EXAMPLES OF ACTIONS ON MANIFOLDS ALMOST DIFFEOMORPHIC TO $V_{n+1,2}$

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In [7] Hirzebruch discusses a relationship between transformation groups, knot theory, and the study of Brieskorn varieties. This interplay originally represented the convergence of the work of K. Jänich [9] and of W. C. and W. Y. Hsiang [8] on classifying the type of O(n)-manifolds called "knot manifolds" with the work of Brieskorn, Milnor, and others [1], [7], [12] on the behaviour of certain complex varieties near isolated singularities. Hirzebruch pointed out that the Brieskorn spheres provide examples of knot manifolds. These examples have since been used in work on smooth actions of other compact Lie groups, notably S^1 and $\mathbb{Z}_p$, on homotopy spheres (e.g. [2]). In this paper, we exhibit analogous examples which differ from Hirzebruch's in three ways. First of all, rather than being concerned with actions on homotopy spheres, in our examples the ambient manifold is almost diffeomorphic to $V_{n+1,2}$, the Stiefel manifold of 2-frames in $\mathbb{R}^{n+1}$. Secondly, it will be necessary to use manifolds defined by weighted homogeneous polynomials (see [12; p. 75] for definition of these) rather than the Brieskorn manifolds. Finally, in our examples the action will be associated with a link in S^3 rather than a knot.

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Throughout this paper "manifold" will mean "smooth, compact, orientable manifold" (with or without boundary), all group actions will be smooth, and " $\cong$ " will mean diffeomorphic. Also, "Σ" will be used to denote a homotopy sphere and bP_{2m} will denote the subgroup of homotopy spheres which bound parallelizable manifolds.

1. The Examples, $K_m^{p,q}$

Consider the weighted homogeneous polynomial $g: \mathbb{C}^{m+1} \to \mathbb{C}$ defined by

$$g(z) = (z_1)^p + (z_1)(z_2)^q + (z_3)^2 + \cdots + (z_{m+1})^2$$

where p and q are odd and $\gcd(p-1,q) = 1$. Let

$$K_m^{p,q} = g^{-1}(0) \cap S^{2m+1},$$

where $S^{2m+1} \subset \mathbb{C}^{m+1}$ is the unit sphere. We will be interested in the examples $K_m^{p,q}$ when $m = 2n$, although similar results also hold if m is odd. We will show that $K_{2n}^{p,q} = V_{2n+1,2} \# \Sigma$, for some $\Sigma \in bP_{4n}$. Then, examining the natural action of $O(2n-1)$ on $K_{2n}^{p,q}$, we will show that $K_{2n}^{p,q}$ is an example of what we shall call a "prime link manifold."

First, we must recall some facts proved in [12]. Let $f: \mathbb{C}^{m+1} \to \mathbb{C}$ be a polynomial such that $f(0) = 0$, the origin is a critical point, and $f^{-1}(0) \cap (D^{2m+2} - 0)$ contains no critical points. (g satisfies these conditions.) If $V_m^f = f^{-1}(0) \cap S^{2m+1}$, then

1) $\varphi : S^{2m+1} - V_m^f \to S^1$ defined by $\varphi(z) = f(z)/\|f(z)\|$, is the projection map of a smooth fibre bundle.

2) V_m^f is an (m-2)-connected, compact, (2m-1)-dimensional Π - manifold.

3) V_m^f bounds the (m-1)-connected, parallelizable 2m - manifold $\bar{F}_\theta$, where $\bar{F}_\theta$ is the closure of a typical fibre. (If $f = g$, we denote this fibre by $F_{2n}^{p,q}$.)

Associated with any such fibre bundle over the circle is a characteristic polynomial $\Delta(t)$. (See [12; p. 67] for the definition of $\Delta(t)$.) A trivial modification of the proof of Theorem 8.5 in [12] shows,

(1.1) <u>Lemma</u>: V_{2n}^f is a homology $V_{2n+1,2}$ (that is $H_{2n-1}(V_{2n}^f) = \mathbb{Z}_2$) if and only if $\Delta(1) = \pm 2$.

<u>Remark</u>: If

$$f(z) = (z_1)^{a_1} + \ldots\ldots + (z_{m+1})^{a_{m+1}}$$

V_m^f is called a <u>Brieskorn manifold</u> and often denoted by $V_m(a_1, \ldots. a_{m+1})$. It is not difficult to show that if the characteristic polynomial associated with $V_{2n}(a_1, \ldots. a_{2n+1})$ satisfies $\Delta(1) = \pm 2$ each of the $a_i = 2$ [3; Prop. 2.3]. Since $V_m(2,2 \ldots. 2)$ can be identified with $V_{m+1,2}$ in a natural way, it follows that the Brieskorn manifolds do not provide non-trivial examples of manifolds homeomorphic to $V_{2n+1,2}$.

Using [12; Theorem 9.6] it is possible to compute $\Delta(t)$

for any manifold defined by a weighted homogeneous polynomial.

(1.2) Lemma: The characteristic polynomial of $K_{2n}^{p,q}$ is

$$\Delta(t) = \frac{(t+1)(t^{pq}+1)}{(t^p+1)}$$

Hence $\Delta(1) = 2$ and so $H_{2n-1}(K_{2n}^{p,q}) = \mathbb{Z}_2$.

We will say that a manifold M^{4n-1} satisfies (A) if and only if

(A) M is a 1-connected Π - manifold with the integral homology of $V_{2n+1,2}$.

Summarizing the above results, we have:

(1.3) Corollary: $K_{2n}^{p,q}$ satisfies (A). Furthermore, it bounds the parallelizable manifold $F_{2n}^{p,q}$.

The following proposition shows that this corollary is all that is needed to prove $K_{2n}^{p,q}$ is homeomorphic to $V_{2n+1,2}$.

(1.4) Proposition: If M^{4n-1} satisfies (A), $n > 2$, then

$$M \equiv V_{2n+1,2} \# \Sigma .$$

If M also bounds a parallelizable manifold, then $\Sigma \in bP_{4n}$.

This proposition is an analog of the fact that 1-connected homology spheres are homotopy spheres. Undoubtedly, it is a special case of a more general theorem (for example, a theorem of Wall's) but we give a direct proof based on the next two lemmas.

Let $E(\gamma)$ denote the total space of the closed 2n-disc bundle over S^{2n} classified by $\gamma \in \pi_{2n-1}(SO(2n))$. Let $E_0(\gamma) = \partial E(\gamma)$ be the associated sphere bundle. Let $\tau \in \pi_{2n-1}(SO(2n))$ classify the tangent bundle (so that $E_0(\pm \tau) = \pm V_{2n+1,2}$) and let σ generate the stable part of $\pi_{2n-1}(SO(2n))$. The proof of the following lemma can essentially be found in [11].

(1.5) <u>Lemma</u> (Kosinski): $E_0(\gamma)$ satisfies (A) if and only if

$$\gamma = \begin{cases} \pm\tau & ; \text{ if } n \text{ is odd} \\ \pm\tau + 2m\sigma; & \text{if } n \text{ is even} \end{cases}$$

where $m \in \mathbb{Z}$.

The substance of the proof of the next lemma is contained in [10].

(1.6) <u>Lemma</u>: If M satisfies (A), then there exists a framed surgery on M so that the resulting manifold is a homotopy sphere.

<u>Proof</u>: Let $L(\lambda, \mu) \in \mathbb{Q}/\mathbb{Z}$ be the rational linking number of two homology classes. (See [10; p. 524].) If λ is the non-zero element of $H_{2n-1}(M) = \mathbb{Z}_2$, then it follows from Poincaré duality for torsion groups that $L(\lambda, \lambda) = \frac{1}{2}$. By Lemmas 6.3 and 6.4 in [10], a framed surgery can be chosen so that H_{2n-1} of the new manifold is definitely smaller than $\mathbb{Z}_2$; hence 0.

<u>Proof of Proposition 1.4</u>: Since M can be obtained by one surgery on a homotopy sphere Σ, we may assume (subtracting and adding Σ) that we get it by a single surgery on S^{4n-1}. The

result of the surgery is completely determined by the isotopy class of an embedding $S^{2n-1} \times D^{2n} \subset S^{4n-1}$. Applying the results of [6], the isotopy class is unique on $S^{2n-1} \times 0$, $(n > 2)$, so by the tubular neighborhood theorem the isotopy class of a bundle map

$$\hat{\gamma}: S^{2n-1} \times D^{2n} \longrightarrow S^{2n-1} \times D^{2n} .$$

It follows that $M \equiv E_0(\gamma)$, where $\hat{\gamma}$ is a characteristic map for $\gamma \in \pi_{2n-1}(SO(2n))$. If n is odd, Lemma 1.5 completes the proof. If n is even, $\gamma = \pm \tau + 2\, m\sigma$. According to [11, 5.7.1],

$$E_0(\pm \tau + 2\, m\sigma) \equiv E_0(\pm \tau) \# m^2\, \Sigma(\sigma, \sigma)$$

where $\Sigma(\sigma, \sigma)$ is a homotopy sphere. Noting that M bounds a π-manifold if and only if Σ does, completes the proof.

(1.5) <u>Remark</u>: If M^{4m+1} is a 1-connected π-manifold with the integral homology of $V_{2m+2,2}$ (i.e., $H_{2m+1}(M) = H_{2m+2}(M) = \mathbb{Z}$), then the same argument shows that either

$$M \equiv V_{2m+2,2} \# \Sigma \quad \text{or}$$
$$M \equiv (S^{2m+1} \times S^{2m}) \# \Sigma$$

This is a special case of a theorem of DeSapio [4].

2. Link Manifolds

Since $K_m^{p,q}$ is defined by the polynomial

$$(z_1)^p + (z_1)(z_2)^q + (z_3)^2 + \dots + (z_{m+1})^2$$

it admits an action of $O(m-1)$ in the usual fashion by operating on the last $m-1$ coordinates. This is an example of an $O(m-1)$-action with exactly three orbit types. When dealing with such actions we shall use the term knot manifold as originally defined. (See [7; 314-10].) In particular, we shall mean that the orbit space of a knot manifold is D^4 and that the fixed point set is a circle. The term link manifold will be used to distinguish the case where the fixed point set consists of more than one circle. (For our purposes, the fixed point set will be precisely two circles.) It is easily checked that $K_m^{p,q}$ is a link manifold. Since the fixed point set of a link manifold is an embedded submanifold of S^3, the boundary of the orbit space, we can think of the fixed point set as a link in S^3.

A well known theorem of K. Jänich [9] shows that for each $m \geq 3$ there is a one-to-one correspondence between smooth unoriented knot classes (S^3, F) and equivariant diffeomorphism classes of $(2m-1)$-dimensional knot manifolds $M^{2m-1}(F)$. Jänich's results also give a corresponding theorem for link manifolds (as communicated to me by Dieter Erle), which is only slightly more complicated.

Two oriented links (S^3, L_i), $i = 1,2$, will be called equivalent if and only if there exists a diffeomorphism of pairs $(S^3, L_1) \longrightarrow (S^3, L_2)$ which preserves the link orientations (but not necessarily the orientation of S^3).

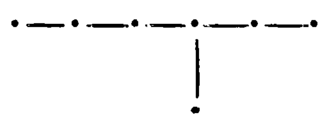
(2.1) Theorem (Classification of link manifolds): If $m \geq 3$, then for each link equivalence class (S^3, L) there corresponds a unique (up to equivariant diffeomorphism) $(2m-1)$-dimensional link manifold $M^{2m-1}(L)$.

So if (S^3, L) is a link with 2 components, then depending on whether or not L is amphicheiral there are either one or two link manifolds with L (unoriented) as fixed point set.

Remark: For no knot (S^3, F) is $M^{4n-1}(F)$ homeomorphic to $V_{2n+1,2}$. For if $H_{2n-1}(M^{4n-1}(F)) = \mathbb{Z}_2$, then it follows from Hirzebruch [7; 314-16] that $\det F = \pm 2$ and that $M^{4n+1}(F)$ is therefore a homotopy sphere. But $M^{4n+1}(F)$ has a $\mathbb{Z}_2$- action with $M^{4n-1}(F)$ as fixed point set, which, by P. A. Smith theory, is a contradiction.

Using Theorem 2.1, we can reduce questions of equivariant diffeomorphism to knot theoretic questions. We have two applications of this technique.

Application 1 (Equivalent link manifolds): Let $M_o(E_7)$ denote the plumbing of 7 copies of the unit tangent S^{m-1}-bundle to S^m along the graph



(the Dynkin diagram of E_7). $O(m-1)$ operates on each copy of the unit tangent bundle, $V_{m+1,2}$, and the plumbing can be taken equivariantly. This gives $M_o(E_7)$ the structure of a link manifold.

On the other hand, $K_m^{p,q}$ is the link manifold corresponding to the oriented link $(S^3, K_1^{p,q})$. $K_1^{p,q}$ is a torus link of two components consisting of an unknotted circle linked q times with the torus knot of type $(p-1,q)$, denoted by $t(p-1,q)$. (See

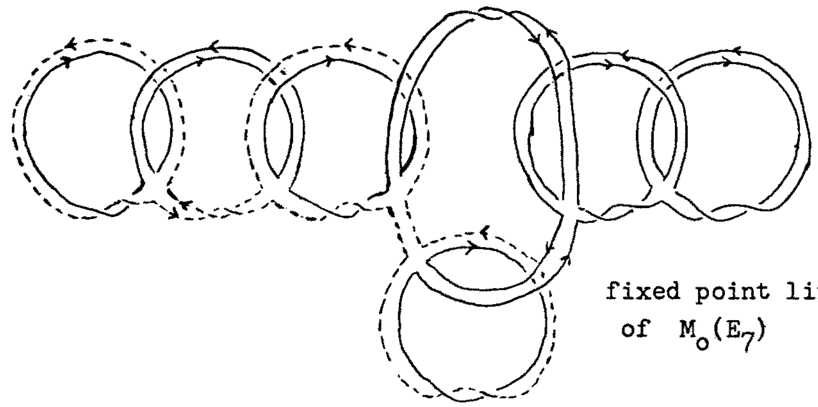
[12, chapter 10].) The author has verified that the fixed point link of $M_o(E_7)$ is equivalent to $K_1^{3,3}$. (These links are pictured on the next page.) Thus, as a consequence of Theorem 2.1:

(2.2) Proposition: If $m \geq 3$, $M_o(E_7)$ is $O(m-1)$-diffeomorphic to $K_m^{3,3}$. (Compare [7, 314-14].)

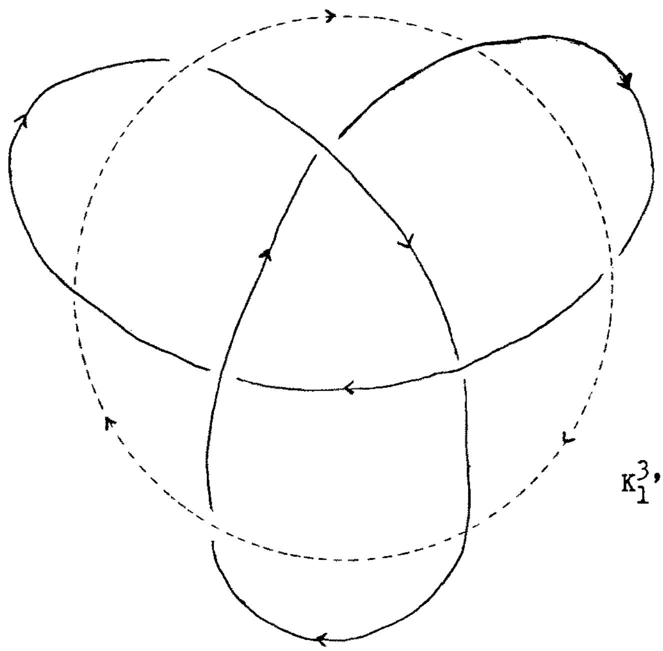
Application 2 (Prime actions): One can construct link manifolds almost diffeomorphic to $V_{m+1,2}$ as equivariant connected sums. For example, the inclusion $O(m-1) \subset O(m+1)$ gives $V_{m+1,2}$ the structure of the link manifold, where the fixed point link consists of two unknotted circles linked once. If the knot manifold $M^{2m-1}(F)$ is a homotopy sphere, the connected sum $V_{m+1,2} \# M^{2m-1}(F)$ can be performed equivariantly at fixed points. Moreover, the link which classifies the connected sum is just the connected sum of the fixed point sets, that is, a trivial knot linked once with F. This leads us to the following definition.

Definition: A link manifold $O(m-1) \times M \longrightarrow M$ is decomposable if the action is the connected sum of two non-trivial actions, i.e., M is equivariantly diffeomorphic to $M_1 \# M_2$ where M_i is a link manifold or a knot manifold, the connected sum is equivariant, and M_i is not equivalent to S^{2m-1} with the standard diagonal action (that is, the knot manifold classified by the trivial knot). If the action is not decomposable, the link manifold is prime. Clearly, this definition can be generalized to G-actions with fixed points, where G is a Lie group which acts in some standard fashion on S^n.

Note, for example, that a knot manifold is prime if and only if the corresponding knot is prime.



fixed point link
of $M_o(E_7)$



$K_1^{3,3}$

If a link manifold M is homeomorphic to $V_{2n+1,2}$ and $M \equiv M_1 \# M_2$ is decomposable, then

$$\mathbb{Z}_2 = H_{2n-1}(M) = H_{2n-1}(M_1) \oplus H_{2n-1}(M_2) .$$

So using Proposition 1.4, one of the M_i must be homeomorphic to $V_{2n+1,2}$ while the other is a homotopy sphere. (Link manifolds are, of course, Π-manifolds and bound Π-manifolds [5], [7; 314-16].)

(2.3) <u>Proposition</u>: $K_{2n}^{p,q}$ is prime.

<u>Proof</u>: Let L denote the link corresponding to $K_{2n}^{p,q}$. If $K_{2n}^{p,q}$ were decomposable, then we would have $L = F \# L'$, where F is a knot and L' is a link of two components, one of which is trivial and the other of which is a knot F' satisfying $t(p-1,q) = F \# F'$. Since torus knots are prime [13; p. 250], F' must be trivial and $F = t(p-1,q)$. But the knot manifold $M^{4n-1}(F)$ corresponding to $t(p-1,q)$ is the Brieskorn manifold $V_{2n}(p-1,q,2_1,\ldots,2)$ which satisfies $H_{2n-1}(M^{4n-1}(F)) = \mathbb{Z}_q$. Since $H_{2n-1}(M^{4n-1}(F))$ is a direct summand of $H_{2n-1}(K_{2n}^{p,q}) = \mathbb{Z}_2$, this is a contradiction.

3. Further Questions and Remarks

Semifree S^1-actions: If $k \leq n-1$, let ρ_k be the representation of S^1 in $O(2n-1)$ which takes $g \in SO(2)$ to the matrix

$$\rho_k(g) = \begin{bmatrix} I_g & & & \\ & \ddots & & \\ & & \ddots & \\ & & & g \end{bmatrix}$$

where there are k copies of g. This defines a semifree

S^1-action on $K_{2n}^{p,q}$. The proof of Theorem 1.5 in [2] also shows,

(3.1) Proposition (Browder-Petrie): If $k = n/2$, $k \leq n-1$ and if the semifree S^1- action defined by ρ_k on $K_{2n}^{p,q}$ and $K_{2n}^{p',q'}$ are equivalent, then $I(\bar{F}_{2n}^{p,q}) = I(\bar{F}_{2n}^{p',q'})$, where I means index.

Link manifolds diffeomorphic to $V_{m+1,2}$, m odd: As pointed out in Remark 1.5, in these dimensions the homology of the ambient manifold does not distinguish between link manifolds diffeomorphic to $V_{m+1,2}$ and those almost diffeomorphic to $S^m \times S^{m-1}$. Suppose that a link manifold M^{2m-1} is a homology $V_{m+1,2}$. Suppose further that the fixed point set has two components linked with linking number ℓ. There is some evidence for the following conjecture.

(3.2) Conjecture: If m is odd,

$$M^{2m-1} \equiv V_{m+1,2} \quad ; \quad \text{if } \ell \text{ is odd}$$

while $$M^{2m-1} \equiv S^m \times S^{m-1} \# \Sigma \, ; \quad \text{if } \ell \text{ is even}$$

$\Sigma \in bP_{2m}$.

Restricting to $O(m-k) \subset O(m-1)$: $O(m-k)$ acts on $K_m^{p,q}$ $(1 \leq k \leq m-4)$ via the inclusion. This again is an action with 3 orbit types, the orbit space is D^{2k+4} and the embedding of the fixed point manifold is $K_k^{p,q} \subset S^{2k+1}$ = the boundary of the orbit space. The question naturally arises - for which values of k is this action on $K_m^{p,q}$ prime? Not much is known about this question, although some observations are made in [3].

We have discussed prime actions in this paper, because this is a particularly interesting question in the study of non-standard G-actions on manifolds almost diffeomorphic to $V_{2n+1,2}$. Since a G-action on such a manifold is decomposable only if one of the manifolds in the connected sum is a homotopy sphere, the non-standard prime G-actions are, in some sense, precisely those actions whose "exoticness" depends on the topological structure rather than the specific differential structure of an exotic $V_{2n+1,2}$.

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