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SOME ASPHERICAL MANIFOLDS

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0. Introduction. Let Y denote the vector space of real, tridiagonal,[1] symmetric, $(n+1) \times (n+1)$ matrices. Let Λ be any set of $n+1$ distinct real numbers. Let P^n denote the set of those matrices in Y with spectrum equal to Λ. In [T], Carlos Tomei proves that the space P^n is a closed n-manifold. Of course, this is hardly surprising; however, Tomei goes on to show that these manifolds have several amazing properties. The most surprising property is that P^n is aspherical: in fact, its universal cover is diffeomorphic to Euclidean n-space. P^1 is a circle. P^2 is a surface of genus two. P^3 is the "double" of a certain hyperbolic 3-manifold of finite volume.[2] The proof of the asphericity of P^n in [T] uses results from [D1] on groups generated by reflections. Before continuing our description of these manifolds, we need to make a few general remarks concerning reflection groups.

Suppose that W is a discrete group acting smoothly and properly on a manifold M and that W is generated by smooth reflections. A *chamber* X for W on M is the closure of a component of the set of nonsingular points. Let S denote the set of reflections on W across the codimension-one faces of X. Then (W, S) is a Coxeter system (cf. [D1]). The manifold M can be reconstructed from the chamber X and the group W: paste together copies of X, one for each element of W, in the obvious fashion. In [D1], we gave simple necessary and sufficient conditions for the result of this pasting construction to be contractible.

There is a natural group generated by reflections on Tomei's manifold P^n. This can be seen as follows. The group $O(n+1)$ acts by conjugation on the vector space of $(n+1) \times (n+1)$ symmetric matrices. The kernel of this action is $\{\pm 1\}$. Let J denote the group $\{\text{diagonal matrices in } O(n+1)\}/\{\pm 1\}$. Obviously, $J \cong (\mathbb{Z}/2)^n$. The subspace Y is J-stable. J acts on Y as the group of all possible sign changes of the off-diagonal entries. Thus, J is a linear reflection group on Y. Since $O(n+1)/\{\pm 1\}$ preserves the spectrum of a symmetric matrix, the submanifold P^n is J-stable and J is a smooth reflection group on it. A fundamental chamber X^n for J on P^n is the intersection of P^n with the set of

[1] To say that a matrix $y = (y_{ij})$ is "tridiagonal" means that $y_{ij} = 0$ whenever $|i - j| > 1$.

[2] This means that there is a compact 3-manifold M^3 such that (a) each component of ∂M^3 is torus. (b) the interior of M^3 is homeomorphic to a hyperbolic 3-manifold of finite volume, and (c) P^3 is the double of M^3.

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those matrices in Y with all off-diagonal entries $\geqslant 0$. Tomei's results boil down to the fact that X^n can be identified with a certain particularly nice convex polyhedron. We shall call this polyhedron the "dual of the Coxeter complex of the symmetric group on $n+1$ letters". (The vertex set of this polyhedron can be identified with the symmetric group.) The fixed point set of a reflection in J intersects X in a disjoint union of several codimension-one faces. There is an obvious way of extracting from this situation an infinite Coxeter group $\tilde{J}$ such that (a) $\tilde{J}$ has exactly one fundamental reflection for each codimension-one face of X^n and (b) there is an epimorphism $\tilde{J} \to J$ with torsion-free kernel. (This is explained in Section 3.) Let $\tilde{P}^n$ denote the result of applying the pasting construction to X^n and $\tilde{J}$. It is easy to see that $\tilde{P}^n$ is a covering space of P^n. Tomei observes that it follows from the results of [D1] that $\tilde{P}^n$ is diffeomorphic to $\mathbb{R}^n$; hence, P^n is aspherical.

The above results suggest many further applications of the pasting construction. One of the most interesting applications is the following. Start by choosing the fundamental chamber to be an n-cube and the Coxeter group to be $W \times J$ where $J = (\mathbb{Z}/2)^n$ and where W is any finite Coxeter group of rank n. Next, pick two opposite vertices v_0 and v_1 of the n-cube. Choose a bijection from a set of fundamental reflections for W to the set of codimension-one faces of the n-cube which contain v_0. Also, choose a bijection from the set of fundamental reflections of J to the remaining codimension-one faces of the n-cube (i.e., to the set of codimension-one faces which contain v_1). The pasting construction yields a smooth closed n-manifold, $M^n(W)$, together with a $W \times J$-action on it. It is not difficult to see that, up to diffeomorphism, this manifold depends only on W (or more precisely, on a Coxeter system with underlying Coxeter group W). The manifolds $M^n(W)$ have the following properties.

(1) The group $W \times J$ is a reflection group on $M^n(W)$.

(a) A chamber for $W \times J$ on $M^n(W)$ is combinatorially isomorphic to an n-cube.

(b) A chamber for W on $M^n(W)$ is isomorphic to a larger n-cube.

(c) A chamber for J on $M^n(W)$ is isomorphic to the "dual of the Coxeter complex of W". (The definition of this is given in Section 6.)

(2) If W is the symmetric group on $n+1$ letters, then $M^n(W)$ is homeomorphic to Tomei's manifold P^n.

(3) The universal cover of $M^n(W)$ is diffeomorphic to $\mathbb{R}^n$.

(4) The only Coxeter group of rank one is $\mathbb{Z}/2$. If W is $\mathbb{Z}/2$, then $M^1(W)$ is a circle. The Coxeter groups of rank two are the dihedral groups. If W is the dihedral group of order $2m$, then $M^2(W)$ is a surface of genus $m-1$ (cf. Remark 4.6).

(5) If $W = W_1 \times W_2$, then $M^n(W) = M^{n_1}(W_1) \times M^{n_2}(W_2)$, where $n_i = \operatorname{rank}(W_i)$ (cf. Remark 4.5). For example, if $W = (\mathbb{Z}/2)^n$, then $M^n(W)$ is a n-torus.

(6) $M^n(W)$ is stably parallelizable (cf. Proposition 1.3).

(7) The homology of $M^n(W)$ can be explicitly computed (cf. Section 5). In particular, $\sum_{i=0}^n H_i(M^n(W); \mathbb{Z})$ is a free abelian group whose rank is equal to the

order of W, and as a W-module, $\Sigma_{i=0}^{n} H_i(M^n(W); \mathbb{Q})$ is isomorphic to the regular representation (cf. Theorem 5.4).

The discussion of $M^n(W)$ and its properties occurs in Sections 1 through 9. In light of these properties we conjecture that if W is not a direct product of Coxeter groups of rank 1 and 2, then $M^n(W)$ is not homeomorphic to a manifold of the form $\Gamma \setminus G/K$ for any Lie group G, maximal compact subgroup K, and torsion-free uniform lattice Γ.

In Sections 10 through 13 we generalize the manifolds P^n in a different direction. Suppose that $\mathfrak{g}$ is a real, semisimple, split Lie algebra of rank n. We shall define a submanifold $P^n(\mathfrak{g})$ of $\mathfrak{g}$. It is a closed n-manifold with the following properties:

(i) $P^n(\mathfrak{g})$ is homeomorphic to $M^n(W)$, where W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ ($\mathfrak{h}$ a split Cartan subalgebra),

(ii) if $\mathfrak{g} = sl(n+1, \mathbb{R})$, then $P^n(\mathfrak{g})$ can be naturally identified with manifold P^n.

The original motivation for considering the manifold P^n of isospectral tridiagonal symmetric matrices came from the study of the mechanical system known as the "Toda lattice". (See [M], [K], as well as, the references therein). After a change of coordinates, the phase space for this mechanical system becomes the set of y in Y with trace$(y) = 0$ and with the off-diagonal entries of y all positive. In these coordinates, the total energy is $\frac{1}{2}$trace(y^2). It follows from an argument of Lax that the coefficients of the characteristic polynomial of y are constants of motion, moreover, the Hamiltonian system is completely integrable. The interior of X^n is an integral submanifold. (Here X^n is the fundamental chamber for J on P^n.) Since the formula for the total energy (i.e., $\frac{1}{2}$trace(y^2)) makes sense on all of Y, the Hamiltonian flow on the interior of X^n extends to all of P^n. In a similar fashion, to each real, semisimple, split Lie algebra $\mathfrak{g}$, one can associate a generalized Toda lattice together with a Hamiltonian flow (or "Toda flow") on $P^n(\mathfrak{g})$. (See [K].) This flow has no 1-dimensional closed orbits and its fixed points are all hyperbolic. It is "perfect" in the sense of Morse theory. In the case of Tomei's manifold P^n, this was first proved by Fried [F].) Moreover, the topological type of this flow is determined by (and determines) the Bruhat order on W. These facts are discussed in Sections 9, 11, and 13.

1. Preliminaries. *Mirror structures.* Suppose that X is a CW-complex. A *mirror structure* on X is a family $\mathcal{M} = (X_s)_{s \in S}$ of subcomplexes indexed by some set S. A subcomplex in this family is a *mirror*. For each point x in X let $S(x)$ denote the set of elements s in S such that x is in X_s. For each subset T of S, put

$$X_T = X \cap \bigcap_{t \in T} X_t \tag{1}$$

$$X^T = \bigcup_{t \in T} X_t. \tag{2}$$

Let Nerve$(\mathcal{M})$ denote the nerve of the covering of X^S by the elements of $\mathcal{M}$.

(Nerve($\mathcal{M}$) is a simplicial complex; its simplices can be identified with those nonempty subsets T of S such that X_T is nonempty.)

The dual of a simplicial complex. Suppose that K is a simplicial complex. Denote by Ver(K) its set of vertices, by Poset(K) its set of simplices partially ordered by inclusion, and by Cone(K) the union of Poset(K) and $\{\phi\}$. We shall often identify a simplex of K with its set of vertices. Let K' denote the barycentric subdivision of K and let K^* denote the cone on K'. (The complexes K' and K^* are the geometric realizations of the posets, Poset(K) and Cone(K), respectively.) For each v in Ver(K) let K_v^* denote the closed star of v in K' (i.e., K_v is the subcomplex consisting of all simplices in K' which contain v). The family of subcomplexes $(K_v^*)_{v \in \mathrm{Ver}(K)}$ is a mirror structure on K^*. We shall denote it by $\mathcal{M}_K$ and call it the *canonical mirror structure* on K^*. Obviously, Nerve($\mathcal{M}_K$) $= K$. The complex K^* together with its canonical mirror structure is called the *dual* of K.

Remark 1.1. If K is a PL-triangulation of an $(n-1)$-sphere, then $K^*(= K_\phi^*)$ is an n-cell and for each simplex T of K, the complex K_T^* is the dual cell of T. For example, if K is the boundary complex of an octahedron, then K^* is a cube.

Coxeter Systems. Suppose that S is a finite set and that m is a function from $S \times S$ to the positive integers union $\{\infty\}$ satisfying the conditions: $m(s,t) = m(t,s)$, $m(s,s) = 1$, and $m(s,t) \geqslant 2$ if $s \neq t$. These numbers give a presentation for a group W as follows: the set of generators is S and the set of relations consists of all words of the form $(st)^{m(s,t)}$, where (s,t) ranges over all elements of $S \times S$ such that $m(s,t) \neq \infty$. It is known (cf. Proposition 4, page 92 in [B]) that

(i) the natural map $S \to W$ is an injection (and hence, that S can be identified with its image in W),

(ii) for each s in S, the order of s in W is 2, and

(iii) for each (s,t) in $S \times S$, the order of st in W is $m(s,t)$ (rather than just dividing $m(s,t)$).

The group W is a *Coxeter group* and the pair (W,S) is a *Coxeter system*. The rank of (W,S) is the cardinality of S. The Coxeter system (W,S) is *finite* if W is a finite group. For any subset T of S, let W_T $(= \langle T\rangle)$ denote the subgroup of W generated by T $(W_\phi = \{1\})$.

Associated to the Coxeter system (W,S), there is a simplicial complex, which we shall denote by Nerve(W,S). The vertex set of Nerve(W,S) is S; its simplices consist of those nonempty subsets T of S such that W_T is a finite group. For example, the nerve of a finite Coxeter system of rank n is a simplex of dimension $n-1$.

Reflection Groups. Suppose that (W,S) is a Coxeter system, that X is a CW-complex, and that $\mathcal{M} = (X_s)_{s\in S}$ is a mirror structure on X indexed by S. Define an equivalence relation $\sim$ on $W \times X$ by $(w,x) \sim (w',x') \Leftrightarrow x = x'$ and $w^{-1}w' \in W_{S(x)}$. Denote the quotient space $(W \times X)/\!\sim$ by $\mathcal{U}(W, X, \mathcal{M})$ (or simply by $\mathcal{U}$ when there is no ambiguity). For any (w,x) in $W \times X$, denote its image in $\mathscr{U}$ by $[w, x]$. We can identify X with the subset of $\mathscr{U}$ consisting of all points of the form $[1, x]$. The space $\mathscr{U}$ is naturally a CW-complex and X is a subcomplex. There is a natural action of W on $\mathscr{U}$ defined by $w'[w, x] = [w'w, x]$. The subcomplex X is a fundamental domain in the strong sense that every W-orbit intersects X in exactly one point. The family of subcomplexes $(wX)_{w \in W}$ are the *chambers* of W on $\mathscr{U}$, and X is the *fundamental* chamber. The action of a conjugate of an element in S on is called a *reflection* on $\mathscr{U}$; an element of S is a *fundamental reflection*.

The isotropy subgroup at $[w, x]$ is $wW_{S(x)}w^{-1}$. The W-action on $\mathscr{U}$ is proper if and only if each isotropy subgroup is finite, i.e., if and only if Nerve($\mathscr{M}$) is a subcomplex of Nerve(W, S) (cf. Lemma 13.4 in [D1]). One of the principal results of [D1] is that $\mathscr{U}$ is contractible if and only if X is contractible and X_T is nonempty and acyclic for each simplex T in Nerve(W, S). In particular, if $\mathscr{U}$ is contractible, then Nerve(W, S) is a subcomplex of Nerve($\mathscr{M}$).

Remark 1.2. We are using slightly different notation and terminology than in [D1]. In [D1] a typical Coxeter system was denoted by (Γ, V) instead of (W, S), Nerve(W, S) was denoted by $K_0(\Gamma, V)$, and X^T was denoted by $X_{\sigma(T)}$. Also, the word "panel" was used in [D1] in place of "mirror."

The Canonical Representation. We recall from [B] or [V] that there is a representation of W as a linear reflection group on $\mathbb{R}^n$, $n = |S|$, such that the elements of S are represented as linear reflections across the faces of a simplicial cone C. We shall call this the *canonical representation* of (W, S). (In [B] it is called the "dual of the canonical representation.") Let C^f denote the union of those faces of C with finite isotropy groups. The following well-known facts are due to Tits (cf. [B]) and Vinberg (cf. [V]): WC, the union of translates of C, is a convex cone, W acts properly on the interior Ω of this cone, and C^f is a chamber for W on Ω.

The existence of the canonical representation has many implications. One of these is the following lemma which we shall need later. (I would like to thank Frank Connolly for pointing out the necessity of proving this lemma.)

LEMMA 1.3. *Any finite subgroup of W is conjugate to a subgroup of the form W_T for some T in* Nerve(W, S).

Proof. Let G be a finite subgroup of W. Consider the W-action on Ω. Since Ω is convex and W acts through affine maps, G must have a fixed point y in Ω. Then $y = wx$ for some $x \in C^f$ and $w \in W$. The isotopy group at x is of the form W_T for some T in Nerve(W, S) and hence, $G \subset wW_Tw^{-1}$.

Manifolds. Suppose that X is a smooth manifold with corners, that as a manifold with corners it is "nice" in the sense of [D1], page 304, that the mirror structure $\mathscr{M}$ is such that the W-action is proper (i.e., Nerve($\mathscr{M}$) $\subset$ Nerve(W, S)), and that each mirror in $\mathscr{M}$ is a codimension-one face of X. As before, put $\mathscr{U} = \mathscr{U}(W, S, \mathscr{M})$. Then $\mathscr{U}$ is a manifold and it can be given a smooth structure which is compatible with the smooth structure on X and in which W acts smoothly; moreover, this smooth structure on $\mathcal{U}$ is unique up to isotopy (cf. [D1], Section 17).

Tangent Bundles. Let π: $\mathcal{U} \to X$ be the orbit map defined by $[w, x] \to x$. The tangent bundle of $\mathcal{U}$, denoted $T\mathcal{U}$, is naturally a W-vector bundle. As a smooth manifold with corners, X has a tangent bundle, denoted TX; it is a vector bundle over X.[3]

PROPOSITION 1.4. *The W-vector bundles $T\mathcal{U}$ and $\pi^*(TX)$ are W-equivariantly stably equivalent.*

(In this proposition, the W-action on X is the trivial action so that the map π: $\mathcal{U} \to X$ is W-equivariant.)

Proof. Again, we consider the canonical representation of W on $\mathbb{R}^n$. Let Ω and C^f be as before and let p: $\Omega \to C^f$ be the orbit map. Choose a face-preserving map h: $X \to C^f$. (This is possible since for each face of X the corresponding face of C^f is nonempty and contractible.) We may assume that h is smooth (as a map of orbifolds) and transverse to the singular set. Consider the smooth map ψ: $X \times \Omega \to \mathbb{R}^n$ defined by $(x, y) \to h(x) - p(y)$. It is easy to see that 0 is a regular value of ψ. Moreover, $\psi^{-1}(0)$ can be identified with $\mathcal{U}$ via the smooth W-equivariant embedding $\mathcal{U} \to X \times \Omega$ defined by $[w, x] \to [x, wh(x)$. The normal bundle of $\mathcal{U}$ in $X \times \mathbb{R}^n$ is W-equivariantly trivial. Since $\mathbb{R}^n$ is W-equivariantly contractible, the inclusion $\mathcal{U} \to X \times \mathbb{R}^n$ is W-homotopic to π: $\mathcal{U} \to X$. It follows that

$$T(X \times \Omega)|\mathcal{U} \cong \pi^*(TX) \times \mathbb{R}^n.$$

On the other hand, the left-hand side of this equation is isomorphic to $T\mathcal{U}$ plus the normal bundle (which is trivial). The proposition follows.

2. A method for constructing reflection groups. Suppose that K is a simplicial complex with vertex set V, that (W, S) is a Coxeter system, and that f: $V \to S$ is a function. Consider the following two conditions on f.

(A) If v and v' are the vertices of an edge in K, then $f(v) \neq f(v')$.

(B) If T is a simplex in K, then the subgroup $W_{f(T)}$ is finite.

Remark 2.1. Condition (B) is equivalent to the condition that f define a simplicial map from K to Nerve(W, S). (This simplicial map will also be denoted by f.) Condition (A) is then equivalent to the condition that the restriction of f to any simplex of K is a bijection.

[3] Following [Th] it is possible to define the "tangent bundle" of X as an orbifold. This is not a vector bundle, rather it is locally isomorphic to quotient of a vector bundle by a finite group. In our case, the "tangent bundle" of X as an orbifold is just $T\mathcal{U}/W$.

Suppose that we are given a function $f\colon V \to S$ satisfying (A) and (B). For each s in S, put

$$K_s^* = \bigcup_{v \in f^{-1}(s)} K_v^*$$

where K^* denotes the dual of K and K_v^* is the canonical mirror dual to a vertex v. Denote the family of subcomplexes $(K_s^*)_{s\in S}$ by $\mathscr{M}_f$. Then $\mathscr{M}_f$ is a mirror structure on K^* indexed by S. If T is a nonempty subset of S with m elements, then K_T^* is the union of "dual faces" $K_{T'}^*$ where T' ranges over the $(m-1)$-simplices in $f^{-1}(T)$.

As in the previous section, we construct the complex $\mathscr{U}(W, K^*, \mathscr{M}_f)$.

In view of condition (B), if K_T^* is nonempty, then T is a simplex in Nerve(W, S). As we mentioned previously, the fact that Nerve$(\mathscr{M}_f)$ is a subcomplex of Nerve(W, S) implies that the W-action on $\mathscr{U}(W, K^*, \mathscr{M}_f)$ is proper.

Remark 2.2. Let $T = f(V)$. Put $\mathscr{U} = \mathscr{U}(W, K^*, \mathscr{M}_f)$ and $\mathscr{U}' = \mathscr{U}(W_T, K^*, \mathscr{M}_f)$. It is easy to see that $\mathscr{U}'$ is connected and that $\mathscr{U}$ is W-equivariantly homeomorphic to the twisted product

$$W \times_{W_T} \mathscr{U}'.$$

In particular, $\mathscr{U}$ is connected if and only if $f\colon V \to S$ is surjective.

Remark 2.3. If K is a PL-triangulation of an $(n-1)$-sphere, then K^* together with the mirror structure $\mathscr{M}_f$ is a "manifold with faces" in the sense of Section 6 of [D1]. It follows that, in this case, $\mathscr{U}$ is an n-manifold (cf. Theorem 15.2 in [D1]). Moreover, if K is a smooth triangulation, then K^* can be given the structure of a smooth orbifold so that $\mathscr{U}$ becomes smooth (cf. Section 17 in [D1]).

3. The universal cover of $\mathscr{U}$. We retain the notation and hypotheses of the previous section. We shall suppose that the function $f\colon V \to S$ is surjective. From the 1-skeleton of K and from the function f, we shall construct a new Coxeter group $\tilde{W}$ with V as its fundamental set of generators. We shall then use the Coxeter system $(\tilde{W}, V)$ together with the canonical mirror structure on K^* to construct the universal cover of $\mathscr{U}(= \mathscr{U}(W, K^*, \mathscr{M}_f))$.

For each pair of distinct vertices v and v' in V, put

$$\tilde{m}(v, v') = \begin{cases} m(f(v), f(v')); & \text{if } \{v, v'\} \text{ is an edge of } K \\ \infty; & \text{otherwise,} \end{cases}$$

where for any two elements s, s' of S, $m(s, s')$ denotes the order of ss' in W. Also, put $\tilde{m}(v, v) = 1$ for each $v \in V$. As in Section 1, these numbers define a Coxeter system $(\tilde{W}, V)$. By construction, the function $f\colon V \to S$ extends to a surjective homomorphism $\tilde{f}\colon \tilde{W} \to W$. Denote the kernel of $\tilde{f}$ by Γ.

LEMMA 3.1.

(i) *The function* $f\ V \to S$ *defines a simplicial map from* Nerve$(\tilde{W}, V)$ *to* Nerve(W, S). *Moreover, for any simplex* T *in* Nerve$(\tilde{W}, V)$, f *takes* T *bijectively onto* $f(T)$ *and* $\tilde{f}$ *takes* $\tilde{W}_T$ *isomorphically onto* $W_{f(T)}$.

(ii) *The complex* K *is a subcomplex of* Nerve$(\tilde{W}, V)$. *Moreover, these complexes have the same* 1*-skeleton.*

(iii) *The group* Γ *is a torsion-free subgroup of* $\tilde{W}$.

Proof. (i) Let T be a subset of V such that $\tilde{W}_T$ is finite. Its homomorphic image $W_{f(T)}$ is, of course, also finite. But this is precisely what is meant by the statement that f defines a simplicial map from Nerve$(\tilde{W}, V)$ to Nerve(W, S). It follows immediately from condition (B) of Section 2 that the restriction of f to each simplex is a bijection. Since $m(v, v') = m(f(v), f(v'))$ for any two vertices v and v' of T, it is clear that $\tilde{f}$ takes the Coxeter system $(\tilde{W}_T, T)$ isomorphically onto $(W_{f(T)}, f(T))$.

(ii) Let T be a simplex of K. By condition (B), $W_{f(T)}$ is finite. Hence, $\tilde{W}_T$ (being isomorphic to $W_{f(T)}$) is also finite. Thus, K is a subcomplex of Nerve$(\tilde{W}, V)$. Their 1-skeletons are equal by construction.

(iii) By Lemma 1.3, any finite subgroup of $\tilde{W}$ is conjugate to a subgroup of the form $\tilde{W}_T$ where T is in Nerve$(\tilde{W}, V)$. Since such subgroups are mapped injectively by $\tilde{f}$, it follows that the kernel of $\tilde{f}$ is torsion-free.

Consider the $\tilde{W}$-complex $\tilde{\mathscr{U}} = \mathscr{U}(\tilde{W}, K^*, \mathscr{M}_K)$ where $\mathscr{M}_K$ denotes the canonical mirror structure on K^*. Let π: $\tilde{\mathscr{U}} \to \mathscr{U}$ be the natural $\tilde{f}$-equivariant projection defined by $[\tilde{w}, x] \to [\tilde{f}(\tilde{w}), x]$.

PROPOSITION 3.2. *The complex* $\tilde{\mathscr{U}}$ *is the universal covering of* $\mathscr{U}$, π *is the covering projection, and* Γ *is the group of covering transformations.*

Proof. Let x be a point in K^* and let $V(x)$ be the set of v in V such that x is in K_v^*. Choose an open neighborhood U_x of x in K^* which meets only the mirrors which contain x (i.e., the mirrors indexed by $V(x)$). Then $W_{f(V(x))}U_x$ is an open neighborhood of x in $\mathscr{U}$. Its inverse image in $\tilde{\mathscr{U}}$ is a disjoint union of translates of $\tilde{W}_{V(x)}U_x$. Since $\tilde{f}$: $\tilde{W}_{V(x)} \to W_{f(V(x))}$ is an isomorphism (by Lemma 3.1(i)) it follows that π evenly covers $W_{f(V(x))}U_x$. Similarly, π evenly covers the translate of such an open set by an element in W. But any point in $\mathscr{U}$ has such a neighborhood. Hence, π is a covering projection. It remains to prove that $\tilde{\mathscr{U}}$ is simply connected. By Corollary 10.2 and Theorem 13.5 in [D1] it suffices to show that (a) K^* is simply connected, (b) each mirror in $\mathscr{M}_K$ is connected, (c) if $\tilde{m}(v, v') \neq \infty$ for distinct vertices v, v' in V, then $K^*_{\{v, v'\}} (= K_v^* \cap K_{v'}^*)$ is nonempty. Each nonempty face of K^* is a cone, hence, contractible. Since this applies to K^* itself and to each mirror, (a) and (b) hold. The face $K^*_{\{v, v\}}$ corresponds to an edge of Nerve$(\tilde{W}, V)$. Since this edge is also in K (cf. Lemma 3.1(ii)), the face is nonempty, i.e., (c) holds.

A simplicial complex L is *determined by its* 1-*skeleton* if it satisfies condition (C) below.

(C) If T is a subset of $\mathrm{Ver}(L)$ such that any two elements of T bound an edge in L, then T is a simplex in L.

Condition (C) is clearly equivalent to the following condition.

(C′) If L' is any simplicial complex such that (a) L is a subcomplex of L' and (b) the 1-skeletons of L and L' and equal, then $L = L'$.

(In other words, L is determined by its 1-skeleton if it is the smallest full subcomplex of the simplex on $\mathrm{Ver}(L)$ which contains the 1-skeleton of L.)

For example, the following simplicial complexes are determined by their 1-skeletons: (1) a simplex, (2) the boundary complex of an m-gon, $m > 3$, (3) the boundary complex of an octahedron or of an icosahedron, (4) the derived complex of any poset (cf. the proof of Lemma 11.3 in [D1]), e.g., the barycentric subdivision of any convex cell complex.

THEOREM 3.3. *If K is determined by its* 1-*skeleton, then* $\mathscr{U}$ *is aspherical.*

Proof. We must show that $\tilde{\mathscr{U}}$ is contractible. According to [D1] it suffices to prove that (i) K^* is contractible, (ii) each nonempty face of K^* is acyclic, and (iii) $\mathrm{Nerve}\,(\mathscr{M}_K) = \mathrm{Nerve}(\tilde{W}, V)$. Since each nonempty face of K^* is a cone, (i) and (ii) hold. Recall that $\mathrm{Nerve}(\mathscr{M}_K) = K$. We have that $K \subset \mathrm{Nerve}(\tilde{W}, V)$ and that their 1-skeletons are equal (cf. Lemma 3.1(ii)). Since K is determined by its 1-skeleton, we see from condition (C′) that (iii) holds. This completes the proof.

Remark 3.4. If K is a PL-triangulation of an $(n-1)$-sphere, and if K is determined by its 1-skeleton, then $\tilde{\mathscr{U}}$ is PL-homeomorphic to $\mathbb{R}^n$. The reason is that the proof that $\tilde{\mathscr{U}}$ is contractible actually shows that $\tilde{\mathscr{U}}$ is an increasing union of n-disks. (See the argument in Remark 10.6 of [D1].) Similarly, if K is a smooth triangulation of S^{n-1}, then $\tilde{\mathscr{U}}$ is diffeomorphic to $\mathbb{R}^n$.

Remark 3.5. Let us make two observations. The first observation is that if, in the above construction, $\tilde{\mathscr{U}}$ is contractible, then $K = \mathrm{Nerve}(\tilde{W}, V)$. Call a Coxeter system (W, S) *right-angled* if for any pair of distinct elements s and t in S either $m(s,t) = 2$ or $m(s,t) = \infty$. The second observation is that the nerve of any right-angled Coxeter system is determined by its 1-skeleton. These two observations sometimes can be combined to show that a complex K is determined by its 1-skeleton. For example, suppose that K is the boundary comples of an n-octahedron, that $W = (\mathbb{Z}/2)^{2n}$, and that f is a bijection. Then K^* is an n-cube, $\mathscr{U}$ is an n-torus, and $\tilde{W}$ is the direct product of n copies of the infinite dihedral group. Since the universal cover $\tilde{\mathscr{U}}$ is $\mathbb{R}^n$ (which is contractible), we deduce that K is determined by its 1-skeleton. Of course, the same conclusion could have been reached by an easier route.

4. Some examples with fundamental chamber an *n*-cube. We shall consider some examples of the construction of Section 2 in the special case where the fundamental chamber is combinatorially equivalent to an n-cube. There are two situations in which these examples are well-known. First, if the Coxeter group W is a direct product of cyclic groups of order two, then the resulting manifold is an n-torus and the translates of the fundamental chamber give a standard tiling of it by n-cubes. Second, if $n = 2$ and the Coxeter group is finite, then the resulting closed 2-manifold will be a surface of genus > 0 tiled by quadrilaterals. When the genus is > 1 such tilings are familiar in 2-dimensional hyperbolic geometry. The examples which we shall discuss below can be regarded as generalizations of the above two types.

Throughout this section K will denote the boundary complex of an n-octahedron. Its dual K^* can be identified with the n-cube. The vertex set of K is the set of vectors in $\mathbb{R}^n$ of the form $\pm e_i$, where $\{e_1, \dots, e_n\}$ is the standard basis. A set of such vertices determines a simplex in K if and only if it contains no pairs of antipodal vertices. It follows that K is determined by its 1-skeleton. (We already knew this, cf. Remark 3.5) According to Theorem 3.3, this implies that the results of our constructions will be aspherical manifolds. These manifolds will be compact if W is finite.

We shall usually identify the vertex set of K with the set $I_n \times \{\pm 1\}$, where $I_n = \{1, 2, \dots, n\}$ and (i, ε) corresponds to εe_i. Recall that the necessary data for the construction consist of a Coxeter system (W, S) together with a surjection f: $I_n \times \{\pm 1\} \to S$ satisfying conditions (A) and (B) of Section 2. In all four of our examples the group W is finite, in which case condition (B) is automatic. In the first three examples, f is a bijection, in which case condition (A) is also automatic.

Example 4.1. Let (W, S) be any finite Coxeter system of rank $2n$ and let f: $I_n \times \{\pm 1\} \to S$ be any bijection. Then $\mathcal{U}(W, K^*, \mathcal{M}_f)$ is a closed aspherical n-manifold. We shall denote it by $U^n(W, f)$.

Let us make two observations concerning how $U^n(W, f)$ depends of the choice of f. If we vary f by a symmetry of K (i.e., by an element of the Coxeter group $\mathbf{B}_n$), then the resulting manifolds are clearly diffeomorphic (in fact, W-equivariantly diffeomorphic). Similarly, if we vary f by a permutation of S corresponding to a diagram automorphism of (W, S), then the resulting manifolds are again diffeomorphic and the W-actions differ only by the corresponding outer automorphism of W. For example, if $W = (\mathbb{Z}/2)^{2n}$, then any choice of f yields the n-torus.

Remark 4.2. Suppose that S can be decomposed as $S_1 \amalg S_2$ where each element of S_1 commutes with each element of S_2. Then W can be written as the direct product of the corresponding groups W_1 and W_2. Suppose further that I_n can be decomposed as $I_n = A \amalg B$ in such a fashion that f decomposes as

$$f = f_A \amalg f_B\colon (A \times \{\pm 1\}) \amalg (B \times \{\pm 1\}) \to S_1 \amalg S_2.$$

Then it is easy to see that $U^n(W, f) = U^p(W_1, f_A) \times U^q(W_2, f_B)$ where $p = |A|$ and $q = |B|$.

Example 4.3. This is a special case of Example 4.1. Suppose that (W, S) is the direct product of two subsystems (W_1, S_1) and (W_2, S_2), each of rank n. Let us further require that the bijection f take $I_n \times \{+1\}$ to S_1 and $I_n \times \{-1\}$ to S_2. ($I_n \times \{+1\}$ is the vertex set of a top-dimensional simplex in K, and $I_n \times \{-1\}$ is the vertex set of the antipodal simplex.) Up to a symmetry of K choosing such an f is equivalent to choosing a bijection $\theta\colon S_1 \to S_2$, where θ and f are related by the formula $\theta(f(i,1)) = f(i,-1)$. In this case, we denote the manifold $U^n(W, f)$ by $M^n(W_1, W_2, \theta)$.

Example 4.4. This example is a further specialization. The only Coxeter group of rank one is $\mathbb{Z}/2$. Suppose that W_2 is the direct product of n copies of $\mathbb{Z}/2$, and that S_2 is the standard set of generators of $(\mathbb{Z}/2)^n$. Let (W, S) be any finite Coxeter system of rank n and put $(W_1, S_1) = (W, S)$. Since any bijection of S_2 to itself corresponds to a diagram automorphism of (W_2, S_2), the manifold $M^n(W_1, W_2, \theta)$ is independent of θ. We shall denote this manifold by $M^n(W)$. It is one of the basic objects of study in this paper.

Remark 4.5. Suppose, in the previous example, that the Coxeter system (W, S) is a direct product of two subsystems (W', S') and (W'', S'') (i.e., $S = S' \amalg S''$ and $W = W' \times W''$). It then follows from Remark 4.2 that

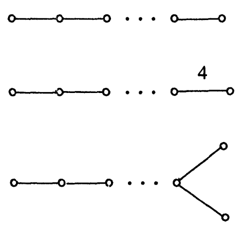
$$M^n(W) = M^{n'}(W') \times M^{n''}(W'') \tag{1}$$

where $n' = |S'|$ and $n'' = |S''|$.

Remark 4.6. In Example 4.4, we associated to each finite Coxeter system (W, S) of rank n, a closed n-manifold $M^n(W)$. The Euler characteristics of these manifolds can be computed. Since the Euler characteristic of the product of two spaces is the product of their Euler characteristics, it suffices, in view of (1), to consider finite irreducible Coxeter groups. Let us first consider the case $n = 2$. In this case, W is a dihedral group of order $2m$ and the Coxeter diagram of W is $\circ\overset{m}{—}\circ$, $m \geqslant 3$. The surface $M^2(W)$ is tiled by $|W \times J|$ $(= 8m)$ copies of a quadrilateral. Hence, there are $8m$ two-cells. Since the isotropy group at each edge is $\mathbb{Z}/2$, the number of edges is $(4)(8m)/2$ $(= 16m)$. Three of the vertices have isotropy subgroup $\mathbb{Z}/2 \times \mathbb{Z}/2$, the fourth vertex has isotropy subgroup W. Hence, the number of vertices is $3(8m)/4 + 4$ $(= 6m + 4)$. Thus

$$\chi\big(M^2(W)\big) = 8m - 16m + 6m + 4 = 4 - 2m. \tag{2}$$

It follows that $M^2(W)$ is a surface of genus $m - 1$. For $n > 2$, the Euler characteristics can be computed in a similar fashion. In each dimension $n > 2$, there are only a finite number of finite irreducible Coxeter systems. There are three families which appear for each $n \geqslant 4$. Their Coxeter diagrams are



$\mathbf{A}_n$ ($n \geqslant 1$ vertices)

$\mathbf{B}_n$ ($n \geqslant 2$ vertices)

$\mathbf{D}_n$ ($n \geqslant 4$ vertices).

(There are only six other groups, the Coxeter diagrams of which are denoted by $\mathbf{H}_3$, $\mathbf{F}_4$, $\mathbf{H}_4$, $\mathbf{E}_6$, $\mathbf{E}_7$, and $\mathbf{E}_8$.) For the three infinite families there are the following amazing formulas:

(3) $$\chi(M^n(\mathbf{A}_n)) = 2^{n+2}(2^{n+2} - 1)B_{n+2}/n + 2$$

(4) $$\chi(M^n(\mathbf{B}_n)) = 2^n E_n$$

(5) $$\chi(M^n(\mathbf{D}_n)) = 2^n E_n + 2^{2n-1}(2^n - 1)B_n$$

where B_n is the nth Bernoulli number and E_n is the nth Euler number. (E_n is 0 if n is odd; if n is even, it is defined by the equation

$$\sec z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} E_{2k} z^{2k}.)$$

Formula (3) is due to Tomei [T]. Formulas (4) and (5) are due to K. Druschel.

Example 4.7. Suppose that (W, S) is a finite Coxeter system of rank n and that g: $I_n \to S$ is any bijection. Let $f = g \circ p$: $I_n \times \{\pm 1\} \to S$, where p: $I_n \times \{\pm 1\} \to I_n$ denotes projection on the first factor. It is clear that f satisfies condition (A) of Section 2. Since any two such f's differ by a symmetry of K, the resulting manifold $\mathscr{U}(W, K^*, \mathscr{M}_f)$ is independent of f (i.e., of g). Denote it by $\hat{M}^n(W)$. (The geometric picture to have in mind as follows: a mirror corresponding to an element s of S is the disjoint union of two opposite codimension-one faces of the n-cube.) In fact, $\hat{M}^n(W)$ is actually W-equivariantly diffeomorphic to the manifold $M^n(W)$ of the previous example. (However, recall that the larger group $W \times (\mathbb{Z}/2)^n$ acts on $M^n(W)$.) This can be seen as follows. Let $X(= K^*)$ be the fundamental chamber for $W \times (\mathbb{Z}/2)^n$ on $M^n(W)$. Consider the action of $W \times \{1\}$ on $M^n(W)$. It is a reflection group with fundamental chamber $(\mathbb{Z}/2)^n X$. After a moment of thought, one sees that $(\mathbb{Z}/2)^n X$ is again a cube and that each mirror is a union of opposite codimension-one faces. It follows that $M^n(W)$ can be obtained by pasting together $|W|$ copies of this cube exactly as we obtained $\hat{M}^n(W)$, i.e., $M^n(W) = \hat{M}^n(W)$. The case where $n = 2$ and W is a dihedral group generated by two elements s_1 and s_2 is illustrated in Figure 1.

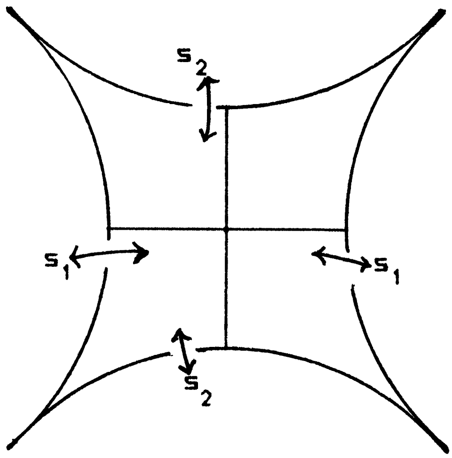


FIGURE 1.

Remark 4.8. By Remark 3.4, the universal cover of each example in this section is diffeomorphic to $\mathbb{R}^n$. Since the tangent bundle of an n-cube is trivial, it follows from Proposition 1.4, that each of these examples is stably parallelizable.

5. The homology of these manifolds. In [D2] the homology of $\mathcal{U}(W, X, \mathcal{M})$ is calculated in terms of the relative homology groups $H_*(X, X^T)$, where T is a subset of S and where X^T is the union of mirrors indexed by T. In order to apply these calculations to the examples of the previous section, we need to know the groups $H_*(X, X^T)$ when X is the n-cube I^n and X^T is a union of codimension-one faces. The answer is provided by the following lemma, which states that such homology groups vanish unless the pair (X, X^T) is isomorphic to $(I^j, \partial I^j) \times I^{n-j}$ for some j, with $0 \leqslant j \leqslant n$. Its proof is left as an easy exercise for the reader. Before giving the precise version of this lemma, we introduce some notation: for any subset A of I_n (where $I_n = \{1, \ldots, n\}$) let $T(A)$ be the subset of $I_n \times \{\pm 1\}$ defined by $T(A) = A \times \{1\} \cup A \times \{-1\}$.

LEMMA 5.1. *Suppose that X is an n-cube with its canonical mirror structure. Let T be a subset of* I_n.

(i) *If* $T \neq T(A)$ *for some subset A of* I_n, *then* $H_*(X, X^T)$ *is zero in each dimension.*

(ii) *Suppose* $T = T(A)$ *where A has j elements. Then*

$$H_i(X, X^T) = \begin{cases} \mathbb{Z}; & i = j \\ 0; & i \neq j \end{cases}$$

Let us recall some notation from [D2]. Suppose that (W, S) is a finite Coxeter system. Denote the word length (with respect to the generating set S) of an element w in W by $l(w)$. Let T be a subset of S. Denote by W^T the set of w in W such that $l(ws) = l(w) - 1$ if s is in T and $l(ws) = l(w) + 1$ if s is in $S - T$. Denote by $\mathbb{Z}(W^T)$ the free abelian group with basis W^T. Define elements ξ_T, η_T, ψ_T in rational group algebra $\mathbb{Q}(W)$ by

$$\xi_T = |W_T|^{-1} \sum_{wW_T} w, \qquad \eta_T = |W_T|^{-1} \sum_{wW_T} (-1)^{l(w)} w$$

$$\psi_T = \xi_{S-T}\eta_T.$$

PROPOSITION 5.2. *Suppose that* (W, S) *is a finite Coxeter system of rank* $2n$ *and that* $f\colon I_n \times \{\pm 1\} \to S$ *is a bijection. For each subset* A *of* I_n *let* $T(A) = A \times \{1\} \cup A \times \{-1\}$ *and let* $T'(A) = f(T(A))$. *The homology of the manifold* $U^n(W, F)$ *defined in Example* 4.1 *is given as follows*:

(i) $H_j(U^n(W, f); \mathbb{Z}) \cong \sum \mathbb{Z}(W^{T'(A)})$.

(ii) *There is an isomorphism of rational representations of* W,

$$H_j(U^n(W, f); \mathbb{Q}) \cong \sum \mathbb{Q}(W)\psi_{T'(A)}.$$

Here the summations in (i) *and* (ii) *run over all subsets* A *of* I_n *of cardinality* j.

Proof. Using Lemma 5.1, we see that (i) and (ii) are special cases of Theorem A′ and Theorem B, respectively, in [D2].

COROLLARY 5.3. *Suppose that* (W_1, S_1) *and* (W_2, S_2) *are finite Coxeter systems of rank* n *and that* $\theta\colon S_1 \to S_2$ *is a bijection. Then the homology of the manifold* $M^n(W_1, W_2, \theta)$ *of Example* 4.3 *is given as follows*

(i) $H_j(M^n(W_1, W_2, \theta); \mathbb{Z}) \cong \sum \mathbb{Z}(W_1^T) \otimes \mathbb{Z}(W_2^{\theta(T)})$.

(ii) *There is an isomorphism of rational representations of* $W_1 \times W_2$,

$$H_j(M^n(W_1, W_2, \theta); \mathbb{Q}) \cong \sum \mathbb{Q}(W_1)\psi_T \otimes \mathbb{Q}(W_2)\psi_{\theta(T)}.$$

Here the summations range over all subsets T *of* S_1 *of cardinality* j.

Let J denote the group $(\mathbb{Z}/2)^n$ (where $\mathbb{Z}/2$ is written multiplicatively as $\{\pm 1\}$). For $1 \leqslant i \leqslant n$, let $p_i\colon J \to \mathbb{Z}/2$ be projection on the ith factor (so that p_i is a character) and let r_i be the element of J which has ith-component -1 and all other components equal to 1. Put $R = \{r_1, \ldots, r_n\}$. In the case of the Coxeter system (J, R) many of our previous constructions can be simplified. Suppose that A is a subset of I_n. Put

$$R(A) = \{r_1 \in R \mid i \in A\}$$

$$r_A = \prod_{i \in A} r_i$$

$$p_A = \prod_{i \in A} p_i\colon J \to \mathbb{Z}/2$$

$\Lambda_A =$ the 1-dimensional $\mathbb{Q}$-vector space with J-action defined by $\varepsilon \cdot \lambda = p_A(\varepsilon)\lambda$ $(\varepsilon \in J,\ \lambda \in \Lambda_A)$.

Clearly, $J^{R(A)} = \{r_A\}$. Also, an easy calculation shows that for any ε in J, we have $\varepsilon\psi_{R(A)} = p_A(\varepsilon)\psi_{R(A)}$. Hence $\mathbb{Q}(J)\psi_{R(A)}$ and Λ_A are isomorphic as J-modules.

Now let us return to the situation of Corollary 5.3 in the special case where $(W_2, S_2) = (J, R)$ and $(W_1, S_1) = (W, S)$, with $S = \{s_1, \dots, s_n\}$. As we pointed out in Example 4.4, the resulting manifold is independent of the bijection θ: $S \to R$. Hence, we may as well assume that $\theta(s_i) = r_i$. For each subset A of I_n, put

$$S(A) = \{s_i \in S \mid i \in A\}.$$

THEOREM 5.4. *Suppose that* (W, S) *is a finite Coxeter system of rank* n. *The homology of the manifold* $M^n(W)$ *of Example* 4.4 *can be calculated as follows (where summations range over the subsets* A *of* I_n *with* j *elements unless otherwise specified).*

(i) $H_j(M^n(W); \mathbb{Z}) \cong \sum \mathbb{Z}(W^{S(A)})$.

(ii) *There is an isomorphism of rational representations of* $W \times J$,

$$H_j(M^n(W); \mathbb{Q}) \cong \sum \mathbb{Q}(W)\psi_{S(A)} \otimes \Lambda_A.$$

(iii) *There is an isomorphism of* W-*spaces*,

$$H_j(M^n(W); \mathbb{Q}) \cong \sum \mathbb{Q}(W)\psi_{S(A)}.$$

(iv) *As a* W-*space*, $H_*(M^n(W); \mathbb{Q})$ $(= \sum_{i=0}^n H_i(M^n(W); \mathbb{Q}))$ *is isomorphic to the regular representation of* W.

Proof. Statements (i) and (ii) are special cases of Corollary 5.3. Statement (iii) follows from (ii). Statement (iv) follows immediately from a theorem of Solomon [S] which states that $\mathbb{Q}(W)$ is the direct sum of left ideals $\mathbb{Q}(W)\psi_{S(A)}$, $A \subset I_n$.

6. Some examples with fundamental chamber the dual of a Coxeter complex. Suppose that (W, S) is a finite Coxeter system of rank n.

The Coxeter complex of (W, S). There is a canonical representation of W on $\mathbb{R}^n$ so that the elements are S are represented by orthogonal linear reflections and so that a fundamental chamber is a simplicial cone. The intersection of this cone with the unit sphere S^{n-1} is a spherical simplex Δ of dimension $n - 1$. The translates of Δ by elements of W give a triangulation of S^{n-1} by spherical simplices. The underlying simplicial complex of this triangulation is denoted by $L(W)$ and called the *Coxeter complex* of (W, S).

Since Δ is a fundamental chamber for W on S^{n-1}, it inherits a mirror structure $\mathscr{M}$ indexed by S: for each s in S, the simplex Δ_s is the intersection of Δ with the hyperplane fixed by s. We have a simplicial isomorphism

$$(1) \qquad L(W) \cong \mathscr{U}(W, \Delta, \mathscr{M}).$$

For any proper subset T of S, the simplex Δ_T is the intersection of Δ with the codimension-one faces of Δ which are indexed by T. It follows that the codimension of Δ_T in Δ is $|T|$ (where $|T|$ denotes the cardinality of T). The stabilizer of Δ_T is W_T. It follows from (1) that every simplex of $L(W)$ has the form $w\Delta_T$ for some subset T of S and some element w in W. Furthermore, $w\Delta_T = w'\Delta_{T'}$ if and only if $T = T'$ and $w^{-1}w' \in W_T$. In other words, the simplices of $L(W)$ of codimension i can be identified with $\coprod W/W_T$, where T ranges over all subsets of S with $|T| = i$. In particular, the top-dimensional simplices of $L(W)$ correspond bijectively with W.

We can identify the vertex set of Δ with S by matching each vertex of Δ with the reflection across the opposite face. (This induces an identification of Δ with Nerve(W, S).) It follows from (1) that the orbit space of $L(W)$ is Δ. The natural projection is a simplicial map, which we shall denote by Type: $L(W) \to \Delta$. We shall denote the composition of the restriction of Type to Ver$(L(W))$ with the natural identification Ver$(\Delta) \cong S$ by

$$\overline{\text{Type}}: \text{Ver}(L(W)) \to S. \tag{2}$$

The dual of a Coxeter complex. Let $L^*(W)$ denote the dual of the simplicial complex $L(W)$ (where "dual" is defined in Section 1). We can identify $L^*(W)$ with an n-dimensional convex polyhedron as follows. Regard $L(W)$ as a triangulation of S^{n-1}. Put a vertex at the center of Δ and each of its translates. The convex polyhedron spanned by these vertices can then be identified with $L^*(W)$. (A detailed description of $L^*(W)$ in the case where W is $\mathbf{A}_n$, the symmetric group on $n+1$ symbols, is given in Section 3, pp. 986–989, of [T].) The vertex set of $L^*(W)$ can be identified with W. The set of i-dimensional faces of $L^*(W)$ then corresponds to the set of all cosets wW_T where T ranges over all subsets of S, with $|T| = i$. (This is because the codimension-i simplices of $L(W)$ are identified with this same set of cosets.) The vertices of the i-face corresponding to wW_T are the elements of this coset.

Since the orthogonal reflection group W permutes the centers of the simplices of $L(W)$, it acts naturally on $L^*(W)$. A fundamental chamber of W on $L^*(W)$ is the intersection of $L^*(W)$ with the fundamental simplicial cone. This polyhedron obviously can be identified with Δ^*. (Recall that Δ^* is the cone on the barycentric subdivision of Δ.) The W-action defines a mirror structure $\mathcal{M}_0$ on Δ^* indexed by S: for each $s \in S$, Δ^*_s is the cone on Δ_s. Clearly,

$$L^*(W) = \mathcal{U}(W, \Delta^*, \mathcal{M}_0). \tag{3}$$

The examples.

Example 6.1. Suppose that (W_1, S_1) and (W_2, S_2) are finite Coxeter systems of rank n and that θ: $S_1 \to S_2$ is a bijection. Define a function f: Ver$(L(W_1)) \to S_2$ by $f = \theta \circ \overline{\text{Type}}$, where $\overline{\text{Type}}$: Ver$(L(W_1)) \to S_1$ is defined as in (2). Since Type: $L(W_1) \to \Delta$ is a simplicial map which is bijective on the vertex set of each simplex and since $\Delta \cong \text{Nerve}(W_2, S_2)$, it follows from Remark 2.1 that f satisfies conditions (A) and (B) of Section 2. Since $L(W_1)$ is a smooth triangulation of S^{n-1}, it follows from Remark 2.3 that the complex $\mathcal{U}(W_2, L^*(W_1), \mathcal{M}_f)$ is a smooth n-manifold. We shall denote this manifold by $N^n(W_1, W_2, \theta)$. The group W_2 acts as a reflection group on $N^n(W_1, W_2, \theta)$ with fundamental chamber $L^*(W_1)$. The group W_1 acts on this fundamental chamber and the action extends naturally to all of $N^n(W_1, W_2, \theta)$. Moreover, these actions commute. Thus, $W_1 \times W_2$ is a reflection group on $N^n(W_1, W_2, \theta)$ with fundamental chamber Δ^*.

In Example 4.3 we defined a manifold $M^n(W_1, W_2, \theta)$ from the same data as above. The group $W_1 \times W_2$ acted as a reflection group with fundamental chamber an n-cube. We shall show that these two constructions are the same. First, however, we shall consider the following special case of the example above.

Example 6.2. Suppose that W_2 is the product of n copies of $\mathbb{Z}/2$ and that $(W_1, S_1) = (W, S)$ is an arbitrary finite Coxeter system of rank n. As in Example 4.4, the resulting manifold $N^n(W_1, W_2, \theta)$ is independent of θ. We shall denote it by $N^n(W)$.

THEOREM 6.3. *Suppose that (W_1, S_1) and (W_2, S_2) are finite Coxeter systems of rank n and that θ: $S_1 \to S_2$ is a bijection. Then the manifold $N^n(W_1, W_2, \theta)$ of Example* 6.1 *is $(W_1 \times W_2)$-equivariantly diffeomorphic to the manifold $M^n(W_1, W_2, \theta)$ of Example* 4.3. *In particular, if $(W_1, S_1) = (W, S)$ then the manifold $N^n(W)$ of Example* 6.2 *is diffeomorphic to $M^n(W)$ of Example* 4.4.

As was mentioned previously, $W_1 \times W_2$ is a reflection group on $M^n(W_1, W_2, \theta)$ with fundamental chamber the n-cube K^* (where K is the boundary complex of the n-octahedron) and $W_1 \times W_2$ is also a reflection group on $N^n(W_1, W_2, \theta)$ with fundamental chamber Δ^*. In other words:

$$M^n(W_1, W_2, \theta) = \mathcal{U}(W_1 \times W_2, K^*, \mathcal{M}) \tag{4}$$

$$N^n(W_1, W_2, \theta) = \mathcal{U}(W_1 \times W_2, \Delta^*, \mathcal{M}') \tag{5}$$

where $\mathcal{M}$ is a certain mirror structure on K^* and $\mathcal{M}'$ is a mirror structure on Δ^*. Both mirror structures are indexed by $S_1 \amalg S_2$. Thus, the content of Theorem 6.3 is that $(K^*, \mathcal{M})$ and $(\Delta^*, \mathcal{M}')$ are isomorphic as complexes with mirror structures. Enumerate the elements of S_1 as $s_1, \dots, s_n$. Put $s_i' = \theta(s_i)$ so that $S_2 = \{s_1', \dots, s_n'\}$. To simplify the bookkeeping let us reindex everything by $I_n \times \{\pm 1\}$ where $(i, 1)$ corresponds to s_i and $(i, -1)$ to s_i'.

First let us consider the mirror structure $\mathcal{M}$ on K^*. The face corresponding to s_i is the dual face to $(i, 1)$ (where $I_n \times \{\pm 1\}$ is regarded as the vertex set of the n-octahedron). Similarly, the face corresponding to s_i' is the dual face to $(i, -1)$. In other words, $\mathcal{M} = \{K^*_{(i,\varepsilon)}\}$ where (i, ε) ranges over $I_n \times \{\pm 1\}$ and where $K^*_{(i,\varepsilon)}$ denotes the face of the n-cube which is dual to the vertex εe_i.

Next, consider the mirror structure $\mathcal{M}'$ on Δ^*. The face corresponding to $s_i \in S$ is the cone on Δ_{s_i}. Let v_i denote the vertex of Δ which is opposite to the face Δ_{s_i} (so that $\mathrm{Ver}(\Delta) = \{v_1, \ldots, v_n\}$). Then $\Delta^*_{(i,1)}$ is the cone on the face of Δ opposite to v_i. The mirror of Δ^* corresponding to s_i' is the intersection of Δ^* with the codimension-one face of $L^*(W_1)$ corresponding to the coset $1(W_1)_{S-\{s_i\}}$. Unwinding a few definitions, we see that this is precisely the dual face of the vertex v_i in Δ^*. That is to say, $\Delta^*_{(i,-1)}$ is the dual face to v_i.

Theorem 6.3 has now been reduced to the following lemma.

LEMMA 6.4. *Let K^* be the n-cube and Δ^* the cone on the barycentric subdivision of an $(n-1)$-simplex. Let $\mathcal{M}$ and $\mathcal{M}'$ be the mirror structures defined above. Then $(K^*, \mathcal{M})$ and $(\Delta^*, \mathcal{M}')$ are isomorphic as complexes with mirror structures.*

Proof. Each face of K^* is a cell, as is each face of Δ^*. By construction, $\mathrm{Nerve}(\mathcal{M}) = K$. Thus, it suffices to show that $\mathrm{Nerve}(\mathcal{M}') = K$. In other words, we must show that for any subset T of $I_n \times \{\pm 1\}$, the intersection of mirrors $\Delta^*_{(i,\varepsilon)}$, $(i, \varepsilon) \in T$, is nonempty if and only if T contains no pair of elements of the form $(i, 1)$ and $(i, -1)$. But this is obvious.

As a corollary of Theorem 6.3 we have the following result.

PROPOSITION 6.5. *The Coxeter complex of a finite Coxeter system is determined by its* 1-*skeleton.*

Proof. This follows easily from Remark 3.4.

Remark 6.6. If the Coxeter diagram of W is a straight line (i.e., if W is $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{H}_3$, $\mathbf{F}_4$ or $\mathbf{H}_4$ or if W is a dihedral group), then W is the group of symmetries of a regular polyhedron. Furthermore, the Coxeter complex $L(W)$ can be identified with the barycentric subdivision of this polyhedron and $L^*(W)$ is the dual to the barycentric subdivision. This remark suggests a further class of examples. Let P be any n-dimensional convex polyhedron and let L denote the barycentric subdivision of ∂P. If v is a vertex of L, then it is the barycenter of a face F_v of P. Let $\dim F_v$ be the dimension of this face and define d: $\mathrm{Ver}(L) \to I_n$ by $d(v) = \dim F_v + 1$. (It is clear that d is the restriction of a simplicial map from L to the simplex on I_n.) Let (W_2, S_2) be any finite Coxeter system of rank n and let θ: $I_n \to S_2$ be a bijection. Then $\theta \circ d$: $\mathrm{Ver}(L) \to S_2$ satisfies conditions (A) and (B). Hence, we obtain a manifold $\mathcal{U}(W_2, L^*, \mathcal{M}_{\theta \circ d})$. It is aspherical since L is determined by its 1-skeleton (cf. Theorem 3.3).

7. Faces of $L^*(W)$ and some submanifolds of $M^n(W)$. In this section we shall describe the fixed point sets of the standard subgroups of J on $M^n(W)$. This description is facilitated by the fact that $J \cong (\mathbb{Z}/2)^n$. We begin with some general remarks about this situation. Suppose that (J, R) is a smooth reflection system on a manifold M with fundamental chamber X and that $J \cong (\mathbb{Z}/2)^n$. For any subset R' of R, let $M_{R'}$ denote the fixed point set of $J_{R'}$ on M and let $X_{R'}$ denote the intersection of X with $M_{R'}$. Then $M_{R'}$ is a smooth submanifold of M of codimension $|R'|$. The group $J_{R-R'}$ centralizes $J_{R'}$ and acts as a reflection group on each component of $M_{R'}$ with fundamental chamber the corresponding component of $X_{R'}$.

We can apply these remarks to the J-action on $M^n(W)$ where, as before, (W, S) is a finite Coxeter system of rank n. First we set up some notation for the faces of the fundamental chamber $L^*(W)$.

The poset of faces of $L^*(W)$ is isomorphic to the poset of cosets of the form wW_T. For each w in W and each subset T of S, let $F(w, T)$ be the face of $L^*(W)$ corresponding to wW_T. ($F(w, T) = F(w', T')$ if and only if $T = T'$ and $w^{-1}w' \in W_T$.) The dimension of $F(w, T)$ is $|T|$. (Thus the number of k-dimensional faces of $L^*(W)$ is $\sum |W/W_T|$, where the summation ranges over all subsets T of S of cardinality k.) Clearly,

$$F(w, T) \cong L^*(W_T). \tag{1}$$

Let θ: $S \to R$ be the bijection occurring in the definition of $M^n(W)$. The mirror of $L^*(W)$ corresponding to an element r in R is the union of all $(n-1)$-dimensional faces of the form $F(w, S - \theta^{-1}(r))$, where $w \in W$. More generally, for any subset T of S, we have that $L^*(W)_{R-\theta(T)}$ is the union of all faces of the form $F(w, T)$. Suppose that $|T| = k$. Let $M^k(w, T)$ denote the component of the fixed set of $J_{R-\theta(T)}$ which contains $F(w, T)$, i.e.,

$$M^k(w, T) = J_{\theta(T)}F(w, T). \tag{2}$$

The group wW_Tw^{-1} acts naturally as a reflection group on $M^k(w, T)$. The manifold $M^k(1, T)$ can be canonically identified with $M^k(W_T)$; moreover, this identification is $W_T \times J_{\theta(T)}$-equivariant. Translation by w^{-1} provides a diffeomorphism from $M^k(w, T)$ to $M^k(1, T)$. Thus,

$$M^k(w, T) \cong M^k(W_T). \tag{3}$$

We shall call $M^k(w, T)$ a *standard* submanifold of $M^n(W)$.

The next result shows that the standard submanifolds are analogous to incompressible surfaces in a 3-manifold.

LEMMA 7.1. *The fundamental group of any standard submanifold is mapped monomorphically into the fundamental group of $M^n(W)$ by the map induced by the inclusion.*

Proof. It clearly suffices to prove this for standard submanifolds of the form $M^k(1, T)$. We showed in Section 3 that the universal cover $\tilde{M}^n(W)$ of $M^n(W)$ admits an action of a reflection group $\tilde{J}$, where $\tilde{J}$ has one fundamental generator for each codimension-one face of $L^*(W)$. The fundamental group Γ of $M^n(W)$ is the kernel of the natural projection $\tilde{J} \to J$. The same construction applied to $L^*(W_T)(\cong F(1, T))$ yields a reflection group $\tilde{J}_{\theta(T)}$ on $\tilde{M}^k(W_T)$. It is clear that $\tilde{J}_{\theta(T)}$ can be identified with the subgroup of $\tilde{J}$ generated by the fundamental generators of J corresponding to those codimension-one faces of $L^*(W)$ which intersect $F(1,T)$ in a codimension-one face of $F(1,T)$. Thus, the fundamental group Γ_T of $M^k(1,T)$ is naturally identified with the intersection $\Gamma \cap \tilde{J}_{\theta(T)}$, a subgroup of Γ. This proves the lemma.

Call a subset T of S a *commuting* subset if any two elements in it commute with each other. If T is a commuting subset of S, then the subgroup W_T is isomorphic to $(\mathbf{Z}/2)^k$, where $k = |T|$; hence, $M^k(W_T)$ is a k-torus. Let $q(W)$ denote the cardinality of the largest commuting subset of S. It is clear that if the Coxeter diagram of W is a straight line, then $q(W) = [(n+1)/2]$. Also, it is easily checked that $q(\mathbf{D}_n) = [(n+2)/2]$, $q(\mathbf{E}_6) = 3$, and $q(\mathbf{E}_7) = q(\mathbf{E}_8) = 4$. This takes care of all finite irreducible Coxeter systems. It is also clear that $q(W_1 \times W_2) = q(W_1) + q(W_2)$. From these facts, we see that we always have $q(W) \geqslant n/2$. An immediate corollary of Lemma 7.1 is the following result.

PROPOSITION 7.2. *The fundamental group of $M^n(W)$ contains a free abelian subgroup of rank $q(W)$. Moreover, $q(W) \geqslant n/2$.*

It seems to be a plausible conjecture that $q(W)$ is the rank of the largest free abelian subgroup of $\pi_1(M^n(W))$.

8. A presentation of the fundamental group of $M^n(W)$. There is a natural CW-structure on $M^n(W)$: the cells are the translates of the faces of $L^*(W)$. The k-skeleton of $M^n(W)$ is then the union of the standard k-dimensional submanifolds. There is one such submanifold for each k-face of $L^*(W)$. From the 2-skeleton, we shall derive an explicit presentation for the fundamental group Γ of $M^n(W)$. There will be one generator for each standard circle. Each standard surface will contribute three relations.

The group $W \times J$ is a reflection group on $M^n(W)$ with fundamental chamber an n-cube. As in Section 3, this leads to a reflection group $\tilde{W}$ on the universal cover $\tilde{M}^n(W)$ with one fundamental generator for each codimension-one face of the n-cube. We can naturally identify this set of fundamental generators with $S \amalg R$. Let $\tilde{f}\colon \tilde{W} \to W \times J$ be the natural epimorphism, so that Γ is the kernel of $\tilde{f}$. (The group $\tilde{J}$, utilized in the proof of Lemma 7.1, is $\tilde{f}^{-1}(\{1\} \times J)$.) Also, the groups W and J can be identified with subgroups of $\tilde{W}$, namely, the subgroups $\tilde{W}_S$ and $\tilde{W}_R$, respectively. We shall give generators and relations for the subgroup Γ of $\tilde{W}$.

First let us set up some notation. Suppose that $S = \{s_1, \dots, s_n\}$, $R = \{r_1, \dots, r_n\}$, and $\theta(s_i) = r_i$. For each i, $1 \leqslant i \leqslant n$, put $W_i = W_{\{s_i\}}$, and for each pair of distinct integers i, j between 1 and n, put $W_{ij} = W_{\{s_i, s_j\}}$. Thus, W_i is a cyclic group of order 2 and W_{ij} is a dihedral group of order $2m_{ij}$, where $m_{ij} = m(s_i, s_j)$.

The generators. For each i, $1 \leqslant i \leqslant n$, and each $w \in W$, define an element α_i^w in $\tilde{W}$ by

$$(1) \qquad \alpha_i^w = w(r_i s_i)^2 w^{-1}.$$

Since $\tilde{f}(r_i)$ and $\tilde{f}(s_i)$ commute in $W \times J$, we have that $\tilde{f}(\alpha_i^w) = 1$, i.e., that α_i^w belongs to Γ. From the equation $s_i(r_i s_i)s_i = (r_i s_i)^{-1}$, we get that

$$\alpha_i^{w'} = \left(\alpha_i^w\right)^{-1}, \quad \text{whenever } w' = ws_i. \tag{2}$$

Moreover, it is clear that the element α_i^w corresponds to a loop around the standard circle $M^1(w, \{s_i\})$. Thus, the family of elements (α_i^w), where i ranges over the integers $1, 2, \dots, n$ and w ranges over a set of representatives for the cosets W/W_i, is a generating set for Γ.

The relations. Fix an element w in W and a pair of distinct integers i and j between 1 and n. To simplify notation put $m = m_{ij}$ (= the order of $s_i s_j$). The face $F(w, \{s_i, s_j\})$ is a $2m$-gon. The standard submanifold $M^2(w, \{s_i, s_j\})$ is made up of the 4 translates of this polygon under the group $J_{\{r_i, r_j\}}$. By Remark 4.6, the Euler characteristic of $M^2(w, \{s_i, s_j\}$ is $4 - 2m$, i.e., $M^2(w, \{s_i, s_j\})$ is a surface of genus $m - 1$. The fundamental group of $M^2(w, \{s_i, s_j\})$ is a subgroup of Γ; we shall denote this subgroup by Γ_{ij}^w. Define elements w_l in W_{ij} by

$$w_{2k} = \left(s_i s_j\right)^k, \qquad w_{2k+1} = \left(s_i s_j\right)^k s_i, \tag{3}$$

so that $W_{ij} = \{w_1, \dots, w_{2m}\}$. Let $\lambda_1, \lambda_2, \dots, \lambda_{2m}$ be the generators of Γ_{ij}^w defined by

$$\lambda_l = \begin{cases} \alpha_j^u, & \text{if } l \text{ is even} \\ \alpha_i^u, & \text{if } l \text{ is odd} \end{cases} \tag{4}$$

where $u = ww_l$. Define words $\beta_{ij}^w, \gamma_{ij}^w, \delta_{ij}^w$ in these generators as follows:

$$\begin{aligned} \beta_{ij}^w &= \lambda_{2m}\lambda_{2m-2} \cdots \lambda_2 \\ \gamma_{ij}^w &= \lambda_{2m-1}\lambda_{2m-3} \cdots \lambda_1 \\ \delta_{ij}^w &= \lambda_{2m}\lambda_{2m-1} \cdots \lambda_1. \end{aligned} \tag{5}$$

Easy calculations show that the relations $\beta_{ij}^w = \gamma_{ij}^w = \delta_{ij}^w = 1$ hold in $\tilde{W}$. Moreover, it is not hard to see that $\langle \lambda_1, \dots, \lambda_{2m}; \beta_{ij}^w, \gamma_{ij}^w, \delta_{ij}^w \rangle$ is a presentation for the surface group Γ_{ij}^w. A geometrical argument for these facts is illustrated in Figure 2, where the case $m = 3$ is pictured. The central hexagon is the face $F(w, \{s_i, s_j\})$. The e_k, $1 \leqslant k \leqslant 6$, are the edges of the hexagon. The quadrilaterals are chambers for $\tilde{W}_{ij}$ on $\tilde{M}^2(w, \{s_i, s_j\})$. The entire star-shaped polygon with $4m$ edges (12 in this case) is a fundamental domain (in the weak sense) for Γ_{ij}^w on the universal cover of the surface. The $4m$ sides of the star-shaped polygon are identified in pairs by the λ_l to form a surface of genus $m - 1$. Under these identifications the vertices of the star-shaped polygon fall into three equivalence classes. The three

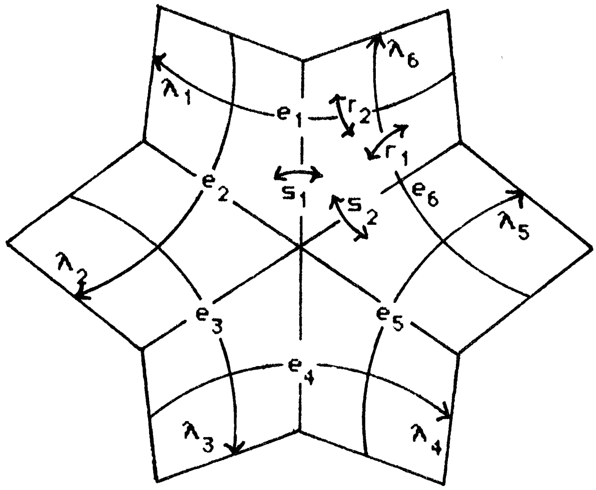


FIGURE 2.

relations given by (5) can be read off from the translates of the polygon which meet at a vertex from each equivalence class.

We should comment on the effect of replacing w in the above discussion by wa for some a in W_{ij}. If a is a "rotation" of the form w_{2k}, then the new λ_l's are obtained from the old ones by a cyclic permutation. If a is "reflection" of the form w_{2k+1}, then the new λ_l's are obtained from the old by composing a cyclic permutation with the transformation $\lambda_1 \to (\lambda_{2m+1-l})^{-1}$. Hence, in either case the set consisting of the generators and their inverses remains unchanged. Also, the new relations are obviously equivalent to the old ones.

In summary we have the following result:

PROPOSITION 8.1. *For each i, $1 \leqslant i \leqslant n$, choose a set A_i of representatives for W/W_i and for $1 \leqslant i < j \leqslant n$, choose a set of representatives A_{ij} for W/W_{ij}. Let G be the set $\{\alpha_i^w\}$, where $1 \leqslant i \leqslant n$ and $w \in A_i$, and let Z be the set $\{\beta_{ij}^u, \gamma_{ij}^u, \delta_{ij}^u\}$ where $1 < i < j \leqslant n$ and $u \in A_{ij}$ and where the elements of Z are regarded as words in the elements of G and the inverses. Then the fundamental group Γ of $\pi_1(M^n(W))$ has a presentation of the form $\langle G; Z\rangle$.*

9. A "perfect" cell structure on $M^n(W)$. For each w in W we shall define a subset $C(w)$ of $M^n(W)$ such that $C(w)$ is homeomorphic to the interior of a cell and such that $M^n(W)$ is the disjoint union of the $C(w)$, $w \in W$.[4] The closure of $C(w)$ is a standard submanifold. This cell structure is compatible with the Bruhat ordering on W: the closure of $C(w)$ is contained in cells of the form $C(v)$, where $v < w$.

[4]This is *not* a CW-structure on $M^n(W)$. The boundary of a cell might be contained in cells of higher dimension.

The number of cells in this subdivision is $|W|$. This number is also the total rank of $H_*(M^n(W))$ (cf. Theorem 5.4). It follows that the cell structure is "perfect" in the sense of Morse theory. That is to say, if we put $p(t) = \Sigma t^{\dim C(w)}$, then $p(t)$ is the Poincaré polynomial of $M^n(W)$.

This analogy with Morse theory is not a coincidence. We shall show in Section 13 that when W is the Weyl group of a split semisimple real Lie algebra, then the cells $C(w)$ are the ascending manifolds of a certain naturally defined Morse-Smale flow on $M^n(W)$.

Combinatorics of Coxeter systems. Here we shall review some standard material concerning the Coxeter system (W, S). Details and further information can be found in [B] and [H].

Denote by $\mathscr{R}$ the set of all conjugates of elements of S. For each w in W, put

(1) $$S(w) = \{s \in S | l(ws) < l(w)\}$$

(2) $$\mathscr{R}(w) = \{r \in \mathscr{R} | l(rw) < l(w)\}.$$ [5]

Then

(3) $$|\mathscr{R}(w)| = l(w).$$

The *Bruhat ordering* is the partial ordering on W defined by: $v < w$ if and only if $\mathscr{R}(v) \subsetneqq \mathscr{R}(w)$.

For each subset T of S, put

(4) $$A_T = \{w \in W | l(wt) > l(w) \text{ for all } t \in T\}$$

(5) $$W^T = \{w \in W | S(w) = T\}.$$

For any $v \in A_T$ and $a \in W_T$, we have that

(6) $$\mathscr{R}(va) = \mathscr{R}(v) \amalg \mathscr{R}(a),$$

and consequently,

(7) $$l(va) = l(v) + l(a).$$

For any w in W there is a unique element in wW_T of shortest length. Furthermore, this element belongs to A_T. Thus, we can write any element w uniquely in the form $w = va$ where $v \in A_T$ and $a \in W_T$. Let p_T: $W \to A_T$ denote the function $w \to v$.

[5] Geometrically, $\mathscr{R}(w)$ is the set of reflections in W such that the corresponding wall separates the fundamental chamber X from its translate wX. The set $S(w)$ is the subset of S such that the walls corresponding to the elements of $wS(w)w^{-1}$ separate X from wX.

Now suppose that the subset T is such that the subgroup W_T is finite. Then there is a unique element of longest length in W_T. Denote it by w_T. It can be characterized as the only element in W_T which satisfies either of the following two conditions:

$$S(w_T) = T, \tag{8}$$

$$\mathscr{R}(w_T) = \mathscr{R} \cap W_T. \tag{9}$$

It follows from (6) and the above discussion that the coset wW_T contains a unique maximum element, namely $p_T(w)w_T$, as well as a unique minimum element, namely $p_T(w)$. The next result follows immediately from these remarks.

LEMMA 9.1. *An element w of W is the maximum in wW_T if and only if $T \subset S(w)$. In particular, w is the maximum in $wW_{S(w)}$.*

The cells. Let $\mathring{F}(w, T)$ denote the interior of $F(w, T)$ (where $F(w, T)$ is the face of $L^*(W)$ corresponding to the coset wW_T). For each w in W, put

$$H(w) = \coprod_{T \subset S(w)} \mathring{F}(w, T). \tag{10}$$

In view of the previous lemma, we have that $H(w)$ is the union of the interiors of those faces whose maximum vertex is w (where we are identifying the vertex $F(w, \phi)$ with w). Since each face of $L^*(W)$ has a maximum vertex, the family $(H(w))_{w \in W}$ is a disjoint partition of $L^*(W)$.

The set $H(w)$ is $\mathring{F}(w, S(w))$ with part of its boundary attached; $H(w)$ contains the vertex w as well as any face of $F(w, S(w))$ which contains w. It follows that $H(w)$ is isomorphic to a simplicial cone of dimension $|S(w)|$. In other words, $H(w)$ is isomorphic to a fundamental chamber for the action of $J_{\theta(S(w))}$ as a linear reflection group on Euclidean space.

Put

$$C(w) = J_{\theta(S(w))} H(w). \tag{11}$$

By the above remarks, $C(w)$ is equivariantly diffeomorphic to Euclidean space of dimension $k (= |S(w)|)$ equipped with a linear $J_{\theta(S(w))}$ action (where $J_{\theta(S(w))} \cong (\mathbb{Z}/2)^k$). The closure of $C(w)$ is clearly $J_{\theta(S(w))} F(w, S(w))$, i.e., it is the standard submanifold $M(w, S(w))$. Hence, $\overline{C(w)} - C(w)$ is the union of the standard submanifolds corresponding to the faces of $F(w, S(w))$ which are not in $H(w)$. The vertices of any face of $F(w, S(w))$ can be identified with the elements of $wW_{S(w)}$. If such a face is not in $H(w)$, then its maximum vertex is some element v in $wW_{S(w)}$, with $v \neq w$. Since w is the maximum of $wW_{S(w)}$, we have $v < w$. Thus, $\overline{C(w)} - C(w)$ is contained in the union of cells of the form $C(v)$ where v is as above. In summary, we have proved the following result.

PROPOSITION 9.2. (i) *Suppose that* $w \in W$ *and that* $|S(w)| = k$, *where* $S(w)$ *is the subject of S defined by* (1). *Then* $C(w)$ *is diffeomorphic to the interior of a k-cell. Moreover,* $\overline{C(w)}$ *is the standard submanifold* $M^k(w, S(w))$.

(ii) *The family* $(C(w))_{w \in W}$ *is a disjoint partition of* $M^n(W)$.

(iii) $\overline{C(w)} - C(w)$ *is contained in the union of cells of the form* $C(v)$, *where* v *ranges over* $wW_{S(w)} - \{w\}$. *In particular, any such* v *is* $< w$.

Remark 9.3. In the summer of 1984, after listening to a lecture of mine, David Fried told me that he could prove that the Toda flow on Tomei's manifold P^n was perfect in the sense of Morse theory. (See Section 13.) These remarks of Fried inspired me to prove the results of this section as well as those in Section 5 and in paper [D2]. After writing this paper, I read Fried's arguments in [F]. The elementary arguments of [F] show that Proposition 9.2 implies that the cell structure on $M^n(W)$ is perfect (without resorting to Theorem 5.4). The crucial fact is that the closure of $C(w)$ is the orientable submanifold $M^k(w, S(w))$ (that it is orientable follows immediately from Proposition 1.4). Similarly, Fried's other arguments concerning the cohomology of $M^n(W)$ go through with little change; in particular, $H^*(M^n(W); \mathbb{Z})$ is generated by $H^1(M^n(W); \mathbb{Z})$.

10. Real semisimple split Lie algebras. In this section we shall set up some notation and review some standard facts.

Let $\mathfrak{g}$ be a real semisimple split Lie algebra and let $\mathfrak{h}$ be a split Cartan subalgebra.[6] Denote by Δ the set of nonzero roots in the dual space $\mathfrak{h}^*$. For each α in Δ, denote the corresponding root space by $\mathfrak{g}_\alpha$.

Cartan decomposition. For each pair of roots $\{\alpha, -\alpha\}$, choose root vectors e_α in $\mathfrak{g}_\alpha$ and $e_{-\alpha}$ in $\mathfrak{g}_{-\alpha}$ such that

$$\langle e_\alpha, e_{-\alpha} \rangle = 1, \tag{1}$$

where $\langle\ ,\ \rangle$ denotes the Killing form. There is an involutive automorphism φ: $\mathfrak{g} \to \mathfrak{g}$, called a *Cartan involution*, defined by the requirements that φ takes $\mathfrak{h}$ to itself by multiplication by -1 and that for each α in Δ, it takes e_α to $-e_{-\alpha}$. Put

$$\mathfrak{k} = \sum (e_\alpha - e_{-\alpha}) \tag{2}$$

$$\mathfrak{p} = \mathfrak{h} + \sum (e_\alpha + e_{-\alpha}), \tag{3}$$

where the summations run over all pairs $\{\alpha, -\alpha\}$. The decomposition $\mathfrak{g} = \mathfrak{k} \otimes \mathfrak{p}$ is called a *Cartan decomposition* of $\mathfrak{g}$.

A fundamental chamber of a system of simple roots. The family of hyperplanes $(\alpha = 0)_{\alpha \in \Delta}$ partitions $\mathfrak{h}$ into simplicial cones. Choose one of these simplicial

[6] To say that $\mathfrak{h}$ is a split Cartan subalgebra means that for each $x \in \mathfrak{h}$, the eigenvalues of ad x are real.

cones. Denote it by C and its interior by $\mathring{C}$. The n-dimensional simplicial cone C can be written uniquely as an intersection of n half-spaces, of the form $\alpha \geqslant 0$, where α is a root. The set of such α is denoted by $\Pi(=\{\alpha_1,\ldots,\alpha_n\})$ and called the system of simple roots corresponding to C. Any α in Δ can be written uniquely as an integral linear combination of elements in Π; furthermore, the coefficients are either all nonnegative or all nonpositive. The α with nonnegative coefficients (resp. nonpositive coefficients) are called the *positive roots* (resp. *negative roots*) and denoted by Δ_+ (resp. Δ_-). For $1 \leqslant i \leqslant n$, let s_i denote the orthogonal linear reflection on $\mathfrak{h}$ across the hyperplane $\alpha_i = 0$. Put $S = \{s_1,\ldots,s_n\}$. The *Weyl group* W of $(\mathfrak{g}, \mathfrak{h})$ is the linear reflection group generated by S. The simplicial cone C is a fundamental chamber for W on $\mathfrak{h}$.

Some vectors in $\mathfrak{g}$. For any α in $\mathfrak{h}^*$, let h_α be the unique element of $\mathfrak{h}$ defined by the equation

$$\langle h_\alpha, x\rangle = \alpha(x), \quad \text{for all } x. \tag{4}$$

It follows from (1) that for all $\alpha \in \Delta$, we have

$$[e_\alpha, e_{-\alpha}] = h_\alpha. \tag{5}$$

For $1 \leqslant i \leqslant n$, define vectors $h_i \in \mathfrak{h}$, $f_i \in \mathfrak{p}$, and $g_i \in \mathfrak{k}$ by

$$h_i = h_{\alpha_i} \tag{6}$$

$$f_i = e_{\alpha_i} + e_{-\alpha_i} \tag{7}$$

$$g_i = e_{\alpha_i} - e_{-\alpha_i}. \tag{8}$$

The subspace Y of generalized tridiagonal matrices. Let Y be the $2n$-dimensional subspace of $\mathfrak{p}$ defined by

$$Y = \mathfrak{h} + \sum_{i=1}^{n} \mathbb{R} f_i. \tag{9}$$

(If $\mathfrak{g} = sl(n+1, \mathbb{R})$), then we can take $\mathfrak{h}$ to be the subspace of diagonal matrices in $\mathfrak{g}$, $\mathfrak{p}$ the subspace of symmetric matrices, and Y the subspace of tridiagonal symmetric matrices.)

Subgroups of the complex adjoint group. Let $G_{\mathbb{C}}$ denote the group of inner automorphisms of the complex Lie algebra $\mathfrak{g} \otimes \mathbb{C}$. Let $H_{\mathbb{C}}$ and $K_{\mathbb{C}}$ denote the subgroups corresponding to the complex subalgebras $\mathfrak{h} \otimes \mathbb{C}$ and $\mathfrak{k} \otimes \mathbb{C}$, respectively. Let G denote the subgroup of $G_{\mathbb{C}}$ which stabilizes the real subalgebra $\mathfrak{g}$.[7]

[7]N.B. The identity component of G is the real adjoint group; however, in general, G will *not* be connected.

Define three more groups by

(10) $$K = G \cap K_{\mathbb{C}},$$

(11) $$H = G \cap H_{\mathbb{C}},$$

(12) $$J = K \cap H.$$

We determine the structure of J. For each α in Δ, let $\hat{\alpha}$: $H_{\mathbb{C}} \to \mathbb{C}^*(= \mathbb{C} - \{0\})$ be the homomorphism defined by the diagram

$$\begin{array}{ccccc} \mathfrak{h} \otimes \mathbb{C} & \xrightarrow{\alpha} & \mathbb{C} & \qquad & z \\ {\scriptstyle \exp}\downarrow & & \downarrow & & \downarrow \\ H_{\mathbb{C}} & \overset{\hat{\alpha}}{\dashrightarrow} & \mathbb{C}^* & & e^{2\pi i z} \end{array}$$

For any h in $H_{\mathbb{C}}$ we have that

(13) $$h \cdot e_\alpha = \hat{\alpha}(h) e_\alpha .$$

Since $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for $(\mathfrak{h} \otimes \mathbb{C})^*$, the map $h \to (\hat{\alpha}_1(h), \ldots, \hat{\alpha}_n(h))$ is an isomorphism from $H_{\mathbb{C}}$ onto $(\mathbb{C}^*)^n$. Moreover, this isomorphism takes J onto the subgroup $\{\pm 1\}^n$. For $1 \leqslant i \leqslant n$, let r_i be the element in $H_{\mathbb{C}}$ which maps to the point in $(\mathbb{C}^*)^n$ with ith coordinate -1 and all other coordinates equal to 1. Put

(14) $$R = \{r_1, \ldots, r_n\}.$$

Obviously $J = \langle R \rangle$. From (13), we have that

(15) $$r_i \cdot e_{\alpha_j} = (-1)^{\delta_{ij}} e_{\alpha_j}$$

(16) $$r_i \cdot f_j = (-1)^{\delta_{ij}} f_j \quad \text{and} \quad r_i \cdot g_j = (-1)^{\delta_{ij}} g_j .$$

The K-action on $\mathfrak{p}$. The group K stabilizes $\mathfrak{p}$. Each K-orbit in $\mathfrak{p}$ intersects $\mathfrak{h}$ in a W-orbit. Each W-orbit in $\mathfrak{h}$ intersects the fundamental chamber C in a single point. That is to say, the inclusions $C \subset \mathfrak{h} \subset \mathfrak{p}$ induce homeomorphisms $C \cong \mathfrak{h}/W \cong \mathfrak{p}/K$. We shall often identify C, $\mathfrak{h}/W$ and $\mathfrak{p}/K$ via these canonical homeomorphisms.

A point x in $\mathfrak{h}$ is *regular* if it lies in the interior of a chamber. A point x in $\mathfrak{p}$ is *regular* if its orbit $K(x)$ intersects $\mathfrak{h}$ in regular elements. Denote by $\mathfrak{h}_{\text{reg}}$ and $\mathfrak{p}_{\text{reg}}$ the set of regular elements in $\mathfrak{h}$ and $\mathfrak{p}$, respectively. Obviously, $\mathring{C} = \mathfrak{h}_{\text{reg}}/W = \mathfrak{p}_{\text{reg}}/K$.

If x is a regular point in $\mathfrak{p}$, then its K-isotropy subgroup is conjugate to J. The space $\mathfrak{p}_{\text{reg}}$ is the union of principal orbits of K on $\mathfrak{p}$. The orbit map $\mathfrak{p}_{\text{reg}} \to \mathring{C}$ is the projection map of a smooth fiber bundle with fiber K/J.

The J-action on Y. Let $(b_1(y), \dots b_n(y), a_1(y), \dots, a_n(y))$ be the linear coordinates of a point y in Y with respect to the basis $\{h_1, \dots, h_n, f_1, \dots, f_n\}$ of Y, i.e.,

$$y = \sum_{i=1}^{n} b_i(y) h_i + \sum_{i=1}^{n} a_i(y) f_i. \tag{17}$$

The group J stabilizes Y. The fixed point set of J on Y is $\mathfrak{h}$. A fundamental generator r_i in R acts on Y as the orthogonal linear reflection across the hyperplane $a_i = 0$. It follows that a fundamental chamber for J on Y is the convex set Y_+ defined by the linear inequalities $a_i \geqslant 0$, $1 \leqslant i \leqslant n$. In other words, the set Y_+ is the Cartesian product of $\mathfrak{h}$ with a simplicial cone.

11. The Toda flow. Let $\lambda\colon \mathfrak{p} \to \mathfrak{k}$ be the linear map with kernel $\mathfrak{h}$ which for each α in Δ_+ sends $e_\alpha + e_{-\alpha}$ to $e_\alpha - e_{-\alpha}$. Consider the linear differential equation on the vector space $\mathfrak{p}$ given by

$$\frac{dx}{dt} = [\lambda(x), x].$$

The associated vector field $\phi(x) = [\lambda(x), x] \in \mathfrak{p} \cong T_x\mathfrak{p}$ is called the *Toda vector field*. We collect some standard facts.

LEMMA 11.1. (i) *The Toda vector field is J-equivariant.*
(ii) *The Toda vector field is tangent to each K-orbit.*
(iii) *Suppose $x \in \mathfrak{p}_{\text{reg}}$. Then $\phi(x) = 0$ if and only if $x \in \mathfrak{h}$.*

Proof. (i) It follows from (15) in Section 10 that for any ε in J, $\lambda(\varepsilon x) = \varepsilon\lambda(x)$. Since J acts on $\mathfrak{g}$ through automorphisms, we have $[\lambda(\varepsilon x), \varepsilon x] = \varepsilon[\lambda(x), x]$, which proves (i).

(ii) For each x in $\mathfrak{p}$ the tangent space of the orbit $K(x)$ is naturally identified with $[\mathfrak{k}, x]$. Since $\lambda(x) \in \mathfrak{k}$, we have that $\phi(x)$ is tangent to $K(x)$, i.e., (ii) holds.

(iii) Since x is regular, the linear map $\mathfrak{k} \to \mathfrak{p}$ given by $z \to [z, x]$ is an injection. Hence, $\phi(x) = 0$ if and only if $\lambda(x) = 0$, i.e., if and only if $x \in \mathfrak{h}$.

Since K is compact, the Toda vector field is complete. The corresponding flow is called the Toda flow.

LEMMA 11.2. *The Toda vector field is tangent to Y.*

Proof. Let Z be the image of Y under λ, i.e., $Z = \sum \mathbb{R} g_i$. The restriction of λ to Y is the linear map defined by

$$\lambda(y) = \sum_{i=1}^{n} a_i(y) g_i \tag{1}$$

where $(b_1(y), \dots b_n(y), a_1(y), \dots, a_n(y))$ are the linear coordinates on Y defined by equation (17) of Section 10. For any h in $\mathfrak{h}$, we have

(2) $$[g_i, h] = \alpha_i(h) f_i.$$

For $i \neq j$, we have $[e_{\alpha_i}, e_{-\alpha_j}] = 0$, while $[e_{\alpha_i}, e_{-\alpha_i}] = h_i$. If follows that

(3) $$[g_i, f_j] = \begin{cases} -[g_j, f_i], & \text{for } i \neq j \\ 2h_i, & \text{for } i = j \end{cases}.$$

From (2) and (3) we get

(4) $$[\lambda(y), y] = 2\sum_{i=1}^{n} a_i(y)^2 h_i + \sum_{i=1}^{n} \tilde{\alpha}_i(y) a_i(y) f_i$$

where $\tilde{\alpha}_i$ is the linear form on Y defined by pre-composing α_i with the orthogonal projection from Y onto $\mathfrak{h}$. In particular, from (4) we see that $[\lambda(y), y] \in Y$, i.e., $\phi(y)$ is tangent to Y.

In the remainder of this section we are interested in restriction of the Toda flow to Y, we shall denote it by ψ_t: $Y \to Y$. The proofs of the next two lemmas are minor modifications of the proofs in Section 2 of [M] in the special case of ordinary tridiagonal matrices (i.e., for $\mathfrak{g} = sl(n+1, \mathbb{R})$).

LEMMA 11.3. *Let $y(t)$ be an integral curve for the Toda vector field on Y. As $t \to \pm\infty$, $y(t)$ converges to points $y(\infty)$ and $y(-\infty)$ in $\mathfrak{h}$.*

By Lemma 11.1(ii), the integral curve $y(t)$ lies in a single K-orbit; hence, $y(\infty)$ and $y(-\infty)$ are also in this K-orbit. This K-orbit intersects $\mathfrak{h}$ in a W-orbit and it intersects C in a single point c. Since $y(\infty)$ and $y(-\infty)$ are in $\mathfrak{h}$, both these points are W-translates of c, i.e., $y(\infty) = wc$ and $y(-\infty) = w'c$ for some w and w' in W. The elements w and w' are determined as follows.

LEMMA 11.4. *Let $y(t)$ be an integral curve for the Toda vector field on Y and let c be the point where the K-orbit of $y(t)$ intersects C. Then $y(-\infty) = c$ and $y(\infty) = w_S c$, where w_S is the element of longest length in W.*

Denote the restriction of the orbit map $\mathfrak{p} \to \mathfrak{p}/K = C$ to Y_{reg} by π: $Y_{\text{reg}} \to \mathring{C}$. The proof of the next result follows as in [T], Lemma 2.2.

LEMMA 11.5 **(Tomei)**. *The map π: $Y_{\text{reg}} \to \mathring{C}$ is the projection map of a smooth fiber bundle.*

12. Some subsets, subgroups and subspaces. The purpose of this section is to set up more notation and to collect some obvious facts. These facts will be stated as lemmas, the proofs of which will be left to the reader. Throughout this section, A will denote a subset of I_n. (Recall that $I_n = \{1, \ldots, n\}$.)

First, we define linear subspaces of $\mathfrak{h}$ as follows:

(1) $$\mathfrak{h}(A) = \sum_{i \in A} \mathbb{R} h_i,$$

(2) $$\mathfrak{h}_A = \sum_{j \in I_n - A} \mathbb{R} t_j,$$

where $\{h_1, \ldots, h_n\}$ is the basis for $\mathfrak{h}$ defined in Section 10, and where $\{t_1, \ldots, t_n\}$ is the basis for $\mathfrak{h}$ which is dual to $\{\alpha_1, \ldots, \alpha_n\}$. Put

(3) $$S(A) = \{s_i \in S \mid i \in A\}.$$

LEMMA 12.1.
(i) *The subspace* $\mathfrak{h}_A$ *of* $\mathfrak{h}$ *is the intersection of the hyperplanes* $\alpha_i = 0$, $i \in A$.
(ii) *The subspaces* $\mathfrak{h}_A$ *and* $\mathfrak{h}(A)$ *are orthogonal complements in* $\mathfrak{h}$.
(iii) *The fixed point set of* $W_{S(A)}$ *on* $\mathfrak{h}$ *is* $\mathfrak{h}_A$.

Define convex subsets of $\mathfrak{h}$ as follows:

(4) $C(A) =$ the intersection of the half-spaces $\alpha_i \geqslant 0$, $i \in A$,

(5) $\mathring{C}(A) =$ the interior of $C(A)$,

(6) $\overline{C}(A) = C(A) \cap \mathfrak{h}(A)$.

LEMMA 12.2.
(i) $\overline{C}(A)$ *is a simplicial cone in* $\mathfrak{h}(A)$.
(ii) $C(A) = \mathfrak{h}_A \times \overline{C}(A)$.
(iii) $C(A)$ *is a chamber for* $W_{S(A)}$ *on* $\mathfrak{h}$ *and* $\overline{C}(A)$ *is a chamber for* $W_{S(A)}$ *on* $\mathfrak{h}(A)$.

It follows that the orbit space of $W_{S(A)}$ on $\mathfrak{h}$ can be identified with $C(A)$. We denote the natural projection by

(7) $$q_A\colon \mathfrak{h} \to C(A).$$

Let $\lambda_A\colon C(A) \to C$ be the projection map defined by the following diagram:

(8)
$$\begin{array}{ccc} & \mathfrak{h} & \\ {}^{q_A}\swarrow & & \searrow^{q} \\ C(A) & \xrightarrow[\lambda_A]{} & C \end{array}$$

where $q\colon \mathfrak{h} \to \mathfrak{h}/W = C$ is the orbit map.

Define subsets of Δ by

(9) $$\Pi(A) = \{\alpha_i \in \Pi \mid i \cap A\}$$

(10) $$\Delta(A) = \{\alpha \in \Delta \mid \alpha \text{ is an integral linear combination of elements of } \Pi(A)\}$$

(11) $$\Delta_+(A) = \Delta_+ \cap \Delta(A).$$

LEMMA 12.3. *The set* $\Delta(A)$ *is naturally a root system in* $\mathfrak{h}(A)^*$; $\Pi(A)$ *is a system of simple roots for* $\Delta(A)$; *and* $(W_{S(A)}, S(A))$ *is the corresponding Coxeter system.*

Next, define subspaces of $\mathfrak{g}$ as follows:

(12) $$\bar{\mathfrak{g}}(A) = \mathfrak{h}(A) + \sum_{\alpha \in \Delta(A)} \mathbb{R} e_\alpha$$

(13) $$\bar{\mathfrak{p}}(A) = \mathfrak{h}(A) + \sum_{\alpha \in \Delta_+(A)} \mathbb{R}(e_\alpha + e_{-\alpha})$$

(14) $$\mathfrak{k}(A) = \sum_{\alpha \in \Delta_+(A)} \mathbb{R}(e_\alpha - e_{-\alpha})$$

(15) $$\mathfrak{g}(A) = \mathfrak{h}_A + \bar{\mathfrak{g}}(A)$$

(16) $$\mathfrak{p}(A) = \mathfrak{h}_A + \bar{\mathfrak{p}}(A).$$

LEMMA 12.4.

(i) *The subalgebra* $\bar{\mathfrak{g}}(A)$ *is a real semisimple split Lie algebra*; $\mathfrak{h}(A)$ *is a split Cartan subalgebra*; *and* $\bar{\mathfrak{g}}(A) = \mathfrak{k}(A) + \bar{\mathfrak{p}}(A)$ *is a Cartan decomposition.*

(ii) *The subalgebra* $\mathfrak{g}(A)$ *is reductive and its center is* $\mathfrak{h}_A$. *Moreover*, $\mathfrak{g}(A)$ *is the centralizer of* $\mathfrak{h}_A$ *in* $\mathfrak{g}$.

Let $K(A)$ denote the intersection of the stabilizer of $\mathfrak{g}(A)$ in G with K. The group $K(A)$ acts naturally on $\bar{\mathfrak{p}}(A)$ and the orbit space can be identified with $\bar{C}(A)$. Similarly, $K(A)$ acts on $\mathfrak{p}(A)$ with orbit space $C(A)$. Denote the natural projection by

(17) $$p_A \colon \mathfrak{p}(A) \to C(A).$$

LEMMA 12.5. *The following diagram commutes,*

$$\begin{array}{ccc} \mathfrak{p}(A) & \subset & \mathfrak{p} \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle p} \\ C(A) & \xrightarrow{\lambda_A} & C \end{array}$$

where λ_A *is defined by diagram* (8).

Define a subset of the set of fundamental generators for J by

$$R(A) = \{r_i \in R \mid i \in A\}. \tag{18}$$

Clearly, $J_{R(A)} = K(A) \cap J$.

Finally, consider the following linear subspaces of the space Y of generalized tridiagonal matrices:

$$\overline{Y}(A) = \mathfrak{h}(A) + \sum_{i \in A} \mathbb{R} f_i, \tag{19}$$

$$Y(A) = \mathfrak{h} + \sum_{i \in A} \mathbb{R} f_i. \tag{20}$$

Obviously, $\overline{Y}(A) = Y \cap \overline{\mathfrak{p}}(A)$ and $Y(A) = Y \cap \mathfrak{p}(A)$.

LEMMA 12.6.

(i) *$\overline{Y}(A)$ is the space of generalized tridiagonal matrices for $\overline{\mathfrak{g}}(A)$ and $Y(A) = \mathfrak{h}_A + \overline{Y}(A)$.*

(ii) *The fixed point set of the group $J_{R-R(A)}$ on Y is $Y(A)$.*

(iii) *The group $J_{R(A)}$ acts as a linear reflection group on $Y(A)$ with fixed point set $\mathfrak{h}$.*

13. The isospectral fiber. The orbit map $\pi\colon Y_{\text{reg}} \to \mathring{C}$ is the projection map of a smooth fiber bundle (cf. Lemma 11.5). Fix a point x in $\mathring{C}$ and denote the fiber of π at x by $P^n(\mathfrak{g})$. In other words, $P^n(\mathfrak{g}) = Y \cap K(x)$. Since J is a subgroup of K, the J-action on Y stabilizes each fiber of π. Furthermore, J is a reflection group on $P^n(\mathfrak{g})$. A fundamental chamber is the manifold with corners $X^n(\mathfrak{g})$ defined by

$$X^n(\mathfrak{g}) = Y_+ \cap P^n(\mathfrak{g}), \tag{1}$$

where Y_+ is a fundamental chamber for J on Y. Let $\mathring{X}^n(\mathfrak{g})$ denote the interior of $X^n(\mathfrak{g})$.

The goal of this section is to prove the following result.

THEOREM 13.1. *The "isospectral fiber" $P^n(\mathfrak{g})$ is J-equivariantly homeomorphic to the manifold $M^n(W)$ of Example* 4.4.

In view of Theorem 6.3 proving this theorem is equivalent to proving the following proposition.

PROPOSITION 13.2. *There is a face-preserving homeomorphism from $X^n(\mathfrak{g})$ to the dual of the Coxeter complex $L^*(W)$.*

Remark. When $\mathfrak{g} = sl(n+1, \mathbb{R})$ this proposition is equivalent to the results in Section 4, pp. 989–993, of [T]. Although Tomei makes no mention of Lie algebras his arguments are similar to the proof given below.

We shall prove this proposition by showing that

(A) the interior of each face of $X^n(\mathfrak{g})$ is diffeomorphic to the interior of a cell, and

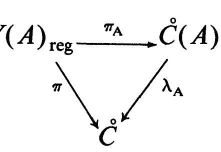
(B) the faces of $X^n(\mathfrak{g})$ intersect in exactly the same fashion as do the faces of $L^*(W)$.

Using (A) and (B) it is easy to construct a face-preserving homeomorphism from $X^n(\mathfrak{g})$ to $L^*(W)$. (Undoubtedly, $X^n(\mathfrak{g})$ and $L^*(W)$ are diffeomorphic as smooth manifolds with corners (or as smooth orbifolds); however, the method of proof indicated above is not well suited for proving this.) The proof of (A) comes down to the following result.

PROPOSITION 13.3 (Moser [M], Kostant [K]). *The interior of the fundamental chamber $\mathring{X}^n(\mathfrak{g})$ is diffeomorphic to $\mathbb{R}^n$.*

This proposition follows from the fact that the Toda flow is a completely integrable Hamiltonian flow on $\mathring{Y}_+$ and a general principle. The general principle (Liouville's Theorem) states that the integral submanifold $\mathring{X}^n(\mathfrak{g})$ is diffeomorphic to a disjoint union of a products of the form $T^s \times \mathbb{R}^{n-s}$ where T^s is a torus. It can then be shown that $\mathring{X}^n(\mathfrak{g})$ is $\mathbb{R}^n$. In [M], Moser gives an explicit diffeomorphism $\mathring{X}^n(\mathfrak{g}) \cong \mathbb{R}^n$ when $\mathfrak{g} = sl(n+1, \mathbb{R})$. A different approach is taken in [K] by Kostant, who works with an arbitrary real semisimple split Lie algebra.

Next, consider the faces of $X^n(\mathfrak{g})$. We shall use the notation of Section 12. Let A be a subset of I_n. The group $K(A)$ acts on $\mathfrak{p}(A)$ with orbit space $C(A)$. Denote the restriction of the orbit map to $Y(A) \cap Y_{\text{reg}}$ by $\pi_A\colon Y(A) \cap Y_{\text{reg}} \to \mathring{C}(A)$. Consider the following commutative diagram,



where λ_A is defined by diagram (8) in Section 12. We see that the intersection of $P^n(\mathfrak{g})$ (or $X^n(\mathfrak{g})$) with the subspace $Y(A)$ has several components, one component for each element of $\lambda_A^{-1}(x)$. The elements of $\lambda_A^{-1}(x)$ can be identified with the right cosets $W_{S(A)} \setminus W$. In fact, we can identify $\lambda_A^{-1}(x)$ with the set of all elements of the form $q_A(wx)$ where $w \in W$ and $q_A\colon \mathfrak{h} \to C(A)$ is the orbit map for $W_{S(A)}$. For each element w in W, put

(2) $$P(w, A) = \pi_A^{-1}\big(q_A(w^{-1}x)\big),$$

and

(3) $$X(w, A) = P(w, A) \cap X^n(\mathfrak{g}).$$

We have the following lemma.

LEMMA 13.4. *Let $d = |A|$ and let $\bar{\mathfrak{g}}(A)$ be the semisimple Lie algebra of rank d defined by equation* (12) *of Section* 12.

(i) $P(w, A) \cong P^d(\bar{\mathfrak{g}}(A))$

(ii) $X(w, A) \cong X^d(\bar{\mathfrak{g}}(A))$

(iii) *The interior of $X(w, A)$ is diffeomorphic to $\mathbb{R}^d$.*

(iv) *The fixed point set of $J_{R-R(A)}$ on $P^n(\mathfrak{g})$ is the intersection of $Y(A)$ with $P^n(\mathfrak{g})$. This fixed point set can also be described as the disjoint union of the d-dimensional submanifolds $P(w, A)$ where w ranges over a set of representatives for the cosets $W/W_{S(A)}$.*

Proof. Statements (i), (ii) and (iv) are obvious; (iii) follows from (ii) and Proposition 13.3.

The next lemma describes the combinatorics of the face structure of $X^n(\mathfrak{g})$. The proof is obvious.

LEMMA 13.5. *Suppose that A and A′ are subsets of I_n and that w and w′ are elements of W. The following three statements are equivalent:*

(a) $P(w, A) \cap P(w', A') \neq \emptyset$

(b) $X(w, A) \cap X(w', A') \neq \emptyset$

(c) $wW_{S(A)} \cap w'W_{S(A')} \neq \emptyset$

Suppose that (c) holds and that $u \in wW_{S(A)} \cap w'W_{S(A')}$, then

(d) $wW_{S(A)} \cap w'W_{S(A')} = uW_{S(A \cap A')}$

and

(e) $X(w, A) \cap X(w', A') = X(u, A \cap A')$.

This completes the proof of statements (A) and (B). It follows that there is a homeomorphism $f\colon X^n(\mathfrak{g}) \to L^*(W)$ such that f makes the face $X(w, A)$ homeomorphically onto the standard face $F(w, S(A))$ of $L(W)$. This proves 13.2 and thereby, Theorem 13.1.

Remark 13.6. Consider the restriction of the Toda flow to a regular K-orbit $K(x)$. There are no closed 1-dimensional orbits. The fixed points are isolated and hyperbolic. The set of fixed points is the W-orbit $K(x) \cap \mathfrak{h}$. It follows that the ascending submanifolds at the fixed points give a cell structure on the "flag manifold" $K(x)$ $(K(x) \cong K/J)$. It is known that these cells correspond to the Bruhat decomposition of K/J. This cell decomposition is perfect in the sense that the closures of the $|W|$ cells give a basis for the homology of K/J with $\mathbb{Z}/2$-coefficients.

Now further restrict the Toda flow to $P^n(\mathfrak{g})$. The fixed points in $K(x)$ all lie in $P^n(\mathfrak{g})$; in fact, they correspond precisely to the vertices of $X^n(\mathfrak{g})(= L^*(W))$. It follows from Lemma 11.4 that the ascending submanifold at the vertex corresponding to w is the cell $C(w)$ described in Section 9. Therefore, the "perfect" cell structure described in Section 9 is, on the one hand, induced by the Toda flow on $P^n(\mathfrak{g})$ and on the other hand, induced by intersecting the Bruhat decomposition of K/J with $P^n(\mathfrak{g})$.

When $\mathfrak{g} = sl(n+1, \mathbb{R})$, Tomei [T] mentions that the restriction of the Toda vector field to $P^n(\mathfrak{g})$ essentially can be identified with the gradient field of a certain Morse function. It seems likely that a formula similar to the one on the bottom of p. 985 of [T] would also yield such a Morse function in the general case.

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