Weighted L^2-cohomology of Coxeter groups

MICHAEL DAVIS

(joint work with Jan Dymara, Tadeusz Januszkiewicz, Boris Okun)

Suppose (W, S) is a Coxeter system and q is a positive real number. $\mathbf{R}^{(W)}$ denotes the Euclidean space of finitely supported real-valued functions on W and $\{e_w\}_{w\in W}$ its standard basis. Define an inner product $\langle\ ,\ \rangle_q$ by

$$\langle e_v, e_w\rangle_q = \begin{cases} q^{l(w)} & \text{if } v = w, \\ 0 & \text{if } v \neq w. \end{cases}$$

$L^2_q(W)$ denotes the Hilbert space completion of $\mathbf{R}^{(W)}$ with respect to this inner product. Deform the multiplication in the group algebra $\mathbf{R}W$ by

$$e_s e_w = \begin{cases} e_{sw} & \text{if } l(sw) > l(w), \\ qe_{sw} + (q-1)e_w & \text{if } l(sw) < l(w). \end{cases}$$

This defines a new multiplication on $\mathbf{R}^{(W)}$ making it into an associative algebra $\mathbf{R}_q W$ called the *Hecke algebra* (see [1, Ex. 22, pp. 56–57]). Define a linear involution $x \to x^*$ of $\mathbf{R}_q W$ by $(e_w)^* := e_{w^{-1}}$. $\mathbf{R}_q W$ acts on L^2_q $(= L^2_q(W))$ by either left or right multiplication. It is easily checked that $*$ is an anti-involution of the algebra $\mathbf{R}_q W$ and that $\langle xy, z\rangle_q = \langle y, x^* z\rangle_q = \langle x, zy^*\rangle_q$. So, $\mathbf{R}_q W$ defines two C^*-algebras of operators (by left or right multiplication); the weak closure of either one is denoted $\mathcal{N}_q$. It is a von Neumann algebra, called the *Hecke - von Neumann algebra*. The *trace* of an element $\phi \in \mathcal{N}_q$ is defined by the usual formula: $\operatorname{tr}\phi := \langle \phi(e_1), e_1\rangle_q$. This extends to definition of a trace for any $\mathcal{N}_q$-endomorphism of a finite direct sum of copies of L^2_q. Hence, if $V \subset \oplus L^2_q$ is any closed $\mathcal{N}_q$-submodule we can define its *von Neumann dimension* by $\dim V := \operatorname{tr} p_V$, where $p_V : \oplus L^2_q \to \oplus L^2_q$ is orthogonal projection onto V.

The von Neumann dimensions of some important $\mathcal{N}_q$-modules are tied to the growth series of W, i.e., to the power series defined by $W(t) := \sum_{w\in W} t^{l(w)}$. Its radius of convergence is denoted by ρ. $W(t)$ is a rational function of t. One way to see this is simply to prove a formula such as

$$\frac{1}{W(t)} = \sum_{T\in\mathcal{S}} \frac{(-1)^{|T|}}{W_T(t^{-1})}, \tag{1}$$

where for any $T \subset S$, $W_T := \langle T\rangle$ is the special subgroup generated by T and $\mathcal{S} := \{T \subset S \mid |W_T| < \infty\}$ is the *poset of spherical subsets*. Formula (1) shows that $W(t)$ is a rational function, since the growth series of any finite group, such as a spherical subgroup W_T, is a polynomial. Hecke algebras (resp. growth series) make sense when q (resp. t) is replaced by a certain I-tuple $\mathbf{q} = (q_i)$ (resp. $\mathbf{t} = (t_i)$), where we are given a function $i : S \to I$ which is constant on conjugacy classes of elements of S. Given $s \in S$, write q_s (resp. t_s) instead of $q_{i(s)}$ (resp. $t_{i(s)}$). Then $q^{l(w)}$ (resp. $t^{l(w)}$) is replaced by $q_w := q_{s_1}\cdots q_{s_n}$ (resp. t_w) where $w = s_1 \cdots s_n$ is any reduced expression for w.

Two important self-adjoint idempotents in $\mathcal{N}_q$ are

$$a_T := \frac{1}{W_T(q)} \sum_{w \in W_T} e_w \quad \text{and} \quad h_T := \frac{1}{W_T(q^{-1})} \sum_{w \in W_T} (-1)^{l(w)} q^{-l(w)} e_w.$$

Right multiplication by a_T (resp. h_T) is a well-defined, bounded linear operator precisely when $q < \rho_T$ (resp. $q > \rho_T^{-1}$), where ρ_T is the radius of convergence of $W_T(t)$. In particular, both are defined when T is spherical. These idempotents define subspaces, $A_T := L^2_q a_T$ and $H_T := L^2_q h_T$ of dimension $1/W_T(q)$ and $1/W_T(q^{-1})$, respectively.

Let Σ be the standard piecewise Euclidean CAT(0) complex on which W acts properly and isometrically (e.g., see [2]). It is defined by pasting together copies of a certain fundamental domain K, one for each element of W. $C^*(\Sigma)$ is its cellular cochain complex, i.e., $C^i(\Sigma) = \{f : \{i\text{-cells}\} \to \mathbf{R}\}$ and $C^i_c(\Sigma) := \{\text{finitely supported f}\}$ is the subcomplex of finitely supported cochains. We can define a weighted inner product on $C^i_c(\Sigma)$ similar to the one discussed above. Given a cell σ of Σ, let $w((\sigma))$ be the element w of shortest length such that $w^{-1}\sigma \subset K$. Assign a weight to the characteristic function e_σ of the cell so that its length is $q^{l(w(\sigma))}$. Its Hilbert space completion is denoted $L^2_q C^i(\Sigma)$. As q varies between 0 and ∞ it interpolates between $C^i(\Sigma)$ and $C^i_c(\Sigma)$. $L^2_q C^i(\Sigma$ inherits the structure of an $\mathcal{N}_q$-module in a straightforward fashion so that the coboundaries are maps of $\mathcal{N}_q$-modules. Its (reduced) cohomology groups are denoted $L^2_q H^*(\Sigma)$. We compute them as $\mathcal{N}_q$-modules for $q \leq \rho$ and for $q > \rho^{-1}$. The von Neumann dimension of $L^2_q H^i(\Sigma)$ is denoted $b^i_q(\Sigma)$ and called the ith *L^2_q-Betti number*. When q is an integer (or when $\mathbf{q}$ is an I-tuple of integers), these numbers are equal to the ordinary L^2-Betti numbers of of any building of type (W, S) and thickness q (or $\mathbf{q}$) with a chamber transitive automorphism group. (Thickness q means that $q+1$ chambers meet along each mirror.) The *L^2_q- Euler characteristic* is the alternating sum of the $b^i_q(\Sigma)$.

Theorem 1. (Dymara [6]). $\chi_q(\Sigma) = 1/W(q)$.

This is proved in the usual fashion by calculating the alternating sum of the dimensions of the spaces of cochains on orbits of cells. There is a cellulation of Σ with one orbit of cells for each $T \in \Sigma$; the dimension of the space of cochains on this orbit is $\dim H_T = 1/W_T(q^{-1})$; so, the theorem follows from (1). The next result of Dymara says that for $q < \rho$, L^2_q-cohomology behaves like ordinary cohomology.

Theorem 2. (Dymara [6]). *For $q < \rho$, $L^2_q H^*(\Sigma)$ is concentrated in dimension* 0 *and is isomorphic to* $\mathbf{R}$ *as a vector space. Its von Neumann dimension is given by* $b^0_q(\Sigma) = 1/W(q)$. *Moreover, for $q > \rho$, $b^0_q(\Sigma) = 0$.*

For each $T \in \mathcal{S}$, let $A_{>T}$ be the $\mathcal{N}_q$-submodule of A_T defined by $A_{>T} = \sum_{U \in \mathcal{S}_{>T}} A_U$. Put $D_T := A_T/A_{>T}$. Using Theorem 2 we prove the following generalization of a result of L. Solomon [8] when W is finite.

The Decomposition Theorem. ([3]). *For $q > \rho^{-1}$ and any $T \in \mathcal{S}$,*

$$A_T = \overline{\bigoplus_{U \in \mathcal{S}_{>T}} D_U}.$$

This leads directly to the following.

The Main Theorem. ([3]). *For $q > \rho^{-1}$, we have an isomorphism of $\mathcal{N}_q$-modules*

$$L^2_q H^*(\Sigma) \cong \bigoplus_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes D_T.$$

Here the fundamental domain K is contractible complex, it is equipped with a family $\{K_s\}_{s \in S}$ of closed contractible subcomplexes ("mirrors") indexed by S. K^{S-T} denotes the union of those mirrors which are indexed by $S-T$. $H^*(K, K^{S-T})$ is ordinary cohomology.

In [2, 4] we carry out a similar calculation for $H^*_c(\Sigma)$. A consequence is that the inclusion $C^*_c(\Sigma) \hookrightarrow L^2_q C^*(\Sigma)$ induces an injection with dense image, $H^*_c(\Sigma) \to L^2_q H^*(\Sigma)$. So, in this sense weighted L^2-cohomology is like compactly supported cohomology in the range $q > \rho^{-1}$. The big question is what happens in the intermediate range $\rho < q < \rho^{-1}$?

References

[1] N. Bourbaki, *Lie Groups and Lie Algebras, Chapters 4–6*, Springer-Verlag, New York, 2002.

[2] M.W. Davis *The cohomology of a Coxeter group with group ring coefficents*, Duke Math. J. **91** (1998), 297–314.

[3] M.W. Davis, J. Dymara, T. Januszkiewicz and B. Okun, *Weighted L^2-cohomology of Coxeter groups*, preprint 2004.

[4] ——— *The cohomology of a Coxeter group with group ring coefficents: II*, preprint 2005.

[5] J. Dymara, *L^2-cohomology of buildings with fundamental class*, Proc. Amer. Math. Soc. **132** (2004), 1839–1843.

[6] ——— *Thin buildings*, to appear in Geometry & Topology.

[7] J. Dymara and T. Januszkiewicz, *Cohomology of buildings and their automorphism groups*, Invent. Math. **150** (2002), 579–627.

[8] L. Solomon, *A decomposition of the group algebra of a finite Coxeter group*, J. of Algebra **9** (1968), 220–239.