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THE EULER CHARACTERISTIC OF A NONPOSITIVELY CURVED, PIECEWISE EUCLIDEAN MANIFOLD

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A conjecture of H. Hopf states that if M^{2n} is a closed, Riemannian manifold of nonpositive sectional curvature, then its Euler characteristic, $\chi(M^{2n})$, should satisfy $(-1)^n\chi(M^{2n}) \geq 0$. In this paper, we investigate the conjecture in the context of piecewise Euclidean manifolds having "nonpositive curvature" in the sense of Gromov's CAT(0) inequality. In this context, the conjecture can be reduced to a local version which predicts the sign of a "local Euler characteristic" at each vertex. In the case of a manifold with cubical cell structure, the local version is a purely combinatorial statement which can be shown to hold under appropriate conditions.

The original conjecture of Hopf, and a similar conjecture for nonnegative curvature (which we shall not be concerned with here), are true in dimensions 2 and 4, by the Chern-Gauss-Bonnet Theorem: in both cases the curvature condition forces the Gauss-Bonnet integrand to have the correct sign. This is immediate in dimension 2. Chern [**Ch**] gives a proof in dimension 4 and attributes the result to Milnor. A result of [**Ge**] shows that, in dimensions ≥ 6, the curvature condition does not force the Gauss-Bonnet integrand to have the correct sign; hence, the same argument does not work in higher dimensions. (However, the hypothesis that the curvature operator is negative semidefinite does force the integrand to have the correct sign.)

Here we are concerned with the analogous conjecture for piecewise Euclidean manifolds. In order to make sense of this, two points bear further discussion: (1) the meaning of "nonpositive curvature" for piecewise Euclidean spaces, and (2) the combinatorial analogue of the Gauss-Bonnet Theorem for such spaces.

As for the first point, "nonpositive curvature" makes sense for a more general class of metric spaces than Riemannian manifolds, namely, it makes sense for "geodesic spaces" (also called "length spaces"). For geodesic spaces the notion of nonpositive curvature is defined via Gromov's "CAT(0) inequality" (see [**G**, p.106]). A piecewise Euclidean cell complex naturally has the structure of a geodesic space (cf. (1.3)) so nonpositive curvature makes sense in this context. Associated to a vertex v of such a cell complex, there is a piecewise spherical complex called the "link of v". It is homeomorphic to the topological link; its metric is determined by the solid angles at v in the cells which contain v. There is an infinitesimal version of nonpositive curvature, due to Gromov, which is equivalent to the previous version: the link of each vertex must be "large". (By definition, a piecewise spherical space is *large* if there is a unique geodesic between any two points of distance less than π.)

In dimension 2, in the case of a piecewise Euclidean surface M^2, these ideas are quite well-known. For a vertex v of M^2, let $\theta(v)$ denote the sum, over all 2-cells containing v, of the interior angles at v. The link of v is then a circle of length $\theta(v)$; it is large if and only if $\theta(v) \geq 2\pi$.

As for the second point, there is a well-known analogue of the Gauss-Bonnet formula for piecewise Euclidean cell complexes. It has the form $\chi(X) = \sum \kappa(v)$, where the summation is over the vertices of X and where $\kappa(v)$ depends only on the link of v. (The precise definition of $\kappa(v)$ is given in (3.4.3).) It is shown, rather convincingly, in [**CMS**] that $\kappa(v)$ is the correct analogue of the Gauss-Bonnet integrand in this context.

In contrast with the smooth case, we conjecture that, in all dimensions, the infinitesimal version of nonpositive curvature should force $\kappa(v)$ to have the correct sign.

For a vertex v of a piecewise Euclidean surface, $\kappa(v) = 1 - (2\pi)^{-1}\theta(v)$. Hence, in dimension 2, our conjecture is true: $\theta(v) \geq 2\pi$ implies $\kappa(v) \leq 0$.

In general, it is difficult to check that a piecewise spherical space L is large. However, if L is a simplicial complex and if each edge of L has length $\pi/2$, then Gromov has given a simple combinatorial condition which is necessary and sufficient for L to be large: it must be a "flag complex". (The definition is given in (2.7).) Such L (i.e., simplicial complexes with edge lengths $\pi/2$) arise as links in any piecewise Euclidean cell complex whose cells are regular cubes. Furthermore, for such L, the quantity $\kappa(L)$ is easily seen to be given by the formula:

$$\kappa(L) = 1 + \sum \left(-\frac{1}{2}\right)^{i+1} f_i,$$

where f_i is the number of i-simplices in L. Hence, the local form of the Hopf Conjecture for piecewise Euclidean cubical manifolds is the following, purely combinatorial conjecture, called Conjecture D in Section 4.

Conjecture D. *Suppose that L is a simplicial complex homeomorphic to S^{2n-1} and that $\kappa(L)$ is defined by the formula above. If L is a flag complex, then $(-1)^n\kappa(L) \geq 0$.*

This conjecture is first interesting in dimension 3. For any triangulation L^3 of S^3, the Lower Bound Theorem of [**W**], states that $f_1 \geq 4f_0 - 10$. In this dimension, Conjecture D is equivalent to the statement that if, in addition, L^3 is a flag complex, then $f_1 \geq 5f_0 - 16$.

Given an arbitrary simplicial complex L, we construct, in Section 6, a finite, piecewise Euclidean, cubical complex so that the link of each vertex is L. If L is a flag complex, then the cubical complex is nonpositively curved. If L is homeomorphic to a sphere, the cubical complex is a manifold. Hence, Conjecture D is equivalent to the Hopf Conjecture for piecewise Euclidean manifolds cellulated by cubes (cf. Proposition 6.5).

There is a beautiful generalization, due to G. Moussong [**M**], of Gromov's combinatorial condition for largeness of piecewise spherical simplicial complexes with edges of length $\pi/2$. Moussong gives an analogous condition in the case where the edge lengths are $\geq \pi/2$ which is partly combinatorial and partly metric (see (2.9), (2.10)). This leads to a conjecture for such L, analogous to Conjecture D, called Conjecture C in Section 4.

Section 5 provides some evidence for Conjectures C and D. First, we show that Conjecture D holds for some wide classes of flag complexes which arise as subdivisions. Then we show in Proposition (5.7) that Conjecture D implies Conjecture C (in the case where the underlying simplicial complex is a flag complex.) The proof uses a formula of [**CMS**] for the first derivative of $\kappa(t)$, where $\kappa(t) = \kappa(L_t)$ for some 1-parameter family L_t of piecewise spherical simplicial complexes.

Finally, in Section 7, we discuss an equivalent form of Conjecture D, of interest to combinatorialists.

1. Piecewise Euclidean and piecewise spherical cell complexes.

1.1. A *Euclidean cell* is a convex polytope in some Euclidean space. A *convex polyhedral cone* in $\mathbb{R}^n$ (with vertex at the origin) is any intersection of a finite number of linear half-spaces which contains no line. The *spherical cell* associated to a convex polyhedral cone is its intersection with S^{n-1}. A convex polyhedral cone is a *simplicial cone* if this intersection is a spherical simplex.

1.1.1. If P is a Euclidean cell and F is a face of P, then the *normal cone* of F in P is the convex polyhedral cone $N(F, P)$ consisting of all inward pointing normal vectors to F. Its dimension is the codimension of F in P. The associated spherical cell is called the *link* of F in P and denoted $Lk(F, P)$. Thus, $Lk(F, P)$ is the "solid angle" of P along F. Similarly, if F is a face in a spherical cell P, then $Lk(F, P)$ is defined to be the link of the associated polyhedral cones.

1.1.2. A *piecewise Euclidean cell complex* is a space X formed by gluing together Euclidean cells via isometries of their faces together with the decomposition of X into cells. A *piecewise spherical cell complex* is defined similarly using spherical cells. We assume throughout that all cell complexes are locally finite. We also assume that there is a positive lower bound on the heights of all cells of X. (The *height* of a cell C is the minimum distance between disjoint faces of C.)

1.1.3. If F_1 and F_2 are faces of a Euclidean (resp. spherical) cell P and $F_1 \subset F_2$, then $Lk(F_1, F_2)$ is canonically identified with a face of $Lk(F_1, P)$. It follows that if F is a cell in a piecewise Euclidean (resp. spherical) complex X, then the set of all $Lk(F, P)$, P a cell of X which contains F, fit together to give a piecewise spherical cell complex

$$Lk(F, X) = \bigcup_{F \subset P} Lk(F, P)$$

called the *link of F in X*.

1.1.4. An n-cell P is *simple* if precisely n codimension-one faces of P meet at each vertex. Equivalently, P is simple if $Lk(F, P)$ is a simplex for each proper face F of P. For example, a simplex is simple, so is a cube; an octahedron is not. If each cell of a piecewise Euclidean cell complex X is simple, then $Lk(F, X)$ is a simplicial cell complex for each cell F in X.

1.1.5. A note on terminology. "Simplicial complex" is not synonymous with "simplicial cell complex". In a simplicial cell complex the intersection of two simplices is a union of faces, while in a simplicial complex such an intersection is either empty or a single simplex.

1.2. A complete metric space Y is a *geodesic space* (or "length space") if, given points y_1, y_2 in Y, there is a path γ from y_1 to y_2 with $\ell(\gamma) = d(y_1, y_2)$. Such a distance minimizing path is called a *geodesic.*

1.3. Suppose X is a piecewise Euclidean or piecewise spherical cell complex. Then "arc length" makes sense in each cell; hence, the length ℓ of a path in X is well-defined. Given points x and y in X, define $d(x, y) = \inf \{\ell(\gamma)\}$, where the infimum is over all paths γ from x to y of finite length. (If x and y are in distinct path components, put $d(x, y) = \infty$.) Under the assumptions of (1.1.2), Moussong ([**M**]) showed that d is a complete, geodesic metric on X. It is locally isometric with the given Euclidean or spherical metrics on the individual cells. A *piecewise Euclidean* (resp. *piecewise spherical*) *space* is the metric space underlying such a piecewise Euclidean (resp. piecewise spherical) cell complex.

2. The infinitesimal version of nonpositive curvature.

2.1. A geodesic space is *nonpositively curved*, abbreviated (NP), if, locally, geodesic triangles satisfy the CAT(0)-inequality (see [**G**, p. 106]). The (NP) condition has some strong consequences. For example, if a space is (NP), then it is locally contractible and its universal covering space is contractible.

2.1.1. A piecewise spherical space is a *large* if there is a unique geodesic between any two points of distance $< \pi$.

The following result is due to Gromov. For a proof, see [**B**] or [**Br**]. (See also Theorem 3.1 and the appendix of [**CD1**].)

Theorem 2.2 (Gromov). *A piecewise Euclidean cell complex is* (NP) *if and only if the link of each vertex is large.*

In some cases, when the cell structure on a piecewise spherical space is simplicial, there are combinatorial conditions which are necessary and sufficient for it to be large. In order to state these conditions, it is first necessary to develop some terminology and elementary facts about spherical simplices.

2.3. Let σ be a spherical n-simplex in S^n with vertex set V, a set of linearly independent unit vectors in $\mathbb{R}^{n+1}$. The *associated cosine matrix*, $c(\sigma) = (c_{vv'}), v, v'$ in V is the symmetric V by V matrix of inner products: $c_{vv'} = v \cdot v'$. We note that $c(\sigma)$ is positive definite.

2.3.1. For $v \neq v'$, let $\ell_{vv'}$ be the length of the edge (v, v') of σ. Put $\ell_{vv} = 0$. Then $c_{vv'} = \cos(\ell_{vv'})$.

2.3.2. Conversely, suppose we are given a V by V symmetric matrix $(\ell_{vv'})$ satisfying: $\ell_{vv} = 0$ and $\ell_{vv'} \in (0, \pi)$ for $v \neq v'$. If the associated cosine matrix c, defined by $c_{vv'} = \cos(\ell_{vv'})$, is positive definite, then we can find a basis V for $\mathbb{R}^{n+1}$, unique up to isometry, which spans a spherical n-simplex with associated cosine matrix c. Hence, $(\ell_{vv'})$ is the matrix of edge lengths of a spherical simplex if and only if the associated cosine matrix is positive definite.

2.4. A spherical simplex is *all right* if each of its edges has length $\pi/2$; it *has size* $\geq \pi/2$ if each of its edges has length $\geq \pi/2$. The associated cosine matrix of an all right n-simplex σ is the identity matrix. It follows that σ is isometric to the regular n-simplex in S^n spanned by the standard basis of $\mathbb{R}^{n+1}$. The associated cosine matrix of a simplex of size $\geq \pi/2$ is "almost negative" in the sense that its off-diagonal entries are ≤ 0.

The following is a result of [**M**].

Lemma 2.4.1. *Let τ be a face of a spherical simplex σ.*
(i) *If σ is all right, then so is $Lk(\tau, \sigma)$.*
(ii) *If σ has size $\geq \pi/2$, then so has $Lk(\tau, \sigma)$.*

Proof. Statement (i) is obvious. To prove (ii) first note that if $\tau_1 \subset \tau_2 \subset \sigma$, the $Lk(Lk(\tau_1, \tau_2), Lk(\tau_1, \sigma))$ is isometric to $Lk(\tau_2, \sigma)$. Hence, by induction, we can reduce to the case $\dim \tau = 0$. So, suppose σ is spanned by $\{v_0, \dots, v_n\}$ and τ is the vertex v_0. Let $p : \mathbb{R}^{n+1} \to (\mathbb{R}v_0)^\perp$ be orthogonal projection. Then $Lk(v_0, \sigma)$ can be identified with the spherical simplex in $(\mathbb{R}v_0)^\perp$ spanned by $u_i = p(v_i)/|p(v_i)|, i = 1, \dots, n$. For $i \neq j$,

$$\begin{aligned} p(v_i) \cdot p(v_j) &= (v_i - (v_i \cdot v_0)v_0) \cdot (v_j - (v_j \cdot v_0)v_0) \\ &= v_i \cdot v_j - (v_i \cdot v_0)(v_j \cdot v_0) \end{aligned}$$

which is ≤ 0 since the cosine matrix for σ is almost negative. It follows that $u_i \cdot u_j \leq 0$ so the cosine matrix for $Lk(v_0, \sigma)$ is almost negative. $\square$

2.5. A piecewise spherical simplicial cell complex is *all right* (resp. *has simplices of size* $\geq \pi/2$) if the corresponding property holds for each of its simplices.

2.5.1. If L is all right or if it has simplices of size $\geq \pi/2$, then, by Lemma (2.4.1), the link of any simplex in L has the same property.

2.6. A piecewise Euclidean cell complex has *nonacute, simple cells* if each of its cells is simple (cf. (1.1.4)) and the face angles in each cell are all $\geq \pi/2$; or equivalently, if the link of every cell is simplicial with simplicies of size $\geq \pi/2$. It is *cubical* if each cell is a regular Euclidean cube. In this case, links are all right simplicial cell complexes.

2.7. Let K be a simplicial complex (no metric structure is assumed). A set V of vertices in K *spans a complete graph* if any two distinct elements of V span an edge in K. A simplicial cell complex K is a *flag complex* if it is a (genuine) simplicial complex (cf. 1.1.5) and if any set of vertices which spans a complete graph actually spans a simplex. Thus, a flag complex is a simplicial complex with no "empty simplices".

2.7.1. Flag complexes have nice properties. For example,
(a) the link of any simplex in a flag complex is again a flag complex, and
(b) the join of two flag complexes is a flag complex.

The following lemma is proved in [**G**, p. 122].

2.8. Gromov's Lemma. *An all right piecewise spherical simplicial cell complex is large if and only if it is a flag complex.*

2.8.1. Hence, a piecewise Euclidean, cubical cell complex is (NP) if and only if the link of each vertex is a flag complex.

2.9. Suppose L is a piecewise spherical simplicial complex and that V is a set of vertices of L which span a complete graph. As in (2.3), there is associated to V a cosine matrix, $(c_{vv'})$, where $c_{vv'}$ is the cosine of the length of the edge (v, v'). A cell complex L is a *metric flag complex* if it is a simplicial complex and if whenever V is a set of vertices such that (i) V spans a complete graph, and (ii) the associated cosine matrix is positive definite, then V spans a simplex of L.

The following difficult generalization of Gromov's Lemma is proved in [**M**].

2.10. Moussong's Lemma. *A piecewise spherical simplicial cell complex with simplices of size $\geq \pi/2$ is large if and only if it is a metric flag complex.*

2.10.1. So, a piecewise Euclidean cell complex with nonacute, simple cells is (NP) if and only if the link of each vertex is a metric flag complex.

3. The Gauss-Bonnet Theorem for piecewise Euclidean spaces.

3.1. Let C be an n-dimensional convex polyhedral cone and σ the associated spherical $(n-1)$-cell. The outward pointing normals to the supporting hyperplanes of C generate another convex polyhedral cone C^* called the *dual cone.* (If $u_1, \ldots, u_k$ are outward pointing normal vectors to the supporting hyperplanes, then C^* is the set of all nonnegative linear combinations of the u_i.) The associated spherical cell $\sigma^* = C^* \cap S^{n-1}$ is the *dual cell* to σ.

3.2. The *angle* of C at 0, denoted by $a(\sigma)$, is the $(n-1)$-dimensional volume of σ, normalized so that the volume of S^{n-1} is 1, i.e.,

$$a(\sigma) = \frac{\text{vol } (\sigma)}{\text{vol } (S^{n-1})}.$$

The *exterior angle* of C at 0, denoted by $a^*(\sigma)$ is the angle of C^*, i.e., $a^*(\sigma) = a(\sigma^*)$.

3.3. Let P be a Euclidean n-cell. After choosing an interior point of P as the origin, one can define the "dual cell" P^* whose vertices are the normal vectors to the codimension-one faces of P. For example, the octahedron is dual to the cube. Radial projection of ∂P^* onto S^{n-1} gives a cellulation of S^{n-1}. The $(n-1)$-cells in this cellulation are associated to the duals of the normal cones at the vertices of P, i.e., each such $(n-1)$-cell is of the form $Lk(v,P)^*$ for some vertex v. It follows that $\sum a^*(Lk(v,P)) = 1$, where the summation is over all vertices v of P. (If $v = P$, then $Lk(v,P) = \emptyset$, so we adopt the convention $a^*(\emptyset) = 1$.)

3.4. Now suppose that X is a finite piecewise Euclidean cell complex. Then

$$\begin{aligned}\chi(X) &= \sum_P (-1)^{\dim P} \\ &= \sum_P (-1)^{\dim P} \sum_v a^*(Lk(v,P))\end{aligned}$$

where P runs over the cells of X and v runs over the vertices of P. Reversing the order of summation gives

$$\begin{aligned}\chi(X) &= \sum_v \sum_{v\in P} (-1)^{\dim P} a^*(\text{Link }(v,P)) \\ &= \sum_v \left(1 + \sum_\sigma (-1)^{\dim\sigma+1} a^*(\sigma)\right)\end{aligned} \tag{3.4.1}$$

where v runs over the vertices of X and σ over the cells of $Lk(v,X)$. (Note that the 1 in the summation arises from the case $P = v$.) For any finite, piecewise spherical cell complex L, define

$$\kappa(L) = 1 + \sum_\sigma (-1)^{\dim\sigma+1} a^*(\sigma) \tag{3.4.2}$$

where σ runs over the cells of L. Then (3.4.1) can be rewritten as

$$\chi(X) = \sum_v \kappa(v), \tag{3.4.3}$$

where $\kappa(v) = \kappa(Lk(v,X))$. This is the desired analogue of the Gauss-Bonnet Theorem for piecewise Euclidean spaces.

3.5. If σ is an all right simplex, then $\sigma^* = \sigma$ and $a^*(\sigma) = a(\sigma) = (\frac{1}{2})^{\dim\sigma+1}$. Hence, if L is an all right, piecewise spherical, simplicial cell complex, then

$$\kappa(L) = 1 + \sum_\sigma \left(-\frac{1}{2}\right)^{\dim\sigma+1}. \tag{3.5.1}$$

3.5.2. Let K be a simplicial cell complex (without metric structure) and $f_i(= f_i(K))$ the number of i-simplices in K. Put

$$\lambda(K) = 1 + \sum_i \left(-\frac{1}{2}\right)^{i+1} f_i.$$

If we endow K with the structure of an all right spherical complex by declaring each simplex to be all right, then $\lambda(K) = \kappa(K)$.

3.6. The following facts concerning κ are proved in [**CMS**, Section 3].

3.6.1. If the underlying metric spaces of L_1 and L_2 are isometric, then $\kappa(L_1) = \kappa(L_2)$.

3.6.2. If L is isometric to the round sphere, then $\kappa(L) = 0$.

3.6.3. Suppose $\sigma^n \subset S^n$ and $\tau^k \subset S^k$ are spherical cells; embed S^n and S^k in orthogonal linear subspaces of $\mathbb{R}^{n+k+2}$; then the *orthogonal join* $\sigma * \tau$ is the spherical $(n+k+1)$-cell spanned by σ and τ (it is the union of all geodesics in S^{n+k+1} from σ to τ). This extends to a definition of the *orthogonal join* $L_1 * L_2$ of two piecewise spherical cell complexes L_1 and L_2 (see the Appendix of [**CD**]). By [**CMS**, p.442],

$$\kappa(L_1 * L_2) = \kappa(L_1)\kappa(L_2).$$

3.6.4. If L is homeomorphic to an even dimensional sphere, then $\kappa(L) = 0$. (Actually for this to be true it is only necessary to assume that $\chi(L) = 2$ and $\chi(Lk(\sigma, L)) = 2$ for each odd dimensional cell σ.)

3.6.5. Let K be a simplicial cell complex and L_t a 1-parameter family of piecewise spherical structures on K. For each edge e in K, let $a_e(t)$ be the length of the corresponding edge e_t in L_t, normalized so that S^1 has length 1. (Thus, $a_e(t) = (2\pi)^{-1}\ell(e_t)$, where ℓ is the usual edge length.) Put $\kappa(t) = \kappa(L_t)$. The following beautiful formula for the first derivative of $\kappa(t)$ is given in [**CMS**, p. 424]

$$\kappa'(t) = -\sum_e a_e'(t)\kappa(Lk(e_t, L_t)). \tag{3.6.6}$$

The proofs of the above facts can be sketched as follows. In [**C**], Cheeger considers another quantity, let us call it $\overline{\kappa}(L)$, defined as the difference of the L_2-Euler characteristic of the cone of radius 1 on L and $\frac{1}{2}$ the L_2-Euler characteristic of L. So $\overline{\kappa}(L)$ depends only on the metric and not the cell structure. Using heat equation methods he derives a formula for $\overline{\kappa}(L)$ solely in terms of interior angles and flags of odd dimensional cells (formula 3.35 in [**CMS**]) and he proves that $\chi(X) = \sum \overline{\kappa}(Lk(v, X))$. In [**CMS**, p. 424] it is proved that $\overline{\kappa}(L) = \kappa(L)$. Since $\overline{\kappa}(L)$ depends only on the metric, (3.6.1) follows. The formulas in (3.6.2), (3.6.4) and (3.6.6) follow easily from the formula for $\overline{\kappa}(L)$; (3.6.3) is an easy computation.

4. Statements of the conjectures.

We are now in position to state precisely the various conjectures which were mentioned in the Introduction. First, there is the analogue of the Hopf Conjecture.

Conjecture A. *If M^{2n} is a nonpositively curved, piecewise Euclidean, closed manifold, then $(-1)^n\chi(M^{2n}) \geq 0$.*

Next, suppose that L^{2n-1} is a $(2n-1)$-dimensional piecewise spherical cell complex which is a *generalized homology sphere*, in the sense that it has the homology of S^{2n-1} and for each k-cell $\sigma, Lk(\sigma, L^{2n-1})$ has the homology of S^{2n-k-2}. For example, if L^{2n-1} is homeomorphic to S^{2n-1}, then it is a generalized homology sphere. In view of Gromov's Theorem (2.2), Conjecture A is implied by the following.

Conjecture B. *If L^{2n-1} is large (cf. (2.1.1)), then $(-1)^n\kappa(L^{2n-1}) \geq 0$, where κ is defined by* (3.4.2).

By Moussong's Lemma (2.10) the following conjecture is a special case of Conjecture B.

Conjecture C. *Suppose that L^{2n-1} is simplicial and has simplices of size $\geq \pi/2$ (cf. (2.5)). If L^{2n-1} is a metric flag complex (cf. (2.9)), then $(-1)^n\kappa(L^{2n-1}) \geq 0$.*

If L is a metric flag complex with simplices of size $\geq \pi/2$, then the same is true for the link of each simplex in L, by Lemma (2.4.1). In the sequel we will want to consider the statement that Conjecture C holds for a particular complex L and for each link in L. It is therefore convenient to state the following equivalent formulation of Conjecture C.

Conjecture C$'$. *If L^{2n-1} is a metric flag complex with simplices of size $\geq \pi/2$, then $(-1)^n\kappa(L^{2n-1}) \geq 0$ and for each simplex σ of codimension $2k, (-1)^k\kappa(Lk(\sigma, L)) \geq 0$.*

Finally, we want to consider the special case where L^{2n-1} is all right. In order to de-emphasize the metric in this case, we let K^{2n-1} be a simplicial cell complex which is a generalized homology sphere, as above, and $\lambda(K)$ the quantity defined in (3.5.2).

Conjecture D. *If K^{2n-1} is a flag complex, then* $(-1)^n\lambda(K^{2n-1}) \geq 0$.

We also have an equivalent reformulation.

Conjecture D′. *If K^{2n-1} is a flag complex, then $(-1)^n\lambda(K^{2n-1}) \geq 0$ and for each simplex σ of codimension $2k, (-1)^k\lambda(Lk(\sigma, K^{2n-1})) \geq 0$.*

5. Evidence.

We begin by discussing some partial results for Conjecture D.

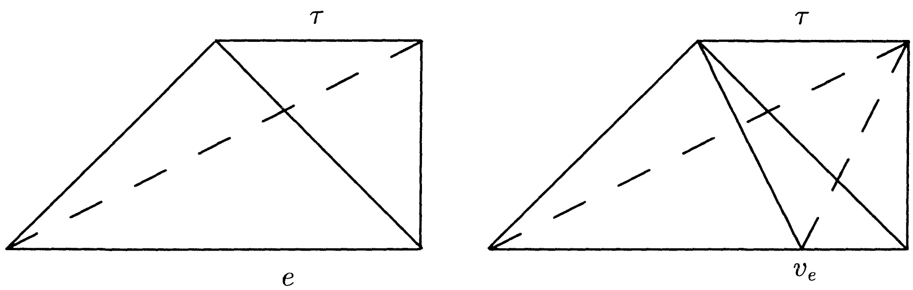
5.1. Let O^m be the $(m+1)$-fold join of S^0 with itself. By (2.7.1), O^m is a flag complex and by (3.6.3), $\lambda(O^m) = 0$. It is alternately described as the boundary complex of the $(m+1)$-dimensional octahedron (or "cross polytope"). It is the simplest flag complex which triangulates the m-sphere. If we endow it with an all right, piecewise spherical structure, then it is isometric to the round sphere. The link of a k-simplex in O^m is isomorphic to O^{m-k-1}. It follows that Conjecture D′ holds for O^{2n-1}.

5.2. Flag complexes arise naturally as derived complexes of posets. (Recall that if A is a poset, then its *derived complex* A' is the abstract simplicial complex with simplices the finite chains in A.) If A is the poset of cells in a cell complex, then the geometric realization of A' can be identified with the barycentric subdivision of the cell complex. In particular, if A is the poset of faces of the boundary complex of a $2n$-cell, then A' is a flag complex homeomorphic to S^{2n-1}. Based on work of R. Stanley, E. Babson showed that Conjecture D′ holds for such A'. (For the argument, see (7.3).) Hence, *Conjecture D′ holds for the barycentric subdivision of the boundary complex of a Euclidean cell.*

5.3. Suppose K is a simplicial complex (no metric assumed). Recall that for any simplex σ of $K, Lk(\sigma, K)$ can be identified with the subcomplex of K consisting of all closed simplices τ, disjoint from σ, such that the join of σ and τ is a simplex of K containing σ.

5.3.1. Let e be an edge of K. We shall now define a subdivision of K, called the *edge subdivision* and denoted by $\mathrm{Sub}_e(K)$. Introduce a new vertex v_e as the midpoint of e, subdividing e into two new edges, say e_+ and e_-. Let σ be a k-simplex containing e. Then $\sigma = e * \tau$ for some simplex τ in

$Lk(e, K)$. Introduce a new $(k-1)$-simplex $\sigma_0 = v_e * \tau$, subdividing σ into two new k-simplices $\sigma_+ = e_+ * \tau$ and $\sigma_- = e_- * \tau$. The simplices which do not contain e are not changed.



$\sigma_+ \cup \sigma_-$ $\qquad\qquad$ σ

5.3.2. If K is a flag complex, then so is $\mathrm{Sub}_e(K)$. Of course, we could have subdivided K by introducing a barycenter of some simplex of dimension > 1 and then coning off; however, this does not preserve the property of being a flag complex.

5.3.3. How are the links in $\mathrm{Sub}_e(K)$ related to links in K? Let σ be a simplex in $\mathrm{Sub}_e(K)$. There are four cases to consider:

(i) σ corresponds to a simplex of K which is not in the closed star of e,

(ii) σ corresponds to a simplex in the link of e,

(iii) there is a simplex $\hat{\sigma}$ of K which contains e and $\sigma = \hat{\sigma}_0$ (or $\sigma = v_e$),

(iv) there is a simplex $\hat{\sigma}$ of K which contains e and $\sigma = \hat{\sigma}_+$ or $\sigma = \hat{\sigma}_-$.

Then

$$Lk(\sigma,\ \mathrm{Sub}_e(K)) = \begin{cases} Lk(\sigma, K) & ;\ \text{in case (i)} \\ \mathrm{Sub}_e(Lk(\sigma, K)) & ;\ \text{in case (ii)} \\ S^0 * Lk(\hat{\sigma}, K) & ;\ \text{in case (iii)} \\ Lk(\hat{\sigma}, K) & ;\ \text{in case (iv)} \end{cases}$$

5.3.4. For any set of simplices E, put

$$\tilde{\lambda}(E) = \sum_{\sigma \in E} \left(-\frac{1}{2}\right)^{\dim \sigma + 1}.$$

If σ is an open simplex and M a simplicial complex, then let $E(\sigma, M)$ be the set of simplices in $\sigma * M$ which do not lie in M. Clearly,

$$\tilde{\lambda}(E(\sigma, M)) = (-\frac{1}{2})^{\dim \sigma + 1} \lambda(M).$$

Lemma 5.3.5. $\lambda(Sub_e(K)) = \lambda(K) - \frac{1}{4}\lambda(Lk(e,K))$.

Proof. To construct $\mathrm{Sub}_e(K)$ from K one removes the open star of e and replaces it by the open star of v_e in $v_e * S^0 * Lk(e,K)$. The contribution of the simplices in the open star of e to $\lambda(K)$ is $\tilde{\lambda}(E(e, LK(e,K))) = \frac{1}{4}\lambda(Lk(e,K))$. The contribution of those in the open star of v_e is

$$\tilde{\lambda}(E(v_e, S^0 * Lk(e,K))) = -\frac{1}{2}\lambda(S^0 * Lk(e,K)) = -\frac{1}{2}\lambda(S^0)\lambda(Lk(e,K)) = 0.$$

The formula follows. □

5.4. Now suppose K^{2n-1} is a flag complex and a generalized homology sphere. Let e be an edge of K.

Proposition 5.4.1. *If Conjecture D holds for K and for $Lk(e,K)$, then it also holds for $Sub_e(K)$.*

Proof. If $(-1)^n\lambda(K) \geq 0$ and $(-1)^{n-1}\lambda(Lk(e,K)) \geq 0$, then

$$(-1)^n\lambda(Sub_e(K)) = (-1)^n\lambda(K) + \frac{1}{4}(-1)^{n-1}\lambda(Lk(e,K)) \geq 0.$$

□

Using this, (5.3.2) and (5.3.3), we immediately deduce the following.

Proposition 5.4.2. *If Conjecture D′ holds for K, then it also holds for any complex obtained from K by repeated edge subdivisions.*

Corollary 5.4.3. *Conjecture D′ holds for any complex obtained from O^{2n-1} by repeated edge subdivisions.*

5.4.5. Proposition (5.4.2) also follows from formula (3.6.6). Let L_0 be K with an all right piecewise spherical structure. Let L_t be the 1-parameter family where the edge corresponding to e has length $(1+t)\pi/2$ and all other edges have length $\pi/2$. By induction on dimension, $(-1)^{n-1}\kappa(Lk(e_t, L_t)) \geq 0$. Hence, by (3.6.6), $(-1)^n\kappa(L_t)$ is a non-decreasing function of t. But L_1 is just the all right structure on $\mathrm{Sub}_e(K)$. Proposition (5.4.2) follows.

5.5. A *k-circuit* in K is a circuit of k edges. A 3-circuit is *empty* if it is not the boundary of a 2-simplex. Similarly a 4-circuit is *empty* if it is not the boundary of the union of two adjacent 2-simplices.

5.5.1. The inverse operation of edge subdivision is "edge collapse." Suppose e is an edge of K which is not part of an empty 3-circuit. (This is automatic if K is a flag complex). To collapse e, one first removes the open star of e. The resulting boundary is isomorphic to $S^0 * Lk(e, K)$. One then collapses this to $Lk(e, K)$. The result is a simplicial complex $\overline{K}_e$ which is a quotient space of K. If K is PL-homeomorphic to S^{2n-1}, then so is $\overline{K}_e$. If K is a flag complex, then $\overline{K}_e$ is also a flag complex if and only if e is not part of an empty 4-circuit. Hence, one can try to simplify K by performing edges collapses on edges which are not part of empty 4-circuits.

5.5.2. At one point we thought it might be possible, given a flag complex K^m, PL-homeomorphic to S^m, to perform edge collapses to reduce it to O^m. This is true for $m = 1, 2$, however, there is a counterexample for $m = 3$. (Conjecture D holds for this counterexample.)

We turn now to Conjecture C′.

5.6. Suppose L_0 and L_1 are two piecewise spherical structures on a simplicial cell complex K. For a simplex σ in K, let $\sigma_i, i = 0, 1$, be the corresponding spherical simplex in L_i. Let $c(\sigma_i), i = 0, 1$, be the associated cosine matrix (cf. (2.3)).

5.6.1. We shall now define a 1-parameter family L_t of piecewise spherical simplicial cell complexes L_t which interpolates between L_0 and L_1. Put $c_t(\sigma) = tc(\sigma_1) + (1-t)c(\sigma_0)$. Since a convex combination of positive definite matrices is positive definite, $c_t(\sigma) > 0$. It follows that, for each $t \in [0, 1]$, there is a corresponding spherical simplex σ_t. These fit together to give L_t. The family L_t is called the *canonical deformation* from L_0 to L_1. If e is an edge of K, then let $\ell_e(t) (= \ell(e_t))$ be the length of the corresponding edge in L_t. Then $\cos(\ell_e(t)) = t\cos(\ell_1) + (1-t)\cos(\ell_0)$, where $\ell_i = \ell_e(i), i = 0, 1$.

5.6.2. In particular, suppose L_0 is all right and L_1 has simplices of size $\geq \pi/2$. Then for each edge e, $\ell_e(0) = \pi/2$ and $\ell_e(t) \in [\pi/2, \pi)$. So, $\ell_e(t) = \cos^{-1}(ct)$, where $c = \cos(\ell_1)$. Hence, $\ell_e'(t) = -c(1 - c^2t^2)^{-\frac{1}{2}}$, which is ≥ 0 (since $c \leq 0$). Similarly, if $a_e(t) = (2\pi)^{-1}\ell_e(t)$ is the normalized length, then $a_e'(t) \geq 0$.

Proposition 5.7. *Let L^{2n-1} be a piecewise spherical simplicial complex with simplices of size $\geq \pi/2$ which is a generalized homology sphere, and let K^{2n-1} be the underlying simplicial complex. Suppose K^{2n-1} is a flag complex. If Conjecture D′ holds for K^{2n-1}, then Conjecture C′ holds for L^{2n-1}.*

Proof. Let L_0 be the all right structure on K, let $L_1 = L$, and let L_t be the canonical deformation. We assume that Conjecture D′ holds for K, i.e., that

Conjecture C$'$ holds for L_0. We may also assume, by induction on n, that Conjecture C holds for the link of each odd dimensional simplex in L_t. In particular, for each edge e_t, $(-1)^{n-1}\kappa(Lk(e_t, L_t)) \geq 0$. By (3.6.6),

$$(-1)^n\kappa'(t) = (-1)^{n-1}\sum a'_e(t)\kappa(Lk(e_t, L_t))$$

which is ≥ 0 (using 5.6.2). Hence, $(-1)^n\kappa(t)$ is nondecreasing. Thus, $(-1)^n\kappa(0) \geq 0$ implies $(-1)^n\kappa(1) \geq 0$. □

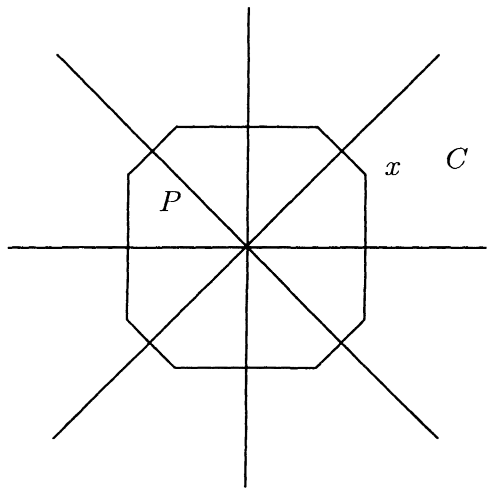
Remark 5.8. In a forthcoming paper [**CD2**] we find further evidence for Conjecture B. Suppose that X^{2n} is a convex polytope of finite volume in hyperbolic $2n$-space. Its "completed polar dual" L^{2n-1} is large. Moreover, $(-1)^n\kappa(L^{2n-1})$ is twice the volume of X^{2n} (suitably normalized). Hence, Conjecture B holds in this case.

6. Examples from Coxeter groups.

In this section we review the work of G. Moussong [**M**] (also, see [**D1**], [**G**, §4.6]). In particular, we show how to associate to each Coxeter system (W, S) a piecewise spherical, simplicial complex Nerve (W, S) and a contractible, nonpositively curved, piecewise Euclidean cell complex $\tilde{Y}$ with W-action so that the link of each vertex of $\tilde{Y}$ is Nerve (W, S). Moreover, any finite, all right, flag complex can be realized as Nerve (W, S) for some (W, S).

6.1. Suppose S is a finite set and $m = (m_{ss'})$ is a symmetric S by S matrix, with entries positive integers or ∞, satisfying $m_{ss} = 1$ and $m_{ss'} \geq 2$, for $s \neq s'$. This gives a presentation for a group $W = \langle S | (ss')^{m_{ss'}} = 1\rangle$. The pair (W, S) is a *Coxeter system* and W is a *Coxeter group*. The *rank* of (W, S) is the cardinality of S. The group W is *right-angled* if $m_{ss'} = 2$ or ∞ for all $s \neq s'$. Associated to m, there is an S by S *cosine matrix* $c = (c_{ss'})$ defined by $c_{ss'} = -\cos(\pi/m_{ss'})$. (Here $\cos(\pi/\infty) = \cos(0) = 1$.)

6.2. The group W is finite if and only if the cosine matrix is positive definite (cf. [**Bo**]). Suppose this is the case. Then W can be represented as a group generated by orthogonal linear reflections on $\mathbb{R}^n$, where n is the rank of (W, S). We recall how this works. Since c is positive definite, one can find a basis $(u_s)_{s\in S}$ for $\mathbb{R}^n$ such that $u_s \cdot u_{s'} = c_{ss'}$. Let r_s be reflection across the linear hyperplane H_s normal to u_s. Then W is identified with the group $\langle r_s\rangle$ generated by the r_s and this group is finite. The *fundamental cone* $C(= C(W, S))$ is the simplicial cone defined by $u_s \cdot x \leq 0, s \in S$. (So the H_s are the supporting hyperplanes of C.) Then C is a fundamental domain for W on $\mathbb{R}^n$.



Its intersection with S^{n-1} is the *fundamental simplex* $\sigma(=\sigma(W,S))$. As in (3.1), define $C^*(=C^*(W,S))$, the *fundamental dual cone*, and its intersection with $S^{n-1}, \sigma^*(=\sigma^*(W,S))$, the *fundamental dual simplex*. Thus, σ^* is the spherical simplex spanned by the u_s and its associated cosine matrix (cf. 2.3) is c. Let $\sigma_s = \sigma \cap H_s$. Note that the dihedral angle of σ along $\sigma_s \cap \sigma_{s'}$ is $\pi/m_{ss'}$ and that the length of the edge in σ^* from u_s to $u_{s'}$ is $\pi(1-(m_{ss'})^{-1})$.

6.2.1. The W-translates of σ give a triangulation of S^{n-1}. The underlying simplicial complex is called the *Coxeter complex* of W.

6.2.2. Choose a function $\epsilon : S \to (0,\infty)$. Let x be the interior point of C defined by $x \cdot u_s = -\epsilon(s)$, i.e., the Euclidean distance from x to H_s is $\epsilon(s)$. The convex hull of the W-orbit of x is denoted by $P(=P(W,S))$ and called a *Coxeter cell.* One checks easily that

(i) the vertices of P are the W-translates of x,

(ii) P is a simple n-cell and the normal cone at any vertex is $C^*(W,S)$, and

(iii) ∂P is the dual of the Coxeter complex. The vertices wx and $w'x$ are connected by an edge if and only if $w' = ws$, for some s in S, and the Euclidean length of such an edge is $2\epsilon(s)$.

6.2.3. Examples

(i) Suppose $S = \{s,s'\}, m = m_{ss'}$, and W is the dihedral group of order $2m$. Then P is a $2m$-gon and it is regular if and only if $\epsilon(s) = \epsilon(s')$.

(ii) Suppose $W = (\mathbb{Z}/2)^n$. Then P is a product of intervals $\prod[-\epsilon(s), \epsilon(s)]$, $s \in S$; P is a regular n-cube if and only if ϵ is constant.

(iii) If W is the symmetric group on $n+1$ letters, then the n-cell P is sometimes called a "permutohedron."

6.2.4. A *Coxeter block* $B(= B(W,S))$ is the convex n-cell defined by, $B = P \cap C$. It is combinatorially equivalent to an n-cube. Call the origin the *inside vertex* of B; its link is σ. Call x the *outside vertex* of B; its link is σ^*. Note that $WB = P$.

6.3. We return to the general situation where W may be infinite. For any subset T of S, let W_T denote the subgroup generated by T. (In fact, (W_T, T) is a Coxeter system.) Associate to (W,S) an abstract simplicial complex Nerve (W,S) as follows: the vertex set is S; a proper subset T of S is a simplex if and only if W_T is finite.

6.3.1. There is a natural piecewise spherical structure on Nerve (W,S) defined as follows. The simplex T is given the structure of the spherical simplex associated to the cosine matrix of (W_T, T). Thus, the simplices of Nerve (W,S) are isometric to the fundamental dual simplices $\tau_T = \sigma^*(W_T, T)$ corresponding to the finite Coxeter subgroups of W. It follows that Nerve (W,S) has simplices of size $\geq \pi/2$ and, by (6.2), that it is a metric flag complex.

6.3.2. If (W,S) is right-angled, then Nerve (W,S) is an all right flag complex. Conversely, given any finite, all right flag complex K, there is a right-angled Coxeter system with Nerve $(W,S) = K$: simply, let S be the vertex set of K and define $m_{ss'}$, for $s \neq s'$, by

$$m_{ss'} = \begin{cases} 2; & \text{if } (s,s') \text{ is an edge} \\ \infty; & \text{if } (s,s') \text{ is not an edge.} \end{cases}$$

6.4. We define a *fundamental chamber* $Y(= Y(W,S))$. Let $N =$ Nerve(W, S), let N' be the barycentric subdivision of N, and Y the cone on N' with cone point x. For each simplex T in N, let v_T be its barycenter. Each point y in N' lies in the open star of a (unique) v_T. Put $W_y = W_T$. For y in $Y - N'$, let W_y be the trivial subgroup.

6.4.1. Define an equivalence relation $\sim$ on $W \times Y$ by: $(w_1, y_1) \sim (w_2, y_2)$ if and only if $y_1 = y_2$ and $w_1 W_{y_1} = w_2 W_{y_2}$. Let $\tilde{Y}(= \tilde{Y}(W,S))$ be the quotient space $(W \times Y)/\sim$. Then $\tilde{Y}$ is naturally a simplicial complex, and W acts properly on $\tilde{Y}$ with orbit space Y.

6.4.2. Put piecewise Euclidean structures on Y and $\tilde{Y}$ as follows. Choose a function $\epsilon : S \to (0, \infty)$. For each simplex T in N, let B_T be the Coxeter block defined as in (6.2.4) by using $\epsilon|_T$. (So the length of the edge from the outside vertex of B_T to the hyperplane corresponding to t is $\epsilon(t)$.) We can identify the subcomplex Cone (τ'_T) of Y with B_T in such a fashion that v_T corresponds to the inside vertex of B_T and x corresponds to the outside vertex. This defines a piecewise Euclidean structure on Y such that $Lk(x, Y)$ is Nerve (W, S) with its natural piecewise spherical structure. The space $\tilde{Y}$ inherits a piecewise Euclidean structure from Y.

6.4.3. The map $y \to (1, y)$ from Y to $W \times Y$ induces an identification of Y with a subcomplex of $\tilde{Y}$. Then $W_T B_T$ is a subcomplex of $\tilde{Y}$ isometric to the Coxeter cell $P(W_T, T)$. It follows that $\tilde{Y}$ is cellulated by the Coxeter cells $wP(W_T, T)$, where $w \in W$ and T is a simplex in N. The vertices of these cells are the W-translates of the cone point x and the link of each vertex is isometric to Nerve (W, S). By Moussong's Lemma (2.10) such links are large. This gives the following result of [**M**].

Theorem 6.4.4 (Moussong). *For any Coxeter system (W, S), the piecewise Euclidean cell complex $\tilde{Y}(W, S)$ is (NP). Moreover, the link of each vertex is isometric to Nerve(W, S).*

6.4.5. We can find finite, piecewise Euclidean cell complexes with the same property by taking the quotient of $\tilde{Y}$ by a torsion-free subgroup Γ of finite index in W. Thus, $X = \tilde{Y}/\Gamma$ also satisfies the conclusions of Theorem (6.4.4).

6.4.6. Since $(\tau_T)^* = \sigma(W_T, T)$ is a fundamental simplex for $W_T, a^*(\tau_T) = |W_T|^{-1}$. Hence,

$$\kappa(\text{Nerve}(W, S)) = \sum (-1)^{Card(T)} |W_T|^{-1} \tag{6.4.7}$$

where the summation is over all subsets T of S such that W_T is finite. We note that the right hand side of (6.4.7) is the usual expression for the Euler characteristic of W (that is, the "orbihedral Euler characteristic" of Y) and that $\chi(X) = [\Gamma : W] \sum (-1)^{Card(T)} |W_T|^{-1}$. (Compare [**D2**].)

6.4.8. Let L be any all right flag complex and let (W, S) be the right-angled Coxeter system corresponding to L (cf. (6.3.2)). Let $\tilde{Y}$ and X be as above. The conclusion is that there is a finite, piecewise Euclidean, cubical cell complex X such that the link of each vertex is isometric to L. This gives the following.

Proposition 6.5. *Conjecture A holds for all piecewise Euclidean manifolds cellulated by regular cubes if and only if Conjecture D holds.*

7. The h-polynomial.

7.1. Combinatorialists, interested in the poset of faces associated to a convex cell, have made the following definitions. Suppose K is a finite simplicial complex of dimension $m-1$, that f_i is the number of i-simplices in K, and that $f_{-1}=1$. The *f-vector* of K is the m-tuple $(f_{-1}, f_0, \dots, f_{m-1})$ and the *h-vector* $(h_0, \dots, h_m)$ is defined by the equation

$$\sum_{i=0}^{m} f_{i-1}(t-1)^{m-i} = \sum_{i=0}^{m} h_i t^{m-i}. \tag{7.1.1}$$

Polynomials $f(t)(=f_K(t))$ and $h(t)(=h_K(t))$ are defined by

$$f(t) = \sum_{i=0}^{m} f_{i-1} t^i \tag{7.1.2}$$

$$h(t) = \sum_{i=0}^{m} h_i t^i. \tag{7.1.3}$$

Formula (7.1.1) can then be rewritten as

$$h(t) = (1-t)^m f\left(\frac{t}{1-t}\right). \tag{7.1.4}$$

7.2. Let $\lambda(K)$ be the quantity defined in (3.5.2). Clearly, $\lambda(K) = f_K\left(-\frac{1}{2}\right)$. By substituting $t=-1$ into (7.1.4) we get

$$h(-1) = 2^m f\left(-\frac{1}{2}\right) = 2^m \lambda(K). \tag{7.2.1}$$

Therefore, $\lambda(K)$ and $h(-1)$ have the same sign. So Conjecture D is equivalent to the following.

Equivalent Form of Conjecture D *Suppose K^{2n-1} is a generalized homology sphere (as in Section 4). If K is a flag complex, then $(-1)^n h_K(-1) \geq 0$.*

7.3. In this paragraph, we discuss an observation of E. Babson (as communicated to us by L. Billera) which proves (5.2). For any "Eulerian poset" A, combinatorialists have defined the "cd-index" $\Phi_A(c,d)$, a certain polynomial in two (noncommuting) variables (e.g. see [**S2**]). It is a refinement of the

h-vector of the derived complex A'. In particular, it follows directly from its definition that

$$\Phi_A(0,-2) = h_{A'}(-1). \tag{7.3.1}$$

Stanley [**S2**] has proved (in slightly greater generality) that if A is the boundary complex of a convex m-cell, then the coefficients of Φ_A are all nonnegative. In particular, for $m = 2n, \Phi_A(0,-2)$ is $(-2)^n$ times the coefficient of d^n. It follows that Conjecture D holds for the barycentric subdivision of the boundary complex of a $2n$-cell.

7.4. The following well-known formulas hold for any K^{m-1} which is a generalized homology $(m-1)$-sphere.

$$h(t) = t^m h(t^{-1}) \tag{7.4.1}$$

$$h_i \geq 0, \qquad \text{for } 0 \leq i \leq n. \tag{7.4.2}$$

Formula (7.4.1) (which means that $h_i = h_{m-i}$) is equivalent to the Dehn-Sommerville Relations. These relations are a consequence of the facts that $\chi(K) = \chi(S^{m-1})$ and that for any i-simplex $\sigma, \chi(Lk(\sigma, K)) = \chi(S^{m-i-2})$. Thus, (7.4.1) holds for any K which is an "Euler sphere" in the above sense. The inequalities in (7.4.2) hold whenever K is a "Cohen-Macaulay complex" in the sense of [**S1**]. In particular, both formulas hold for generalized homology spheres.

Conjecture E. *If K^{m-1} is a generalized homology sphere and a flag complex, then $h(t)$ has no roots of modulus 1, except possibly -1.*

Lemma 7.5. *Conjecture E implies Conjecture D.*

Proof. Suppose that $m = 2n$. Factor h as a product of monic polynomials, $h = h_1h_2$, where the roots of h_1 are real and those of h_2 are not. We list the non-real roots of h as: $\beta_1, \dots \beta_k, \overline{\beta}_1, \dots, \overline{\beta}_k$. Since $h_2(t) = \prod(t-\beta_i)(t-\overline{\beta}_i), h_2(-1) = \prod(-1-\beta_i)(-1-\overline{\beta}_i) = \prod|-1-\beta_i|^2$ which is ≥ 0. Hence, we need only show $h_1(-1)$ has the correct sign. By (7.4.1), if γ is a root then so is γ^{-1}. If $|\beta| \neq 1$, then $\beta^{-1} \neq \overline{\beta}$. Supposing that Conjecture E holds, we see that k is even. By (7.4.2) the real roots of h are negative. If $h(-1) = 0$, then Conjecture D holds. So, suppose $h(-1) \neq 0$. List the real roots of h as: $\alpha_1, \dots, \alpha_{n-k}, (\alpha_1)^{-1}, \dots (\alpha_{n-k})^{-1}$, where $-1 < \alpha_i < 0$. Since

$$h_1(-1) = \prod(-1-\alpha_i)\prod(-1-(\alpha_i)^{-1}),$$

the sign of $h_1(-1)$ is $(-1)^{n-k}$ which is equal to $(-1)^n$ (since k is even). $\square$

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