right-angled buildings of a given type (W, S) are isomorphic if and only if they have the same thickness vector.
- Any regular right-angled building comes from a BN–pair. In other words, its full automorphism group G is chamber transitive and if B denotes the stabilizer of a given chamber and N the stabilizer of some apartment containing that chamber, then there is a set of generators S for $W := N/N \cap B$ so that (G, B, N, S) is a Tits system.

Hecke algebras and functions on $B\backslash G/B$ This paragraph is taken from [4, Exercise 22, pp 56–57].

Suppose G is a topological group and B is a compact open subgroup. Let $C(G)$ denote the vector space of continuous real-valued functions on G. Let $\alpha\colon G \to G/B$ and $\beta\colon G \to B\backslash G/B$ be the natural projections. Define subspaces $H \subseteq L \subseteq C(G)$ by

$$L := \alpha^* \mathbf{R}^{(G/B)} \quad \text{and} \quad H := \beta^* \mathbf{R}^{(B\backslash G/B)},$$

where, as in Section 4, for any set X, $\mathbf{R}^{(X)}$ denotes the vector space of finitely supported functions on X.

For each $gB \in G/B$, let $a_{gB} \in L$ be defined by $a_{gB}(x) = 1$ for $x \in gB$ and $a_{gB}(x) = 0$ for $x \notin gB$. Since (a_{gB}) is a basis for L, there is a unique linear form on L such that $a_{gB} \to 1$ for all $gB \in G/B$. We denote this form by $\varphi \to \int \varphi$ (since it coincides with the Haar integral normalized by the condition that $\int a_B = 1$).

If $\varphi \in L$ and $\psi \in H$, then for each $x \in G$, the function $\theta_x\colon G \to \mathbf{R}$, defined by $\theta_x(y) = \varphi(y)\psi(y^{-1}x)$, belongs to L. The function $\varphi * \psi\colon x \to \int \varphi(y)\psi(y^{-1}x)dy$ also belongs to L. Moreover, if $\varphi \in H$, then $\varphi * \psi \in H$. The map $(\varphi, \psi) \to \varphi * \psi$ makes H into an algebra and L into a right H–module. H is called the *Hecke algebra* of G with respect to B.

Next, suppose that G is a chamber transitive automorphism group on a building and that $r\colon G/B \to W$ is defined by taking the W–distance from the chamber corresponding to B. Let $\gamma := r \circ \alpha\colon G \to W$ and $J := \gamma^*(\mathbf{R}^{(W)}) \subseteq H$.

Remark If (G, B, N, S) is a Tits system, then $\overline{r}\colon B\backslash G/B \to W$ is a bijection and hence, $J = H$.

Lemma 13.4 *Suppose, as above, that a given building admits a chamber transitive automorphism group G (so G/B is the set of chambers). Let $\mathbf{q}$ be the thickness vector. Then we have the following:*

(i) *J is a subalgebra of H.*

(ii) *$J \cong \mathbf{R_q}[W]$, the Hecke algebra of Section 4.*

Proof Since G is chamber transitive, $\gamma^*\colon \mathbf{R}^{(W)} \to J$ is an isomorphism of vector spaces. So, we only need to check that γ^* is an algebra homomorphism from $\mathbf{R_q}[W]$ to H. Let $f_w = \gamma^*(e_w)$. Then f_w is the characteristic function of $\{g \in G \mid r(gB) = w\}$. In particular, for each $s \in S$, f_s is the characteristic function of $G_s - B$. We want to see that

$$f_w * f_s = \begin{cases} f_{ws}, & \text{if } l(ws) > l(w), \\ q_s f_{ws} + (q_s - 1) f_w, & \text{if } l(ws) < l(w). \end{cases}$$

By definition of convolution,

$$(f_w * f_s)(g) = \int_G f_w(x) f_s(x^{-1}g)dx = \int_G f_w(gu) f_s(u^{-1})du = \int_{G_s - B} f_w(gu)du,$$

which is equal to the Haar measure of the set

$$U_g := \{u \in G_s - B \mid r(guB) = w\}.$$

Let $C_0 := g_0 B$ be the chamber which is s–adjacent to gB and which is closest to B. There are q_s other chambers adjacent to gB. We list them as: $C_1 = g_1 B, \dots, C_{q_s} = g_{q_s} B$. So, for $i > 0$, $r(C_i) = r(C_0)s$. Notice that if $u \in G_s - B$, then guB is s–adjacent to gB and therefore, guB is equal to some C_i. So, if $r(guB) = w$, then $r(gB) = w$ or ws. In other words, if $r(gB) \notin \{w, ws\}$, then $(f_w * f_s)(g) = 0$. We now consider two cases. Each case further divides into two subcases depending on whether $r(gB) = w$ or ws.

Case 1 $l(w) < l(ws)$. In this case $r(C_0) = w$ and $r(C_i) = ws$ for $i > 0$.

(a) Suppose $r(gB) = w$. Then $gB = C_0$ and $guB = C_i$ for $i > 0$, so that $r(guB) = ws$. Thus, $U_g = \varnothing$ and $(f_w * f_s)(g) = 0$.

(b) Suppose $r(gB) = ws$. Then $gB = C_k$, for some $k > 0$, and

$$U_g = \{u \in G_s - B \mid guB = C_0\} = (G_s - B) \cap g^{-1} g_0 B.$$

Since gB and $g_0 B$ are s–adjacent and not equal, $g^{-1} g_0 B \subseteq G_s - B$, so that $U_g = g^{-1} g_0 B$ has measure 1. Therefore, $(f_w * f_s)(g) = 1$. So, in Case 1, $f_w * f_s = f_{ws}$.

Case 2 $l(w) > l(ws)$. In this case $r(C_0) = ws$ and $r(C_i) = w$ for $i > 0$.

(a) Suppose $r(gB) = w$. Then $gB = C_k$ for some $k > 0$. So, the set

$$U_g = \bigcup_{0<i} \{u \in G_s - B \mid guB = C_i\} = \bigcup_{0<i\neq k} g^{-1} g_i B$$

has measure $q_s - 1$.

(b) Suppose $r(gB) = ws$. Then $gB = C_0$, and the set

$$U_g = \bigcup_{0<i} \{u \in G_s - B \mid guB = C_i\} = \bigcup_{0<i} g^{-1} g_i B$$

has measure q_s. So, in Case 2, $f_w * f_s = q_s f_{ws} + (q_s - 1) f_w$. □

The geometric realization of a building Suppose Φ is a building with associated Coxeter system (W, S). As in Section 2, let K be the geometric realization of $\mathcal{S}$ and Σ the geometric realization of $W\mathcal{S}$. By (6–4), $\Sigma = \mathcal{U}(W, K)$, where $\mathcal{U}(W, K) = (W \times K)/\sim$ and where $\sim$ is the equivalence relation defined in the beginning of Section 6. Following [14, pp 117–118], define the *geometric realization* of Φ to be

$$\mathcal{U}(\Phi, K) = (\Phi \times K)/\sim, \qquad (13\text{–}1)$$

where $(\varphi, x) \sim (\varphi', x')$ if and only if $x = x'$ and φ, φ' belong to the same $S(x)$–residue. ($S(x)$ is defined in (6–2).)

Since K only involves the spherical subsets of S, $\mathcal{U}(\Phi, K)$ only involves the spherical residues of Φ. It follows that if Φ has finite thickness, then $\mathcal{U}(\Phi, K)$ locally finite.

We often write X as a shorthand for $\mathcal{U}(\Phi, K)$.

The von Neumann algebra of G Next suppose G is a chamber transitive group of automorphisms of Φ and that B is the stabilizer of some fixed chamber ζ. G acts as a group of homeomorphisms of X, so give it the compact-open topology. Then B is a compact open subgroup. Let μ be Haar measure on G, normalized by the condition that $\mu(B) = 1$.

We have the left regular representation of G on $L^2(G)$. The *von Neumann algebra* $\mathcal{N}(G)$ consists of all G–equivariant bounded linear endomorphisms of $L^2(G)$.

Any $\alpha \in \mathcal{N}(G)$ is represented by convolution with some distribution f_α. This distribution need not be a function. For example, if α is the identity map on $L^2(G)$, then $f_\alpha = \delta_1$ (the Dirac delta). One would like to define the "trace" of α to be $f_\alpha(1)$

whenever f_α is a function. However, since f_α is well-defined only up to sets of measure 0, we must proceed slightly differently.

Suppose α is a nonnegative self-adjoint element of $\mathcal{N}(G)$. Let β be its square root. If f_β is a L^2 function, then put

$$\mathrm{tr}_{\mathcal{N}(G)}\,\alpha := \|f_\beta\| := \left(\int_G f_\beta(x)^2 d\mu\right)^{1/2}.$$

This extends in the usual fashion to give a "trace" on $(n\times n)$–matrices with coefficients in $\mathcal{N}(G)$. For a closed G–stable subspace V of $\bigoplus L^2(G)$, using orthogonal projection $\pi_V\colon \bigoplus L^2(G) \to \bigoplus L^2(G)$, the *von Neumann dimension* of V is defined by

$$\dim_{\mathcal{N}(G)} V := \mathrm{tr}_{\mathcal{N}(G)}\,\pi_V.$$

We identify $L^2(\Phi) = L^2(G/B)$ with the subspace of $L^2(G)$ consisting of the functions which are constant on each left coset gB, $g \in G$. Orthogonal projection from $L^2(G)$ onto $L^2(G/B)$ is given by convolution with the characteristic function of B. In view of the assumption that $\mu(B) = 1$,

$$\dim_{\mathcal{N}(G)} L^2(G/B) = 1.$$

The map $r\colon G/B \to W$ defined by the W–distance from the base chamber induces a bounded linear map $L^2_{\mathbf{q}}(W) \to L^2(G/B)$ which we shall also denote by r. Since this map takes bounded elements of $L^2_{\mathbf{q}}(W)$ to bounded elements of $L^2(G/B)$, we get the following version of Lemma 13.4.

Lemma 13.5 *The map $r\colon L^2_{\mathbf{q}}(W) \to L^2(G/B)$ induces a monomorphism of von Neumann algebras $r\colon \mathcal{N}_{\mathbf{q}} \to \mathcal{N}(G)$. (In particular, r commutes with the $*$ anti-involutions on $\mathcal{N}_{\mathbf{q}}$ and $\mathcal{N}(G)$.)*

$L^2C^*(X)$ denotes the Hilbert space of square summable simplicial cochains on X and $\mathcal{H}^*(X)$ denotes the subspace of harmonic cocycles. (Of course, the $\mathcal{H}^*(X)$ are isomorphic to reduced cohomology groups of the cochain complex $L^2C^*(X)$.) Supposing G is a chamber transitive automorphism group, we have

$$L^2C^i(X) = \bigoplus_{\sigma\in K^{(i)}} L^2(G/G_\sigma) \subset \bigoplus_{\sigma\in K^{(i)}} L^2(G),$$

where $G_\sigma := G_{S(\sigma)}$ is the stabilizer of the i–simplex σ. ($S(\sigma)$ is the spherical subset defined in (6–3).) One then defines the *L^2–Betti numbers of X with respect to G* by

$$b^i(X;G) = \dim_{\mathcal{N}(G)} \mathcal{H}^i(X),$$

The map $r\colon X \to \Sigma$ induces a map on cochains which we denote by the same letter, ie r is a cochain map from $L^2_{\mathbf{q}}C^*(\Sigma)$ to $L^2C^*(X)$. We also have "transfer maps" on chains and cochains. On the level of chains, the transfer map sends a cell c of Σ to $r^{-1}(c)/\operatorname{Card}(r^{-1}(c))$. The transfer map $t\colon L^2C^*(X) \to L^2_{\mathbf{q}}C^*(\Sigma)$ is defined on the level of cochains by

$$t(f)(c) := \frac{1}{\operatorname{Card}(r^{-1}(c))} \sum f(c'),$$

where the sum is over all $c' \in r^{-1}(c)$. (The orientations on the c' are induced from the orientation of c.) Note that $\operatorname{Card}(r^{-1}(c)) = \mu_{\mathbf{q}}(c)$, where $\mu_{\mathbf{q}}$ is the measure on Wc defined in (7–1) (ie if $c = w\sigma$ with w $(\varnothing, S(\sigma))$–reduced, then $\mu_{\mathbf{q}}(c) = q_w$).

Remark Suppose X is the geometric realization of a building associated to a Tits system (G, B, N, S). Then $L^2_{\mathbf{q}}C^*(\Sigma)$ can be identified with the B–invariant cochains $L^2C^*(X)^B$ and the map $r\colon L^2_{\mathbf{q}}C^*(\Sigma) \to L^2C^*(X)$ with the inclusion of the B–invariant cochains. The map $t\colon L^2C^*(X) \to L^2_{\mathbf{q}}C^*(\Sigma)$ is then identified with averaging over B. In other words, if Σ is identified with a subspace of X via some section of $r\colon X \to \Sigma$, then

$$t(f)(c) = \int_{x\in B} f(xc)d\mu.$$

Lemma 13.6 (i) $t \circ r = \mathrm{id}\colon L^2_{\mathbf{q}}C^i(\Sigma) \to L^2_{\mathbf{q}}C^i(\Sigma)$.

(ii) *The maps r and t are adjoint to each other.*

(iii) *These maps take harmonic cocycles to harmonic cocycles.*

Proof Statement (i) is obvious.

(ii) For $f \in L^2_{\mathbf{q}}C^i(\Sigma)$ and $f' \in L^2C^i(X)$, we have

$$\begin{aligned}\langle r(f), f'\rangle &= \sum_{c'\in X^{(i)}} [r(f)(c')][f'(c')] = \sum_{c\in\Sigma^{(i)}} \sum_{c'\in r^{-1}(c)} f(r(c'))f'(c')\\ &= \sum_{c\in\Sigma^{(i)}} f(c) \sum_{c'\in r^{-1}(c)} f'(c')\\ &= \sum_{c\in\Sigma^{(i)}} \operatorname{Card}(r^{-1}(c))[f(c)][t(f')(c)] = \sum_{c\in\Sigma^{(i)}} \mu_{\mathbf{q}}(c)[f(c)][t(f')(c)]\\ &= \langle f, t(f')\rangle.\end{aligned}$$

(iii) Since $r\colon L^2_{\mathbf{q}}C^*(\Sigma) \to L^2C^*(X)$ is induced by the simplicial map $r\colon X \to \Sigma$, it takes cocycles to cocycles. We must show it also takes cycles to cycles. If $c' \in X^{(i-1)}$

and $d' \in X^{(i)}$ and if the incidence number $[c' : d']$ is nonzero, then it is equal to $[r(c') : r(d')]$. Hence,

$$\partial(r(f))(c') = \sum [c' : d'] f(r(c')) = \sum [c : d] \frac{\mu_{\mathbf{q}}(c)}{\mu_{\mathbf{q}}(d)} f(c) = \partial^{\mathbf{q}}(f)(c),$$

where $c = r(c')$, $d = r(d')$ and the last equality comes from the definition given in equation (7–5). So, $\partial^{\mathbf{q}}(f) = 0$ implies that $\partial(r(f)) = 0$. Since t is the adjoint of r, it also must take cocycles to cocycles and cycles to cycles. □

Consider the diagram:

$$\begin{array}{ccc} L^2_{\mathbf{q}} C^*(\Sigma) & \xrightarrow{r} & L^2 C^*(X) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle P} \\ L^2_{\mathbf{q}} \mathcal{H}^*(\Sigma) & \xrightarrow{r} & \mathcal{H}^*(X) \end{array}$$

where p and P denote the projections onto harmonic cocycles.

Lemma 13.7 $P \circ r = r \circ p$.

Proof Let $x \in L^2_{\mathbf{q}} C^*(\Sigma)$. It is enough to show that $P \circ r(x) - r \circ p(x)$ is orthogonal to any harmonic cocycle $h \in \mathcal{H}^*(X)$. We have: $\langle P \circ r(x), h \rangle = \langle r(x), P(h) \rangle = \langle r(x), h \rangle$. Hence,

$$\langle P \circ r(x) - r \circ p(x), h \rangle = \langle r(x - p(x)), h \rangle = \langle x - p(x), h \rangle = 0,$$

where the second and third equalities follow, respectively, from parts (ii) and (iii) of Lemma 13.6. □

Theorem 13.8 *Suppose Φ is a building with a chamber transitive automorphism group G and with thickness vector $\mathbf{q}$. Then the L^2–Betti numbers of X $(= \mathcal{U}(\Phi, K))$ equal the $L^2_{\mathbf{q}}$–Betti numbers of Σ, ie*

$$b^i(X; G) = b^i_{\mathbf{q}}(\Sigma).$$

Remark This theorem is proved in [27, Fact 3.5] in the case where the building comes from an BN–pair. Here we use Lemma 13.7 to weaken the hypothesis to the case of an arbitrary chamber transitive group G. The key technique of [27] of integrating over B is replaced by the use of the transfer map t.

Proof of Theorem 13.8 For each simplex σ in the fundamental chamber K, consider the commutative diagram:

$$\begin{array}{ccc} L^2_{\mathbf{q}}(W) & \xrightarrow{r} & L^2(G/B) \\ \downarrow & & \downarrow \\ L^2_{\mathbf{q}}(W/W_{S(\sigma)}) & \xrightarrow{r} & L^2(G/G_{S(\sigma)}) \end{array}$$

where $S(\sigma) := \{s \in S \mid \sigma \subset K_s\}$, where $W_{S(\sigma)}$ and $G_{S(\sigma)}$ are the isotropy subgroups of σ in W and G, respectively, where the vertical maps are orthogonal projections and where r $(= r^*)$ is the map induced by $r\colon G/B \to W$. Let $e_B \in L^2(G/B)$ denote the characteristic function of B and let e_σ be its orthogonal projection in $L^2(G/G_{S(\sigma)})$. (e_σ is the characteristic function of $G_{S(\sigma)}$ renormalized to have norm 1.) We note that e_B is the image of the basis vector $e_1 \in L^2(W)$ under r and e_σ is the image of $a_{S(\sigma)}$. We have the commutative diagram:

$$\begin{array}{ccccccc} \bigoplus L^2_{\mathbf{q}}(W) & & & \xrightarrow{r} & & & \bigoplus L^2(G) \\ \downarrow & & & & & & \downarrow \\ \bigoplus L^2_{\mathbf{q}}(W/W_{S(\sigma)}) & = & L^2_{\mathbf{q}}C^i(\Sigma) & \xrightarrow{r} & L^2C^i(X) & = & \bigoplus L^2(G/G_{S(\sigma)}) \\ & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle P} & & \\ & & L^2_{\mathbf{q}}\mathcal{H}^i(\Sigma) & \xrightarrow{r} & \mathcal{H}^i(X) & & \end{array}$$

where the sums are over all $\sigma \in K^{(i)}$. Let $\mathbf{e} \in \bigoplus L^2(G/G_{S(\sigma)})$ denote the vector $(e_\sigma)_{\sigma\in K^{(i)}}$ and let $\mathbf{a} \in \bigoplus L^2_{\mathbf{q}}(W/W_{S(\sigma)})$ be the vector $(a_{S(\sigma)})_{\sigma\in K^{(i)}}$. (So, $r(\mathbf{a}) = \mathbf{e}$.) Using Lemma 13.6, we get

$$\begin{aligned} b^i(X;G) &:= \dim_{\mathcal{N}(G)} \mathcal{H}^i(X) \\ &= \langle P(\mathbf{e}), \mathbf{e}\rangle = \langle Pr(\mathbf{a}), r(\mathbf{a})\rangle = \langle rp(\mathbf{a}), r(\mathbf{a})\rangle \\ &= \langle p(\mathbf{a}), tr(\mathbf{a})\rangle = \langle p(\mathbf{a}), \mathbf{a}\rangle = \dim_{\mathcal{N}_{\mathbf{q}}} L^2_{\mathbf{q}}\mathcal{H}^i(\Sigma) \\ &:= b^i_{\mathbf{q}}(\Sigma). \end{aligned}$$

□

The Decomposition Theorem for $L^2(G/B)$ As above, G is a chamber transitive automorphism group of a building Φ. For each $T \in \mathcal{S}$, let

$$\hat{A}_T := L^2(G/G_T) = L^2(G)^{G_T}$$

be the subspace of $L^2(G/B)$ consisting of the square summable functions on G which are constant on each coset gG_T. Set

$$\widehat{D}_{S-T} := \widehat{A}_T \cap \Big(\sum_{U \in \mathcal{S}_{>T}} \widehat{A}_U \Big)^{\perp}.$$

$\widehat{D}_{S-T}$ is a closed G–stable subspace in the regular representation. (It corresponds to the $\mathcal{N}_{\mathbf{q}}$–module D_{S-T} defined in Section 9.)

Theorem 13.9 (The Decomposition Theorem for $L^2(G/B)$) *Suppose G is a chamber transitive automorphism group of a building Φ and B is the stabilizer of a chamber. If the thickness vector $\mathbf{q}$ lies in $\mathcal{R}^{-1}$, then*

$$\sum_{T \in \mathcal{S}} \widehat{D}_{S-T}$$

is a dense subspace of $L^2(G/B)$ and a direct sum decomposition.

Given a module M and a collection of submodules $(M_\alpha)_{\alpha \in \mathcal{A}}$, the statement that $(M_\alpha)_{\alpha \in \mathcal{A}}$ gives a direct sum decomposition of M can be interpreted as a statement about chain complexes as follows. Set

$$C_1 := \bigoplus_{\alpha \in \mathcal{A}} M_\alpha \quad \text{and} \quad C_0 := M,$$

where $\bigoplus$ means external direct sum. Let $\partial\colon C_1 \to C_0$ be the natural map. This gives a chain complex, $C_* := \{C_0, C_1\}$, with nonzero terms only in degrees 0 and 1. The statement that the internal sum $\sum M_\alpha$ is direct is equivalent to the statement that ∂ is injective, ie that $H_*(C_*)$ vanishes in dimension 1. The statement that the M_α span M is equivalent to the statement that ∂ is onto, ie that $H_*(C_*)$ vanishes in dimension 0. Similarly, if M and the M_α are Hilbert spaces, then the statement that M_α is dense in M is equivalent to the statement that the reduced homology $\mathcal{H}_*(C_*)$ vanishes in dimension 0.

Proof of Theorem 13.9 The map r from Lemma 13.5 takes A_T to $\widehat{A}_T$ and D_{S-T} to $\widehat{D}_{S-T}$. Define chain complexes $\widehat{C}_* = \{\widehat{C}_0, \widehat{C}_1\}$ and $C_* = \{C_0, C_1\}$ by

$$\widehat{C}_1 := \bigoplus_{T \in \mathcal{S}} \widehat{D}_{S-T} \quad \text{and} \quad \widehat{C}_0 := L^2(G/B),$$

$$C_1 := \bigoplus_{T \in \mathcal{S}} D_{S-T} \quad \text{and} \quad C_0 := L^2_{\mathbf{q}}(W),$$

where the boundary maps $\widehat{C}_1 \to \widehat{C}_0$ and $C_1 \to C_0$ are the natural maps. By the Decomposition Theorem for $L^2_{\mathbf{q}}$ (Theorem 9.11), $\mathcal{H}_*(C_*)$ vanishes identically. So, by the proof of Theorem 13.8, $\mathcal{H}_*(\widehat{C}_*)$ has dimension 0 with respect to $\mathcal{N}(G)$ and hence, also vanishes identically. The theorem then follows from the previous paragraph. □

Decoupling cohomology As in Section 6, suppose we are given a finite CW complex Z and a family of subcomplexes $(Z_s)_{s\in S}$. As in (13–1), given a building Φ, define its Z*–realization* to be

$$\mathcal{U}(\Phi, Z) = (\Phi \times Z)/\sim,$$

where $(\varphi, x) \sim (\varphi', x')$ if and only if $x = x'$ and φ, φ' belong to the same $S(x)$–residue.

The proof of Theorem 10.3 goes through to give the following two results.

Theorem 13.10 *Suppose Φ is a building with a chamber transitive automorphism group G and that its thickness vector $\mathbf{q}$ lies in $\mathcal{R}^{-1}$. Then there is an isomorphism of orthogonal G–representations:*

$$\mathcal{H}^*(\mathcal{U}(\Phi, Z)) \cong \bigoplus_{T\in\mathcal{S}} H^*(Z, Z^{S-T}) \otimes \widehat{D}_{S-T}.$$

Corollary 13.11 (Compare [20] and [28, Corollary 8.2 and Proposition 8.5].) *Suppose Φ is a building with a chamber transitive automorphism group G and that its thickness vector $\mathbf{q}$ lies in $\mathcal{R}^{-1}$. Then, for $X = \mathcal{U}(\Phi, K)$, there is an isomorphism of orthogonal G–representations:*

$$\mathcal{H}^*(X) \cong \bigoplus_{T\in\mathcal{S}} H^*(K, K^{S-T}) \otimes \widehat{D}_{S-T}.$$

14 The case where L is a sphere

A simplicial complex Λ is a *generalized homology m–sphere* (for short, a GHS^m) if it is a homology m–manifold having the same homology as S^m. This is equivalent to the condition that, for each $T \in \mathcal{S}(\Lambda)$, $\mathrm{Lk}(T, \Lambda)$ has the same homology as $S^{m-\mathrm{Card}(T)}$.

Similarly, a pair $(\Lambda, \partial\Lambda)$ is a *generalized homology m–disk* (for short, a GHD^m) if it is an acyclic homology m–manifold with boundary.

From now on, when we say that a complex is a generalized homology sphere or disk or that it is a homology manifold, *we only require that it be one with respect to homology* *with real coefficients*. (This is all that is needed to insure that Poincaré duality holds for the (weighted) L^2–cohomology of various related complexes.)

If the nerve L of (W,S) is homeomorphic to S^{n-1}, then Σ is a contractible n–manifold. If L is PL-homeomorphic to S^{n-1}, then each face K_T of the fundamental chamber K is a PL-disk of codimension $\mathrm{Card}(T)$. Similarly, if L is a GHS^{n-1}, then Σ is a contractible homology n–manifold and each K_T is a contractible $GHD^{n-\mathrm{Card}(T)}$. (See [14; 16].)

For the remainder of this section suppose that L is a GHS^{n-1}.

Poincaré duality It is proved in [27] that $L^2_{\mathbf{q}}\mathcal{H}_*(\Sigma)$ satisfies Poincaré duality, where the duality changes $\mathbf{q}$ to $\mathbf{q}^{-1}$. We repeat the argument below.

For each $T \in \mathcal{S}$ and $w \in W$, the subcomplex wK_T is the "dual cell" to the Coxeter cell $w\langle T\rangle$ (defined in Section 6 and Section 7). (Strictly speaking, wK_T is not a cell unless $\mathrm{Lk}(T,L)$ is a PL-sphere; however, since $(K_T,\partial K_T)$ is a $GHD^{n-\mathrm{Card}(T)}$, the wK_T behave homologically as if they were dual cells.) The chain complex obtained by partitioning Σ into these "dual cells" is denoted $L^2_{\mathbf{q}}C_*(\Sigma_{ghd})$ in [27]. It is naturally identified with the cochain complex $L^2_{\mathbf{q}}C^{n-*}(A_\varnothing)$ associated to the cosheaf A on L, defined in Section 8. By Lemma 8.1(ii), $L^2_{\mathbf{q}}C_*(\Sigma_{cc})$ is identified with the chain complex $L^2_{\mathbf{q}}C_*(H_\varnothing)$ associated to the cosheaf H on L. It is proved in [27] that the chain complexes $L^2_{\mathbf{q}}C_*(\Sigma_{ghd})$ and $L^2_{\mathbf{q}}C_*(\Sigma_{cc})$ are both chain homotopy equivalent to $L^2_{\mathbf{q}}C_*(\Sigma)$, the chain complex defined via the standard simplicial structure on Σ. (This simplicial structure is a common subdivision of Σ_{ghd} and Σ_{cc}.) Hence, all three complexes have the same homology. The map $L^2_{\mathbf{q}}C^{n-*}(\Sigma_{ghd}) \to L^2_{\mathbf{q}^{-1}}C_*(\Sigma_{cc})$, induced by $wK_T \to w\langle T\rangle$ is a chain isomorphism. (When viewed as a map $L^2_{\mathbf{q}}C_*(A_\varnothing) \to L^2_{\mathbf{q}^{-1}}C_*(H_\varnothing)$, it is induced by the j–isomorphism of Section 5.) So, we have proved the following.

Proposition 14.1 [27, Theorem 6.1] *Suppose the nerve L of (W,S) is a GHS^{n-1}. Then there is j–equivariant isomorphism from the Hilbert $\mathcal{N}_{\mathbf{q}}$–module $L^2_{\mathbf{q}}\mathcal{H}_k(\Sigma)$ to the Hilbert $\mathcal{N}_{\mathbf{q}^{-1}}$–module $L^2_{\mathbf{q}^{-1}}\mathcal{H}_{n-k}(\Sigma)$ (where j is the isomorphism of Section 5). Hence, $b^k_{\mathbf{q}}(\Sigma) = b^{n-k}_{\mathbf{q}^{-1}}(\Sigma)$.*

Remark The same type of Poincaré duality (exchanging $\mathbf{q}$ with $\mathbf{q}^{-1}$) holds for $\mathcal{U}(W,Z)$, whenever Z is compact and $\mathcal{U}(W,Z)$ is a homology manifold. In other words, it holds provided that, for each $T \in \mathcal{S}$, $(Z_T,\partial Z_T)$ is a compact homology manifold with boundary (see [12; 15]).

Corollary 14.2 [9] *Suppose the nerve L of (W,S) is a GHS^{n-1}. Then the growth series of W is $(-1)^n$–reciprocal, ie*

$$\frac{1}{W(\mathbf{q})} = \frac{(-1)^n}{W(\mathbf{q}^{-1})}.$$

Proof Take the alternating sums of the dimensions on both sides of the equation of Proposition 14.1. By Proposition 7.4, the left hand side gives $\chi_{\mathbf{q}}(\Sigma)$ and the right hand side $(-1)^n\chi_{\mathbf{q}^{-1}}(\Sigma)$. □

The next result is proved in [27] as a corollary of Proposition 14.1. (It is also a consequence of Theorem 10.4.)

Corollary 14.3 [27, Corollary 10.4] *Suppose the nerve L of (W,S) is a GHS^{n-1}.*

(i) *If $\mathbf{q}\in\overline{\mathcal{R}}$, then $L^2_{\mathbf{q}}\mathcal{H}_*(\Sigma)$ is concentrated in dimension 0; moreover,*

$$L^2_{\mathbf{q}}\mathcal{H}_0(\Sigma)\cong A_S,$$

where A_S is the representation of $\mathbf{R_q}[W]$ on $\mathbf{R}$ via the symmetric character α_S of Definition 5.6.

(ii) *If $\mathbf{q}\in\overline{\mathcal{R}^{-1}}$, then $L^2_{\mathbf{q}}\mathcal{H}_*(\Sigma)$ is concentrated in dimension n and*

$$L^2_{\mathbf{q}}\mathcal{H}_n(\Sigma)\cong H_S,$$

where H_S is the representation of $\mathbf{R_q}[W]$ on $\mathbf{R}$ via the alternating character β_S of Definition 5.6.

Remark If L is a GHS^{n-1}, then $(K,\partial K)$ is a GHD^n where $\partial K := K^S$. Since $H^1(K;\mathbf{Z}/2)=0$, K is orientable. So, we can choose orientations for the n–simplices of K so that their sum is a relative cycle, ξ_K. Its homology class $[K]\in H_n(K,\partial K)$ is the *fundamental class* of K. By Theorem 10.4, $L^2_{\mathbf{q}}\mathcal{H}_n(\Sigma)$ is spanned by $[K]h_S$. (This was proved in [26].) A representative for this class is obtained by taking the fundamental cycle ξ_K and then harmonizing it to $\xi_K h_S$.

Example 14.4 ($\dim L = 1$) Suppose L is a k–gon. In other words, suppose we are given a Coxeter matrix on a set S, so that its nerve L is a circle and so that $\mathrm{Card}(S)=k$. This means, first of all, that the 1–skeleton of L is a k–gon. When $k=3$, for L to be equal to its 1–skeleton, a further condition is needed. Suppose $S=\{s_1,s_2,s_3\}$, $m_{ij}:=m_{s_is_j}$ and $\alpha_{ij}:=\pi/m_{ij}$, where $\{i,j,k\}=\{1,2,3\}$. The condition is that $\alpha_{12}+\alpha_{23}+\alpha_{13}\le\pi$. When this holds, the W–action on Σ is isomorphic to the action of a group of isometries on the Euclidean or hyperbolic plane generated by the reflections across the edges of a k–gon. (The Euclidean case occurs only when $k = 4$ and W is right-angled or when $k = 3$ and $\alpha_{12} + \alpha_{23} + \alpha_{13} = \pi$.)

If $\mathbf{q} \in \mathcal{R}$, then $L^2_{\mathbf{q}}\mathcal{H}_*(\Sigma)$ is concentrated in dimension 0; if $\mathbf{q} \in \mathcal{R}^{-1}$, it is concentrated in dimension 2; if $\mathbf{q} \notin \mathcal{R} \cup \mathcal{R}^{-1}$, then it is concentrated in dimension 1 (since it vanishes in dimensions 0 and 2). In each case, the nonzero Betti number is given by $\pm\chi_{\mathbf{q}}$.

Corollary 14.5 *Suppose that W is a Euclidean reflection group, ie that it can be represented as a cocompact group generated by isometric reflections on $\mathbf{R}^n$. Suppose further that $\mathbf{q} \geq \mathbf{1}$. Then $L^2_{\mathbf{q}}\mathcal{H}_*(\Sigma)$ is concentrated in the top dimension, $* = n$. (It is 0 if $\mathbf{q} = \mathbf{1}$.)*

Proof By Proposition 3.10 (or Remark 3.11), when t is a single indeterminate, the reciprocal of the radius of convergence of $W(t)$ is 1. It follows that $\{\mathbf{q} \mid \mathbf{q} > \mathbf{1}\} \subseteq \mathcal{R}^{-1}$.

Since $\Sigma \cong \mathbf{R}^n$, it follows from [15, Theorem B] that L is a GHS^{n-1}. In fact, L is a triangulation of S^{n-1}. (When (W, S) is irreducible, L is isomorphic to boundary complex of an n–simplex, by [4, Proposition 8, p 90]; when it is not irreducible it is a join of such complexes.) So, the result is a consequence of the previous proposition. □

Combining this with Corollary 13.11, we get the following (known) result.

Corollary 14.6 *Suppose that X is a Euclidean building with a chamber transitive automorphism group. Then its reduced L^2–cohomology is concentrated in the top dimension.*

A generalization of the Singer Conjecture Corollary 14.3 states that when L is a GHS^{n-1}, $L^2_{\mathbf{q}}\mathcal{H}_*(\Sigma)$ is concentrated in dimension 0 for $\mathbf{q} \in \overline{\mathcal{R}}$ and in dimension n for $\mathbf{q} \in \overline{\mathcal{R}^{-1}}$. What about the intermediate range, $\mathbf{q} \notin \overline{\mathcal{R}} \cup \overline{\mathcal{R}^{-1}}$? By Remark 7.6, in this range, $L^2_{\mathbf{q}}\mathcal{H}_0(\Sigma) = 0$ and by Poincaré duality, $L^2_{\mathbf{q}}\mathcal{H}_n(\Sigma) = 0$. For $\mathbf{q} = \mathbf{1}$, $L^2_{\mathbf{q}}\mathcal{H}_*(\Sigma)$ is the ordinary reduced L^2–homology $\mathcal{H}_*(\Sigma)$. In this case, the Singer Conjecture predicts that $\mathcal{H}_*(\Sigma)$ vanishes except in dimension $\frac{n}{2}$. There is considerable evidence for this version of the Singer Conjecture, at least in the case where (W, S) is right-angled. For example, it holds for $n = \dim \Sigma \leq 4$ and, when L is a barycentric subdivision, for $n = 6, 8$. (See Davis and Okun [21; 22].)

This suggests that the following generalization of the Singer Conjecture for Coxeter groups should hold for weighted L^2–homology.

Conjecture 14.7 (The Generalized Singer Conjecture) *Suppose L is a GHS^{n-1}. If $\mathbf{q} \leq \mathbf{1}$ and $k > \frac{n}{2}$, then $L^2_{\mathbf{q}}\mathcal{H}_k(\Sigma) = 0$.*

By Poincaré duality, this is equivalent to the conjecture that if $\mathbf{q} \geq \mathbf{1}$ and $k < \frac{n}{2}$, then $L^2_{\mathbf{q}}\mathcal{H}_k(\Sigma) = 0$.

In Section 16 we prove Conjecture 14.7 (as Theorem 16.13) in the case where W is right-angled and $n \leq 4$.

To further simplify the discussion, suppose $\mathbf{q} = q$, a single indeterminate. Then by Corollary 14.2, the roots of χ_q $(= 1/W(q))$ are symmetric about 1, ie if q is a root, then so is q^{-1}.

At one point, the following scenario (which is stronger than Conjecture 14.7) seemed plausible:

(a) χ_q has exactly n positive real roots (counted with multiplicity).

(b) $L^2_q\mathcal{H}_*(\Sigma)$ is always concentrated in a single dimension. The dimension jumps each time q passes a root of χ_q and the size of the jump is the multiplicity of the root.

In fact, both (a) and (b) are false. Gal [30] has given counterexamples to (a) in dimensions ≥ 6. We shall explain why (b) is false in dimensions $n \geq 4$ in Section 17 below.

15 Properties of weighted L^2–homology in the right-angled case

The usual L^2–cohomology of Σ is the case $\mathbf{q} = \mathbf{1}$. In [21] the first and fourth authors studied this case when (W, S) was right-angled. (Recall Definition 13.2: (W, S) is *right-angled* if $m_{st} = 2$ or ∞ for all pairs $\{s, t\}$ of distinct elements in S.) Much of [21] extends in a straightforward fashion from $\mathbf{q} = \mathbf{1}$ to the case of a general $\mathbf{q}$. The purpose of this section is to rewrite parts of [21] in the general case.

If (W, S) is right-angled, then its nerve L is a flag complex. (A simplicial complex Λ is a *flag complex* if any finite set of vertices in Λ which are pairwise connected by edges span a simplex of Λ.) Conversely, given any finite flag complex L, there is a right-angled Coxeter group W_L with nerve L. (The set of generators S for W_L is the vertex set of L and $m_{st} = 2$ if and only if $\{s, t\}$ spans an edge of L.) For further explanations, see Davis [12; 16] and Davis and Okun [21].

In this section, as well as in Section 16 and Section 17, *all simplicial complexes will be flag complexes and all subcomplexes will be full subcomplexes.* Given a finite flag complex L, let Σ_L be the complex on which W_L acts. As usual, $\mathbf{q}$ is an I–tuple of positive real numbers. For each $i \in \mathbb{N}$, we have a Hilbert $\mathcal{N}_{\mathbf{q}}$–module, $L^2_{\mathbf{q}}\mathcal{H}_i(\Sigma_L)$. Similarly, to each pair (L, A), we can associate the Hilbert $\mathcal{N}_{\mathbf{q}}$–module, $L^2_{\mathbf{q}}\mathcal{H}_i(\Sigma_L, W_L\Sigma_A)$.

We introduce some useful notation which reflects this situation.

Notation

(15–1) $$\mathfrak{h}^{\mathbf{q}}_i(L) := L^2_{\mathbf{q}}\mathcal{H}_i(\Sigma_L) \quad \mathfrak{h}^i_{\mathbf{q}}(L) := L^2_{\mathbf{q}}\mathcal{H}^i(\Sigma_L)$$

(15–2) $$\mathfrak{h}^{\mathbf{q}}_i(A) := L^2_{\mathbf{q}}\mathcal{H}_i(W_L\Sigma_A)$$

(15–3) $$\mathfrak{h}^{\mathbf{q}}_i(L, A) := L^2_{\mathbf{q}}\mathcal{H}_i(\Sigma_L, W_L\Sigma_A)$$

(15–4) $$b^i_{\mathbf{q}}(A) := \dim_{\mathcal{N}_{\mathbf{q}}}(\mathfrak{h}^{\mathbf{q}}_i(A))$$

(15–5) $$b^i_{\mathbf{q}}(L, A) := \dim_{\mathcal{N}_{\mathbf{q}}}(\mathfrak{h}^{\mathbf{q}}_i(L, A))$$

(15–6) $$\chi_{\mathbf{q}}(L) := \sum(-1)^i b^i_{\mathbf{q}}(A).$$

The notation in (15–2) and (15–4) will not lead to confusion, since $L^2_{\mathbf{q}}\mathcal{H}_i(W_L\Sigma_A)$ is the induced representation from $L^2_{\mathbf{q}}\mathcal{H}_i(\Sigma_A)$ and therefore, $b^i_{\mathbf{q}}(W_L\Sigma_A) = b^i_{\mathbf{q}}(\Sigma_A)$, where the left hand side of this equation denotes a dimension calculated with respect to $\mathcal{N}_{\mathbf{q}}(W_L)$ while the right hand side is with respect to $\mathcal{N}_{\mathbf{q}}(W_A)$.

Basic algebraic topology The next theorem is a compilation of properties of $\mathfrak{h}^{\mathbf{q}}_i(L, A)$ which were proved in [21] for the case $\mathbf{q} = \mathbf{1}$.

Theorem 15.1 (Compare [21, Section 7.2].)

(a) (Exact sequence of the pair) *The sequence*
$$\to \mathfrak{h}^{\mathbf{q}}_i(A) \to \mathfrak{h}^{\mathbf{q}}_i(L) \to \mathfrak{h}^{\mathbf{q}}_i(L, A) \to$$
is weakly exact.

(b) (Excision) *Let T be a set of vertices of A such that the open star of any vertex in T is contained in the interior of A. Then*
$$\mathfrak{h}^{\mathbf{q}}_i(L, A) \cong \mathfrak{h}^{\mathbf{q}}_i(L - T, A - T).$$

(c) (Mayer–Vietoris sequence) *Suppose $L = L_1 \cup L_2$ and $A = L_1 \cap L_2$, where L_1 and L_2 (and therefore, A) are full subcomplexes of L. Then*
$$\to \mathfrak{h}^{\mathbf{q}}_i(A) \to \mathfrak{h}^{\mathbf{q}}_i(L_1) \oplus \mathfrak{h}^{\mathbf{q}}_i(L_2) \to \mathfrak{h}^{\mathbf{q}}_i(L) \to$$
is weakly exact.

(d) *With* L_1, L_2 *and* A *as in* (c),

$$\mathfrak{h}_i^{\mathbf{q}}(L, A) \cong \mathfrak{h}_i^{\mathbf{q}}(L_1, A) \oplus \mathfrak{h}_i^{\mathbf{q}}(L_2, A).$$

(e) (The Künneth Formula: the Betti numbers of a join)

$$b_{\mathbf{q}}^k(L_1 * L_2) = \sum_{i+j=k} b_{\mathbf{q}}^i(L_1) b_{\mathbf{q}}^j(L_2).$$

(f) (Atiyah's Formula)

$$\chi_{\mathbf{q}}(L) = \sum_{T \in \mathcal{S}} \prod_{s \in T} \frac{-q_s}{1+q_s} = \frac{1}{W_L(\mathbf{q})}.$$

(g) (0–dimensional homology [27])

$$b_{\mathbf{q}}^0(L) = \begin{cases} 0 & \text{if } \mathbf{q} \notin \mathcal{R}, \\ \frac{1}{W(\mathbf{q})} & \text{if } \mathbf{q} \in \mathcal{R}. \end{cases}$$

and $b_{\mathbf{q}}^i(L) = 0$ *for* $i > 0$, $\mathbf{q} \in \mathcal{R}$.

(h) (Pseudomanifolds [26, Theorem 10.3]) *Suppose* L *is a* $(n-1)$*–dimensional pseudomanifold. Then* Σ_L *is an* n*–dimensional pseudomanifold and, since the* 1*–skeleton of* Σ_L *is the Cayley graph of* W_L*, each component of the complement of codimension* 2 *skeleton of* Σ_L *is infinite. So, if* $\mathbf{q} \notin \mathcal{R}^{-1}$*, then* $b_{\mathbf{q}}^n(L) = 0$*. (If* $\mathbf{q} \in \mathcal{R}^{-1}$ *and in addition,* L *is orientable and the complement of its codimension* 2*–skeleton is connected, then* $b_{\mathbf{q}}^n(L) = 1/W(\mathbf{q}^{-1})$.

(i) (The empty set) *Since* $\Sigma_\varnothing$ *is a point,*

$$b_{\mathbf{q}}^i(\varnothing) = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

(j) (A k–simplex) *Given a* k*–simplex* σ, $W_\sigma \cong (\mathbf{Z}_2)^{k+1}$ *and* $\Sigma_\sigma = [-1,1]^{k+1}$. *Hence, for* $\mathbf{q} = q$*, a single indeterminate:*

$$b_q^i(\sigma) = \begin{cases} \left(\frac{1}{1+q}\right)^{k+1} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

(k) (The Betti numbers of a disjoint union) *Suppose* L *is the disjoint union of* L_1 *and* L_2*. Then, for* $i \geq 2$,

$$b_{\mathbf{q}}^i(L) = b_{\mathbf{q}}^i(L_1) + b_{\mathbf{q}}^i(L_2).$$

For $\mathbf{q} \notin \mathcal{R}_{L_1} \cup \mathcal{R}_{L_2}$,

$$b^1_{\mathbf{q}}(L) = b^1_{\mathbf{q}}(L_1) + b^1_{\mathbf{q}}(L_2) + 1.$$

Proof Properties (a) through (e) follow from general principles as in [21]. Property (f) is Proposition 7.4; (g) is proved in Section 7 as Proposition 7.5; (h) is proved in [26] (it also follows from Theorem 10.4); properties (i) and (j) are special cases of (g). Property (k) follows from (c) (the Mayer–Vietoris sequence); the last sentence of (k) follows after noting that $L_1 \cap L_2 = \varnothing$ has nonzero Betti number, $b^0_{\mathbf{q}}(\varnothing) = 1$ and that, by (g) $b^0_{\mathbf{q}}(L_1) = b^0_{\mathbf{q}}(L_2) = 0$. □

In the next proposition we assume that I is a singleton so that $\mathbf{q}$ is a single parameter q. We extend some simple calculations of [21] from $q = 1$ to the case where q is arbitrary.

Proposition 15.2 (Compare [21, Section 7.3].) *Suppose* $\mathbf{q} = q$, *a positive real number.*

(a) (The Betti numbers of k points) *Let* P_k *denote the disjoint union of* k *points. If* $k \geq 2$, *then*

$$b^0_q(P_k) = \begin{cases} \frac{1-(k-1)q}{1+q} & \text{if } q < \frac{1}{k-1}, \\ 0 & \text{if } q \geq \frac{1}{k-1}. \end{cases}$$

$$b^1_q(P_k) = \begin{cases} 0 & \text{if } q < \frac{1}{k-1}, \\ \frac{(k-1)q-1}{1+q} & \text{if } q \geq \frac{1}{k-1}. \end{cases}$$

In particular,

$$b^0_q(S^0) = b^0_q(P_2) = \begin{cases} \frac{1-q}{1+q} & \text{if } q < 1, \\ 0 & \text{if } q \geq 1, \end{cases}$$

$$b^1_q(S^0) = b^1_q(P_2) = \begin{cases} 0 & \text{if } q < 1, \\ \frac{q-1}{1+q} & \text{if } q \geq 1. \end{cases}$$

(b) (The Betti numbers of a suspension) *The "suspension" of* L *is defined by* $SL := S^0 * L$. *Then*

$$b^i_q(SL) = \begin{cases} \frac{1-q}{1+q} b^i_q(L) & \text{if } q < 1, \\ \frac{q-1}{1+q} b^{i-1}_q(L) & \text{if } q \geq 1, \end{cases}$$

for all i.

(c) (The boundary complex of an n–octahedron) *Let*

$$O_n := \underbrace{S^0 * \cdots * S^0}_{n}.$$

Then

$$b_q^0(O_n) = \begin{cases} \left(\frac{1-q}{1+q}\right)^n & \text{if } q < 1, \\ 0 & \text{if } q \geq 1, \end{cases}$$

$$b_q^i(O_n) = 0, \text{ for } 1 \leq i \leq n-1 \text{ and for all } q,$$

$$b_q^n(O_n) = \begin{cases} 0 & \text{if } q \leq 1, \\ \left(\frac{q-1}{1+q}\right)^n & \text{if } q > 1. \end{cases}$$

(d) (The Betti numbers of a cone)

$$b_q^i(CL) = \frac{1}{q+1} b_q^i(L),$$

$$b_q^{i+1}(CL, L) = \frac{q}{1+q} b_q^i(L).$$

Moreover, the sequence of the pair (CL, L) breaks up into short exact sequences:

$$0 \to \mathfrak{h}_{i+1}^q(CL, L) \to \mathfrak{h}_i^q(L) \to \mathfrak{h}_i^q(CL) \to 0.$$

Proof Since Σ_{P_k} is 1–dimensional, $b_q^i(P_k) = 0$ for $i > 1$. By Theorem 15.1(g), $\mathfrak{h}_i^{\mathbf{q}}(P_k)$ is concentrated in one dimension. Since $\chi_q(P_k) = (1-(k-1)q)/(1+q)$, the calculation in (a) follows.

The calculations of Betti numbers in (b), (c) and (d) follow immediately from part (a) and Theorem 15.1(e). The proof of the last sentence of (d) is similar to Lemma 7.3.3 of [21]. $\mathfrak{h}_i^q(L)$ means $L_q^2\mathcal{H}_i(\Sigma_L \times \mathbf{Z}_2)$ $(= L_q^2\mathcal{H}_i(\Sigma_L) \otimes L_q^2(\mathbf{Z}_2))$. $\mathfrak{h}_i^q(CL)$ can be identified with the subspace $L_q^2\mathcal{H}_i(\Sigma_L) \otimes A_s$, where s is the generator corresponding to the cone point. The map $\mathfrak{h}_i^q(L) \to \mathfrak{h}_i^q(CL)$ is then identified with orthogonal projection onto this subspace. □

16 W is right-angled and L is a sphere

In the right-angled case, Conjecture 14.7 can be attacked using the techniques of [21]. In this case, the arguments of [21] are sufficient to prove the conjecture for $n \leq 4$. We give the details below.

Poincaré duality If a pair $(D, \partial D)$ of flag complexes is a generalized homology disk, then Σ_D is a polyhedral homology manifold with boundary (its boundary being $W_D \Sigma_{\partial D}$). Hence, it satisfies a relative version of Poincaré duality.

Proposition 16.1 (Compare [21, Section 7.4].)

(i) *If L is a GHS^{n-1}, then $b^i_{\mathbf{q}}(L) = b^{n-i}_{\mathbf{q}^{-1}}(L)$.*

(ii) *If $(D, \partial D)$ is a GHD^{n-1}, then $b^i_{\mathbf{q}}(D, \partial D) = b^{n-i}_{\mathbf{q}^{-1}}(D)$.*

(iii) *If $(D, \partial D)$ is a GHD^{n-1}, then the homology and cohomology sequences of the pair $(D, \partial D)$ are isomorphic under Poincaré duality in the sense that the following diagram commutes up to sign,*

$$\begin{array}{ccccccccc}
\to & \mathfrak{h}^{\mathbf{q}}_{i+1}(D, \partial D) & \to & \mathfrak{h}^{\mathbf{q}}_{i}(\partial D) & \to & \mathfrak{h}^{\mathbf{q}}_{i}(D) & \to & \mathfrak{h}^{\mathbf{q}}_{i}(D, \partial D) & \to \\
 & \updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong & \\
\to & \mathfrak{h}_{\mathbf{q}^{-1}}^{n-i-1}(D) & \to & \mathfrak{h}_{\mathbf{q}^{-1}}^{n-i-1}(\partial D) & \to & \mathfrak{h}_{\mathbf{q}^{-1}}^{n-i}(D, \partial D) & \to & \mathfrak{h}_{\mathbf{q}^{-1}}^{n-i}(D) & \to
\end{array}$$

where the vertical isomorphisms are given by Poincaré duality.

Suppose that $L = D_1 \cup D_2$ and $M = D_1 \cap D_2$. Also suppose that L is a GHS^{n-1} and that (D_1, M) and (D_2, M) are GHD^{n-1}'s. By Theorem 15.1(d), we have $\mathfrak{h}^i_{\mathbf{q}^{-1}}(L, M) \cong \mathfrak{h}^i_{\mathbf{q}^{-1}}(D_1, M) \oplus \mathfrak{h}^i_{\mathbf{q}^{-1}}(D_2, M)$. Similarly to Proposition 16.1(iii), the homology Mayer–Vietoris sequence of $L = D_1 \cup D_2$ is isomorphic, via Poincaré duality, to the exact sequence of the pair (L, M) in cohomology. In other words, the following diagram commutes up to sign,

$$\begin{array}{ccccccc}
\to & \mathfrak{h}^{\mathbf{q}}_{i+1}(L) & \to & \mathfrak{h}^{\mathbf{q}}_{i}(M) & \to & \mathfrak{h}^{\mathbf{q}}_{i}(D_1) \oplus \mathfrak{h}^{\mathbf{q}}_{i}(D_2) & \to \\
 & \updownarrow \cong & & \updownarrow \cong & & \updownarrow \cong & \\
\to & \mathfrak{h}_{\mathbf{q}^{-1}}^{n-i-1}(L) & \to & \mathfrak{h}_{\mathbf{q}^{-1}}^{n-i-1}(M) & \to & \mathfrak{h}_{\mathbf{q}^{-1}}^{n-i}(D_1, M) \oplus \mathfrak{h}_{\mathbf{q}^{-1}}^{n-i}(D_2, M) & \to
\end{array}$$

where the first row is the Mayer–Vietoris sequence, the second is the exact sequence of the pair and the vertical isomorphisms are given by Poincaré duality. We record the special case of this where $n = 2k + 1$ and $i = k$ as the following lemma.

Lemma 16.2 (Compare [21, Lemma 7.4.6].) *With hypotheses as above, suppose $n = 2k + 1$. Then the map $i_*\colon \mathfrak{h}^{\mathbf{q}^{-1}}_k(M) \to \mathfrak{h}^{\mathbf{q}^{-1}}_k(L)$ induced by the inclusion is dual (under Poincaré duality) to the connecting homomorphism $\partial_*\colon \mathfrak{h}^{\mathbf{q}}_{k+1}(L) \to \mathfrak{h}^{\mathbf{q}}_k(M)$ in the Mayer–Vietoris sequence.*

Proof In this special case, the previous diagram becomes the following:

$$\begin{array}{ccccc}
\mathfrak{h}^{\mathbf{q}}_{k+1}(L) & \xrightarrow{\partial_*} & \mathfrak{h}^{\mathbf{q}}_{k}(M) & \longrightarrow & \mathfrak{h}^{\mathbf{q}}_{k}(D_1)\oplus\mathfrak{h}^{\mathbf{q}}_{k}(D_2) \\
\updownarrow\cong & & \updownarrow\cong & & \updownarrow\cong \\
\mathfrak{h}^{k}_{\mathbf{q}^{-1}}(L) & \xrightarrow{i^*} & \mathfrak{h}^{k}_{\mathbf{q}^{-1}}(M) & \longrightarrow & \mathfrak{h}^{k+1}_{\mathbf{q}^{-1}}(D_1,M)\oplus\mathfrak{h}^{k+1}_{\mathbf{q}^{-1}}(D_2,M)
\end{array}$$

□

Vanishing Conjectures We now consider several conjectures, $\mathbf{I}(n)$, $\mathbf{III}(n)$, $\mathbf{III}'(n)$ and $\mathbf{V}(n)$, concerning the reduced $L^2_{\mathbf{q}}$–homology of Σ_L, where L is a generalized homology sphere. (The notation $\mathbf{I}(n)$, $\mathbf{III}(n)$, $\mathbf{V}(n)$, is taken from [21]; the "n" refers to the dimension of Σ_L, so that $\dim L = n-1$.)

$\mathbf{I}(n)$ *If L is a GHS^{n-1} and $\mathbf{q}\le\mathbf{1}$, then $b^i_{\mathbf{q}}(L)=0$ for $i>n/2$.*

Given $(D,\partial D)$, a generalized homology disk, denote by $\widehat{D}$ (or L) the GHS formed by gluing on $C(\partial D)$ (the cone on ∂D) to D along ∂D. If v denotes the cone point, then $\partial D = L_v$ (the link of v in $\widehat{D}$) and $C(\partial D)=CL_v$. Conversely, given a GHS, call it L, and a vertex v, we obtain a GHD, with $D=L-v$ (the full subcomplex of L spanned by the vertices $\neq v$) and with $\partial D = L_v$.

Next we consider a seemingly weaker version of $\mathbf{I}(2k+1)$.

$\mathbf{III}(2k+1)$ *Suppose (D,L_v) is a GHD^{2k} and $\widehat{D}=D\cup CL_v$ as above. If $\mathbf{q}\le\mathbf{1}$, then, in the Mayer–Vietoris sequence, the map*

$$j_*\oplus h_*\colon \mathfrak{h}^{\mathbf{q}}_k(L_v)\to\mathfrak{h}^{\mathbf{q}}_k(D)\oplus\mathfrak{h}^{\mathbf{q}}_k(CL_v)$$

is a monomorphism.

By Lemma 16.2, $\mathbf{III}(2k+1)$ is equivalent to the following.

$\mathbf{III}'(2k+1)$ *Suppose (D,L_v) is a GHD^{2k} and $\widehat{D}=D\cup CL_v$ as above. If $\mathbf{q}\ge\mathbf{1}$, then the map $i_*\colon\mathfrak{h}^{\mathbf{q}}_k(L_v)\to\mathfrak{h}^{\mathbf{q}}_k(L)$, induced by the inclusion, is the zero homomorphism.*

The following is a stronger version of $\mathbf{I}(n)$.

$\mathbf{V}(n)$ *Suppose L is a GHS^{n-1} and A is any full subcomplex.*

- *If $n=2k$ is even and $\mathbf{q}\le\mathbf{1}$, then $b^i_{\mathbf{q}}(L,A)=0$ for all $i>k$.*
- *If $n=2k+1$ is odd and $\mathbf{q}\le\mathbf{1}$, then $b^i_{\mathbf{q}}(A)=0$ for all $i>k$.*

By [26], $\mathbf{I}(1)$ and $\mathbf{I}(2)$ hold.

Next we list some obvious implications among these conjectures.

Lemma 16.3 (Compare [21, Section 8].)

(a) $\mathbf{I}(2k+1) \implies \mathbf{III}(2k+1)$.

(b) $\mathbf{V}(n) \implies \mathbf{I}(n)$.

(c) $\mathbf{V}(2k)$ *implies that for any full subcomplex A of L (a GHS^{2k-1}), we have*

$$b^i_{\mathbf{q}}(A) = 0 \text{ for all } i > k \text{ and } \mathbf{q} \le \mathbf{1}.$$

Proof (a) is obvious: if $\mathbf{I}(2k+1)$ holds, then the $\mathfrak{h}^{\mathbf{q}}_*(L)$ terms in the Mayer–Vietoris sequence all vanish, so the map $j_* \oplus h_*$ in $\mathbf{III}(2k+1)$ is a weak isomorphism.

(b) If $n = 2k$, take $A = \varnothing$ to get $b^i_{\mathbf{q}}(L) = 0$ for $i > k$. If $n = 2k+1$, take $A = L$, to get $b^i_{\mathbf{q}}(L) = 0$ for $i > k$.

(c) Assume $\mathbf{V}(2k)$ holds. By (b), $b^i_{\mathbf{q}}(L) = 0$ for $i > k$. Hence, in the exact sequence of the pair

$$\mathfrak{h}^{\mathbf{q}}_{i+1}(L, A) \to \mathfrak{h}^{\mathbf{q}}_i(A) \to \mathfrak{h}^{\mathbf{q}}_i(L),$$

the first and third terms vanish for all $i > k$. □

Lemma 16.4 $\mathbf{III}(2k+1) \implies \mathbf{III}(2l+1)$ *for all $l \le k$.*

Proof The proof is the same as in [21, 8.8.1, p 41]. Suppose (D, L_v) is a GHD^{2l}, with $l < k$. Let A be the join of $k-l$ copies of an m–gon, $m \ge 5$ and assign to A a thickness vector $\mathbf{q} = \mathbf{1}$. If $\mathbf{III}(2l+1)$ fails for D, then $\mathbf{III}(2k+1)$ fails for $D * A$ (the join of D and A). □

Inductive arguments We describe the program of [21] for proving Conjecture $\mathbf{V}(n)$. The idea is to use a double induction: first, induction on the dimension n and second, depending on the parity of n, induction either on the number of vertices of A or on the number of vertices in $L - A$. In this section we always assume $\mathbf{q} \le \mathbf{1}$.

As in [21], we set up some notation for the induction on the number of vertices. Suppose A and B are full subcomplexes of L, the vertex sets of which differ by only one element, say v. In other words, $B = A - v$, for some $v \in \mathcal{S}^{(1)}(A)$. Let A_v and L_v denote the link of v in A and L, respectively. Thus, $A = B \cup CA_v$ and $CA_v \cap B = A_v$. We note that L_v is a GHS of one less dimension than L and that A_v is a full subcomplex of L_v.

Lemma 16.5 (Compare [21, Lemma 9.2.1].) $\mathbf{V}(2k-1) \implies \mathbf{V}(2k)$.

Proof Suppose $\mathbf{V}(2k-1)$ holds. Let (L, A) be as in $\mathbf{V}(2k)$ and let $B = A - v$. Assume, by induction on the number of vertices in $L - A$, that $\mathbf{V}(2k)$ holds for (L, A). (The case $A = L$ being trivial.) We want to prove it also holds for (L, B), ie that $b^i_{\mathbf{q}}(L, B) = 0$ for $i > k$. Consider the exact sequence of the triple (L, A, B):

$$\to \mathfrak{h}^{\mathbf{q}}_i(A, B) \to \mathfrak{h}^{\mathbf{q}}_i(L, B) \to \mathfrak{h}^{\mathbf{q}}_i(L, A) \to$$

Suppose $i > k$. By inductive hypothesis, $b^i_{\mathbf{q}}(L, A) = 0$. By excision (Theorem 15.1(b)), $b^i_{\mathbf{q}}(A, B) = b^i_{\mathbf{q}}(CA_v, A_v)$. By Theorem 15.1(e),

$$b^i_{\mathbf{q}}(CA_v, A_v) = \frac{q_v}{q_v + 1} b^{i-1}_{\mathbf{q}}(A_v).$$

Since $\mathbf{V}(2k-1)$ holds for (L_v, A_v) and since $i - 1 > k - 1$, $b^{i-1}_{\mathbf{q}}(A_v) = 0$. So, $0 = b^i_{\mathbf{q}}(CA_v, A_v) = b^i_{\mathbf{q}}(A, B)$. Consequently, $b^i_{\mathbf{q}}(L, B) = 0$. □

Essentially the same argument proves the following lemma.

Lemma 16.6 (Compare [21, Lemma 9.2.2].) *Assume that $\mathbf{V}(2k)$ holds. Suppose that a flag complex L is a polyhedral homology manifold of dimension $2k$ and that A is a full subcomplex. Then $b^i_{\mathbf{q}}(L, A) = 0$ for $i > k + 1$ and $\mathbf{q} \le \mathbf{1}$.*

Proof We proceed as in the previous proof. If $B = A - v$, then

$$b^i_{\mathbf{q}}(A, B) = b^i_{\mathbf{q}}(CA_v, A_v) = \frac{q_v}{q_v + 1} b^{i-1}_{\mathbf{q}}(A_v).$$

Since we are assuming $\mathbf{V}(2k)$ holds, Lemma 16.3(c) implies that $b^{i-1}_{\mathbf{q}}(A_v) = 0$ for $i > k + 1$. Hence, if we assume by induction that the lemma holds for (L, A), then it also holds for (L, B). □

Lemma 16.7 (Compare [21, Lemma 9.2.3].)

$$(\mathbf{V}(2k) \text{ and } \mathbf{III}(2k+1)) \implies \mathbf{V}(2k+1).$$

Proof Assume $\mathbf{V}(2k)$ and $\mathbf{III}(2k+1)$ hold. Let (L, A) be as in $\mathbf{V}(2k+1)$ and let $B = A - v$. Assume, by induction on the number of vertices in B, that $\mathbf{V}(2k+1)$ holds for B. (The case $B = \varnothing$ being trivial.) We want to prove that it also holds for A, ie that $b^i_{\mathbf{q}}(A) = 0$ for $i > k$.

First suppose that $i > k + 1$. Consider the Mayer–Vietoris sequence for $A = B \cup CA_v$:

$$\mathfrak{h}^{\mathbf{q}}_i(B) \oplus \mathfrak{h}^{\mathbf{q}}_i(CA_v) \to \mathfrak{h}^{\mathbf{q}}_i(A) \to \mathfrak{h}^{\mathbf{q}}_{i-1}(A_v).$$

By $\mathbf{V}(2k)$ and Lemma 16.3(c), $b^{i-1}_{\mathbf{q}}(A_v) = 0$ (since $i-1 > k$) and hence, $b^i_{\mathbf{q}}(CA_v) = 0$ (by Theorem 15.1(c)). By inductive hypothesis, $b^i_{\mathbf{q}}(B) = 0$, and consequently, $b^i_{\mathbf{q}}(A) = 0$.

For $i = k+1$, we compare the Mayer–Vietoris sequence of $A = B \cup CA_v$ with that of $L = D \cup CL_v$ (where $D = L - v$):

$$\begin{array}{ccccccc} & & & & \mathfrak{h}^{\mathbf{q}}_{k+1}(L_v, A_v) & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{h}^{\mathbf{q}}_{k+1}(A) & \longrightarrow & \mathfrak{h}^{\mathbf{q}}_{k}(A_v) & \xrightarrow{j'_* \oplus h'_*} & \mathfrak{h}^{\mathbf{q}}_{k}(B) \oplus \mathfrak{h}^{\mathbf{q}}_{k}(CA_v) \\ & & & & f_* \downarrow & & \downarrow \\ & & & & \mathfrak{h}^{\mathbf{q}}_{k}(L_v) & \xrightarrow{j_* \oplus h_*} & \mathfrak{h}^{\mathbf{q}}_{k}(D) \oplus \mathfrak{h}^{\mathbf{q}}_{k}(CL_v) \end{array}$$

By $\mathbf{V}(2k)$, $b^{k+1}_{\mathbf{q}}(L_v, A_v) = 0$; hence, f_* is injective. By $\mathbf{III}(2k+1)$, $j_* \oplus h_*$ is injective. Hence, $j'_* \oplus h'_*$ is injective and therefore, $b^{k+1}_{\mathbf{q}}(A) = 0$. □

One of the main results of [21] has the following analog.

Theorem 16.8 (Compare [21, Theorem 9.3.1].) *Statement* $\mathbf{III}(2k-1)$ *implies that* $\mathbf{V}(n)$ *holds for all* $n \leq 2k$.

Proof By Lemma 16.4, $\mathbf{III}(2k-1)$ implies $\mathbf{III}(2l-1)$, for all $l \leq k$. Suppose, by induction on n, that $\mathbf{V}(n-1)$ holds for some $n \leq 2k$. If $n-1$ is odd, then by Lemma 16.5, $\mathbf{V}(n-1)$ implies $\mathbf{V}(n)$. If $n-1$ is even, then by Lemma 16.7, $\mathbf{V}(n-1)$ and $\mathbf{III}(n)$ imply $\mathbf{V}(n)$. □

The Conjecture in dimension 3 We begin with a discussion of triangulations of S^2. (Details can be found in [21, Section 10.2].)

For $j = 1, 2$, suppose that L_j is a flag triangulation of S^2 and that s_j is a vertex of valence 4 in L_j. Choose an identification of the link of s_1 with that of s_2. (They are both 4–gons.) Define a new triangulation $L_1 \square L_2$ of S^2 by gluing together the 2–disks $L_1 - s_1$ and $L_2 - s_2$ along their boundaries.

Conversely, suppose C is an empty 4–circuit in L. Then C separates L into two 2–disks, D_1 and D_2. Let L_1 and L_2 denote the result of capping off D_1 and D_2, respectively (where "capping off" means adjoining a cone on the boundary). Then $L = L_1 \square L_2$.

Lemma 16.9 (Compare [21, Lemma 10.2.7].) *$b^2_{\mathbf{q}}(L_1 \Box L_2) = b^2_{\mathbf{q}}(L_1) + b^2_{\mathbf{q}}(L_2)$ for $\mathbf{q} \le \mathbf{1}$.*

Proof This follows from the Mayer–Vietoris sequence and Proposition 15.2(c)–(d). □

Andreev [1; 2] determined the possible fundamental polytopes for any reflection group on $\mathbf{H}^3$ of cofinite volume. The right-angled case of the Andreev's Theorem is the following.

Theorem 16.10 (Andreev's Theorem) *Suppose that L is a flag triangulation of S^2 and that*

(i) *L has no empty 4–circuits, and*

(ii) *L is not the suspension of a 4– or 5–gon.*

Let T denote the set of valence 4 vertices of L and let $K_{[S-T]}$ be the dual of the cellulation $[S-T]$ of S^2 obtained by replacing stars of vertices of T by square cells. Then $K_{[S-T]}$ can be realized as an ideal, right-angled convex polytope in $\mathbf{H}^3$. (The ideal vertices correspond to the square faces of $[S-T]$, ie to the vertices of valence 4 in S.) The resulting hyperbolic reflection group is the right-angled Coxeter group W_{S-T}.

Next we show that $\mathbf{III}'(3)$ is true for right-angled reflection groups on $\mathbf{H}^3$.

Equidistant hypersurfaces Suppose a Coxeter group W acts by reflections on hyperbolic $(2k+1)$–space $\mathbf{H}^{2k+1}$ with a fundamental polytope K of finite volume.

Let $\mathbf{H}^{2k}$ be a wall. We claim that the map $L^2_{\mathbf{q}}\mathcal{H}_k(\mathbf{H}^{2k}) \to L^2_{\mathbf{q}}\mathcal{H}_k(\mathbf{H}^{2k+1})$, induced by inclusion, is the zero map for $\mathbf{q} \ge \mathbf{1}$.

Our argument uses weighted L^2–de Rham cohomology theory. We will show that the map $L^2_{\mathbf{q}}\mathcal{H}^k(\mathbf{H}^{2k+1}) \to L^2_{\mathbf{q}}\mathcal{H}^k(\mathbf{H}^{2k})$, induced by restriction of forms, is the zero map. To define these terms we first need a "weight function" on $\mathbf{H}^{2k+1}$ which we can then use to define a new inner product on the vector space C^∞ j–forms on $\mathbf{H}^{2k+1}$.

Given any measurable nonnegative function $f\colon H^{2k+1} \to [0,\infty)$, one can modify the volume form on $\mathbf{H}^{2k+1}$ by multiplying by f and then define a new norm on C^∞ j–forms ω by

$$\|\omega\|_f^2 = \int_{\mathbf{H}^{2k+1}} \|\omega\|_p^2 \, f(p)\, dV,$$

where $\|\omega\|_p$ denotes the pointwise norm. $\|\omega\|_f$ is called the *L^2_f–norm* of ω.

Let K be a fundamental polytope for W on $\mathbf{H}^{2k+1}$. As usual, $\mathbf{q}$ is an I–tuple of positive real numbers. For any point p in $\mathbf{H}^{2k+1}$, put $f(p) = q_w$ when $p \in wK$. Of course, this expression is ambiguous for $p \in w\,\partial K$. Nevertheless, choose some convention to remove the ambiguity, for example, that w is the element of minimum word length with $p \in wK$. Then f is the *word length weight function* on $\mathbf{H}^{2k+1}$. It is a sort of step function in that it is constant on the interior of each chamber.

When K is compact, the arguments of [25] go through to show that the cellular weighted L^2–cohomology of Σ can be calculated using weighted de Rham cohomology, ie

$$L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma) \cong L^2_{\mathbf{q}}\mathcal{H}^k(\mathbf{H}^{2k+1}),$$

where the right hand side is defined using L^2_f forms with f the word length weight function defined above. When K is not compact but has finite volume we can reach the same conclusion by using Cheeger and Gromov [10].

Next let $\mathbf{H}^{2k}$ be a supporting wall of K (ie $\mathbf{H}^{2k}$ is a wall determined by a codimension one face of K). Put coordinates (x,y) on $\mathbf{H}^{2k+1}$ by letting $y \in \mathbf{R}$ be the oriented distance from p to the nearest point $x \in \mathbf{H}^{2k}$. Let N_y be the hypersurface in $\mathbf{H}^{2k+1}$ consisting of the points of (oriented) distance y from $\mathbf{H}^{2k}$. Let $p_y\colon N_y \to \mathbf{H}^{2k}$ be the projection which takes a point in N_y to the closest point in $\mathbf{H}^{2k}$. Then p_y is a homothety. Let $\phi_y\colon \mathbf{H}^{2k} \to N_y$ be its inverse. Also, let $i\colon \mathbf{H}^{2k} \to \mathbf{H}^{2k+1}$ and $i_y\colon N_y \to \mathbf{H}^{2k+1}$ be the inclusions. Thus, i and $i_y \circ \phi_y$ are properly homotopic.

Let $g(x,y) = f(x,0)$. Note that $f(x,y) \geq g(x,y)$.

Let ω be a closed L^2_f-k–form on $\mathbf{H}^{2k+1}$. We claim that the restriction $i^*(\omega)$ of ω to $\mathbf{H}^{2k}$ represents the zero class in reduced L^2_f–cohomology. Suppose, to the contrary, that $[i^*(\omega)] \neq 0$. Then $\|i^*(\omega)\|_g \geq \|[i^*(\omega)]\|_g \geq 0$, where $\|[i^*(\omega)]\|_g$ denotes the norm of the harmonic representative of the cohomology class $[i^*(\omega)]$. Since ϕ_y is a conformal diffeomorphism, it follows that it preserves norms of middle-dimensional forms: $\|\phi_y^*(i_y^*(\omega))\|_g = \|i_y^*(\omega)\|_g$. Since i and $i_y \circ \phi_y$ are properly homotopic, $[\phi_y^*(i_y^*(\omega))] = [i^*(\omega)]$, so it follows that $\|i_y^*(\omega)\|_g \geq \|[i^*(\omega)]\|_g$. Now, since $i_y^*(\omega)$ is just a restriction of ω, we have a pointwise inequality $\|\omega\|_x \geq \|i_y^*(\omega)\|_x$. Therefore, using Fubini's Theorem, we obtain

$$\begin{aligned}\|\omega\|_g^2 &= \int_{\mathbf{H}^{2k+1}} \|\omega\|_x^2\, g(x,y)\, dV = \int_{\mathbf{R}}\int_{N_y} \|\omega\|_x^2\, g(x,y)\, dA\, dy \\ &\geq \int_{\mathbf{R}}\int_{N_y} \|i_y^*(\omega)\|_x^2\, g(x,y)\, dA\, dy = \int_{\mathbf{R}} \|i_y^*(\omega)\|_g^2\, dy \geq \int_{\mathbf{R}} \|[i^*(\omega)]\|_g^2\, ds = \infty.\end{aligned}$$

Since $\|\omega\|_f \geq \|\omega\|_g$, this contradicts our assumption that the L^2_f–norm of ω is finite and thereby completes the proof.

In dimension 3 we get the following.

Theorem 16.11 *Suppose that L is a flag triangulation of S^2 satisfying the conditions of the Andreev's Theorem. Then* $\mathbf{III}'(3)$ *is true for* L.

Proof If $v \in T$, then, by Proposition 15.2(c), $b^1_{\mathbf{q}}(L_v) = 0$ so $\mathbf{III}'(3)$ is automatic. If $v \notin T$, then the result follows from the Andreev's Theorem and the previous paragraphs. □

Theorem 16.12 $\mathbf{I}(3)$ *is true: if L is a triangulation of the 2–sphere as a flag complex, then*

$$b^i_{\mathbf{q}}(L) = 0 \text{ for } i > 1 \quad \text{and} \quad \mathbf{q} \leq \mathbf{1}.$$

Proof If L is the suspension of a 4– or 5–gon, then the theorem follows from Proposition 15.2(b). If L is not the suspension of a 4–gon or a 5–gon and if it has no empty 4–circuits, then the theorem follows from Theorem 16.11, Lemma 16.7 and the fact that $I(2)$ holds [26].

In every other case, L has an empty 4–circuit which we can use to decompose it as, $L = L_1 \square L_2$, as before. Since L_1 and L_2 each have fewer vertices than does L, this process must eventually terminate. So, the theorem follows from Lemma 16.9. □

Since $\mathbf{I}(3)$ is true, Theorem 16.8 (together with Lemma 16.3(a)) yields the following.

Theorem 16.13 (Compare [21, Theorem 11.1.1].) $\mathbf{V}(n)$ *is true for* $n \leq 4$.

If L is a flag triangulation of S^3, then $\mathbf{V}(4)$, Poincaré duality and [27] imply:

- for $\mathbf{q} \in \overline{\mathcal{R}}$, $\mathfrak{h}^{\mathbf{q}}_*(L)$ is concentrated in dimension 0,
- for $\mathbf{q} \leq 1$ and $\mathbf{q} \notin \mathcal{R}$, $\mathfrak{h}^{\mathbf{q}}_*(L)$ is concentrated in dimensions 1 and 2,
- for $\mathbf{q} > 1$ and $\mathbf{q} \notin \mathcal{R}^{-1}$, $\mathfrak{h}^{\mathbf{q}}_*(L)$ is concentrated in dimensions 2 and 3,
- for $\mathbf{q} \in \overline{\mathcal{R}^{-1}}$, $\mathfrak{h}^{\mathbf{q}}_*(L)$ is concentrated in dimension 4.

17 Failure of concentration in the intermediate range

In this section I is a singleton (so that q is a single parameter) and W is right-angled. We retain the notation and conventions of Section 15.

The h–polynomial Combinatorialists have associated two polynomials to a finite simplicial complex L: its "f–polynomial," $f_L(t)$, and its "h–polynomial," $h_L(t)$. The first is defined by

$$f_L(t) := \sum_{T \in \mathcal{S}(L)} t^{\mathrm{Card}(T)} = \sum_{i=0}^{n} f_{i-1} t^i, \tag{17–1}$$

where f_m is the number of m–simplices of L, $f_{-1} = 1$ and $\dim L = n-1$. The second one is defined by

$$h_L(t) := (1-t)^n f_L\left(\frac{t}{1-t}\right). \tag{17–2}$$

If a Coxeter system (W,S) is right-angled, then for each spherical subset T, we have $W_T \cong (\mathbf{Z}/2)^{\mathrm{Card}(T)}$. So, $W_T(t) = (1+t)^{\mathrm{Card}(T)}$. Hence,

$$\frac{1}{W_T(t)} = \left(\frac{1}{1+t}\right)^{\mathrm{Card}(T)} \quad \text{and} \quad \frac{1}{W_T(t^{-1})} = \left(\frac{t}{1+t}\right)^{\mathrm{Card}(T)}. \tag{17–3}$$

Proposition 17.1 *Suppose (W,S) is a right-angled Coxeter system and that its nerve L is $(n-1)$–dimensional. Then*

$$\frac{1}{W(t)} = \frac{h_L(-t)}{(1+t)^n}.$$

Proof By Lemma 3.3 (iv) and (17–3),

$$\frac{1}{W(t)} = \sum_{T \in \mathcal{S}} \frac{\varepsilon(T)}{W_T(t^{-1})} = \sum_{T \in \mathcal{S}} \left(\frac{-t}{1+t}\right)^{\mathrm{Card}(T)} = f_L\left(\frac{-t}{1+t}\right) = \frac{h_L(-t)}{(1+t)^n}. \qquad \square$$

In the next proposition we record some properties of $h_L(t)$.

Proposition 17.2 *Suppose L is a GHS^{n-1}. Let $h_L(t) = \sum h_i t^i$ be its h–polynomial. Then the following holds:*

(i) *h_L is a polynomial of degree n. The constant term h_0 is 1.*

(ii) *$h_L(t) = t^n h_L(t^{-1})$. (This means that the coefficient sequence $(h_0, \ldots, h_n)$ is palindromic. It also implies that $t \to t^{-1}$ is a symmetry of the set of roots of h_L.)*

(iii) *Each $h_i \geq 0$.*

(iv) *If L is also assumed to be 3–dimensional and a flag complex, then all four roots of $h_L(t)$ are real.*

Statements (i),(ii) and (iii) are well-known; proofs can be found in Bronsted [6]. Statement (iv) is proved in Boros [3]. We give a simple argument for it below.

Proof of (iv) Put $\widehat{h}(t) = h_L(-t)$. By Proposition 17.1, $1/W_L(t) = \widehat{h}(t)/(1+t)^n$. The Flag Complex Conjecture is that for $n-1 = 2k-1$, $(-1)^k/W_L(1) \geq 0$, ie $(-1)^k\widehat{h}(1) \geq 0$ [21; 16]. Let ρ be the radius of convergence of $W_L(t)$. By Corollary 3.8, ρ is a root of $\widehat{h}$ and it is the smallest root in absolute value. By (ii), ρ^{-1} is also a root of $\widehat{h}$ and it is the largest in absolute value.

Now suppose $\dim L = 3$. To prove (iv), it suffices to show the four roots of $\widehat{h}$ are positive reals. The Flag Complex Conjecture is known to hold in this dimension [21], ie $\widehat{h}(1) \geq 0$. We know that ρ and ρ^{-1} are roots and also that $\widehat{h}(t) > 0$ for $t \in [0, \rho)$ or $t \in (\rho^{-1}, \infty)$. If the other two roots of $\widehat{h}$ don't lie in $[\rho, \rho^{-1}]$, then $\widehat{h}$ must be negative on that interval, contradicting the fact that $\widehat{h}(1) \geq 0$. □

For any full subcomplex A of L, set $r_A := \rho_A^{-1}$, where, as before, ρ_A is the radius of convergence of $W_A(t)$. Since W_A is a subgroup of W_L, $\rho_A \geq \rho_L$; hence, $r_A \leq r_L$.

Next, suppose that M is a GHS^{n-2} and a full subcomplex of L (so, M is a homology submanifold of codimension one in L). Then M separates L into two generalized homology $(n-1)$–disks, say, A and B. Thus, $\partial A = \partial B = M$ and $L = A \cup B$. Let CM denote the cone on M. Let $\widehat{A}$ (resp. $\widehat{B}$) denote the result of gluing CM onto A (resp. B) along M.

Lemma 17.3 *With hypotheses as above, suppose $q < \min\{r_L, r_{\widehat{A}}, r_{\widehat{B}}\}$ and $q > r_M$. Then*

$$b_q^{n-1}(L) \geq \frac{q-1}{q+1} b_q^{n-1}(M) > 0.$$

Proof Since $q > 1$, by Proposition 15.2(d), we have

$$b_q^k(CM) = b_q^0(\text{point}) b_q^k(M) = \frac{1}{1+q} b_q^k(M). \tag{17–4}$$

By Remark 7.6, since $q < r_L$, $\mathfrak{h}_n^q(L) \cong \mathfrak{h}_0^q(L) = 0$. By Corollary 14.3, since $q > r_M$, $\mathfrak{h}_k^q(M) = 0$ for $k \neq n-1$. Hence, the Mayer–Vietoris sequence (Theorem 15.1(c)) for $L = A \cup B$ gives a weakly exact sequence:

$$0 \longrightarrow \mathfrak{h}_{n-1}^q(M) \longrightarrow \mathfrak{h}_{n-1}^q(A) \oplus \mathfrak{h}_{n-1}^q(B) \longrightarrow \mathfrak{h}_{n-1}^q(L) \longrightarrow 0$$

So,

$$b_q^{n-1}(L) = b_q^{n-1}(A) + b_q^{n-1}(B) - b_q^{n-1}(M). \tag{17–5}$$

A similar Mayer–Vietoris sequence for $\widehat{A} = A \cup CM$ gives

$$b_q^{n-1}(\widehat{A}) = b_q^{n-1}(A) + b_q^{n-1}(CM) - b_q^{n-1}(M),$$

which we rewrite as

$$\begin{aligned} b_q^{n-1}(A) &= b_q^{n-1}(\widehat{A}) - b_q^{n-1}(CM) + b_q^{n-1}(M) \\ &= b_q^{n-1}(\widehat{A}) + \frac{q}{1+q} b_q^{n-1}(M), \end{aligned} \tag{17–6}$$

where the second equality is from (17–4). Similarly,

$$b_q^{n-1}(B) = b_q^{n-1}(\widehat{B}) + \frac{q}{1+q} b_q^{n-1}(M). \tag{17–7}$$

Combining (17–5), (17–6) and (17–7), we get

$$\begin{aligned} b_q^{n-1}(L) &= b_q^{n-1}(\widehat{A}) + \frac{q}{1+q} b_q^{n-1}(M) + b_q^{n-1}(\widehat{B}) + \frac{q}{1+q} b_q^{n-1}(M) - b_q^{n-1}(M) \\ &= b_q^{n-1}(\widehat{A}) + b_q^{n-1}(\widehat{B}) + \frac{q-1}{1+q} b_q^{n-1}(M) \\ &\geq \frac{q-1}{1+q} b_q^{n-1}(M) > 0, \end{aligned}$$

where the last inequality holds because $q > 1$ and $b_q^{n-1}(M) > 0$ (since $q > r_M$). □

Lemma 17.4 (Failure of concentration in dimension 4) *Suppose that L is a triangulation of S^3 as a flag complex, that a full subcomplex M is isomorphic to the boundary complex of an octahedron and that M divides L into two 3–disks A and B nontrivially, ie neither A nor B is a cone on M. Suppose further that $\chi_1(L) \neq 0$. Let p be the second largest root of $h_L(-t)$, and let $r = \min\{p, r_{\widehat{A}}, r_{\widehat{B}}\}$. Then $r > 1$ and for $1 < q < r$, $b_q^2(L)$ and $b_q^3(L)$ are both nonzero.*

Proof We want to use Lemma 17.3 for $n = 4$. Since W_M is the product of 3 copies of the infinite dihedral group, its growth series is given by

$$W_M(t) = \left(\frac{1+t}{1-t}\right)^3.$$

So, $\rho_M = 1 = r_M$.

Suppose $r_{\widehat{A}} = 1$. Then $r_A = 1$ and by Proposition 3.10, W_A splits as $W_0 \times W_1$, where W_0 is a Euclidean reflection group and W_1 is finite. Since $M = \partial A$, the only possibility is $W_0 = W_M$ and $W_1 = \mathbf{Z}/2$, ie $A = CM$, which we have ruled out by hypothesis. Similarly, for B. Thus, $\min\{r_{\widehat{A}}, r_{\widehat{B}}\} > 1$. Since $\chi_1(L) \neq 0$, $p \neq 1$ and by

[21], $\chi_1(L) > 0$. So, $\chi_q(L)$ is positive on the interval (p^{-1}, p) and therefore, also on the subinterval $(1, r)$.

By Lemma 17.3, for $1 < q < \min\{r_L, r_{\widehat{A}}, r_{\widehat{B}}\}$, $b_q^3(L) > 0$. On the interval (p^{-1}, p) we have $b_q^4(L) = 0 = b_q^0(L)$ as well as $\chi_q(L) > 0$ and this forces $b_q^2(L) > 0$. Therefore, for $1 < q < r$, $\mathfrak{h}_*^q(L)$ is nonzero in dimensions 2 and 3. □

Example 17.5 (Existence) Here we show that there is a flag triangulation L of S^3 together with a full subcomplex $M \subset L$ so that the conditions of Lemma 17.4 are satisfied. Let P_m denote an m–gon (ie a triangulation of S^1 with m vertices). Let I_l denote the triangulation of an interval with l vertices. Let $A_{k,m}$ denote a triangulation of the annulus $S^1 \times [0, 1]$ such that its two boundary components are P_k and P_m and such that there are no interior vertices. (This does not determine the triangulation, but it does determine the number of i–simplices in $A_{k,m}$ for $i = 0, 1, 2$.) Form the suspension $SA_{k,m} := S^0 * A_{k,m}$. It has two boundary components: SP_k and SP_m. Fill in SP_m with $I_4 * P_m$ to get a triangulation A of D^3, ie

$$A := SA_{k,m} \cup_{SP_m} (I_4 * P_m).$$

If $k = 4$, then $\partial A = SP_4$, which is the boundary complex M of an octahedron. Hence, we can double A along its boundary to get a triangulation L of S^3 (so, $B = A$).

By Theorem 15.1(f), $$\chi_q(L) = \frac{1}{W(q)} = f_L\left(\frac{-q}{1+q}\right),$$

where f_L was defined in (17–1). This formula is the basic method used for computing Euler characteristics. It gives

$$\chi_q(P_m) = 1 - \frac{mq}{(1+q)} + \frac{mq^2}{(1+q)^2} = \frac{1-(m-2)q+q^2}{(1+q)^2},$$
$$\chi_q(I_4) = 1 - \frac{4q}{(1+q)} + \frac{3q^2}{(1+q)^2} = \frac{1-2q}{(1+q)^2}.$$

We compute the number of simplices in $A_{k,m}$. Each triangle of $A_{k,m}$ has exactly one of its edges on the boundary and each interior edge is on the boundary of two triangles. Hence, there are $k+m$ triangles in $A_{k,m}$ and $k+m$ interior edges. So, $f_0(A_{k,m}) = k+m$, $f_1(A_{k,m}) = 2(k+m)$, $f_2(A_{k,m}) = k+m$ and

$$\chi_q(A_{k,m}) = 1 - \frac{(k+m)q}{(1+q)} + \frac{2(k+m)q^2}{(1+q)^2} - \frac{(k+m)q^3}{(1+q)^3}$$
$$= (1-(k+m-3)q+3q^2+q^3)/(1+q)^3.$$

Therefore, we have the equalities:

$$\begin{aligned}
\chi_q(SA_{k,m}) &= \chi_q(S^0)\chi_q(A_{k,m}) = \frac{1-q}{1+q}\chi_q(A_{k,m}) \\
&= (1-(k+m-2)q+(k+m)q^2-2q^3-q^4)/(1+q)^4 \\
\chi_q(I_4 * P_m) &= \chi_q(I_4)\chi_q(P_m) \\
&= (1-mq+(2m-3)q^2-2q^3)/(1+q)^4 \\
\chi_q(SP_m) &= \chi_q(S^0)\chi_q(P_m) \\
&= (1-(m-2)q+(m-2)q^3-q^4)/(1+q)^4
\end{aligned}$$

So,
$$\begin{aligned}
\chi_q(A) &= \chi_q(SA_{k,m}) + \chi_q(I_4 * P_m) - \chi_q(SP_m) \\
&= (1-(k+m)q+(k+3m-3)q^2-(m+2)q^3)/(1+q)^4.
\end{aligned}$$

Taking $k=4$, $\chi_q(A) = (1-(m+4)q+(3m+1)q^2-(m+2)q^3)/(1+q)^4$; hence,

$$\begin{aligned}
\chi_q(\widehat{A}) &= \chi_q(A) - \left(\frac{q}{1+q}\right)\chi_q(M) = \chi_q(A) - \left(\frac{q}{1+q}\right)\left(\frac{1-q}{1+q}\right)^3 \\
&= (1-(m+5)q+(3m+4)q^2-(m+5)q^3+q^4)/(1+q)^4.
\end{aligned}$$

When $m=10$, the numerator is

$$h_{\widehat{A}}(-q) = 1-15q+34q^2-15q^3+q^4,$$

which has roots $.08$, $.48$, 2.10 and 12.34 (rounded off to two decimal places). Similarly,

$$\chi_q(L) = 2\chi_q(A) - \chi_q(M) = (1-26q+62q^2-26q^3+q^4)/(1+q)^4,$$

which has roots $.04$, $.48$, 2.08 and 23.40. So, the numbers in Lemma 17.4 are $r_{\widehat{A}} = r_{\widehat{B}} = 12.34$ and $r = p = 2.08$. In particular, since $r > 2$, the right-angled building with $q = 2$ has nonvanishing L^2–homology in dimensions 2 and 3.

18 Remarks about other groups

Suppose Γ is a countable discrete group and $|\ |$ is a "norm" on it, ie $|\ |$ is a function from Γ to $[0,\infty)$ such that $|\alpha\beta| \le |\alpha| + |\beta|$. For example, $|\ |$ might be defined by $|\gamma| = l(\gamma)$ where $l\colon \Gamma \to \mathbf{Z}$ is word length with respect to a finite set of generators S. Suppose further that Γ acts properly and cellularly on a CW complex X and that a subcomplex $D \subset X$ is a "fundamental domain" in the sense that it contains at least one cell from each Γ–orbit of cells. Given a cell $\sigma \subset X$, define $d(\sigma)$, its *distance from* D, by $d(\sigma) := \min\{l(\gamma) \mid \sigma \subset \gamma D\}$.

As before, given a positive real number q, define an inner product $\langle\ ,\ \rangle_q$ on $\mathbf{R}^{(\Gamma)}$ by $\langle e_\gamma, e_{\gamma'}\rangle_q := q^{|\gamma|}\delta_{\gamma\gamma'}$. Let $L^2_q(\Gamma, |\ |)$ be its completion. Similarly, define an inner product on compactly supported cellular i–cochains, $C^i_c(X)$, by $\langle e_\sigma, e_{\sigma'}\rangle_q := q^{l(\sigma)}\delta_{\sigma\sigma'}$ and let $L^2_q C^i(X)$ be its completion. Using the usual coboundary operator δ, we get the weighted L^2–cohomology spaces, $L^2_q H^*(X)$. Let ∂^q_i denote the adjoint of $\delta\colon L^2_q C^{i-1}(X) \to L^2_q C^i(X)$. The ∂^q_i give us a chain complex and allow us to define the weighted L^2–homology, $L^2_q H_*(X)$.

The infinite sum
$$\Gamma(t) := \sum_{\gamma\in\Gamma} t^{|\gamma|}.$$

converges for t in a some neighborhood of 0 in $[0,\infty)$. $\Gamma(t)$ is the *growth function* of $(\Gamma, |\ |)$. It is a power series if $|\ |$ is integer-valued (eg if it is given by a word length). Let $\mathcal{R}$ be the region of convergence of $\Gamma(t)$. Suppose X is connected. The argument in the proof of Proposition 7.5 shows that any 0–cocycle is constant and that if $q \in \mathcal{R}$, the only constant which is square summable is 0. Hence, $L^2_q H^0(X) \cong \mathbf{R}$ if $q \in \mathcal{R}$ and is 0 if $q \notin \mathcal{R}$.

Γ acts on these vector spaces; however, it does not act via isometries.

The usual boundary map ∂ gives us a different chain complex structure (on the same underlying Hilbert spaces $L^2_q C^i(X)$).

As in Lemma 7.1, we have the isometry $\theta\colon L^2_q C_i(X) \to L^2_{1/q} C_i(X)$ defined by mapping e_σ to $q^{d(\sigma)} e_\sigma$ intertwines ∂^q with ∂. Hence, it induces an isomorphism $\theta_*\colon H_*(L^2_q C_*(X), \partial) \to L^2_{1/q} H_*(X)$.

As in Remark 7.2, we have natural inclusions of cochain complexes:
$$C^i_c(X) \hookrightarrow L^2_q C^i(X) \hookrightarrow C^i(X).$$

There is also a version for chain complexes (using ordinary boundary map, ∂):
$$C_i(X) \hookrightarrow L^2_q C_i(X) \hookrightarrow C^{lf}_i(X),$$

where $C^{lf}_i(X)$ denotes the infinite cellular chains on X. Using the isometry θ, we get a monomorphism of chain complexes

$$C_i(X) \hookrightarrow L^2_{1/q} C_i(X) \xrightarrow{\theta} L^2_q C_i(X), \qquad (18\text{–}1)$$

where the boundary maps in the first two terms are the usual ones and where the boundary map in the third term is ∂^q. We then have the following version of Theorem 12.1.

Conjecture 18.1

(i) *For $q \in \mathcal{R}$, the canonical map $L^2_q H^i(X) \to H^i(X;\mathbf{R})$ is a monomorphism. Moreover, the map $H_i(X;\mathbf{R}) \to L^2_q H_i(X)$, induced by* (18–1)*, is a monomorphism with dense image.*

(ii) *For $q^{-1} \in \mathcal{R}$, the canonical map $H^i_c(X;\mathbf{R}) \to L^2_q H^i(X)$ is a monomorphism with dense image.*

Quite possibly it will be necessary to add more hypotheses for this conjecture to be true. For example, we might need to assume that the Γ–action is cocompact and that the norm is given by a word length with respect to a set of generators induced by the choice of fundamental domain D.

The missing feature from this picture is that for a general group Γ there is no analog of the Hecke algebra and no analog of the Hecke–von Neumann algebra $\mathcal{N}_{\mathbf{q}}$. So, in the general situation we don't know how to define "dimension" and we don't have weighted L^2–Betti numbers. Nevertheless, in some situations it is still possible to assign a "dimension" to these weighted L^2–cohomology spaces and obtain weighted L^2–Betti numbers. The condition that is needed for these numbers to be well-defined is that the Γ–action on X has a strict fundamental domain.

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