ASPHERICAL MANIFOLDS WITHOUT SMOOTH OR PL STRUCTURE

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ABSTRACT. One construct closed aspherical PL-manifolds which are not homotopy equivalent to closed smooth manifolds. Examples of closed aspherical TOP-manifolds which are not homeomorphic to closed PL-manifolds are also given.

A space X is **aspherical** if it is homotopy equivalent to a CW-complex and if its universal covering is contractible (in other words : X is homotopy equivalent to the Eilenberg-McLane space $K(\pi_1(X),1)$). Closed aspherical manifolds form an interesting class of aspherical spaces. Classical examples come from Lie group theory and differential geometry and are smooth manifolds. A new kind of construction of closed aspherical manifolds appeared in [D1], giving rise to closed aspherical smooth manifolds with universal coverings not homeomorphic to $\mathbf{R}^n$. Using these techniques, we prove in this note the following results :

Theorem 1 For each $n \geqslant 13$, there exists an aspherical closed PL-manifold M of dimension n which does not have the homotopy type of a closed smooth manifold.

We prove Theorem 1 by showing that the Spivak bundle of M admits no linear reduction.

Theorem 2 For each $n \geqslant 8$, there exists an aspherical closed topological manifold M of dimension n such that M is not homeomorphic to a closed PL-manifold.

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We do not know whether M has the homotopy type of a closed PL-manifold (See Remark 4).

Recall that a group G is a **Poincaré duality group** if there exists a ZG-module structure Z on the abelian group of integers and a class $e \in H_n(G;Z)$ so that the cap product homomorphism $-\cap e : H^i(G;B) \longrightarrow H_{n-i}(G;B \otimes_Z Z)$ is an isomorphism for all G-modules B (definition of Bieri and Eckmann [BE]). The fundamental groups of the manifolds M of our theorem are the first examples of Poincaré duality groups G such that K(G,1) is not homotopy equivalent to a closed smooth manifold. Recall that a strong version of the Novikov conjecture says that for a Poincaré duality group G, the space K(G,1) should be homotopy equivalent to a closed topological manifold (see Remark 2).

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The proof of Theorem 1 is given is Section 2, that of Theorem 2 in Section 3, while Section 1 is devoted to recalling some facts about the construction of [D1] (and also [D2]).

1. CONSTRUCTING ASPHERICAL MANIFOLDS WITHOUT CAT-STRUCTURES USING REFLECTION GROUPS

This section is a development of [D1, Remark 15.9]. The notations and the terminology are from [D1], except that, following W. Thurston's terminology, we use the word "mirror" instead of "panel".

Let Q be (finite) CW-complex, P a subcomplex of Q and L a triangulation of P. Replacing L by its barycentric subdivision if necesary, we can assume that L is "determined by its 1-skeleton". (This means that for any set T of vertices in L, if any two distinct elements of T bound an edge, then T spans a simplex of L.) Under the assumption that L is determined by its 1-skeleton, there is a Coxeter system (Γ,V) with L as associated complex (see [D1, Section 11]). In particular, V is the set of vertices of L. For instance, one could take Γ generated by V with the relations

$$v^2 = 1 \quad \text{for all } v \in V$$
$$(vw)^2 = 1 \text{ if } v \text{ and } w \text{ bound an edge in } L.$$

The "canonical mirror structure on L" is then defined as follows. For each $v \in V$, let P_v denote the closed star of v in the barycentric subdivision L' of L; each P_v is called a **mirror**. Form the complex $X = Q \times \Gamma/\sim$, where $(x,g)\sim(y,h)$ if $x = y$ and $h^{-1}g$ is in the subgroup of Γ generated by those v such that $x \in P_v$. We recall the following facts from [D1] :

(1.1.a) $X = \bigcup X_i$, with $X_1 = Q$ and X_i is the union of X_{i-1} with with a translated of Q (we write $X_i = X_{i-1} \cup Q$ for simplicity). The intersection $X_{i-1} \cap Q$ is a union of mirrors and can be identified with the closed star in L' of a certain simplex in L [D1, Section 8]. In particular, $X_{i-1} \cap Q$ is contractible.

(1.1.b) The group Γ operates properly on X as a reflection group with finite isotropy groups, and $X/\Gamma = Q$.

(1.1.c) As a Coxeter group, Γ contains a torsion-free subgroup Γ' of finite index. Then $X \to X/\Gamma' = M$ is a covering projection.

This has the following immediate consequences :

(1.2.a) If P is a polyhedral homology m-manifold, then each $X_{i-1} \cap Q$ is a compact contractible polyhedral homology m-manifold with boundary. (Recall that a polyhedral homology m-manifold is a m-dimensional simplicial complex such that the link of any k-simplex has the homology of S^{m-k-1}).

(1.2.b) If P is a topological manifold, then each $X_{i-1} \cap Q$ is a compact contractible manifold with boundary.

(1.2.c) If P is a PL-manifold and L is a PL-triangulation of P, then each $X_{i-1} \cap Q$ is a PL m-cell.

From now on, we suppose that (Q,P) is a Poincaré pair of formal dimension n and that the simplicial complex L (=P) is a polyhedral homology manifold. In this special case, (1.1) and (1.2) above have the following consequences :

(1.3.a) Each X_i is a Poincaré space, X is an (infinite) Poincaré space and M is a closed Poincaré space.

(1.3.b) If Q is a topological n-manifold with boundary $\partial Q = P$, then each X_i is a manifold with boundary, X is a manifold and M is a closed topological manifold.

(1.3.c) If Q is a triangulated manifold and L is a triangulation of ∂Q, then the barycentric subdivision of the triangulation of Q extends to a triangulation of M. In other words, if Q has a TRI-structure in the sense of [GS], then the closed manifold M inherits a TRI-structure.

(1.3.d) Similarly, if Q is a PL-manifold and L is a PL-triangulation of ∂Q, then X is a PL-manifold, Γ acts through PL-automorphisms, and hence M is a PL-manifold.

(1.3.e) If Q is a smooth manifold and L is a smooth triangulation of ∂Q, then Q can be given the structure of a smooth orbifold (see [D1, Section 17]), and hence M is a smooth manifold.

Essentially, (1.3) says that if Q is a CAT-manifold, then so is M (where CAT = DIFF, PL, TRI or TOP). Moreover, M contains Q as a codimension zero Poincaré space or submanifold.

Finally, we deduce the following facts :

(1.4) If Q is aspherical, then M is aspherical (since $X_i \cap Q$ is contractible, X is aspherical and $X \to M$ is a covering projection).

(1.5) If the map $\nu_Q : Q \longrightarrow BG$ classifying the Spivak bundle of Q does not lift through BCAT (CAT = DIFF, PL, or TRI), then neither does $\nu_M : M \longrightarrow BG$ (since $\nu_M|Q = \nu_Q$) and then M does not have the homotopy type of a CAT-manifold.

(1.6) Suppose that Q is a topological manifold. If the map $\tau_Q : Q \longrightarrow BTOP$ classifying the stable tangent micro-bundle of Q does not lift through BCAT (CAT = DIFF, PL, or TRI), then neither does $\nu_M : M \longrightarrow BTOP$ and M is not homeomorphic to a CAT-manifold.

<u>**Remark**</u> : The argument of [D2, Proposition 1.4] shows that there is a Γ-equivariant embedding $f : X \hookrightarrow Q \times \mathbf{R}^N$ with <u>trivial</u> normal bundle, where Γ acts on $\mathbf{R}^N$ as a linear reflection group. Moreover, the composition of f with the projection on the first

factor is the orbit map $\pi : X \longrightarrow Q$. It follows that $\nu_M : M \longrightarrow BCAT$ factors as $M \xrightarrow{\pi} Q \xrightarrow{\nu_Q} BCAT$. This gives a sort of a converse statement to (1.4)-(1.6).

2. PROOF OF THEOREM 1

Observe that, if M is a closed aspherical PL-manifold with Spivak bundle $\nu_M : M \longrightarrow BG$ admitting no linear reduction, so is $M \times S^1$. Therefore, in order to prove Theorem 1, it is enough, using (1.5) and (1.6), to construct a compact aspherical PL-manifold Q of dimension 13 such that $\nu_Q : Q \longrightarrow BG$ has no linear reduction.

Our technique to construct such manifolds Q is the following. Let a be an element of $\pi_k(BPL)$ so that its image α in $\pi_k(BG)$ is not in the image of the natural homomorphism $\pi_k(BO) \longrightarrow \pi_k(BG)$. One has $BPL = B\widetilde{PL} = \lim(B\widetilde{PL}_i)$, where $B\widetilde{PL}_i$ is the classyfing space for PL-block bundles of rank i [RS1]. Let $a_r : S^k \longrightarrow B\widetilde{PL}_r$ represent a. Let T^k be the torus of dimension k. Take a degree one map $T^k \longrightarrow S^k$ and compose it with a_r to get $b_r : T^k \longrightarrow B\widetilde{PL}_r$, or with α to get $\beta : T^k \longrightarrow BG$. By the classification of "abstract regular PL-neighbourhoods" over a PL-manifold [RS1, Corollary 4.7], there is a compact PL-manifold Q of dimension k + r containing T^k as a codimension r-submanifold such that :

1) Q collapses onto T^k. Therefore, Q is aspherical.

2) The map b_r classifies the normal block-bundle of T^k into Q.

As T^k is parallelizable, it follows from 2) that the composition $Q \longrightarrow T^k$ with β classifies the inverse of the Spivak bundle ν_Q of Q. Suppose that ν_Q admits a lifting through BO. The spaces BO and BG are known to be infinite loop spaces, so one can write $BO = \Omega(\Omega^{-1}BO)$ and $BG = \Omega(\Omega^{-1}BG)$. If ν_Q lifts through BO, the adjoint map $ad(\nu_Q) : \Sigma T^k \longrightarrow \Omega^{-1}BG$ would lift through $\Omega^{-1}BO$. But ΣT^k is homotopy equivalent to $S^{k+1} \vee A$, where A is a wedge of spheres of dimension k, and $ad(\nu_Q)|S^{k+1}$ is $ad(-\alpha)$. This contradicts the fact that α does not lift through BO.

We now give an example of an element a with the above properties. The group $\pi_9(BG) = \pi_8(G) = \pi_8^s$ is isomorphic to $Z_2 \oplus Z_2$. Observe that an element of $\pi_i(BG)$ lifts to $\pi_i(BG_{k+1})$ if and only if the corrersponding element of π_{i-1}^s lifts to $\pi_{i+k-1}(S^k)$. Therefore, the generators of $\pi_9(BG)$ are $\bar{\nu}$ coming from $\pi_9(BG_7)$ and ε coming from $\pi_9(BG_4)$ [To, Theorem 7.1]. The homomorphism $Z_2 = \pi_9(BO) \longrightarrow \pi_9(BG)$ can be identified with the J-homomorphism $J : \pi_8(SO) \longrightarrow \pi_8^s$. The group $\pi_7(SO)$ is infinite cyclic, generated by w, and $J(w) = \sigma$, where σ is represented by the Hopf map $S^{15} \longrightarrow S^8$. Let η be the non-zero element of π_1^s. One has $J(w \circ \eta) = J(w) \circ \eta = \sigma \circ \eta = \eta \circ \sigma = \bar{\nu} + \varepsilon$ (for the last equality, see [To, Theorem 14.1]). Take $\alpha = \varepsilon$. The homomorphism $\pi_9(BPL) \longrightarrow \pi_9(BG)$ is onto since $\pi_8(G/PL) = Z$. Therefore, there exists an element $a \in \pi_9(B\widetilde{PL})$ having image α. The following diagram

$$\begin{array}{ccc} B\widetilde{PL}_r & \longrightarrow & B\widetilde{PL} \\ \downarrow & & \downarrow \\ BG_r & \longrightarrow & BG \end{array}$$

is a pull-back diagram for $r \geqslant 3$ [RS2, Theorem 1.10]. Therefore, a is the image of an element $a_4 \in \pi_9(B\widetilde{PL}_4)$. Thus, the above construction of Q can be performed with dimQ = 13, which proves Theorem 1.

3. PROOF OF THEOREM 2

If M is a closed aspherical TRI-manifold so that the topological stable tangent microbundle $\tau_M : M \longrightarrow BTOP$ does not lift through BPL, then $M \times S^1$ has the same property. Therefore, using (1.6), it is enough to construct a compact aspherical TRI-manifold Q of dimension 7 so that τ_Q does not lift through BPL.

Consider the diagram

$$\begin{array}{ccccc} \pi_4(BTRI) & \longrightarrow & \pi_3(TRI/PL) & \longrightarrow & \pi_3(PL) = 0 \\ \downarrow & & \downarrow & & \\ \pi_4(BTOP) & \longrightarrow & \pi_3(TOP/PL) & & \end{array}$$

The homomorphism $\Omega_H^3 \cong \pi_3(TRI/PL) \longrightarrow \pi_3(TOP/PL) = \mathbb{Z}_2$ can be identified with the Rohlin invariant [GS, Section 6] and is therefore surjective. On the other hand, the homomorphism $\pi_4(BPL) \longrightarrow \pi_4(BG)$ is also surjective, since $\pi_3(G/PL) = 0$. Therefore, there exists a map $\bar{a} : S^4 \longrightarrow BTRI$ inducing a non-zero class in $\pi_3(TOP/PL)$ and the zero class in $\pi_4(BG)$. Denote by a the composition of $\bar{a}$ with the map $BTRI \longrightarrow BTOP$. The diagram

$$\begin{array}{ccc} BTOP_r & \longrightarrow & BTOP \\ \downarrow & & \downarrow \\ BG_r & \longrightarrow & BG \end{array}$$

is a pull-back diagram for $r \geq 3$ [RS3, Corollary 2.5]. Therefore, one can find $a_3 : S^4 \longrightarrow BTOP_3$ giving a when composed with $BTOP_3 \longrightarrow BTOP$. Take a degree one map $T^4 \longrightarrow S^4$ and compose it with a_3 to get $b_3 : T^4 \longrightarrow BTOP_3$. Using the classification of topological abstract regular neighbourhoods [RS3, Theorem 3.2], one shows, as in the proof of Theorem 1, that there is a compact topological manifold Q of dimension 7 containing T^4 as a codimension 3 submanifold and satisfying :

1) The inclusion $T^4 \subset Q$ is a homotopy equivalence.

2) $b : Q \simeq T^4 \longrightarrow BTOP$ is homotopic to τ_Q.

As in the proof of Theorem 1, one shows that τ_Q admits no lifting through BPL. But, as τ_Q lifts through BTRI, Q admits a TRI-structure [GS, Theorems 1 and 1.5]. We have thus constructed a manifold Q with the required properties.

REMARKS :

1) We do not know whether 13 and 7 are the smallest dimensions for which Theorems 1 and 2 are respectively true.

2) It is tempting to use the above method to construct an aspherical Poincaré complex which is not homotopy equivalent to a closed topological manifold. This would contradict a (folklore) strong version of the Novikov conjecture. The problem would be to find a fundamental chamber Q which is a Poincaré complex, so that the Spivak bundle ν_Q admits no TOP-reduction, but with $P = \partial Q$ homotopy equivalent to a closed

polyedral homology manifold.

3) Other examples of aspherical manifolds for Theorems 1 and 2 are obtainable as follows : in the proofs, replace the degree one map $T^k \longrightarrow S^k$ by a map $f : K \longrightarrow S^k$ inducing an isomorphism on integral homology, where K is a finite aspherical polyhedron of dimension k (K and f exist by [Ma]). The manifold Q will then be a thickening of K with $\tau_Q = a \circ f$, which exists in the stable range.

4) By obstruction theory, if K is a complex of dimension 4, any map $K \longrightarrow BG$ which lifts through BTOP admits a lifting through BPL. Therefore, it is not possible to assert that the manifolds M of Theorem 2 are not homotopy equivalent to closed PL-manifolds. But if a homotopy equivalence $f : M' \longrightarrow M$ existed with M' a closed PL-manifold, then f would yeld a homotopy equivalence between aspherical closed manifolds which is not homotopic to a homeomorphism. This would be a negative answer to a question of A. Borel.

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