SOME GROUP ACTIONS ON HOMOTOPY SPHERES OF DIMENSION SEVEN AND FIFTEEN

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0. Introduction. This paper is based on the simple observation that every 4-plane bundle over S^4 admits a natural action of $SO(3)$ by bundle maps and that every 8-plane bundle over S^8 admits a natural action of the compact Lie group G_2 by bundle maps. (These actions are easy to see once one remembers that $SO(3)$ is the group of automorphisms of the quaternions and that G_2 is the group of automorphisms of the Cayley numbers.) The study of these actions is closely connected to several well-known phenomena in differential topology and compact transformation groups.

The most obvious connection is to Milnor's original construction of exotic 7-spheres as 3-sphere bundles over S^4, [26]. Milnor proved that if the Euler class of such a bundle is a generator of $H^4(S^4; \mathbf{Z})$, then its total space is homeomorphic to S^7. He also defined a numerical invariant of the diffeomorphism type and used it to detect an exotic differential structure on some of these sphere bundles. Subsequently, Eells and Kuiper [12] introduced a refinement of this invariant, called the μ-invariant. Using the μ-invariant, they proved that the sphere bundle M_3^7 (Milnor's notation) is a generator for the group of homotopy 7-spheres. These constructions also work for 7-sphere bundles over S^8, [32], and the manifold M_3^{15} is a generator for bP_{16}, the group of homotopy 15-spheres which bound π-manifolds. It is a routine matter to check that Milnor's arguments and their subsequent refinements work G-equivariantly where $G = SO(3)$ or G_2, and we shall do this in Sections 2 and 3. In particular, each sphere bundle with the correct Euler class is G-homeomorphic to an orthogonal action on S^{2n+1}, where $2n + 1 = 7$ or 15. Moreover, distinct sphere bundles have distinct oriented G-diffeomorphism types. The proof of the second fact uses an equivariant version of the μ-invariant. As originally defined this invariant takes values in $\mathbf{Q}/\mathbf{Z}$. However, it is well-known that in the presence of a G-action, with $S^1 \subset G$, its value in $\mathbf{Q}$ is well-defined. Using

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this fact we shall show in Theorem 3.3, that the G-manifold (M_3^{2n+1}, G) generates the infinite cyclic group (under equivariant connected sum) of oriented G-diffeomorphism types of smooth G-actions on homotopy $(2n + 1)$-spheres all of which are G-homeomorphic to a standard linear action on S^{2n+1}.

A second connection is to the group $\Sigma^{6,3}$ of knotted 3-spheres in S^6 and to the work of Montgomery-Yang [28], [29] on semi-free circle actions on homotopy 7-spheres with fixed point set the standard 3-sphere. Let $\tilde{C}_7(SO(2))$ be the abelian group of oriented equivariant diffeomorphism classes of such actions. The results of Montgomery-Yang together with those of Levine [24] show that this group is isomorphic to $\Sigma^{6,3}$. The isomorphism is defined by sending $(\Sigma^7, SO(2)) \in \tilde{C}_7(SO(2))$ to its "orbit knot" $(\Sigma/SO(2), \Sigma^{SO(2)})$. According to Haefliger [16], [17], the group $\Sigma^{6,3}$ is infinite cyclic. Restricting the $SO(3)$-action on M_3^7 to the subgroup $SO(2) \subset SO(3)$ gives an action of $SO(2)$ on M_3^7. This action is readily seen to be semi-free and to have fixed point set S^3. We shall show in Section 7 that the orbit knot of this action is actually a generator for $\Sigma^{6,3}$ and hence, that $(M_3^7, SO(2))$ is a generator for $\tilde{C}_7(SO(2))$. Thus, each of the Montgomery-Yang examples actually come from the restriction of a $SO(3)$-action.

A third connection is to the construction of Gromoll-Meyer [15] of a metric of non-negative sectional curvature on a exotic 7-sphere. They produced an example showing that M_3^7 admits such a metric. In their example the group $SO(3) \times O(2)$ acts as isometries. The $SO(3)$-action on M_3^7 coincides with the one which we have been discussing above. (In fact, this paper grew out of an effort to explain the $SO(3)$-action of Gromoll-Meyer.) The $O(2)$-action also appears in a more general context. We shall show in Section 1 that every 4-plane bundle over S^4 (or every 8-plane bundle over S^8) admits a canonical action of $GL(2, \mathbf{R})$ by bundle maps which commute with the action of $SO(3)$ (or of G_2). The action of the subgroup $O(2) \subset GL(2, \mathbf{R})$ on M_3^7 also coincides with the Gromoll-Meyer $O(2)$-action.

Actually the various connections these actions make with the abovementioned constructions are only peripheral concerns of this paper. Our primary concern is to investigate the role these actions on sphere bundles play in the general theory of biaxial actions on homotopy spheres developed in [2], [3], [5], [6], [7], [11], [13], [14], [19], [20]. The word "biaxial" refers to a smooth action of a group $G \subset O(n)$ on a $(2n + m)$-manifold such that the action is modeled on (i.e., locally isomorphic to) the linear

G-action on the unit sphere in $\mathbf{R}^n \oplus \mathbf{R}^n \oplus \mathbf{R}^{m+1}$ via two times the standard representation of $O(n)$ plus the $(m + 1)$-dimensional trivial representation. If G is transitive on the Stiefel manifold $V_{n,2}$, then the orbit space B of such an action on a homotypy sphere Σ^{2n+m} is a contractible $(m + 3)$-manifold with boundary and the fixed point set B_0 is a $\mathbf{Z}/2\mathbf{Z}$-homology m-sphere embedded in ∂B. Thus, the study of such actions is closely related to the study of knots in codimension 2. If $G = U(n) \subset O(2n)$, $n \geq 2$, or $G = SU(n)$, $n \geq 3$, then the orbit space of Σ^{4n+m} is a contractible $(m + 4)$-manifold and the fixed point set is an integral homology m-sphere. Thus, such actions are related to knots in codimension 3. For G equal to $O(n)$, with $n \geq 2$, $SO(n)$, $n \geq 4$, or $\mathrm{Spin}(7) \subset O(8)$, any biaxial G-action on a homotopy sphere Σ^{2n+m} has the following nice property:

1) (Σ, G) is equivalent to a pullback of its linear model.

This implies that

2) Σ equivariantly bounds a parallelizable manifold, and that
3) Σ is determined by its "orbit triple" $(B, \partial B, B_0)$.

These facts are proved in [6], [19], [20]. Since $SO(3)$ is transitive on $V_{3,2}$ and G_2 is transitive on $V_{7,2}$, one might expect that similar results hold in these cases; however, the proofs break down. For $G = SO(3)$ or G_2, our G-actions on sphere bundles are easily seen to be biaxial. However, for these actions all three of the above properties fail to hold. Moreover, if the sphere bundle is not the actual Hopf-bundle these actions do not extend to biaxial $O(n)$-actions, $n = 3$ or 7. There are similar nice results for biaxial actions of $U(n)$, $n \geq 2$, and $SU(n)$, $n \geq 4$; in particular, for such actions properties 1) and 2) hold. However, the corresponding results are false for $SU(3)$. In fact, in Section 7 we shall show that the action of $SU(3) \subset G_2$ on the sphere bundle M_3^{15} is a counterexample to both these properties.

For $G = SO(3)$ or G_2, 1) can be replaced by the following property:

1)′ (Σ, G) is equivalent to a pullback of the natural G-action on the quaternionic projective plane if $G = SO(3)$, or on the Cayley projective plane if $G = G_2$.

This is proved in [9], where it is also shown that biaxial $SU(3)$-actions are pullbacks of the Cayley projective plane. Under certain conditions (see Proposition 4.9), 1)′ insures that (Σ, G) equivariantly bounds a spin manifold. In particular, in dimensions 7 and 15 the equivariant μ-invariant is always defined. We use this fact in Theorem 5.3 to completely classify biaxial $SO(3)$-actions on homotopy 7-spheres: every such action can be written uniquely as the equivariant connected sum of some number of copies of $(M_3^7, SO(3))$ together with the restriction of some biaxial $O(3)$-action. A similar result holds for G_2-actions on 15-spheres.

An interesting feature of biaxial $O(n)$-actions on homotopy spheres is that non-linear examples occur in nature, [1], [17]. After Milnor's seminal work [26], several other constructions of exotic spheres were discovered. The most natural of these is to represent them as the boundary of some plumbing manifold or equivalently as a Brieskorn manifold, [17], [25]. Kervaire-Milnor proved that every homotopy sphere which bounds a parallelizable manifold could be constructed in this fashion. Hirzebruch observed in [17] that these plumbing manifolds and certain Brieskorn manifolds support canonical biaxial $O(n)$-actions.[1] Since the $SO(3)$- and G_2-actions on sphere bundles generally do not arise as restrictions of $O(n)$-actions, they give a new class of examples. Thus, the essentially different natures of the two known natural constructions of exotic spheres (either as exotic Hopf-bundles or as boundaries of plumbing manifolds) is reflected in the study of their transformation groups. The fact that exotic Hopf-bundles occur only in fiber dimensions 3 and 7 is mirrored by the fact that $SO(3)$ and G_2 play a distinguished role in the theory of biaxial actions.

This distinguished role was already evident in the first work on this subject, by Bredon in [2] and [3]. He constructed certain biaxial $O(n)$-actions without fixed points on $(2n - 1)$-manifolds. Later these actions were shown to coincide with the natural $O(n)$-actions on the Brieskorn manifolds $\Sigma^{2n-1}(k, 2, \ldots, 2)$, with k odd. (See Example 4.2.) For n odd these manifolds are homotopy spheres. If we restrict the action to $G \subset O(n)$ for $G = O(n)$, $n \geq 2$, $SO(n)$, $n \geq 4$, or Spin(7), then for distinct k these actions can be seen to be of distinct G-diffeomorphism types. However, Bredon showed that for k odd, the $SO(3)$-action on $\Sigma^5(k, 2, 2, 2)$ is equivalent to the linear action on S^5 and similarly for the G_2-action on $\Sigma^{13}(k, 2, \ldots, 2)$. Thus, in the presence of fixed points the unusual feature of biaxial actions of $G = SO(3)$ or G_2 is that there are G-actions which do not extend to $O(n)$, while in the absence of fixed points the unusual feature is that different $O(n)$-actions become equivalent when restricted to G.

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[1] These two constructions give the same class of examples.

I would like to thank Glen Bredon for bringing the Gromoll-Meyer example to my attention and for pointing out its significance to the study of biaxial actions.

1. Natural Actions on Some Vector Bundles Over Projective Lines. Let Λ stand for either the real, complex, quaternion, or Cayley numbers (denoted by $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, or $\mathbf{O}$, respectively) and let $n = \dim_{\mathbf{R}} \Lambda - 1$. Let $x \to \bar{x}$ denote the canonical anti-involution on Λ. A real inner product on Λ is defined by $x \cdot y = \frac{1}{2}(x\bar{y} + y\bar{x})$. The Λ-projective line ΛP^1 is formed by identifying two copies of Λ via the diffeomorphism of $\Lambda - \{0\}$

$$u \to u' = u/|u|^2.$$

Clearly, $\Lambda P^1 \cong S^{n+1}$.

For each pair of integers (h, j), let $E_{h,j}(\Lambda)$ be the manifold formed by identifying two copies of $\Lambda \times \Lambda$ via the diffeomorphism of $(\Lambda - \{0\}) \times \Lambda$

$$(u, v) \to (u', v') = \left(\frac{u}{|u|^2}, \frac{u^h v u^j}{|u|^{h+j}} \right). \tag{1.1}$$

The natural projection $E_{h,j}(\Lambda) \to \Lambda P^1$ sending (u, v) to u and (u', v') to u' gives $E_{h,j}(\Lambda)$ the structure of a $(n + 1)$-dimensional vector bundle over ΛP^1.

Remarks. (1.2) The non-associativity of the Cayley numbers causes no problems in (1.1). Indeed, since any two Cayley, numbers u and v generate a proper subalgebra of $\mathbf{O}$ (hence, an associative subalgebra), the expression $u^h v u^j$ is unambiguous.

(1.3) The bundle $E_{h,j}(\Lambda)$ carries a natural inner product induced from the standard inner product on the second factor of $\Lambda \times \Lambda$. Let $D_{h,j}(\Lambda)$ be the unit disk-bundle and let $S_{h,j}(\Lambda) = \partial D_{h,j}(\Lambda)$ be the unit sphere bundle.

(1.4) Isomorphism classes of $(n + 1)$-plane bundles over $\Lambda P^1 = S^{n+1}$ are in one-to-one correspondence with $\pi_n(O(n + 1))$, where

$$\pi_n(O(n+1)) \cong \begin{cases} \mathbf{Z}/2\mathbf{Z}; & n = 0 \\ \mathbf{Z}; & n = 1 \\ \mathbf{Z} \oplus \mathbf{Z}; & n = 3 \quad \text{or} \quad n = 7. \end{cases}$$

Let $\varphi_{hj}: S^n \to O(n+1)$ be the map defined by $\varphi_{hj}(u)(v) = u^h v u^j$, with $u \in S^n \subset \Lambda$ and $v \in \Lambda \cong \mathbf{R}^{n+1}$, and let $\Phi: \mathbf{Z} \oplus \mathbf{Z} \to \pi_n(O(n+1))$ be the homomorphism which sends (h, j) to the homotopy class of φ_{hj}. The map Φ is surjective and it is an isomorphism if $n = 3$ or 7. It follows that *every $(n+1)$-plane bundle over ΛP^1 is isomorphic to one of the form $E_{hj}(\Lambda)$.*

Suppose $\Lambda \neq \mathbf{R}$. Let $\alpha \in H^{n+1}(\Lambda P^1; \mathbf{Z}) \cong \mathbf{Z}$ be the canonical generator. The Euler class of $E_{hj}(\Lambda)$ is given by

$$\chi(E_{hj}(\Lambda)) = (h+j)\alpha. \tag{1.5}$$

The nontrivial Pontriagin classes are given by

$$\begin{aligned} p_1(E_{hj}(\mathbf{H})) &= \pm 2(h-j)\alpha \\ p_2(E_{hj}(\mathbf{O})) &= \pm 6(h-j)\alpha \end{aligned} \tag{1.6}$$

These formulas are an easy calculation given the observation that χ and p_i are both linear in (h, j). (See [26] and [31].)

Let G^Λ be the group of $\mathbf{R}$-algebra automorphisms of Λ. Then

$$G^\Lambda \cong \begin{cases} \{1\}; & \Lambda = \mathbf{R} \\ \mathbf{Z}/2\mathbf{Z}; & \Lambda = \mathbf{C} \\ SO(3); & \Lambda = \mathbf{H} \\ G_2; & \Lambda = \mathbf{O}. \end{cases} \tag{1.7}$$

There is a natural action of G^Λ on $E_{hj}(\Lambda)$: for $g \in G^\Lambda$, define

$$g \cdot (u, v) = (g(u), g(v))$$

$$g \cdot (u', v') = (g(u'), g(v')).$$

These formulas are obviously compatible with the identification (1.1). Hence, we have the following result.

THEOREM 1.8. *Let Λ be a real division algebra of dimension $n+1$ (of course, $n = 0, 1, 3,$ or 7). Every $(n+1)$-plane bundle over S^{n+1}*

admits a natural action of G^Λ by bundle maps covering the canonical action on the base (thought of as ΛP^1).

Remarks. (1.9) By construction, $E_{h,j}(\Lambda)$ is covered by two G^Λ-invariant open sets of the form $(\Lambda \times \Lambda, G^\Lambda)$. This implies that the G^Λ-action is biaxial in the sense of Section 4.

(1.10) The G^Λ-action leaves the natural inner product on $E_{h,j}(\Lambda)$ invariant. Hence, the associated unit disk bundle $D_{h,j}(\Lambda)$ and the sphere bundle $S_{h,j}(\Lambda)$ are both smooth G^Λ-manifolds.

The vector bundle $E_{h,j}(\Lambda) \to \Lambda P^1$ admits further symmetries. In fact, it admits a commuting action of $GL(2, \mathbf{R})$ by bundle maps covering the natural action on ΛP^1 by Möebius transformations. This is obvious for $\Lambda = \mathbf{R}$ or $\mathbf{C}$: for in these cases $E_{h,j}(\Lambda)$ is just the $(h + j)$-fold tensor product of the Hopf-bundle and the Hopf-bundle admits such an action. In the general case we must define this action explicitly.

Suppose $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbf{R})$. The action of γ on ΛP^1 is defined by

$$\gamma \cdot u = (au + b)(cu + d)^{-1}$$

$$\gamma \cdot u' = (c + du')(a + bu')^{-1}.$$

Its action on $E_{h,j}(\Lambda)$ is given by

$$\gamma \cdot (u, v) = (\gamma \cdot u, (cu + d)^h v (cu + d)^j / |cu + d|^{h+j})$$

$$\gamma \cdot (u', v') = (\gamma \cdot u', (a + b\bar{u}')^h v' (a + b\bar{u}')^j / |a + b\bar{u}'|^{h+j}).$$

It is trivial to check that these formulas are compatible with the identification (1.1). Also, it is clear that translation by γ is G^Λ-equivariant and that translation by γ leaves the inner product on $E_{h,j}(\Lambda)$ invariant. It remains to check that these formulas actually define an action of $GL(2, \mathbf{R})$. We must show that if $\gamma_3 = \gamma_1\gamma_2$, then $\gamma_3 \cdot (u, v) = \gamma_1 \cdot (\gamma_2 \cdot (u, v))$. For $i = 1, 2, 3$, let

$$\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}.$$

Then

$$(1.11) \qquad \gamma_1 \cdot (\gamma_2 \cdot (u, v)) = (\gamma_1\gamma_2 \cdot u, \alpha^h(\beta^h v \beta^j)\alpha^j / |\alpha|^{h+j} |\beta|^{h+j}),$$

where

$$\beta = c_2 u + d_2 \quad \text{and}$$

$$\alpha = c_1(\gamma_2 \cdot u) + d_1 = c_1(a_2 u + b_2)(c_2 u + d_2)^{-1} + d_1.$$

(Since α, β and v belong to the subalgebra generated by u and v, non-associativity causes no problem in equation (1.11).) We have that

$$\begin{aligned}\alpha\beta &= [c_1(a_2 u + b_2)(c_2 u + d_2)^{-1} + d_1](c_2 u + d_2)\\ &= (c_1 a_2 + d_1 c_1)u + (c_1 b_2 + d_1 d_2)\\ &= c_3 u + d_3.\end{aligned}$$

Since α and β belong to the subalgebra generated by u, they commute. Hence,

$$\alpha^h(\beta^h v \beta^j)\alpha^j = (\alpha\beta)^h v (\alpha\beta)^j.$$

Substituting this into the formula into (1.11) gives

$$\begin{aligned}\gamma_1 \cdot \gamma_2(u, v) &= (\gamma_3 u, (c_3 u + d_3)^h v (c_2 u + d_3)^j / |c_3 u + d_3|^{h+j})\\ &= \gamma_3 \cdot (u, v).\end{aligned}$$

Therefore, we have shown that every $(n + 1)$-plane bundle over ΛP^1 admits a natural action of $G^\Lambda \times GL(2, \mathbf{R})$ by bundle maps which leave the standard inner product invariant. To get an action of a compact group we should restrict our attention to the action of the subgroup $G^\Lambda \times O(2)$.

Remarks. (1.12) In [15], Gromoll and Meyer give a different construction of an action of $SO(3) \times O(2)$ on the sphere bundle $S_{2,-1}(\mathbf{H}) \to S^4$. Their action is the same as the one constructed above.

(1.13) There is a bundle isomorphism $E_{hj}(\Lambda) \to E_{h'j'}(\Lambda)$ covering a diffeomorphism on the base provided $(h, j) = \pm(h', j')$ or $(h, j) = \pm(j', h')$. Moreover, these isomorphisms are $G^\Lambda \times GL(2, \mathbf{R})$ equivariant. Such an isomorphism $E_{hj}(\Lambda) \to E_{-h-j}(\Lambda)$ can be defined by

$$(u, v) \to (\bar{u}, v), \qquad (u', v') \to (\bar{u}', v').$$

Similarly, there is an isomorphism $E_{hj}(\Lambda) \to E_{jh}(\Lambda)$ defined by

$$(u, v) \to (\bar{u}, \bar{v}), \qquad (u', v') \to (\bar{u}', \bar{v}').$$

Notice that the first map reverses the orientation of the base and that the second reverses the orientations of both base and fiber. Note in particular that these isomorphisms show that the G^{Λ}-manifolds $S_{hj}(\Lambda)$ and $S_{h'j'}(\Lambda)$ are equivariantly diffeomorphic whenever $(h, j) = \pm(h', j')$ or $\pm(j', h')$.

2. The μ-Invariant. Suppose that M^{4q-1} is a smooth manifold such that $H_{2q-1}(M, \mathbf{Q}) = H_{2q}(M, \mathbf{Q}) = 0$, $H^1(M, \mathbf{Z}/2\mathbf{Z}) = 0$,[2] and such that M^{4q-1} bounds a spin manifold W^{4q}. In this situation Eells-Kuiper [12] defined the μ-invariant of M, a certain rational number whose value modulo $\mathbf{Z}$ depends only on M. If W^{4q} supports an effective G-action with $S^1 \subset G$, then this rational number is actually a well-defined invariant of the oriented G-manifold (M^{4q-1}, G). (See page 351 in [1].) This is due to the fact that the $\hat{A}$-genus vanishes for a closed spin manifold with effective S^1-action.

If $\Lambda = \mathbf{H}$ or $\mathbf{O}$ and if $h \neq -j$, then the disk bundle $D_{hj}(\Lambda)$ is a spin manifold of index ± 1 and its boundary $S_{hj}(\Lambda)$ satisfies the necessary homological conditions. The formulas in [12], pages 101 and 106, yield

$$\mu(S_{hj}(\mathbf{H}) = [(p_1)^2 - 4\tau]/2^7 \cdot 7$$

$$= [2^2(h-j)^2 \pm 4]/2^7 \cdot 7$$

$$\mu(S_{hj}(\mathbf{O}) = [5040(p_2)^2 - 181440\tau]/2^{15} \cdot 3^4 \cdot 5 \cdot 7 \cdot 127$$

$$= [5040 \cdot 6^2 \cdot (h-j)^2 \pm 181440]/2^{15} \cdot 3^4 \cdot 5 \cdot 7 \cdot 127,$$

where p_i is the i^{th} Pontriagin class of $D_{hj}(\Lambda)$ and $\tau = \pm 1$ is its index. Note that the sign of τ is the same as that of $h + j$.

THEOREM 2.1. *Suppose $\Lambda = \mathbf{H}$ or $\mathbf{O}$. Then the G^{Λ}-manifolds $S_{hj}(\Lambda)$ and $S_{h'j'}(\Lambda)$ are equivariantly diffeomorphic if and only if $(h, j) = \pm(h', j')$ or $(h, j) = \pm(j', h')$. Such an equivariant diffeomorphism can preserve orientation if and only if $(h, j) = (h', j')$ or (j', h').*

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[2] These conditions can be weakened.

Proof. Suppose there is an orientation-preserving equivariant diffeomorphism $S_{hj}(\Lambda) \to S_{h'j'}(\Lambda)$. (We can always reduce to the oriented case by composing with an equivariant diffeomorphism from 1.13.) Since the Euler class of $S_{hj}(\Lambda)$ is $(h + j)\alpha$, its middle dimensional cohomology is cyclic of order $h + j$ (or ∞ if $h + j = 0$). Hence, $h + j$ is an invariant of the oriented homeomorphism type of $S_{hj}(\Lambda)$. If $h = -j$, then for distinct h the manifolds $S_{hj}(\Lambda)$ have distinct Pontriagin classes, hence, distinct oriented homeomorphism types. If $h \neq -j$, then the μ-invariant is defined. Since this has the form $a(h - j)^2 + b$, we have that $(h - j)^2 = (h' - j')^2$, i.e., $(h - j) = \pm(h' - j')$. The theorem follows.

3. The Manifolds M_k^{2n+1} and Milnor's Morse Function. Let $n = \dim_{\mathbf{R}} \Lambda - 1$. Suppose that $h + j = 1$ and that $k = h - j$. Following [26] we define M_k^{2n+1} to be the $(2n + 1)$-manifold $S_{hj}(\Lambda)$ with its natural G^Λ-action. Also, let $S^{2n+1} \subset \Lambda^2$ be the unit sphere with the linear G^Λ-action.

THEOREM 3.1. *For each integer k, the G^Λ-manifold M_k^{2n+1} is equivariantly homeomorphic to S^{2n+1} with the linear action.*

Proof. (Milnor [26]): Milnor proved the non-equivariant version of this by defining a Morse function f on M_k^{2n+1} with only two critical points. The proof works equivariantly. The manifold M_k^{2n+1} is the union of two copies of $\Lambda \times S^n$ with coordinates (u, v) and (u', v'). Introduce new coordinates on the second copy (u'', v') by $u'' = u'(v')^{-1}$. The function $f : M_k^{2n+1} \to \mathbf{R}$ is given by

$$f(x) = \frac{\mathrm{Re}(v)}{(1 + |u|^2)^{1/2}} = \frac{\mathrm{Re}(u'')}{(1 + |u''|^2)^{1/2}}.$$

It has precisely two critical points, both non-degenerate, at $(u, v) = (0, \pm 1)$. Since the quantities $\mathrm{Re}(v)$, $|u|^2$, $\mathrm{Re}(u'')$, and $|u''|^2$ are all G^Λ-invariant, so is f. Thus, M_k^{2n+1} is G^Λ-equivariantly homeomorphic to the union of two copies of a $(2n + 1)$-disk with linear action.

By way of contrast, as an immediate consequence of Theorem 2.1 we have the following result.

THEOREM 3.2. *Suppose that $\Lambda = \mathbf{H}$ or $\mathbf{O}$. Then (M_k^{2n+1}, G^Λ) is equivariantly diffeomorphic to $(M_\ell^{2n+1}, G^\Lambda)$ if and only if $k = \pm \ell$.*

For homotopy 7-spheres the μ-invariant takes values in $1/28\,\mathbf{Z}$, while for homotopy 15-spheres it takes values in $1/8128\,\mathbf{Z}$. The formulas of Section 2 reduce to

$$\mu(M_k^7) = h(h-1)/56$$

$$\mu(M_k^{15}) = h(h-1)/15{,}256.$$

(These formulas occur on pages 101 and 106 of [12].) Notice that for $k = 3$, we have $h = 2$, and $\mu(M_3^7) = 1/28$, $\mu(M_3^{15}) = 1/8128$.

The group of homotopy 7-spheres, Θ_7, is cyclic of order 28. Also, $\Theta_7 = b_8$, i.e., every homotopy 7-sphere bounds a π-manifold. In dimension 15 it is known that Θ_{15} has order 15,256 and that the cyclic subgroup bP_{16} has index two. By work of Wall [33], a homotopy 15-sphere bounds a parallelizable manifold if and only if it bounds a 7-connected manifold. Hence, $M_3^{15} \in bP_{16}$. It follows from the above calculation of $\mu(M_3^{2n+1})$ that M_3^7 is a generator of $bP_8 = \Theta_7$ and that M_3^{15} is a generator of bP_{16}.

The fixed point set of G^Λ on M_3^{2n+1} is a circle. We can therefore take the connected sum of any number of copies of $\pm(M_3^{2n+1}, G^\Lambda)$ equivariantly, by performing the connected sum operation at a fixed point. Since (M_3^{2n+1}, G^Λ) is equivariantly homeomorphic to (S^{2n+1}, G^Λ) so is any equivariant connected sum. Since the μ-invariant is additive we see that for $\Lambda = \mathbf{H}$ or $\mathbf{O}$, (M_3^{2n+1}, G^Λ) generates an infinite cyclic group of oriented G^Λ-actions on homotopy $(2n+1)$-spheres. Also, the natural map from this group to bP_{2n+2} is just reduction mod 28, $\mathbf{Z} \to \mathbf{Z}/28\,\mathbf{Z}$ if $2n+1 = 7$, or reduction mod 8128, $\mathbf{Z} \to \mathbf{Z}/8128\,\mathbf{Z}$ if $2n+1 = 15$. In summary we have the following result.

THEOREM 3.3. *Let $\Lambda = \mathbf{H}$ or $\mathbf{O}$ and let $G = G^\Lambda$. The G-manifold (M_3^{2n+1}, G) generates an infinite cyclic group (under connected sum) of oriented G-actions on homotopy $(2n+1)$-spheres. (Here $2n+1 = 7$ or 15.) Every element of this group is equivariantly homeomorphic to the orthogonal G-action on S^{2n+1}. Moreover, every homotopy sphere in bP_{2n+2} admits an infinite number of distinct actions in this group.*

4. The General Theory of Biaxial Actions. In this section we shall review some material from [6], [7], [9], [19] and [20] and deduce some consequences.

Let G be a closed subgroup of $O(n)$ such that G acts transitively on both the sphere S^{n-1} and the Stiefel manifold $V_{n,2} = O(n)/O(n-2)$.

For example, we might take $G = O(n)$, $n \geq 2$, or $G = SO(n)$, $n \geq 3$. Another possibility is to take $G = G^\Lambda$, the group of automorphisms of Λ, where $\Lambda = \mathbf{H}$ or $\Lambda = \mathbf{O}$. Let W^Λ be the orthogonal complement of $\mathbf{R}$ in Λ. Choose an isomorphism $W^\Lambda \cong \mathbf{R}^n$ where $n = 3$ if $\Lambda = \mathbf{H}$ or $n = 7$ if $\Lambda = \mathbf{O}$. This defines a representation $G^\Lambda \hookrightarrow O(n)$. If $\Lambda = \mathbf{H}$, then its image is $SO(3)$ and we just have the standard 3-dimensional representation. If $\Lambda = \mathbf{O}$, then its image is denoted by G_2. In either case it is not difficult to check that G is transitive on $V_{n,2}$. If $G \subset O(n)$ is connected and acts transitively on $V_{n,2}$, then it is known [4] that the only other possibility is a certain 8-dimensional representation $\mathrm{Spin}(7) \subset O(8)$.

The natural action of G on $\mathbf{R}^n \oplus \mathbf{R}^n$ is called the *linear biaxial G-action.* Let $H = G \cap O(n-1)$ and let $K = G \cap O(n-2)$. The orbit of $(x, y) \in \mathbf{R}^n \oplus \mathbf{R}^n$ is one of three types: (i) a fixed point if $(x, y) = (0, 0)$, (ii) an $(n-1)$-sphere, G/H, if x and y are linearly dependent, or (iii) a Stiefel manifold, G/K, if x and y are linearly independent. The orbit space of G on $\mathbf{R}^n \oplus \mathbf{R}^n$ can be canonically identified with $B^{\mathbf{R}}(2)$, the space of 2 by 2 positive semi-definite symmetric real matrices. The orbit map is given by

$$(x, y) \to \begin{pmatrix} x \cdot x & x \cdot y \\ y \cdot x & y \cdot y \end{pmatrix}.$$

The space $B^{\mathbf{R}}(2)$ is a solid 3-dimensional cone. The vertex is the image of the origin, the remainder of the boundary is the image of the spherical orbits and the interior is the image of the orbits of type G/K (that is, the principal orbits).

A smooth G-manifold (M, G) is *biaxial* if it is stably modeled on $(\mathbf{R}^n \oplus \mathbf{R}^n, G)$. This just means that the orbit types are G/G, G/H and G/K, that the normal representation at a fixed point is equivalent to $(\mathbf{R}^n \oplus \mathbf{R}^n, G)$, and that the normal representation at a spherical orbit is equivalent to $(\mathbf{R}^{n-1}, H)$. A biaxial G-manifold M will therefore have three strata. We index these by $\{0, 1, 2\}$, where 0 corresponds to G/G, 1 to G/H, and 2 to G/K. By the *i-stratum of M* (denoted by $\dot{M}_i$) we shall mean the union of orbits of type i in M. By the *closed* i-stratum of M (denoted by M_i) we shall mean the i-stratum of M with the lower strata "blown up." (To obtain M_i one attaches to $\dot{M}_i$ a boundary made up of the i-stratum of the normal sphere bundles of the lower strata as in [11] or [21].) Denote by $B(M)$ the orbit space of M and by $\dot{B}_i(M)$ and $B_i(M)$ the orbit spaces of $\dot{M}_i$

and M_i, respectively. Since $B^{\mathbf{R}}(2)$ is homeomorphic to a 3-dimensional Euclidean half-space, $B(M)$ is homeomorphic to a manifold with boundary. The interior of $B(M)$ is the image of the principal orbits. The fixed point set $B_0(M)$ is a submanifold of the boundary of codimension 2.

Example 4.1. Consider the linear action of G on $\mathbf{R}^n \oplus \mathbf{R}^n \oplus \mathbf{R}^{m+1}$ where G acts trivially on the third factor. This action is clearly biaxial with orbit space $B^{\mathbf{R}}(2) \times \mathbf{R}^{m+1}$. Also, we may consider the G-action on the unit sphere $S^{2n+m} \subset \mathbf{R}^n \oplus \mathbf{R}^n \oplus \mathbf{R}^{m+1}$. This is obviously also biaxial. It is easy to see that the orbit space $B(S^{2n+m})$ is homeomorphic to a $(m+3)$-disk. The fixed point set $B_0(S^{2n+m}) = S^m$ is an unknotted sphere embedded in the boundary. All these actions will be referred to as *linear biaxial actions.*

Example 4.2. Let $(\zeta, \omega, z_1, \ldots, z_n)$ be complex coordinates for $\mathbf{C}^{n+2}$ and let f be the polynomial

$$f(\zeta, \omega, z_1, \ldots, z_n) = (\zeta)^p + (\omega)^2 + (z_1)^2 + \cdots + (z_n)^2.$$

The Brieskorn manifold $\Sigma^{2n+1}(p, 2, \ldots, 2)$ is defined as the intersection of $f^{-1}(0)$ and S^{2n+3}. Let $U(n)$ act linearly on S^{2n+3} by operating on the last n coordinates. If $G \subset O(n)$ is transitive on $V_{n,2}$, then the representation $G \subset O(n) \subset U(n)$ gives a linear biaxial action on S^{2n+3}. The submanifold $\Sigma^{2n+1}(p, 2, \ldots, 2) \subset S^{2n+3}$ is clearly G-invariant and biaxial. It is interesting to note that $\Sigma^{2n+1}(p, 2, \ldots, 2)$ also admits a commuting action of S^1 defined by $e^{i\theta} \cdot (\zeta, \omega, z_1, \ldots, z_n) = (e^{i2\theta}\zeta, e^{ip\theta}\omega, z_1, \ldots, z_n)$. It is well known that the orbit space of $\Sigma^{2n+1}(p, 2, \ldots, 2)$ is homeomorphic to D^4 and that the fixed point set $\Sigma^1(p, 2)$ is a torus link of type $(p, 2)$ embedded in the boundary 3-sphere. The S^1-action of $\Sigma^{2n+1}(p, 2, \ldots, 2)$ induces a strata-preserving S^1-action on the orbit space. We leave it as an exercise for the reader to check that this S^1-action is equivariantly homeomorphic to the linear S^1-action given by the representation $\mu_2 + \mu_p : SO(2) \to SO(2) \times SO(2) \subset SO(4)$, where $\mu_n : SO(2) \to SO(2)$ is the homomorphism $g \to g^n$.

Example 4.3. Suppose that $\Lambda = \mathbf{H}$ or $\mathbf{O}$ and that $G = G^\Lambda$ is the group of automorphisms of Λ. The linear G-action on Λ^2 is biaxial. Hence, so is the G-action on $E_{h,j}(\Lambda)$, since by construction $E_{h,j}(\Lambda)$ is the union of two invariant open sets of the form (Λ^2, G). Obviously, the restriction of this action to the sphere-bundle $S_{h,j}(\Lambda)$ is also biaxial. It is not difficult to calculate the orbit spaces of $E_{hj}(\Lambda)$, and of $S_{hj}(\Lambda)$. One uses the facts that $B(\Lambda^2) \cong B^{\mathbf{R}}(2) \times \mathbf{R}^2$ and that $E_{hj}(\Lambda)$ is formed by gluing two copies of Λ^2 via (1.1). The reader is encouraged to check that the orbit space of $S_{hj}(\Lambda)$ is homeomorphic to D^4 and that the fixed point set, $S_{hj}(\mathbf{R})$, is the boundary of a band with $h + j$ half twists. Thus, $S_{hj}(\mathbf{R})$ is a torus link of type $(h + j, 2)$ embedded in the boundary of D^4. We also observed in Section 1 that $S_{hj}(\Lambda)$ admits a commuting action of $O(2) \subset GL(2, \mathbf{R})$. This induces a strata-preserving action of $O(2)$ on the orbit space $B(S_{hj}(\Lambda))$. We again leave it to the reader to check that this action is equivariantly homeomorphic to the linear $O(2)$-action given by $\mu_2 + \mu_{h+j}: O(2) \to O(2) \times O(2) \subset O(4)$, where $\mu_n: O(2) \to O(2)$ is the homomorphism with kernel the cyclic subgroup of order n. Comparing this with Example 4.2, we see that *the G-manifolds $S_{hj}(\Lambda)$ and $\Sigma^{2n+1}(h + j, 2, \ldots, 2)$, $n = 3$ or 7, have the same orbit spaces* (even with respect to the induced $SO(2)$-actions!) Another way to convince oneself of this is to observe that $S_{hj}(\Lambda) = S_{hj}(\mathbf{C}) = L^3(h + j, 1) = \Sigma^3(h + j, 2, 2)$. Next suppose $h + j = \pm 1$. The torus knot of type (1, 2) is the unknot. Hence if $h + j = \pm 1$, *then the orbit space of $S_{hj}(\Lambda)$ is isomorphic to the orbit space of the linear G-action on S^{2n+1}*. (Essentially, we already proved this in Theorem 3.1.)

Now let us return to our general considerations of a biaxial G-manifold M. The equivariant normal bundle of M_i in M may be regarded as a bundle over $B_i(M)$. (This is explained in [8].) The structure group of this bundle can be reduced to a certain compact Lie group S_i, which is defined and computed in [9]. The associated principal bundle is denoted by $P_i(M) \to B_i(M)$. Of particular importance is $P_2(M) \to B_2(M)$. This is just the principal bundle associated to the bundle of principal orbits. Its structure group S_2 is actually $N_G(K)/K$. The biaxial G-manifold M is said to be *trivializable* if $P_2(M) \to B_2(M)$ is a trivial fiber bundle. A *trivialization* of M is the homotopy class $[f]$ of a bundle trivialization $f: P_2(M) \to S_2$.

We denote a trivialized biaxial G-manifold by $(M, G, [f])$. Such an object $(X, G, [g])$ is *universal* if given any $(M, G, [f])$ there is an equivariant stratified map $\tilde{\varphi}: M \to X$ such that the following diagram

$$\begin{tikzcd} P_2(M) \arrow[rr, "P_2(\tilde{\varphi})"] \arrow[dr, "f"'] & & P_2(X) \arrow[dl, "g"] \\ & S_2 & \end{tikzcd}$$

commutes up to an equivariant homotopy. Moreover, the map $\tilde{\varphi}$ is required to be unique up to a homotopy through equivariant stratified maps. If $(X, G, [g])$ is universal, then any $(M, G, [f])$ is equivalent to a pullback of X via a stratified map $\varphi: B(M) \to B(X)$. (We may take φ to be the map induced by $\tilde{\varphi}$.)

In the following table we list some groups $G \subset O(n)$ which act transitively on $V_{n,2}$. The isotropy groups H and K are given as well. We have divided these groups G into two types depending on the structure groups S_i. We have also listed the corresponding universal G-manifold in each case.

TABLE 4.4

	G	H	K	The universal G-manifold
Type I	$O(n)$; $n \geq 2$	$O(n-1)$	$O(n-2)$	$\mathbf{R}^n \oplus \mathbf{R}^n$
	$SO(n)$; $n \geq 4$	$SO(n-1)$	$SO(n-2)$	$\mathbf{R}^n \oplus \mathbf{R}^n$
	Spin(7); $n = 8$	G_2	$SU(3)$	$\mathbf{R}^8 \oplus \mathbf{R}^8$
Type II	$SO(3)$; $n = 3$	$SO(2)$	$\{1\}$	$\mathbf{H}P^2$
	G_2; $n = 7$	$SU(3)$	$SU(2)$	$\mathbf{O}P^2$

The Structure Groups

Type I: $S_0 = O(2)$, $S_1 = O(1) \times O(1)$, $S_2 = O(2)$
Type II; $S_0 = O(2)$, $S_1 = O(2)$, $S_2 = SO(3)$.

THEOREM 4.5 (Bredon). *If G is of type I, then the linear model $(\mathbf{R}^n \oplus \mathbf{R}^n, G, [g])$ is universal for trivialized biaxial G-actions.*[3]

This result was essentially proved by Bredon in [6]. A proof is also given in [9]. (Actually in [9] the result is only stated for $G = O(n)$, $n \geq 2$ and for $G = SO(n)$, $n \geq 4$; however, the same argument works for Spin(7).)

If G is of type II, then $G = G^\Lambda$ for $\Lambda = \mathbf{H}$ or $\mathbf{O}$. Notice that there is a natural action of the group of automorphisms of Λ on ΛP^4, the Λ-projective plane. It is proved in [9] that this action is biaxial, trivializable and universal. Thus, for G of type II, Theorem 4.5 is replaced by the following result.

[3] Since $S_2 = O(2)$ has two components, there are two choices for the trivialization $[g]$. Either one works.

[4] In this case there is only one choice for $[g]$.

THEOREM 4.6. *If $G = G^\Lambda$ is of type II, then $(\Lambda P^2, G^\Lambda, [g])$ is universal for trivialized biaxial actions.* [2]

Remark 4.7. The orbit space $B(\Lambda P^2)$ may be identified with $T^{\mathbf{R}}(3)$, the space of 3 by 3 positive semi-definite real symmetric matrices of trace one. This space is homeomorphic to D^5. The fixed point set $B_0(\Lambda P^2) = \mathbf{R}P^2$ is embedded in the boundary 4-sphere in such a fashion that $\dot{B}_1(\Lambda P^2) = S^4 - \mathbf{R}P^2$ is a 2-plane bundle over $\mathbf{R}P^2$. Thus, both singular strata are homotopy equivalent to $\mathbf{R}P^2$. It is interesting to note that $T^{\mathbf{R}}(3)$ is also the orbit space of the linear triaxial action of $O(n)$ on S^{3n-1}, $n \geq 3$. Moreover, the structure groups associated to the strata are the same in both cases. Also, $(S^{2n-1}, O(n))$ is universal for trivialized triaxial $O(n)$-actions without fixed points. It follows that the category of trivialized biaxial G^Λ-manifolds is equivalent to the category of trivialized triaxial $O(n)$-manifolds without fixed points for any $n \geq 3$. I don't know of any real use for this observation; however, it does motivate the construction given below in the proof of part 1) of Theorem 4.9.

Many interesting consequences of Theorem 4.5 are deduced in [6]. We state some of these in the following result. (See also [7] and [11].)

THEOREM 4.8 (Bredon). *Suppose that G is of type I and that $(M, G, [f])$ is a trivialized biaxial G-manifold.*

1) *M equivariantly bounds a trivialized biaxial G-manifold $(W, G, [f'])$ such that $B(W)$ is homeomorphic to $B(M) \times I$.*
2) *If the tangent bundle $T(B(M)$ is trivial, then $T(M)$ is equivariantly stably trivial.*
3) *In particular, if $T(B(M))$ is trivial then by* 1) *$T(B(W))$ will also be trivial; and hence, $T(W)$ will be equivariantly stably trivial.*

The arguments used to prove this can be modified to deduce the following result for $G = SO(3)$ or G_2.

THEOREM 4.9. *Suppose that $G = G^\Lambda$ is of type II and that $(M, G, [f])$ is a trivialized biaxial G-manifold. Let $\varphi: B(M) \to B(\Lambda P^2)$ be the classifying map.*

1) *If $\varphi | \partial B(M): \partial B(M) \to \partial B(\Lambda P^2) = S^4$ is nullhomotopic, then M equivariantly bounds a trivialized biaxial G-manifold $(W, G, [f'])$ such that $B(W)$ is homeomorphic to $B(M) \times I$.*
2) *If $T(B(M)$ is trivial, then $T(M)$ is equivariantly stably equivalent to a pullback of $T(\Lambda P^2)$.*

3) *If* $\varphi | \partial B(M)$ *is nullhomotopic and if* W *is as in* 1), *then* $T(W)$ *is equivariantly stably equivalent to a pullback of* $T(\Lambda P^2)$. *In particular,* W *is a spin manifold.*

Proof. 1) The argument is similar to the one given on page 94 of [7]. Let $B^{\mathbf{R}}(3)$ be the space of 3 by 3 positive semi-definite symmetric matrices. Then $B^{\mathbf{R}}(3)$ is a cone on $T^{\mathbf{R}}(3) = B(\Lambda P^2)$. Let $E \in B^{\mathbf{R}}(3)$ be the identity matrix. Let $x_0 = \frac{1}{3}E \in T^{\mathbf{R}}(3)$ and let T_1 be the submanifold of $T^{\mathbf{R}}(3)$ defined by requiring that $x \neq x_0$ and that the two smaller eigenvalues of x be equal. We may assume that $\varphi : B(M) \to T^{\mathbf{R}}(3) \cong D^5$ is transverse to x_0 and to T_1. Since $\varphi | \partial B(M) : \partial B(M) \to \partial T^{\mathbf{R}}(3) \cong S^4$ is nullhomotopic, we may also assume that $\varphi^{-1}(x_0)$ is empty. Let $s : B^{\mathbf{R}}(3) \times [0, \infty) \to B^{\mathbf{R}}(3)$ be the map $(x, t) \to x + tE$ and define $B(W)$ by the pullback diagram

$$\begin{tikzcd} B(W) \arrow[r, "\varphi'"] \arrow[d] & B^{\mathbf{R}}(3) \times [0, \infty) \arrow[d, "s"] \\ B(M) \arrow[r, "\varphi"] & T^{\mathbf{R}}(3) \subset B^{\mathbf{R}}(3). \end{tikzcd}$$

Let $0 \in B^{\mathbf{R}}(3)$ denote the cone point. Since φ misses x_0, φ' misses $(0, \frac{1}{3}) \in B^{\mathbf{R}}(3) \times [0, \infty)$. Define $p : (B^{\mathbf{R}}(3) - \{0\}) \times [0, \infty) \to T^{\mathbf{R}}(3)$ by $(x, t) \to (\text{trace } x)^{-1}x$ and set $\varphi'' = p \circ \varphi' : B(W) \to T^{\mathbf{R}}(3)$. The space $B(W)$ is locally isomorphic to the orbit space of a biaxial action and the map φ'' is stratified. (The condition that $\varphi^{-1}(x_0) = \emptyset$ is necessary to insure that $B(W)$ have no stratum corresponding to the cone point of $B^{\mathbf{R}}(3)$.) Set $W = (\varphi'')^*(\Lambda P^2)$. The fibers of s over the interior of $T^{\mathbf{R}}(3)$ are intervals, while over $\partial T^{\mathbf{R}}(3)$ they are points. Since the fibers of $B(W) \to B(M)$ have a similar description, we conclude that $B(W)$ is homeomorphic to $B(M) \times I$.

2) Since M is equivalent to $\varphi^*(\Lambda P^2) = \{(z, b) \in \Lambda P^2 \times B(M) | \varphi(b) = \pi(z)\}$, it is clear that it is embedded in $\Lambda P^2 \times B(M)$ with trivial normal bundle. It follows immediately that $T(M)$ is stably equivalent to $\varphi^*(T(\Lambda P^2))$.

3) Since $B(W)$ is homeomorphic to $B(M) \times I$, $T(B(W))$ is trivial if and only if $T(B(M)$ is trivial. Hence, 3) follows immediately from 1) and 2) and the fact that ΛP^2 is a spin manifold.

5. Biaxial Actions on Homotopy Spheres. Here we will show that the results of the previous section can be applied to actions on homotopy spheres.

Suppose that B is a space which is locally differentiably modeled on $B^{\mathbf{R}}(2) \times \mathbf{R}^m$, i.e., suppose that B is locally the orbit space of a biaxial $O(n)$-action. Then B will have three strata. The triple $(B, \partial B, B_0)$ is called an *orbit triple*. We shall be interested in those triples satisfying the following additional conditions:

$$(5.1)_n \quad \begin{cases} \text{a) } B \text{ is a compact contractible } (m+3)\text{-manifold with boundary.} \\ \text{b) } B_0 \text{ is a } \mathbf{Z}/2\mathbf{Z}\text{-homology } m\text{-sphere embedded in } \partial B. \\ \text{c)}_{\text{even}} \text{ If } n \text{ is even, then } B_0 \text{ is an integral homology } m\text{-sphere.} \\ \text{c)}_{\text{odd}} \text{ If } n \text{ is odd, then the double branched cover of } \partial B \text{ along} \\ \qquad B_0 \text{ is an integral homology } (m+2)\text{-sphere.} \end{cases}$$

Let $G \subset O(n)$ be one of the groups listed in Table 4.4 and let $H = G \cap O(n-1)$, $K = G \cap O(n-2)$. Let Σ be a homotopy sphere on which G acts biaxially and let $B(\Sigma)$ be its orbit space. If Σ^G is empty, then it is known that Σ must be of dimension $2n - 1$ and that $B(\Sigma)$ is a 2-disk. This case has been analyzed completely in [2]. If Σ^G is nonempty of dimension m, then it is clear that Σ has dimension $2n + m$. Henceforth we shall assume that $\dim \Sigma = 2n + m$, $m \geq 0$.

PROPOSITION 5.2. *Suppose that $G \subset O(n)$ is a group listed in Table 4.4 and that G acts biaxially on a homotopy sphere Σ^{2n+m}. Then the orbit triple of (Σ, G) satisfies $(5.1)_n$.*

Proof. Using $\mathbf{Z}/2\mathbf{Z}$-tori, Smith Theory implies that Σ^G and Σ^H are $\mathbf{Z}/2\mathbf{Z}$-homology spheres; hence b) holds. Using the maximal torus of G, Smith Theory implies that if n is even, then Σ^G is an integral homology sphere, while if n is odd Σ^H is an integral homology sphere. Since $\Sigma^G = B_0(\Sigma)$, c)$_{\text{even}}$ holds, and since Σ^H is the double branched cover of $\partial B(\Sigma)$ along $B_0(\Sigma)$, we also have c)$_{\text{odd}}$. In [30] Oliver proved that the orbit space of a compact Lie group action on Euclidean space is contractible. Applying this to the complement of a fixed point in Σ, condition a) follows easily.

COROLLARY 5.3. *Let (Σ^{2n+m}, G) be a homotopy sphere with biaxial action, then (Σ^{2n+m}, G) is a trivializable biaxial G-manifold, i.e., $P_2(\Sigma) \to B_2(\Sigma)$ is a trivial fiber bundle.*

Proof. $B_2(\Sigma)$ is contractible.

COROLLARY 5.4. *With hypotheses as above, the tangent bundle of $B(\Sigma)$ is trivial.*

Proof. $B(\Sigma)$ is contractible.

Let (X, G) be the universal trivializable biaxial G-manifold, i.e., $X = \mathbf{R}^n \oplus \mathbf{R}^n$ if G is of type I or $X = \Lambda P^2$ if G is of type II. In view of Theorems 4.5 and 4.6 and Corollary 5.3, (Σ, G) is equivalent to the pullback of X via some stratified map $\varphi: B(\Sigma) \to B(X)$. As a converse to Proposition 5.2 we have the following result. (For a proof, see [7] or [19].)

PROPOSITION 5.5. *Suppose that $G \subset O(n)$ is a group listed in Table 4.4, that $(B, \partial B, B_0)$ satisfies $(5.1)_n$, and that $\varphi: B \to B(X)$ is a stratified map where (X, G) is universal. Then the pullback $\varphi^*(X)$ is a homotopy $(2n + m)$-sphere with biaxial G-action.*

Therefore, the problem of classifying biaxial G-actions on homotopy spheres is equivalent to the problem of classifying triples $(B, \partial B, B_0)$ satisfying $(5.1)_n$ together with stratified homotopy classes of maps $\varphi: B \to B(X)$. For a fixed triple $(B, \partial B, B_0)$ satisfying $(5.1)_n$ it is not hard to see that there are exactly two stratified homotopy classes of maps $\varphi: B \to B^{\mathbf{R}}(2)$ and that they differ by an automorphism of $B^{\mathbf{R}}(2)$ which is induced by an equivariant linear automorphism of the linear model. Hence, we recover the well-known result of the Hsiangs [19] and Jänich [20]: *if G is of type I, then there is a one-to-one correspondence between equivariant diffeomorphism classes of biaxial G-actions on homotopy $(2n + m)$-spheres and diffeomorphism classes of orbit triples $(B, \partial B, B_0)$ satisfying $(5.1)_n$.*

For $G = SO(3)$ or G_2 there may be more than one equivalence class of map $B \to B(\Lambda P^2)$. (We know that this must be the case, since by Theorem 3.2 and Example 4.3, the homotopy spheres (M_k^{2n+1}, G), $k \in \mathbf{N}$, all have the same orbit triple yet belonging to distinct equivariant diffeomorphisms classes.)

By an *orientation* for (Σ, G) we mean an orientation for Σ^G and one for Σ^H. Specifying these orientations is equivalent to specifying orientations for $B_0(\Sigma)$ and $\partial B(\Sigma)$ (or $B(\Sigma)$). Depending on whether n is even or odd either the orientation for Σ^G or the one for Σ^H will determine an orientation for Σ. By an *oriented equivalence* $\Sigma \to \Sigma'$ we shall mean an equivariant diffeomorphism which preserves both orientations.

Let $C_{2n+m}(G)$ be the abelian semi-group of oriented equivalence classes of biaxial G-actions on homotopy $(2n + m)$-spheres. (The semi-group operation is connected sum.) Let $D_{n,m}$ be the abelian semi-group of oriented diffeomorphism classes of oriented orbit triples $(B^{m+3}, \partial B, B_0)$ satisfying condition $(5.1)_n$. Let $p_G: C_{2n+m}(G) \to D_{n,m}$ be the semi-group homomorphism which sends Σ to $B(\Sigma)$. Also, let $i_G: C_{2n+m}(O(n)) \to$

$C_{2n+m}(G)$ be the map which sends $(\Sigma, O(n))$ to (Σ, G). For groups G of type I, the result of the Hsiangs-Jänich states that p_G is an isomorphism. Hence, i_G must also be an isomorphism. Thus, for G of type I every biaxial G-action extends to biaxial $O(n)$-action.

Now suppose that $G = G^\Lambda$ is of type II. Let $A_{2n+m}(G)$ denote the kernel of $p_G: C_{2n+m}(G) \to D_{n,m}$. Thus, $A_{2n+m}(G)$ consists of those (Σ^{2n+m}, G) with orbit triples equal to that of the linear biaxial action (S^{2n+m}, G).

PROPOSITION 5.6. *$A_{2n+m}(G)$ is an abelian group.*

Proof. If G is of type I, then $A_{2n+m}(G)$ is trivial. Hence, we may assume that G is of type II. We must show that every $(\Sigma, G) \in A_{2n+m}(G)$ is invertible. Let $(-\Sigma, G)$ denote the result of reversing the orientation (i.e., both the orientation of Σ^H and Σ^G are reversed). The usual argument shows that $\Sigma \# (-\Sigma)$ equivariantly bounds a contractible G-manifold V^{2n+m+1}. Since $B(\Sigma^{2n+m}) \cong B(S^{2n+m})$, we have that $B(V^{2n+m+1}) \cong B(D^{2n+m+1})$. Suppose that $D^{2n+m+1} \cong U \subset V^{2n+m+1}$ and $D^{2n+2} \cong W \subset \Lambda P^2$ are linear disk neighborhoods of fixed points in V and ΛP^2, respectively. We have that V is the pullback of ΛP^2 via some stratified map $\varphi: B(V) \to B(\Lambda P^2)$. We may clearly assume that $\varphi | B(U)$ is given by the composition

$$B(U) = B^{\mathbf{R}}(2) \times \mathbf{R}^{m+1} \xrightarrow{\text{proj}} B^{\mathbf{R}}(2) \hookrightarrow B(W) \subset B(\Lambda P^2).$$

Since $B(\overline{V - U}) \cong B(S^{2n+m} \times I)$ there is no obstruction to deforming φ by a stratified homotopy rel $B(U)$ so that its image lies in $B^{\mathbf{R}}(2)$. Hence, V is the pullback of the linear model. It follows that (V^{2n+m+1}, G) must be equivalent to the linear biaxial action (D^{2n+m+1}, G) and hence, that the action on $\Sigma \# (-\Sigma) = \partial V$ is also linear.

6. Actions of $SO(3)$ on Homotopy 7-Spheres and of G_2 on Homotopy 15-Spheres. In this section G is of type II, i.e., $G = SO(3)$ or G_2 and $n = 3$ or 7. We shall also assume that $m = 1$, i.e., that the fixed point set is 1-dimensional.

Suppose that $(\Sigma^{2n+1}, G) \in C_{2n+1}(G)$ is an oriented biaxial action on a homotopy sphere. Let $B = B(\Sigma)$ and $B_0 = \dot{B}_0(\Sigma)$. By 5.2, B is a contractible 4-manifold, B_0 is a circle, and the double branched cover of ∂B along B_0 is an integral homology 3-sphere. We shall now define three different integer-valued additive invariants of (Σ^{2n+1}, G).

The index of the orbit knot. The index of an oriented knotted circle B_0 in an oriented integral homology 3-sphere ∂B is defined as the signature of the symmetrization of a Seifert linking form for the knot. Since the double branched cover of this knot is required to be an integral homology sphere, this signature is divisible by 8. Define $\sigma(\Sigma)$ to be one eighth the index of the orbit knot $(\partial B, B_0)$. It is clearly an additive invariant of the action on Σ.

The μ-invariant. By 4.6 and 5.3, Σ^{2n+1} is equivalent to a pullback of ΛP^2 via a map $\varphi: B \to B(\Lambda P^2)$. Since ∂B is 3-dimensional, $\varphi \mid \partial B: \partial B \to \partial B(\Lambda P^2) = S^4$ is null-homotopic. Hence, by 4.9 and 5.4, Σ^{2n+1} equivariantly bounds a spin manifold W^{2n+2}. As in Section 2 it follows that the rational number $\mu(\Sigma)$ is a well-defined additive invariant of the action. We can normalize it to be an integer by setting $\hat{\mu}(\Sigma) = a_{2n+1}\mu(\Sigma)$ where $a_7 = 28$ and $a_{15} = 8128$.

The obstruction to being a pullback of the linear model. The fixed point set of G on ΛP^2 is $\mathbf{R}P^2$. Let $x_0 \in \mathbf{R}P^2$ be a fixed point and let $D^{2n+2} \subset \Lambda P^2$ be a linear disk centered at x_0. We may identify the linear disk D^{2n} with $D^{2n} \times \{0\} \subset D^{2n+2} \subset \Lambda P^2$. Its orbit space $B(D^{2n})$ consists of the elements in $B^{\mathbf{R}}(2)$ of trace less than or equal to one. Hence, $B(D^{2n}) \subset B(\Lambda P^2)$. We wish to analyze the obstruction to making a stratum-by-stratum deformation of $\varphi: B \to B(\Lambda P^2)$ through stratified maps so that its image lies in $B(D^{2n})$.

First we want to make two observations. For any closed biaxial G-manifold M, $\partial B_1(M)$ is a circle bundle over $B_0(M)$. For $i = 0, 1$, let $\varphi_i: B_i \to B_i(\Lambda P^2)$ denote the restriction of φ to the i-stratum, and let $\partial\varphi_1: \partial B_1 \to \partial B_1(\Lambda P^2)$ denote the restriction of φ_1 to ∂B_1. Since φ is stratified, $\partial\varphi_1$ is a map of circle bundles covering φ_0.

Next we claim that there is essentially a unique stratified map $\theta: B \to B(D^{2n})$. There is only one choice for the map on the 0-stratum, $\theta_0: B_0 \to B_0(D^{2n})$, since its range is the point x_0. Let $S^1_{x_0} \subset B_1(D^{2n})$ be the fiber of $\partial B_1(\Lambda P^2) \to B_0(\Lambda P^2)$ at x_0. The 1-stratum of $B(D^{2n})$ is diffeomorphic to the product $S^1_{x_0} \times I$. Since B_1 is a knot complement, $H^1(B_1; \mathbf{Z}) \cong \mathbf{Z}$. Also, $\partial B_1 \cong B_0 \times S^1$. Let γ be a generator of $H^1(B_1; \mathbf{Z})$ and let $\partial\theta_1: \partial B_1 \to S^1_{x_0}$ be a map of circle bundles representing $j^*(\gamma) \in H^1(\partial B_1; \mathbf{Z})$, where $j: \partial B_1 \hookrightarrow B_1$ is the inclusion. Clearly, $\partial\theta_1$ extends to a map $\theta_1: B_1 \to B_1(D^{2n}) = S^1_{x_0} \times I$ representing γ. Since $B_2(D^{2n})$ is a 3-ball, the maps θ_0 and θ_1 can be extended to a stratified map $\theta: B \to B(D^{2n})$. It is clear that, up to the choice of generator γ, any stratified map $B \to B(D^{2n})$ must be stratified homotopic to the map θ constructed above.

We shall now examine the obstruction to deforming an arbitrary stratified map $\varphi : B \to B(\Lambda P^2)$ to $i \circ \theta : B \to B(\Lambda P^2)$, where $i : B(D^{2n}) \hookrightarrow B(\Lambda P^2)$ is the inclusion. First consider the restriction of φ to the 0-stratum. Since $B_0 = S^1$ and since ∂B is a homology 3-sphere, the normal bundle of B_0 in ∂B is trivial. On the other hand, the normal bundle of $B_0(\Lambda P^2) = \mathbf{R}P^2$ in $\partial B(\Lambda P^2) = S^4$ is non-orientable. Since $\varphi_0 : S^1 \to \mathbf{R}P^2$ is covered by a map of normal bundles, it must be null-homotopic. Hence, we may assume that $\varphi_0(B_0) = x_0$. The 1-stratum of $B(\Lambda P^2)$ is a 2-disk bundle over $\mathbf{R}P^2$. Let $q : B_1(\Lambda P^2) \to \mathbf{R}P^2$ be the projection and let $i : S^1_{x_0} \hookrightarrow \partial B_1(\Lambda P^2)$ be the inclusion of a fiber. It is easy to check that $q \circ i : S^1_{x_0} \to \mathbf{R}P^2$ is the map which is non-trivial on π_1. Let $\bar{\varphi}_1 = q \circ \varphi_1 : B_1 \to \mathbf{R}P^2$ and let $\bar{\theta}_1 = q \circ i \circ \theta_1 : B_1 \to \mathbf{R}P^2$. It follows from the above remarks that $\bar{\varphi}_1 | \partial B_1$ is homotopic to $\bar{\theta}_1 | \partial B_1$. Hence, we may assume that these restrictions are equal. Notice that $\bar{\varphi}_1$ and $\bar{\theta}_1$ induce the same map on fundamental groups since they both must represent the non-trivial element of $H^1(B_1; \mathbf{Z}/2\mathbf{Z})$. The problem of deforming φ to θ therefore reduces to the problem of finding a homotopy of $\bar{\varphi}_1$ to $\bar{\theta}_1$ rel ∂B_1. The obstructions lie in $H^i(B_1, \partial B_1; \pi_i(\mathbf{R}P^2))$. This group is obviously zero for $i = 0, 1$. At first glance it appears to be non-zero if $i = 2$. However, since $\pi_1(\mathbf{R}P^2)$ acts non-trivially on $\pi_2(\mathbf{R}P^2) = \mathbf{Z}$, it follows that the obstruction actually lies in $H^2(B_1, \partial B_1; \mathbf{Z}^{\text{twisted}})$, where the coefficients are twisted via the homomorphism $(\bar{\theta}_1)_* : \pi_1(B_1) \to \pi_1(\mathbf{R}P^2)$. Since the double branched cover of ∂B along B_0 is an integral homology 3-sphere an easy computation shows that $H^2(B_1, \partial B_1; \mathbf{Z}^{\text{twisted}}) = 0$. Therefore, the only possible non-zero obstruction group is $H^3(B_1, \partial B_1; \pi_3(\mathbf{R}P^2))$. Since $\pi_1(\mathbf{R}P^2)$ acts trivially on $\pi_3(\mathbf{R}P^2) \cong \mathbf{Z}$, the coefficients are untwisted. Hence, there is a primary difference class in $H^3(B_1, \partial B_1; \mathbf{Z}) \cong \mathbf{Z}$. It is the complete obstruction to deforming $\bar{\varphi}_1$ to $\bar{\theta}_1$ rel ∂B_1. Define $\tau(\Sigma) \in \mathbf{Z}$ to be this cohomology class evaluated on the fundamental class of B_1. It is clear *that* (Σ^{2n+1}, G) *is equivalent to a pullback of the linear model if and only if* $\tau(\Sigma) = 0$. If Σ^{2n+1} is the pullback of the linear model via $\theta : B \to B(D^{2n})$, then θ can also be used to pull back the linear $O(n)$-action on D^{2n}. Hence, *the G-action on Σ^{2n+1} extends to a biaxial $O(n)$-action if and only if* $\tau(\Sigma) = 0$. We shall leave it as an exercise for the reader to show that τ is additive with respect to equivariant connected sum.

Recall that $A_{2n+1}(G) \subset C_{2n+1}(G)$ is the group consisting of those (Σ^{2n+1}, G) with $B(\Sigma^{2n+1}) = B(S^{2n+1})$. Since τ induces a bijection between the set of stratified homotopy classes of maps $B(S^{2n+1}) \to B(\Lambda P^2)$ and the integers, we have the following result.

PROPOSITION 6.1. *The homomorphism* $\tau: A_{2n+1}(G) \to \mathbf{Z}$ *is an isomorphism.*

Recall that M_3^{2n+1} is the sphere bundle $S_{2,-1}(\Lambda)$ with its canonical action of $G = G^\Lambda$. As we pointed out in Example 4.3 the orbit space of M_3^{2n+1} is isomorphic to that of the linear biaxial action on S^{2n+1}, i.e., $M_3^{2n+1} \in A_{2n+1}(G)$.

PROPOSITION 6.2. *The infinite cyclic group* $A_{2n+1}(G)$ *is generated by* M_3^{2n+1}.

Proof. The homomorphism $\hat{\mu}: A_{2n+1}(G) \to \mathbf{Z}$ takes M_3^{2n+1} to 1.

PROPOSITION 6.3. $C_{2n+1}(G) \cong A_{2n+1}(G) \oplus C_{2n+1}(O(n))$. *Thus, every biaxial G-action on a homotopy* $(2n + 1)$*-sphere can be written uniquely as the equivariant connected sum of the restriction of an* $O(n)$*-action and some number of copies of* M_3^{2n+1}.

Proofs. We must show that any $\Sigma \in C_{2n+1}(G)$ can be written in the form $\Sigma' \# \Sigma''$ where Σ' has linear orbit space and where the G-action on Σ'' extends to an $O(n)$-action. By 6.1 we can choose $\Sigma' \in A_{2n+1}(G)$ so that $\tau(\Sigma') = \tau(\Sigma)$. Since $\tau(\Sigma \# (-\Sigma')) = 0$, the G-action on $\Sigma \# (-\Sigma')$ extends to $O(n)$. Let Σ'' be the image of Σ under the canonical projection $p_{O(n)}^{-1} \circ p_G: C_{2n+1}(G) \to C_{2n+1}(O(n))$ defined in Section 5. Since $(\Sigma \# (-\Sigma'), O(n))$ and $(\Sigma'', O(n)$ have the same orbit space, the theorem of the Hsiangs and Jänich implies that they are equivalent. By adding Σ' to both sides and using 5.6, we deduce that Σ is equivalent to $\Sigma' \# \Sigma''$.

At this point we have defined three homomorphisms σ, $\hat{\mu}$, and τ from $C_{2n+1}(G)$ to the integers. We have shown that $\tau | C_{2n+1}(O(n))$ is the zero map. Since the index of the unknot is zero, we have that $\sigma | A_{2n+1}(G)$ is also the zero map. By 4.8, 5.3, and 5.4 any $\Sigma \in C_{2n+1}(O(n))$ equivariantly bounds a parallelizable manifold V^{2n+2}. By [5], [13], the index of V^{2n+2} is equal to the index of the orbit knot. By [12], the μ-invariant of Σ^{2n+1} is the index of V^{2n+2} divided by $8a_{2n+1}$. It follows that $\hat{\mu}$ and σ agree on $C_{2n+1}(O(n))$. By 6.2 the restrictions of $\hat{\mu}$ and τ to $A_{2n+1}(G)$ agree up to sign. By choosing our generator $\gamma \in H^1(B_1; \mathbf{Z}) \cong \mathbf{Z}$ correctly, we may assume that $\tau | A_{2n+1}(G) = \hat{\mu} | A_{2n+1}(G)$. Hence, $\hat{\mu} = \tau + \sigma$. We therefore have the following result.

PROPOSITION 6.4. $\tau = \hat{\mu} - \sigma: C_{2n+1}(G) \to \mathbf{Z}$.

By 4.8 and the Brieskorn-Hirzebruch examples, the group of homotopy $(2n + 1)$-spheres which admit biaxial $O(n)$-actions is precisely bP_{2n+2}. Therefore the group of homotopy $(2n + 1)$-spheres which admit biaxial

G-actions contains bP_{2n+2}. Since $bP_8 = \Theta_7$, every homotopy 7-sphere admits a biaxial $SO(3)$-action. *A priori* the corresponding result for G_2 could be false since bP_{16} is a subgroup of index 2 in Θ_{15}. However, as we pointed out in Section 2, M_3^{15} bounds a parallelizable manifold (non-equivariantly). Therefore, as a consequence of 6.3, we have the following result.

PROPOSITION 6.5. *The set of homotopy* 15-*spheres which admit biaxial G_2-actions is precisely bP_{16}.*

It is natural to ask to what extent these results can be generalized to higher dimensions. We shall end this section by stating a few open problems.

PROBLEM 1. *Calculate $A_{2n+m}(G)$ and $C_{2n+m}(G)$ for $G = SO(3)$ or G_2 and $m > 1$.*

PROBLEM 2. *Is $C_{2n+m}(G) \cong A_{2n+m}(G) \oplus C_{2n+m}(O(n))$?*

PROBLEM 3. *Suppose $\Sigma^{2n+m} \in C_{2n+m}(G)$ and $\varphi : B(\Sigma) \to B(\Lambda P^2)$ is the classifying map. Is $\varphi_0 : B_0(\Sigma) \to B_0(\Lambda P^2) = \mathbf{R}P^2$ null-homotopic?*

Solving Problem 3 is certainly the first step for solving Problems 1 and 2. We know that $B_0(\Sigma)$ is a $\mathbf{Z}/2\mathbf{Z}$-homology m-sphere. The question is complicated by the fact that $H_i(B_0(\Sigma); \mathbf{Z})$ may be odd torsion. If $m = 2$, then the fact that φ_0 can be covered by a bundle map $\partial\varphi_1 : \partial B_1(\Sigma) \to \partial B_1(\Lambda P^2)$ implies that φ_0 is null-homotopic. If $m > 2$ and if $H_*(B_0(\Sigma); \mathbf{Z})$ has no odd torsion, then I believe that the fact that $\partial\varphi_1$ can be extended to φ_1 implies that φ_0 is null-homotopic. If φ_0 is null-homotopic, then $\varphi | \partial B(\Sigma)$ is certainly also null-homotopic. Hence, Theorem 4.9 together with an affirmative answer to Problem 3 would imply that every biaxial (Σ^{2n+m}, G) is equivariantly equal to the boundary of a spin manifold.

In regard to Problem 1, it should not be difficult to calculate $A_{2n+m}(G)$ in terms of $\pi_{m+1}(\mathbf{R}P^2)$ and $\pi_{m+2}(\mathbf{R}P^2)$.

Another interesting question is the following.

PROBLEM 4. *Let $S : C_{2n+m}(G) \to \Theta_{2n+m}$ be the homomorphism which sends the G-action to the underlying homotopy sphere. Calculate S. In particular, is it possible for $SO(3)$ or G_2 to act biaxially on a homotopy sphere which doesn't bound a parallelizable manifold?*

There are some further open problems related to the sphere bundles $S_{hj}(\Lambda)$ and the Brieskorn manifolds $\Sigma^{2n+1}(h + j, 2, \ldots, 2)$.

PROBLEM 5. *Fix an integer m. According to Example 4.3 the G-manifolds belonging to $\{S_{hj}(\Lambda) | h + j = m\} \cup \{\Sigma^{2n+1}(m, 2, \ldots, 2)\}$*

all have the same orbit triple $(B, \partial B, B_0)$ *where* B_0 *is a torus link of type* $(m, 2)$ *in* $\partial B = S^3$. *Classify these G-manifolds up to equivariant homeomorphism.*

Tamura [32] has done some work on the corresponding non-equivariant question. In particular, it is known that these manifolds are not all homeomorphic. (If $m = 0$ they have distinct rational Pontriagin classes.) Suppose that (M, G) is a biaxial G-manifold with orbit triple $(B, \partial B, B_0)$ as above. According to [1], page 282, it suffices to classify the set of possible $(M - M_0, G)$'s since any equivariant homeomorphism on the complement of the fixed point set will extend across it. To answer this question we need only compute the set of possible homotopy classes of maps on the 1-stratum $\varphi_1 : B_1 \to \mathbf{R}P^2$. (Here B_1 is the complement of a certain torus link in S^3.) The map on the fundamental group $\pi_1(B_1) \to \mathbf{Z}/2\mathbf{Z}$ must be non-zero on any meridian, but other than this condition, $\varphi_1 : B_1 \to \mathbf{R}P^2$ can be arbitrary. Let $\alpha \in H^2(\mathbf{R}P^2, \mathbf{Z}^{\text{twisted}})$ be a generator. It seems possible that the question can be settled by looking at $\varphi_1^*(\alpha) \in H^2(B_1, \mathbf{Z}^{\text{twisted}})$ and also that this class determines the Pontriagin class of M.

Problem 6. *Is* $\Sigma^{2n+1}(m, 2, \ldots, 2)$ *ever G-diffeomorphic to a sphere bundle* $S_{h,j}(\Lambda)$ *with* $h + j = m$?

Since $\Sigma^{2n+1}(m, 2, \ldots, 2)$ equivariantly bounds a parallelizable manifold of index $(m - 1)$, we can compute its μ-invariant. We have already computed the μ-invariant of $S_{h,j}(\Lambda)$ in Section 2. It is easy to check that these numbers are equal only if the peculiar condition $(h - j)^2 = h + j$ is satisfied.

7. Actions of $U(n)$, $SU(n)$ and $SO(2)$. There is a theory of biaxial $U(n)$- and $SU(n)$-actions which is closely analogous to the theory discussed in Sections 4, 5 and 6. For $U(n)$, $n \geq 2$, and for $SU(n)$, $n \geq 4$, this theory is due to Bredon [6]. In these cases the key point is that the linear model is universal. As one might expect this is false for $SU(3)$. In this case the natural action on the Cayley projective plane is universal. In this section we shall discuss these results and some of their implications for $SU(3)$-actions on homotopy 15-spheres and $SO(2)$-actions on homotopy 7-spheres.

Let H stand for $U(n)$, $n \geq 2$, or $SU(n)$, $n \geq 3$. A smooth H-manifold M^{4n+m} is *biaxial* if it is stably modeled on two times the standard linear action, i.e., on $(\mathbf{C}^n \oplus \mathbf{C}^n, H)$. A biaxial H-manifold M has three types of orbits: as before, we shall index the strata by $\{0, 1, 2\}$. We shall denote the orbit space of M by $A(M)$. The linear orbit space $A(\mathbf{C}^n \oplus \mathbf{C}^n)$ may be identified with $B^{\mathbf{C}}(2)$ the space of complex hermitian, positive semi-definite, 2 by 2 matrices. This space is a cone over a 3-disk. It follows that $A(M^{4n+m})$ is homeomorphic to a $(m + 4)$-manifold with boundary and that the fixed point set $A_0(M)$ is a closed m-manifold embedded in $\partial A(M)$ in codimension three.[5]

The linear action of $U(n)$ on $\mathbf{C}^n \oplus \mathbf{C}^n$ is the restriction of the action of $O(2n)$ on $\mathbf{R}^{2n} \oplus \mathbf{R}^{2n} \cong \mathbf{C}^n \oplus \mathbf{C}^n$. Hence, if $O(2n)$ acts biaxially on M^{4n+m}, then restriction gives a biaxial $U(n)$-action $(M^{4n+m}, U(n))$. The complex numbers are a subalgebra of the Cayley numbers. The subgroup of G_2 which fixes $\mathbf{C}$ is isomorphic to $SU(3)$. The linear action of $SU(3)$ on $\mathbf{C}^{\perp} \subset \mathbf{O}$ is equivalent to $SU(3)$ on $\mathbf{C}^3$. Hence the restriction of a biaxial G_2-action to $SU(3)$ is again biaxial.

Example 7.1. Restriction to the subgroup $SU(3) \subset G_2$ gives a biaxial $SU(3)$-action on the Cayley projective plane. The fixed point set is $\mathbf{C}P^2$. In [9] it is proved that the orbit space $A(\mathbf{O}P^2)$ may be identified with $T^{\mathbf{C}}(3)$, the space of complex hermitian, positive semi-definite, 3 by 3 matrices of trace equal to one. This space is homeomorphic to D^8. The 0-stratum is $\mathbf{C}P^2$ and this stratum is embedded in $S^7 = \partial D^8$ in such a way that its complement (the 1-stratum) is a 3-disk bundle over $\mathbf{C}P^2$.

Example 7.2. Restriction gives a biaxial $SU(3)$-action on the sphere bundle $S_{hj}(\mathbf{O})$. The fixed point set is $S_{hj}(\mathbf{C}) = L^3(h + j, 1)$. Hence if $h + j = \pm 1$ the fixed point set is S^3. In particular, $A_0(M_3^{15}) = S^3$. Also, it is straightforward to check that $A(S_{hj}(\mathbf{O}))$ is a 7-disk.

If $H = U(n)$, $n \geq 2$, or $SU(n)$, $n \geq 4$ we say that it is of *type I*. If $H = SU(3)$ then it is of *type II*. For H of type I the universal trivialized biaxial H-manifold is the linear model $(\mathbf{C}^n \oplus \mathbf{C}^n, H)$, [6], [9]; for H of type II the universal action is the one discussed in Example 7.1, $(\mathbf{O}P^2, SU(3))$, [9].

Suppose now that H acts biaxially on a homotopy sphere Σ^{4n+m} and set $A = A(\Sigma)$. By using Smith Theory and Oliver's Theorem [30] we see that the following conditions (analogous to 5.1) hold:

(7.3) a) A is a compact contractible $(m + 4)$-manifold with boundary.
b) A_0 is an integral homology m-sphere embedded in ∂A.

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[5]This is one important difference from the case of biaxial $O(n)$-actions. Recall that for such actions the fixed point set is embedded in the boundary of the orbit space in codimension two.

The pair $(\partial A, A_0)$ is called the *orbit knot.*

It follows from condition a), that any (Σ^{4n+m}, H) is equivalent to a pullback of the corresponding universal H-manifold.

Suppose that H is of type I and that $\varphi: A \to B^{\mathbf{C}}(2) = A(\mathbf{C} \oplus \mathbf{C})$ is the classifying map for (Σ^{4n+m}, H). We may assume that φ is transverse to a ray emanating from the vertex in $\partial B^{\mathbf{C}}(2)$. The inverse image of this ray will be a "Seifert surface" V^{m+1} for A_0 in ∂A, i.e., V^{m+1} will be a framed submanifold of ∂A with $\partial V^{m+1} = A_0$. Thus, in contrast to Proposition 5.5, for H of type I we see that the orbit knot of (Σ^{4n+m}, H) must satisfy the following additional condition:

(7.4) $\quad A_0$ bounds a framed submanifold V^{m+1} in ∂A.

This result was first proved in [14]. Its significance is that while every knot in codimension two has a Seifert surface, there are knots in codimension three which fail to satisfy (7.4). Bredon observed in [6] that stratified homotopy classes of maps $\varphi: A \to B^{\mathbf{C}}(2)$ are in one-to-one correspondence with framed cobordism classes of Seifert surfaces for A_0. In particular, in order for there to exist a stratified map $\varphi: A \to B^{\mathbf{C}}(2)$, A must satisfy (7.4). Therefore, we have the following result.

THEOREM 7.5 (Bredon [6]). *Suppose H is of type I. For a local orbit space A to be the orbit space of biaxial H-action on a homotopy sphere it is necessary and sufficient that the triple $(A, \partial A, A_0)$ satisfy conditions* (7.3) *and* (7.4). *Moreover, equivalence classes of such (Σ^{4n+m}, H) over A are in one-to-one correspondence with framed cobordism classes of Seifert surfaces for A_0.*

As a consequence of this result and a construction similar to that in the proof of Proposition 4.9, we also have the following theorem of Bredon [6].

THEOREM 7.6 (Bredon). *Suppose that H is of type I and that (Σ^{4n+m}, H) is a biaxial H-action on a homotopy sphere. Let $V^{m+1} \subset \partial A(\Sigma)$ be the Seifert surface for $A_0(\Sigma)$ associated to a classifying map $\varphi: A(\Sigma) \to B^{\mathbf{C}}(2)$. Then (Σ^{4n+m}, H) equivariantly bounds a biaxial H-action (V^{4n+m+1}, H) on a parallelizable manifold V^{4n+m+1}. Moreover, the index of V^{4n+m+1} equals that of V^{m+1}. (In fact, their intersection forms are equal.)*

For $H = SU(3)$ the corresponding result, analogous to Proposition 4.9, is the following.

PROPOSITION 7.7. *Suppose that $H = SU(3)$ and that (Σ^{12+m}, H) is a biaxial H-action on a homotopy sphere. Let $\varphi: A(\Sigma) \to A(\mathbf{O}P^2)$ be the classifying map. If $\varphi | \partial A(\Sigma): \partial A(\Sigma) \to S^7$ is null-homotopic, then Σ^{12+m} equivariantly bounds a biaxial action on a spin manifold (W^{12+m}, H).*

As in Section 5, define $C_{4n+m}(H)$ to be the abelian semi-group of oriented equivalence classes of biaxial H-actions on homotopy $(4n + m)$-spheres. Also, let $D_m^{\mathbf{C}}$ be the abelian semi-group of orbit triples $(A, \partial A, A_0)$ which satisfy conditions (7.3). There is a semi-group homomorphism

$$p_H: C_{4n+m}(H) \to D_m^{\mathbf{C}}$$

which associates to each action its orbit triple.

Let $\Theta^{m+3,3}$ be the set of diffeomorphism classes of pairs (S^{m+3}, Σ^m) where Σ^m is a homotopy m-sphere. If $m = 3$ we shall require that $\Sigma^3 = S^3$. Thus, $\Theta^{6,3} = \Sigma^{6,3}$, the set of knotted 3-spheres in S^6. It is known that $\Theta^{m+3,m}$ is an abelian group under connected sum. Let $\Theta_0^{m+3,m}$ be the subgroup consisting of those knots which have Seifert surfaces (i.e., satisfy 7.4). Levine [23] and Haefliger [16], [17] have gone a long way towards calculating the groups $\Theta^{m+3,m}$. In particular, the subgroup $\Theta_0^{m+3,m}$ is cyclic of order 1, 2 or ∞. Also, $\Theta^{6,3} = \Sigma^{6,3}$ is infinite cyclic and $\Theta_0^{6,3} = \Sigma_0^{6,3}$ is a subgroup of index two, [17].

We have that $\Theta^{m+3,m} \subset D_m^{\mathbf{C}}$. Define $\tilde{C}_{4n+m}(H)$ to be the pre-image of $\Theta^{m+3,m}$ under p_H. Thus $\tilde{C}_{4n+m}(H)$ consists of those (Σ^{4n+m}, H) with $A_0(\Sigma)$ simply connected and $\partial A(\Sigma) = S^{m+3}$. The proof of the following result is similar to that of Proposition 5.6.

PROPOSITION 7.8. *$\tilde{C}_{4n+m}(H)$ is an abelian group.*

If H is of type I, then by 7.5 the image of $\tilde{C}_{4n+m}(H)$ under p_H is $\Theta_0^{m+3,m}$. In fact it is proved in [6] that for $m > 1$ and for H of type I, $p_H: \tilde{C}_{4n+m}(H) \to \Theta_0^{m+3,m}$ is an isomorphism.

Biaxial Actions of $SU(3)$ on Homotopy 15-Spheres. We now specialize to the case $H = SU(3), m = 3$. We have the commutative diagram:

$$\begin{array}{ccc} \tilde{C}_{15}(SU(3)) & \to & \Sigma^{6,3} \cong \mathbf{Z} \\ \cup & & \cup \\ \tilde{C}_{15}(U(3)) & \xrightarrow{\cong} & \Sigma_0^{6,3} \cong 2\mathbf{Z}. \end{array}$$

PROPOSITION 7.9. *The group $\tilde{C}_{15}(SU(3))$ is isomorphic either to $\mathbf{Z}$ or to $\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$.*

Proof. We will show that the kernel of $p_H : \tilde{C}_{15}(SU(3)) \to \Sigma^{6,3} \cong \mathbf{Z}$ is either 0 or $\mathbf{Z}/2\mathbf{Z}$. The argument is similar to the proof of 6.1. Suppose that $(A, \partial A, A_0) = (D^7, S^6, S^3)$ is an orbit triple. We will show that there exists at most two stratified homotopy classes of maps $A \to A(\mathbf{O}P^2)$. Suppose that $\varphi, \varphi' : A \to A(\mathbf{O}P^2)$ are two such maps. First observe that any map $\varphi_0 : S^3 \to \mathbf{C}P^2$ is null-homotopic. Hence, we may assume that the bundle maps $\partial\varphi_1, \partial\varphi_1' : \partial A_1 \to \partial A_1(\mathbf{O}P^2)$ differ by a framing of the 2-sphere bundle $\partial A_1 \cong S^3 \times S^2$; i.e., by an element of $\pi_3(SO(3))$. We have that $A_1(\mathbf{O}P^2)$ is a 3-disk bundle over $\mathbf{C}P^2$. Let $q : A_1(\mathbf{O}P^2) \to \mathbf{C}P^2$ be the projection. Since the maps $q \circ \partial\varphi_1$ and $q \circ \partial\varphi_1'$ must both extend to $A_1 = D^4 \times S^2$, it follows that the element of $\pi_3(SO(3))$ is zero. Thus we may assume that $\partial\varphi_1 = \partial\varphi_1'$. We must next try to construct a homotopy from $q \circ \varphi_1$ to $q \circ \varphi_1' : A_1 \to \mathbf{C}P^2$ where on the boundary the homotopy is allowed to vary only through framings of the 2-sphere bundle $\partial A_1 \cong S^3 \times S^2$. Relative to ∂A_1 the only obstruction to finding such a homotopy lies in $H^6(A_1 \partial A_1; \pi_6(\mathbf{C}P^2)) = H^6(D^4 \times S^2, S^3 \times S^2; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$. Hence, the obstruction to finding the original homotopy is at most $\mathbf{Z}/2\mathbf{Z}$. Once the homotopy is found on the 1-stratum there is no problem to extending it across the top stratum.[6]

Let $F_{15}(SU(3))$ denote the quotient of $\tilde{C}_{15}(SU(3))$ by its (possible) 2-torsion. By Proposition 7.7, every $\Sigma^{15} \in \tilde{C}_{15}(SU(3))$ bounds a spin manifold. Hence, we can use the μ-invariant to define a homomorphism $\hat{\mu} : \tilde{C}_{15}(SU(3)) \to \mathbf{Z}$, as in Section 6. This induces a homomorphism $\hat{\mu} : F_{15}(SU(3)) \to \mathbf{Z}$ on the quotient group. Analogous to Proposition 6.2 we have the following result.

PROPOSITION 7.10. *The infinite cyclic group $F_{15}(SU(3))$ is generated by $(M_3^{15}, SU(3))$ and $\hat{\mu}$ is an isomorphism.*

Proof. The homomorphism $\hat{\mu}$ takes $(M_3^{15}, SU(3))$ to 1.

Since $F_{15}(SU(3)) \to \Sigma^{6,3}$ and $\tilde{C}_{15}(U(3)) \to \Sigma_0^{6,3}$ are both injective we have that the natural map $\tilde{C}_{15}(U(3)) \to F_{15}(SU(3))$ is also an injection.

PROPOSITION 7.11. *$\tilde{C}_{15}(U(3))$ is a subgroup of index two in $F_{15}(SU(3))$.*

Proof. Suppose that $\Sigma^{15} \in \tilde{C}_{15}(U(3))$. Let V^4 be a Seifert surface for $(\partial A(\Sigma), A_0(\Sigma)) = (S^6, S^3)$. By Rohlin's Theorem, the index of V^4, $I(V^4)$,

[6]The proof actually shows that the kernel of p_H is the group $\pi_4(\mathfrak{M}, SO(3))$, where $\mathfrak{M}$ is the space of those maps $S^2 \to \mathbf{C}P^2$ which induce an isomorphism on $H_2(\ ;\mathbf{Z})$. The computation of $\tilde{C}_{15}(SU(3))$ reduces to determining whether $\pi_4(\mathfrak{M}, SO(3))$ is 0 or $\mathbf{Z}/2\mathbf{Z}$.

is divisible by 16. By 7.6, Σ^{15} equivariantly bounds a parallelizable V^{16} with index $I(V^4)$. It follows that $\hat{\mu}(\Sigma^{15}) = \frac{1}{8} I(V^4) \equiv 0 \pmod 2$.

COROLLARY 7.12. *The biaxial $SU(3)$-action on M_3^{15} does not extend to a biaxial $U(3)$-action.*

COROLLARY 7.13. *The orbit knot of $(M_3^{15}, SU(3))$ is a generator for $\Sigma^{6,3}$, the group of knotted 3-spheres in S^6.*

Proof. Consider the diagram

$$\begin{array}{ccc} F_{15}(SU(3)) & \rightarrow & \Sigma^{6,3} \\ \uparrow & & \uparrow \\ \tilde{C}_{15}(U(3)) & \xrightarrow{\cong} & \Sigma_0^{6,3}. \end{array}$$

All four groups are isomorphic to $\mathbf{Z}$. The vertical maps are both multiplication by 2. Hence, the top map must be an isomorphism.

Biaxial $SO(2)$-Actions on Homotopy 7-Spheres. A biaxial $SO(2)$-action is just a semi-free circle action with fixed point set of codimension 4. Such actions on homotopy spheres have been studied by Montgomery-Yang [27], [28], [29] and Levine [24]. If $(\Sigma^{4+m}, SO(2))$ is a biaxial $SO(2)$-action on a homotopy sphere, then its fixed point set Σ_0 is an integral homology m-sphere and its orbit space Σ^* is a homotopy $(m+3)$-sphere, [27]. Moreover the set of oriented equivariant diffeomorphism classes of such actions is in one-to-one correspondence with the set of oriented diffeomorphism classes of pairs (Σ^*, Σ_0) where Σ^* is an oriented homotopy $(m+3)$-sphere and Σ_0 is an integral homology m-sphere, [24]. Thus, every such pair occurs as an orbit knot.

Regard $SO(3)$ as the automorphism of the quaternions. The subgroup which fixes $\mathbf{C}$ is $SO(2)$. Thus, if $(M, SO(3))$ is any biaxial $SO(3)$-action, the restriction of the action to $SO(2)$ is again biaxial.

The subgroup of G_2 which fixes $\mathbf{H}$ is $SU(2)$. Denote the normalizer of $SU(2)$ in G_2 by $N_{G_2}(SU(2))$. Then $N_{G_2}(SU(2))/SU(2) \cong SO(3)$. It is easy to see that for any biaxial G_2-manifold M, the orbit space M/G_2 coincides with the orbit space of $M^{SU(2)}$ under $SO(3)$. We also have that

$$N_{SU(3)}(SU(2))/SU(2) \cong SO(2).$$

If $(M, SU(3))$ is a biaxial $SU(3)$-manifold then $M^{SU(2)}$ intersects all the singular orbits. It follows easily that $M^{SU(2)}/SO(2) \cong \partial A(M)$.

Example 7.14. Consider the action of $SO(2)$ on $M_3^7 = S_{2,-1}(\mathbf{H})$. We have that $(M_3^7, SO(2)) = ((M_3^{15})^{SU(2)}, SO(2))$. By the above remarks the orbit knot of $(M_3^7, SO(2))$ coincides with that $(M_3^{15}, SU(3))$. By 7.13 this orbit knot is a generator of $\Sigma^{6,3}$.

Let $\tilde{C}_7(SO(2))$ be the abelian group of oriented equivalence classes of biaxial $SO(2)$-actions on homotopy 7-spheres with orbit knot lying in $\Sigma^{6,3}$. This group was first considered by Montgomery-Yang [28]. As we have seen, the natural map $p_{SO(2)}: \tilde{C}_7(SO(2)) \to \Sigma^{6,3}$ is an isomorphism. As an immediate consequence of 7.13 and 7.14 we have the following result.

PROPOSITION 7.15. *The infinite cyclic group $\tilde{C}_7(SO(2))$ is generated by $(M_3^7, SO(2))$.*

COROLLARY 7.16. *Every semi-free circle action on a homotopy 7-sphere with fixed point set S^3 is the restriction of a biaxial $SO(3)$-action in $A_7(SO(3))$.*

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REFERENCES

[1] Bredon, G., *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972.
[2] ______, Transformation groups on spheres with two types of orbits, *Topology*, **3** (1965), 103-113.
[3] ______, Examples of differentiable group actions, *Topology*, **3** (1965), 115-122.
[4] ______, Exotic actions on spheres, "Proc. Conf. Transformation Groups," New Orleans 1967, 47-76. Springer-Verlag, Berlin and New York, 1968.
[5] ______, Regular $O(n)$-manifolds, suspension of knots, and knot periodicity, *Bull. Amer. Math. Soc.*, **78** (1973), 87-91.
[6] ______, Biaxial actions, mimeographed notes, Rutgers University, 1973.
[7] Davis, M., *Multiaxial Actions on Manifolds*, Springer Lecture Notes, no. 643. Springer-Verlag, Berlin and New York, 1978.
[8] ______, Smooth G-manifolds as collections of fiber bundles, *Pac. J. of Math.*, **77** no. 2 (1978), 315-363.
[9] ______, Universal G-manifolds, *Amer. J. Math.*, **103** (1981), 103-141.
[10] ______ and W. C. Hsiang, Concordance classes of regular $U(n)$ and $Sp(n)$ actions on homotopy spheres, *Ann. of Math.*, **105** (1977), 325-341.
[11] ______, W. C. Hsiang and J. W. Morgan, Concordance classes of regular $O(n)$ actions on homotopy spheres. *Acta Math.*, **144** (1980), 153-221.
[12] Eells, J. and N. H. Kuiper, An invariant for certain smooth manifolds, *Annali Mat.*, **60** (1962), 93-110.

[13] Erle, D., Die quadratische Form eines Knotens und ein Satz uber Knotenmannegfaltigkeiten, *J. Reine Agnew. Math.*, **236** (1969), 174–218.

[14] ______ and W. C. Hsiang, On certain unitary and symplectic actions with three orbit types, *Amer. J. Math.*, **94** (1972), 289–308.

[15] Gromoll, D. and W. Meyer, An exotic sphere with nonnegative sectional curvature, *Ann. of Math.*, **100** (1974), 447–490.

[16] Haefliger, A., Differentiable embeddings of S^n in S^{n+q} for $q > 2$, *Ann. of Math.*, **83** (1966), 402–436.

[17] ______, Knotted $(4k - 1)$-spheres in $6k$-space, *Ann. of Math.*, **75** (1962), 452–466.

[18] Hirzebruch, F., Singularities and exotic spheres, Seminar Bourbaki 19 (1966/67).

[19] Hsiang, W. C. and W. Y. Hsiang, Differentiable actions of compact connected classical groups: I, *Amer. J. Math.*, **89** (1967), 705–786.

[20] Jänich, K., Differenzierbare Mannigfaltigkeiten mit Rand als Orbitraume differenzierbarer G-Mannigfaltigkeiten ohne Rand, *Topology*, **5** (1966), 301–329.

[21] ______, On the classification of $O(n)$-manifolds, *Math. Ann.*, **176** (1968), 53–76.

[22] Kervaire, M. and J. Milnor, Groups of homotopy spheres I, *Ann. of Math.*, **77** (1963), 504–537.

[23] Levine, J., A classification of differentiable knots, *Ann. of Math.*, **82** (1965), 15–50.

[24] ______, Semifree circle actions on spheres, *Inv. Math.*, **22** (1973), 161–186.

[25] Milnor, J., Singular points on complex hypersurfaces, *Ann. of Math. Studies*, no. 61, Princeton Univ. Press, Princeton, 1969.

[26] ______, On manifolds homeomorphic to the 7-sphere, *Ann. of Math.*, **64** (1956), 399–405.

[27] Montgomery, D. and C. T. Yang, Differentiable transformation groups on homotopy spheres. *Michigan Math. J.*, **14** (1967), 33–46.

[28] ______, Differentiable actions on homotopy seven spheres, *Trans. Am. Math. Soc.*, **122** (1966), 480–496.

[29] ______, Differentiable actions on homotopy seven spheres, II. *Proc. Conf. Transformation Groups*, New Orleans 1967, 125–133. Springer-Verlag, Berlin and New York, 1968.

[30] Oliver, R., A proof of the Conner Conjecture, *Ann. of Math.*, **103** (1976), 637–644.

[31] Shimada, N., Differentiable structures on the 15-sphere and Pontriagin classes of certain manifolds, *Nagoya Math. J.*, **12** (1957), 37–69.

[32] Tamura, I., Homeomorphy classification of total spaces of sphere bundles over spheres, *J. Math. Soc. Japan*, **10** (1958), 29–43.

[33] Wall, C. T. C., Classification of $(n - 1)$-connected $2n$-manifolds, *Ann. of Math.*, **75** (1962), 163–189.

References

[28] **Differentiable Actions on Homotopy Seven Spheres**
D. Montgomery; C. T. Yang
Transactions of the American Mathematical Society, Vol. 122, No. 2. (Apr., 1966), pp. 480-498.
Stable URL:
http://links.jstor.org/sici?sici=0002-9947%28196604%29122%3A2%3C480%3ADAOHSS%3E2.0.CO%3B2-T