RUTH CHARNEY AND MICHAEL DAVIS

RECIPROCITY OF GROWTH FUNCTIONS OF COXETER GROUPS*

ABSTRACT. The growth series $W(t)$ of a Coxeter system (W, S) is always a rational function. We prove that for a very general class of infinite Coxeter groups, this function satisfies $W(t^{-1}) = \pm W(t)$.

Suppose that (W, S) is a Coxeter system ([1, p. 11]) with S finite. (In particular, W is a Coxeter group and S is a distinguished set of involutions which generate W.) For each w in W, $l(w)$ denotes its word length with respect to the generating set S.

The *growth function* of (W, S) is the power series in t defined by

$$W(t) = \sum_{w \in W} t^{l(w)}.$$

For a survey on these series, see [4]. It is known that $W(t)$ is always a rational function. In this paper, we show that for a very general class of infinite Coxeter groups, this rational function satisfies $W(t^{-1}) = \pm W(t)$.

If X is any subset of W, put

$$X(t) = \sum_{w \in X} t^{l(w)}.$$

If σ is a subset of S, then W_σ denotes the subgroup of W generated by σ. (By convention, $W_\phi = \{1\}$.)

In what follows, set theoretic inclusion is denoted by '$\leqslant$' and strict inclusion by '$<$'. Also, if Y is a finite set, then put

$$\varepsilon(Y) = (-1)^{\mathrm{Card}(Y)}.$$

The following lemma is proved in [8]. (Part (A) was originally proved in [7].) The proof is also outlined in [1, Exer. 26, p. 45].

LEMMA 1. (A) *Suppose that W is finite and that the element of longest length in W has length m. Then $W(t)$ is a polynomial and the following two formulas hold*:

$$W(t) = t^m W(t^{-1}) \tag{1}$$

$$t^m = \sum_{\sigma \leqslant S} \varepsilon(\sigma) \frac{W(t)}{W_\sigma(t)}. \tag{2}$$

*Partially supported by NSF grant DMS-8905378.

Geometriae Dedicata **39**: 373–378, 1991.

(B) *If W is infinite, then the following formula holds*:

$$(3) \qquad 0 = \sum_{\sigma \leqslant S} \varepsilon(\sigma) \frac{1}{W_\sigma(t)}.$$

This lemma implies that $W(t)$ is the power series expansion of some rational function of t (which we shall continue to denote by $W(t)$).

DEFINITION. The *nerve of* (W, S), denoted by N, is the partially ordered set of those subsets σ of S such that the group W_σ is finite. The partial ordering is by inclusion.

For any $\sigma \in N$, put $L_\sigma = N_{>\sigma}$, where $N_{>\sigma} = \{\tau \in N \mid \sigma < \tau\}$. The poset $N_{>\phi}$ (consisting of the nonempty subsets σ of S such that W_σ is finite) is called the *proper nerve* of (W, S). Clearly, $N_{>\phi}$ is a simplicial complex. (More precisely, it is isomorphic to the poset of simplices of a simplicial complex with vertex set S.) Moreover, for $\sigma \neq \phi$, the simplicial complex L_σ can be identified with the link of σ in $N_{>\phi}$.

REMARK. Any finite polyhedron can occur as the proper nerve of some Coxeter system. In fact, if X is a finite simplicial complex, then there is a Coxeter system whose proper nerve is the barycentric subdivision of X ([2, Lemma 11.3, p. 313]).

For any finite poset K, $\chi(K)$ denotes the Euler characteristic of its geometric realization.

In the next lemma, we give two formulas. Although both are easy consequences of Lemma 1, we postpone the proof to the end of the paper. The first formula is due to Steinberg ([8, Cor. 1.29, p. 14]); the second is the main technical contribution of this paper.

LEMMA 2.

$$(4) \qquad \frac{1}{W(t^{-1})} = \sum_{\sigma \in N} \frac{\varepsilon(\sigma)}{W_\sigma(t)}$$

$$(5) \qquad \frac{1}{W(t)} = \sum_{\sigma \in N} \frac{1 - \chi(L_\sigma)}{W_\sigma(t)}.$$

COROLLARY.

$$(6) \qquad \frac{1}{W(\infty)} = 1 - \chi(N_{>\phi})$$

$$(7) \qquad \frac{1}{W(1)} = \chi(W).$$

Formula (7) is due to Serre ([6, Prop. 17, p. 112]).

Proof. Substituting $t = 0$ into (4), we get

$$\frac{1}{W(\infty)} = \sum_{\sigma \in N} \varepsilon(\sigma)$$
$$= 1 - \sum_{\sigma \in N_{>\phi}} (-1)^{\dim \sigma}$$
$$= 1 - \chi(N_{>\phi}).$$

This proves (6). To prove (7) we recall a construction of [2]. The group W acts properly on a contractible complex $\mathcal{U}$ with fundamental chamber $|N|$ (the geometric realization of the poset N). The orbit space can also be identified with $|N|$. The isotropy subgroup at a point in $|N|$ is of the form W_σ, for some $\sigma \in N$. In fact, the set of points with isotropy group W_σ is precisely $|N_{\geqslant \sigma}| - |N_{>\sigma}|$. Since $|N_{\geqslant \sigma}|$ is a compact cone, its Euler characteristic is 1. Hence, the alternating sum of the cells in $|N_{\geqslant \sigma}| - |N_{>\sigma}|$ is

$$1 - \chi(|N_{>\sigma}|) = 1 - \chi(L_\sigma).$$

The Euler characteristic of W (a rational number) is the 'orbihedral Euler characteristic' of $|N|$, i.e.

$$\chi(W) = \sum_{\sigma \in N} \frac{1 - \chi(L_\sigma)}{\mathrm{Card}(W_\sigma)}$$
$$= \sum_{\sigma \in N} \frac{1 - \chi(L_\sigma)}{W_\sigma(1)}$$
$$= 1/W(1) \quad \text{by (5)}.$$

Hence, (7) holds. □

DEFINITION. Let $\delta = \pm 1$. We say that $W(t)$ is *δ-reciprocal* if $W(t^{-1}) = \delta W(t)$.

On the basis of a result of Serre ([6, Prop. 26(d), p. 145]) as well as results of Floyd and Plotnick [3] and Parry [5], it is natural to conjecture that $W(t)$ should be $(-1)^n$-reciprocal whenever W acts properly and cocompactly as a group generated by reflections on a contractible n-manifold. It follows from [2] that this conjecture is equivalent to the conjecture that W should be $(-1)^n$-reciprocal whenever $|N_{>\phi}|$ is a 'generalized homology $(n-1)$-sphere' (in the sense that for each $\sigma \in N$, $|L_\sigma|$ has the homology of a sphere of dimension $n - 2 - \dim \sigma$). The result of Serre mentioned above proves this in the case that $N_{>\phi}$ is the boundary of an n-simplex. According to the introduction of [3], Floyd, Parry and Plotnick can prove it in the above generality. Actually, a stronger result holds: all that one needs is that $|N_{>\phi}|$

resemble S^{n-1} up to Euler characteristics, that is, for each $\sigma \in N$, we require that L_σ have the same Euler characteristic as $S^{n-2-\dim\sigma}$.

DEFINITION. Let K be a locally finite simplicial complex. For each $\sigma \in K$, let $L_\sigma(K)$ denote the link of σ in K. By convention, put $L_\phi(K) = K$ and $\dim \phi = -1$. Let $\delta = \pm 1$. Then K is an *Euler complex of type* δ if

$$\chi(L_\sigma(K)) = 1 + \delta(-1)^{\dim\sigma}$$

for all $\sigma \in K$. If, in addition, K is a finite complex and the above equation also holds for $\sigma = \phi$ then K is an *Euler sphere of type* δ.

Thus, an Euler sphere of type $(-1)^n$ resembles an ordinary $(n-1)$-sphere up to Euler characteristics.

REMARK. A finite Euler complex K of type $(+1)$ must have $\chi(K) = 0$ and, hence, is automatically an Euler sphere. (*Proof*: $|K| = \amalg\, |K_{\geq\sigma}| - |K_{>\sigma}|$; hence, $\chi(K) = \Sigma\, 1 - \chi(L_\sigma(K)) = \Sigma\, 1 - (1 + (-1)^{\dim\sigma}) = -\Sigma(-1)^{\dim\sigma} = -\chi(K)$.)

THEOREM. If $N_{>\phi}$ *is an Euler sphere of type* δ, *then* $W(t)$ *is* δ*-reciprocal.*

Proof. By hypothesis, for all $\sigma \in N$, $1 - \chi(L_\sigma) = 1 - (1 + \delta(-1)^{\dim\sigma})$ $\delta(-1)^{\mathrm{Card}\,\sigma} = \delta\varepsilon(\sigma)$. Hence, Lemma 2 gives $1/W(t) = \delta/W(t^{-1})$. □

COROLLARY. *If* W *acts properly and cocompactly, as a group generated by reflections on a contractible n-manifold, then* $W(t)$ *is* $(-1)^n$*-reciprocal.*

REMARKS. (i) There is a partial converse to the theorem. If $W(t)$ is δ-reciprocal, then $1 = 1/W(0) = \delta/W(\infty) = \delta(1 - \chi(N_{>\phi}))$ by (6). Hence, $\chi(N_{>\phi}) = 1 - \delta$.

(ii) If $N_{>\phi}$ is only required to be an Euler complex of type (-1) (rather than an Euler sphere), then $1/W(t) + 1/W(t^{-1})$ is the constant function $2 - \chi(N_{>\phi})$.

(iii) If $N_{>\phi}$ is an Euler sphere of type δ, then the contractible complex $\mathcal{U}$ (mentioned in the proof of the corollary) is an Euler complex of type $(-\delta)$.

(iv) If $W(t)$ is (-1)-reciprocal, then, by (7), $\chi(W) = 0$.

(v) One can define a growth function $W(\mathbf{t})$ in a family of indeterminates $\mathbf{t} = (t_i)_{i\in I}$ where I is the set of conjugacy classes of elements of S ([6, p. 144]). Our arguments work equally well in this generality, in particular, the obvious analogs of Lemma 2 and the above theorem hold in this more general situation.

It remains to prove Lemma 2. The proof will use the following fact.

LEMMA 3. *Suppose that* A *is a proper subset of a finite set* X. *Then*

$$\sum_{\substack{Y \\ A \leq Y \leq X}} \varepsilon(Y) = 0.$$

As for the proof of Lemma 3, it suffices to remark that the lemma is equivalent to the fact that the Euler characteristic of a simplex is 1.

PROOF OF LEMMA 2. *Case* 1: W is finite. By (1), $1/W(t^{-1}) = t^m/W(t)$ and by (2), $t^m/W(t) = \Sigma_{\sigma \leqslant S}\, \varepsilon(\sigma)/W_\sigma(t)$. Hence, (4) holds in this case. For $\sigma \neq S$, L_σ is a simplex, while $L_S = \phi$ Hence, $\chi(L_\sigma) = 1$ for $\sigma \neq S$, while $\chi(L_S) = 0$. Thus, when W is finite all the terms but one on the right-hand side of (5) vanish and (5) reduces to the tautology: $1/W(t) = 1/W(t)$

Case 2: W is infinite. Formula (3) can be rewritten as:

$$\text{(8)} \qquad \frac{1}{W(t)} = -\varepsilon(S) \sum_{Y \neq S} \frac{\varepsilon(Y)}{W_Y(t)},$$

or

$$\text{(9)} \qquad \frac{1}{W(t^{-1})} = -\varepsilon(S) \sum_{Y \neq S} \frac{\varepsilon(Y)}{W_Y(t^{-1})}.$$

The proof proceeds by induction on Card(S). By Case 1 and the inductive hypothesis we may assume that (4) and (5) hold for $W_Y(t)$, $Y < S$. First consider (4). Using (9), we get

$$\text{(10)} \qquad \frac{1}{W(t^{-1})} = -\varepsilon(S) \sum_{Y<S} \varepsilon(Y) \sum_{\sigma \in N \cap Y} \frac{\varepsilon(\sigma)}{W_\sigma(t)}.$$

The coefficient of $1/W_\sigma(t)$ on the right-hand side of (10) is:

$$-\varepsilon(S)\varepsilon(\sigma) \sum_{\substack{Y \\ \sigma \leqslant Y < S}} \varepsilon(Y)$$

and by Lemma 3, this coefficient is $\varepsilon(\sigma)$. Thus, (10) can be rewritten as (4). The proof of (5) is similar. Using (8),

$$\text{(11)} \qquad \frac{1}{W(t)} = -\varepsilon(S) \sum_{Y<S} \varepsilon(Y) \sum_{\sigma \in N \cap Y} \frac{1-\chi(L_\sigma(Y))}{W_\sigma(t)}$$

where $L_\sigma(Y) = L_\sigma \cap Y$. The coefficient of $1/W_\sigma(t)$ on the right-hand side of (11) is

$$-\varepsilon(S) \sum_{\substack{Y \\ \sigma \leqslant Y < S}} \varepsilon(Y)(1 - \chi(L_\sigma(Y)).$$

We want to prove this coefficient is equal to $1 - \chi(L_\sigma)$; that is, we must prove

$$\text{(12)} \qquad \sum_{\substack{Y \\ \sigma \leqslant Y \leqslant S}} \varepsilon(Y)(1 - \chi(L_\sigma(Y)) = 0.$$

By Lemma 3, $\Sigma_{\sigma \leqslant Y \leqslant S} \varepsilon(Y) = 0$, hence, (12) is equivalent to

$$\sum_{\sigma \leqslant Y \leqslant S} \varepsilon(Y)\chi(L_\sigma(Y)) = 0. \tag{13}$$

Let τ be a simplex in L_σ. The contribution of τ to the left-hand side of (13) is

$$(-1)^{\dim \tau} \sum_{\tau \leqslant Y \leqslant S} \varepsilon(Y)$$

which, by Lemma 3 again, vanishes. Thus, (12) holds and hence (11) can be rewritten as (5).

REFERENCES

1. Bourbaki, N., *Groupes et Algèbres de Lie*, Chapîtres 4–6, Hermann, Paris, 1968.
2. Davis, M., 'Groups generated by reflections and aspherical manifolds not covered by Euclidean space', *Ann. Math.* **117** (1983), 293–324.
3. Floyd, W. J. and Plotnick, S. P., 'Symmetries of planar growth functions', *Invent. Math.* **93** (1988), 501–543.
4. Paris, L., 'Growth series of Coxeter groups', to appear in Proceedings of Workshop on Group Theory from a Geometrical Viewpoint, I.C.T.P., Trieste, 1990.
5. Parry, W., 'Growth series of Coxeter groups and Salem numbers', Preprint, 1990.
6. Serre, J.-P., 'Cohomologie des groupes discrets', in *Prospects in Mathematics, Ann. Math. Studies* **70**, Princeton Univ. Press, Princeton, 1971, pp. 77–169.
7. Solomon, L., 'The orders of the finite Chevalley groups', *J. Algebra* **3** (1966), 376–393.
8. Steinberg, R., 'Endomorphisms of linear algebraic groups', *Mem. Amer. Math. Soc.* **80** (1968).

Authors' address:

Ruth Charney and Michael Davis,
Department of Mathematics,
100 Mathematics Building,
The Ohio State University,
231 West 18th Avenue,
Columbus, Ohio 43210-1174,
U.S.A.

(Received, September 15, 1990; revised version, January 21, 1991)