Killing Potentials with Geodesic Gradients on Kähler Surfaces

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ABSTRACT. We classify compact Kähler surfaces with nonconstant Killing potentials such that all integral curves of their gradients are reparametrized geodesics.

1. INTRODUCTION

Let τ be a Killing potential on a Kähler manifold (M,g), by which one means a C^∞ function $\tau : M \to \mathrm{I\!R}$ such that $J(\nabla\tau)$ is a Killing field on (M,g). We say that τ has a *geodesic gradient* if all nontrivial integral curves of $\nabla\tau$ are reparametrized geodesics, or—equivalently (Section 4)—if $dQ \wedge d\tau = 0$, where $Q = g(\nabla\tau, \nabla\tau)$.

There are many known examples of nonconstant Killing potentials with geodesic gradients on compact Kähler manifolds. They include the soliton functions of the Kähler-Ricci solitons discovered by Koiso [10] and, independently, Cao [2]; special Kähler-Ricci potentials [4, § 7], [5, §§5–6]; and functions on complex projective spaces obtained as ratios of suitable real quadratic forms (Example 4.5).

Theorem 5.3 of this paper classifies all triples (M,g,τ) formed by a compact Kähler surface (M,g) and a nonconstant Killing potential $\tau : M \to \mathrm{I\!R}$ with a geodesic gradient, which is not a special Kähler-Ricci potential. It turns out that M must then be a holomorphic $\mathbb{C}\mathrm{P}^1$ bundle over a Riemann surface Σ, while g and τ are obtained, via an explicit Calabi-style construction, from a Riemannian metric h on Σ, a function Q on a closed interval $\mathbf{I}$, subject only to specific positivity and boundary conditions, and a nonconstant mapping $\gamma : \Sigma \to \mathrm{I\!R}\mathrm{P}^1 \smallsetminus \mathbf{I}$ (where $\mathbf{I} \subset \mathrm{I\!R} \subset \mathrm{I\!R}\mathrm{P}^1$). The objects $\Sigma, h, \mathbf{I}, Q$ and γ, being geometric invariants of M, g and τ, may be used to parametrize the moduli space of the triples (M,g,τ) in question.

On the other hand, the author and Maschler provided, in [5], a complete description of special Kähler-Ricci potentials on compact Kähler manifolds. Combined with that result, Theorem 5.3 leads to a classification, in Theorem 6.1, of all compact Kähler surfaces admitting nonconstant Killing potentials with geodesic gradients. They are biholomorphic to total spaces of $\mathbb{C}\mathrm{P}^1$ bundles, or to $\mathbb{C}\mathrm{P}^2$.

2. PRELIMINARIES

All manifolds, mappings and tensor fields, including Riemannian metrics and functions, are assumed to be of class C^∞. A (sub)manifold is by definition connected.

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Let Ric be the Ricci tensor of a torsion-free connection ∇ on a manifold M. Any vector field v on M satisfies the Bochner identity

$$\text{(1)}\qquad \text{a) } d\,\mathrm{div}\,v = \mathrm{div}\,\nabla v - \mathrm{Ric}(\,\cdot\,,v), \qquad \text{that is, b) } v^k{}_{,kj} = v^k{}_{,jk} - R_{jk}v^k.$$

The coordinate form (1.b) arises by contraction in $l = k$ from the Ricci identity $v^l{}_{,jk} - v^l{}_{,kj} = R_{jkq}{}^l v^q$, which in turn is nothing else than the definition of the curvature tensor R. For such M, ∇ and v, we treat ∇v as the endomorphism of the tangent bundle acting on vector fields w by $w \mapsto \nabla_v w$, and then $\mathrm{div}\,v = \mathrm{tr}\,\nabla v$.

Whenever (M, g) is a Riemannian manifold, the symbol ∇ will denote both the Levi-Civita connection of g and the g-gradient. If $\tau : M \to \mathrm{I\!R}$, we have

$$\text{(2)}\qquad 2\nabla d\tau(v,\,\cdot\,) = dQ, \quad \text{where } v = \nabla\tau \text{ and } Q = g(v,v),$$

as one sees noting that, in local coordinates, $(\tau_{,k}\tau^{,k})_{,j} = 2\tau_{,kj}\tau^{,k}$.

Given a submanifold Σ of a Riemannian manifold (M, g) and $\varepsilon \in (0,\infty)$, we denote by $N\Sigma$ the normal bundle of Σ, by $N^\varepsilon\Sigma$ the (disjoint) union of radius ε open balls around 0 in the normal spaces of Σ, by $B_\varepsilon(\Sigma)$ the set of points of M lying at distances less than ε from Σ, also called the *ε-neighborhood* of Σ in (M, g), by $\mathcal{D} \subset TM$ is the domain of the exponential mapping Exp of (M, g), and by $\mathrm{Exp}^\perp : \mathcal{D} \cap N\Sigma \to M$ the *normal exponential mapping* of Σ, that is, the restriction of Exp to $\mathcal{D} \cap N\Sigma$. Thus, $N^\varepsilon\Sigma \subset N\Sigma$ and $B_\varepsilon(\Sigma) \subset M$ are open submanifolds.

***Remark* 2.1.** As shown by Kobayashi [7], if u is a Killing vector field on a Riemannian manifold (M, g), the connected components of the zero set of u are mutually isolated totally geodesic submanifolds of even codimensions. Every point of any such component Σ obviously has a neighborhood Σ' in Σ with the property that, for some $\varepsilon \in (0,\infty)$, the domain of $\mathrm{Exp}^\perp$ contains $N^\varepsilon\Sigma'$ and $\mathrm{Exp}^\perp$ maps $N^\varepsilon\Sigma'$ diffeomorphically onto an open set $U \subset M$. Whenever Σ', ε and U are chosen as above, the inverse of the diffeomorphism $\mathrm{Exp}^\perp$ sends u restricted to U to a vector field $\hat{u}$ on $N^\varepsilon\Sigma'$ which is vertical (tangent to the open-ball fibres $N^\varepsilon_y\Sigma$, $y \in \Sigma'$) and, in each fibre $N^\varepsilon_y\Sigma$, coincides with the linear vector field provided by the endomorphism $[\nabla u]_y$ of T_yM restricted to $N_y\Sigma$.

This is immediate since $\mathrm{Exp}^\perp$ maps short line segments emanating from 0 in $N^\varepsilon_y\Sigma$ onto geodesics, and so the local flow of u in the submanifold $\mathrm{Exp}^\perp(N^\varepsilon_y\Sigma)$ corresponds, via $\mathrm{Exp}^\perp$, to the linear local flow near 0 in $N_y\Sigma$ generated by $[\nabla u]_y$.

***Remark* 2.2.** Let Σ be a compact submanifold of a Riemannian manifold (M, g). If $\varepsilon \in (0,\infty)$ is sufficiently small, then the domain of $\mathrm{Exp}^\perp$ contains $N^\varepsilon\Sigma$ and $\mathrm{Exp}^\perp$ maps $N^\varepsilon\Sigma$ diffeomorphically onto $B_\varepsilon(\Sigma)$. For any such ε, the squared distance from Σ is a C^∞ function on $B_\varepsilon(\Sigma)$, corresponding under the diffeomorphism $\mathrm{Exp}^\perp$ to the squared-norm function on $N^\varepsilon\Sigma$, and its g-gradient is tangent to all normal geodesics of lengths less that ε emanating from Σ, all of which are distance-minimizing.

The last claim follows from the generalized Gauss lemma, cf. [6, p. 26], in exactly the same way as the ordinary Gauss lemma is used to establish a special case of this claim, in which Σ consists of a single point.

The following well-known fact will be needed at the very end of Section 12.

***Lemma* 2.3.** *Let $(\hat{M},\hat{g})$ and (M,g) be complete Riemannian manifolds with open subsets $\hat{M}'\subset\hat{M}$ and $M'\subset M$ such that both $\hat{M}\smallsetminus\hat{M}'$ and $M\smallsetminus M'$ are unions of finitely many compact submanifolds of codimensions greater than one. Any isometry of $(\hat{M}',\hat{g})$ onto (M',g) can then be uniquely extended to an isometry of $(\hat{M},\hat{g})$ onto (M,g). If, in addition, $(\hat{M},\hat{g})$ and (M,g) are Kähler manifolds and the isometry $\hat{M}'\to M'$ is a biholomorphism, then so is the extension $\hat{M}\to M$.*

Proof. See, for instance, [5, Lemma 16.1]. □

***Remark* 2.4.** One easily verifies that a Riemannian manifold (M,g) is complete if and only if every curve $(b,c)\ni t\mapsto x(t)\in M$ of finite length has limits as $t\to b$ and $t\to c$.

***Remark* 2.5.** We treat $\mathrm{I\!R}$ as a subset of $\mathrm{I\!RP}^1$ via the usual embedding $\tau\mapsto[\tau,1]$ (in homogeneous coordinates). For algebraic operations involving $\infty=[1,0]\in\mathrm{I\!RP}^1$ and elements of $\mathrm{I\!R}\subset\mathrm{I\!RP}^1$, the standard conventions apply; thus, $p/\infty=0$ and $q/0=p+\infty=\infty$ if $p\in\mathrm{I\!R}$ and $q\in\mathrm{I\!R}\smallsetminus\{0\}$.

3. KILLING POTENTIALS

The symbols J and ω always stand for the complex-structure tensor of a given Kähler manifold (M,g) and for its Kähler form, with $\omega=g(J\,\cdot\,,\,\cdot\,)$. Real-holomorphic vector fields on M then are the sections v of TM such that $\pounds_vJ=0$. This is equivalent to $[J,\nabla v]=0$, the commutator $[\,,]$ being applied here to vector-bundle morphisms $TM\to TM$. In fact, $\pounds_vJ=[J,\nabla v]$, since $\nabla J=0$.

A C^∞ function τ on a Kähler manifold is a Killing potential (Section 1) if and only if $v=\nabla\tau$ is a real-holomorphic vector field, cf. [4, Lemma 5.2]. In this case,

$$d_v\Delta\tau\;=\;2\,\mathrm{div}\,\nabla_vv\;-\;2|\nabla v|^2,\quad\text{where }\;v=\nabla\tau. \tag{3}$$

In fact, the Bochner identity (1.a) with $v=\nabla\tau$ reads $d\Delta\tau=\mathrm{div}\,\nabla d\tau-\mathrm{Ric}(\,\cdot\,,v)$. Multiplying both sides by 2 and then subtracting the well-known equality

$$d\Delta\tau\;=\;-2\mathrm{Ric}(\,\cdot\,,v),\quad\text{ with }\;v=\nabla\tau, \tag{4}$$

valid whenever τ is a Killing potential [1], cf. [4, formula (5.4)], we obtain $d\Delta\tau=2\,\mathrm{div}\,\nabla d\tau$. Hence $d_v\Delta\tau=2v^k{}_{,jk}v^j=2(v^k{}_{,j}v^j)_{,k}-2v^k{}_{,j}v^j{}_{,k}$, as required.

***Remark* 3.1.** Given a Killing potential τ on a Kähler manifold (M,g), let us consider the vector fields $v=\nabla\tau$ and $u=Jv$. Then

(a) v,u are both real-holomorphic, and commute,
(b) u is a Killing field.

Specifically, (b) amounts to the definition of a Killing potential at the beginning of Section 1, and (a) is well known [4, formula (5.1.b) and Lemma 5.2].

On compact Kähler manifolds, Killing potentials have yet another characterization: their gradients are precisely the same as J-images of Killing fields with zeros. See the Appendix.

A *special Kähler-Ricci potential* [4, § 7] on a Kähler manifold (M,g) is any nonconstant Killing potential τ such that, at points where $d\tau\neq0$, all nonzero vectors orthogonal to $\nabla\tau$ and $J(\nabla\tau)$ are eigenvectors of both $\nabla d\tau$ and Ric.

***Remark* 3.2.** Let τ and f be functions on a manifold M such that τ is nonconstant and $f = \chi \circ \tau$ with some C^∞ function $\chi : \mathbf{I} \to \mathrm{I\!R}$, where $\mathbf{I} = \tau(M)$ is the range of τ. We then say that f is *a C^∞ function of τ*.

***Remark* 3.3.** In view of (2) and (4), a nonconstant Killing potential τ on a Kähler manifold (M, g) is a special Kähler-Ricci potential if and only if every point with $d\tau \neq 0$ has a neighborhood on which both $Q = g(\nabla\tau, \nabla\tau)$ and $\Delta\tau$ are C^∞ functions of τ.

4. GEODESIC GRADIENTS: THE SIMPLEST EXAMPLES

Let ∇ be a connection in the tangent bundle TM of a manifold M. A *geodesic vector field* relative to ∇ is any vector field v on M such that, for some function $\psi : M' \to \mathrm{I\!R}$ defined on the open set $M' \subset M$ on which $v \neq 0$,

$$\nabla_v v = \psi v \qquad \text{everywhere in } M', \tag{5}$$

or, equivalently, such that the integral curves of v are reparametrized ∇-geodesics.

We say that a function $\tau : M \to \mathrm{I\!R}$ on a Riemannian manifold (M, g) has a *geodesic gradient* if $v = \nabla\tau$ is a geodesic vector field for the Levi-Civita connection ∇ of g. It is clear from (2) and (5) that this amounts to the condition

$$dQ \wedge d\tau = 0, \qquad \text{where} \quad Q = g(\nabla\tau, \nabla\tau), \tag{6}$$

which is in turn the same as requiring Q to be, locally in M', a function of τ.

***Remark* 4.1.** If v is a geodesic vector field for a connection ∇ on M, then so is μv for any function $\mu : M \to \mathrm{I\!R}$.

***Example* 4.2.** Each of the following assumptions about a given Riemannian manifold (M, g) and a function $\tau : M \to \mathrm{I\!R}$ implies that τ has a geodesic gradient.

(a) Some group of isometries of (M, g) with principal orbits of codimension 1 leaves τ invariant.
(b) $\dim M = 1$.
(c) $\tau = \chi \circ \rho$ for some function ρ on (M, g) that has a geodesic gradient and some $\chi : \mathbf{I} \to \mathrm{I\!R}$, where $\mathbf{I} \subset \mathrm{I\!R}$ is an interval containing the range $\rho(M)$.
(d) (M, g) is the ε-neighborhood, for any sufficiently small $\varepsilon \in (0, \infty)$, of a given compact submanifold Σ in a Riemannian manifold, and τ is the squared distance from Σ.
(e) (M, g) is a Riemannian product and τ is a function with a geodesic gradient on one of the factor Riemannian manifolds, treated as a function on M.

For (a) this is a direct consequence of (6), as the gradients of τ and Q are both normal to the orbits; (b) leads to (a) for the trivial group; and the claims in (c) – (d) easily follow from Remarks 4.1 and 2.2, while the case of (e) is obvious.

***Example* 4.3.** A a nonconstant function τ with a geodesic gradient exists on every Riemannian manifold (M, g), and may be chosen so that 0 is a regular value of τ, and $\tau^{-1}(0)$ is any prescribed compact submanifold Σ of codimension 1 which disconnects M (such as a sphere embedded in a coordinate domain).

In fact, for ε as in Remark 2.2 and a unit normal vector field w along Σ, the assignment $(y,t) \mapsto \exp_y tw_y$ defines a diffeomorphism $\Sigma \times (-\varepsilon, \varepsilon) \to B_\varepsilon(\Sigma)$. As the function $\rho : B_\varepsilon(\Sigma) \to \mathrm{I\!R}$ sending $\exp_y tw_y$ to t has a geodesic gradient (cf. Remark 2.2), we may set $\tau = \chi \circ \rho$, as in (iii), with $\chi : \mathrm{I\!R} \to \mathrm{I\!R}$ that is nondecreasing, constant on both $(-\infty, -\delta)$ and (δ, ∞) for some $\delta \in (0, \varepsilon)$, and equal to the identity on a neighborhood of 0.

***Example* 4.4.** Every special Kähler-Ricci potential on a Kähler manifold (Section 3) has a geodesic gradient, which is immediate as (2) then implies (6).

***Example* 4.5.** For fixed nonnegative integers k, l, m with $m = k + l + 1 \geq 2$, let g be the Fubini-Study metric on $M = \mathbb{C}\mathrm{P}^m$. Then $\tau : M \to \mathrm{I\!R}$ defined by the assignment $[x,y] \mapsto |y|^2/(|x|^2 + |y|^2)$, where $[x,y]$ are the homogeneous coordinates, while $x \in \mathbb{C}^{k+1}$ and $y \in \mathbb{C}^{l+1}$, is a nonconstant Killing potential with a geodesic gradient. More precisely, it is easy to verify that Q in (6) equals $4(1-\tau)\tau$, so that the critical points of τ form the union of two disjoint linear varieties $\mathbb{C}\mathrm{P}^k$ and $\mathbb{C}\mathrm{P}^l$ in $\mathbb{C}\mathrm{P}^m$, corresponding to the subspaces $\mathbb{C}^{k+1} \times \{0\}$ and $\{0\} \times \mathbb{C}^{l+1}$ of $\mathbb{C}^{m+1}$.

***Remark* 4.6.** Let τ be a function with a geodesic gradient exists on a Riemannian manifold. For any nonconstant integral curve $t \mapsto x(t)$ of the gradient $v = \nabla\tau$, the τ-image of the curve has the form (b, c), with $-\infty \leq b < c \leq \infty$. Since τ is an increasing function of t, it can be used as a new curve parameter. In terms of τ, the length of the curve obviously equals $\int_b^c Q^{-1/2}\, d\tau$, where $Q = g(v,v)$.

5. FURTHER EXAMPLES AND A CLASSIFICATION THEOREM

The following construction generalizes that of [5, §5] (in the case $m = 2$), and gives rise to compact Kähler surfaces (M, g) with nonconstant Killing potentials τ, which have geodesic gradients, but, in contrast with [5, §5], need *not* be special Kähler-Ricci potentials. For a detailed comparison with [5, §5], see Remark 5.1 below.

One begins by fixing a nonuple

$$\mathbf{I},\ a,\ \Sigma,\ h,\ \mathcal{L},\ (\ ,\ ),\ \mathcal{H},\ \gamma,\ Q \tag{7}$$

consisting of the following objects:

- (i) a nontrivial closed interval $\mathbf{I} = [\tau_{\min}, \tau_{\max}]$ of the variable τ,
- (ii) a real number $a > 0$,
- (iii) a compact Kähler manifold (Σ, h) of complex dimension 1,
- (iv) a C^∞ function $Q : \mathbf{I} \to \mathrm{I\!R}$ equal to 0 at the endpoints of $\mathbf{I}$, positive on its interior $\mathbf{I}^\circ$, with $dQ/d\tau = 2a$ at $\tau_{\min}$ and $dQ/d\tau = -2a$ at $\tau_{\max}$,
- (v) a C^∞ mapping $\gamma : \Sigma \to \mathrm{I\!R}\mathrm{P}^1 \smallsetminus \mathbf{I}$, with $\mathbf{I} \subset \mathrm{I\!R} \subset \mathrm{I\!R}\mathrm{P}^1$ as in Remark 2.5,
- (vi) a C^∞ complex line bundle $\mathcal{L}$ over Σ with a Hermitian fibre metric $(\ ,\ )$,
- (vii) the horizontal distribution $\mathcal{H}$ of a connection in $\mathcal{L}$ making $(\ ,\ )$ parallel and having the curvature form $\Omega = -a(\tau_* - \gamma)^{-1}\omega^{(h)}$,

where $\omega^{(h)}$ is the Kähler form of (Σ, h). Thus, $\Omega = 0$ at points at which $\gamma = \infty$. Note that, in (iii), (Σ, h) is nothing else than a closed oriented real surface endowed with a Riemannian metric.

In addition to the data (7), let us fix a C^∞ diffeomorphism $\mathbf{I}^\circ \ni \tau \mapsto r \in (0,\infty)$ such that $dr/d\tau = ar/Q$, and a "base point" $\tau_* \in \mathbf{I}$. We choose τ_* to be the midpoint of $\mathbf{I}$, which is just an arbitrary normalization. See Remark 5.2.

We use the symbol $\mathcal{V}$ for the vertical distribution $\operatorname{Ker} d\pi$ on the total space of the bundle (also denoted by $\mathcal{L}$), $\pi : \mathcal{L} \to \Sigma$ being the bundle projection. From now on the norm function $r : \mathcal{L} \to [0,\infty)$ of $(\,,\,)$ is treated, simultaneously, as an independent variable ranging over $[0,\infty)$, so that our fixed diffeomorphism $\tau \mapsto r$ turns τ, and hence Q as well, into functions $\mathcal{L} \to \mathrm{I\!R}$.

Next we define a Riemannian metric g on $M' = \mathcal{L} \smallsetminus \Sigma$, where Σ is identified with the zero section, by $g = (\tau_* - \gamma)^{-1}(\tau - \gamma)h$ or $g = h$ on $\mathcal{H}$, $g = (ar)^{-2}Q\,\mathrm{Re}(\,,\,)$ on $\mathcal{V}$, and $g(\mathcal{H},\mathcal{V}) = \{0\}$. Tensors on Σ are denoted by the same symbols as their pullbacks to M', so that γ stands here for $\gamma \circ \pi$ and h for $\pi^* h$. On $\mathcal{H}$, the first formula is to be used in the π-preimage of the set in Σ on which $\gamma \neq \infty$, and the second one on its complement. Note that C^∞-differentiability of the algebraic operations in $\mathrm{I\!RP}^1$, wherever they are permitted (cf. Remark 2.5) implies that g is of class C^∞.

Obviously, (M', g) is an almost Hermitian manifold for the almost complex structure J obtained by requiring that the subbundles $\mathcal{V}$ and $\mathcal{H}$ of TM' be J-invariant and, for any $x \in M'$, the restriction of J_x to $\mathcal{V}_x$, or $\mathcal{H}_x$, coincide with the complex structure of the fibre $\mathcal{L}_{\pi(x)}$ or, respectively, with the $d\pi_x$-pullback of the complex structure of Σ.

Let M be the $\mathbb{C}\mathrm{P}^1$ bundle over Σ resulting from the projective compactification of $\mathcal{L}$. Our g, τ and J then have C^∞ extensions to a metric, function, and almost complex structure on M denoted, again, by g, τ and J. In fact, such extensions exist for the distributions $\mathcal{V}$ and $\mathcal{H}$. Our claim thus follows since, according to the conclusion made in [5, §5] for $m = 1$, the function τ restricted to the subset $\mathcal{L}_y \smallsetminus \{0\}$ of a single fibre of $\mathcal{L}_y$ of $\mathcal{L}$, and the metric $(ar)^{-2}Q\,\mathrm{Re}(\,,\,)$ on $\mathcal{L}_y \smallsetminus \{0\}$, can both be smoothly extended to the Riemann-sphere compactification of $\mathcal{L}_y$.

For the section v of the vertical distribution $\mathcal{V}$ on $\mathcal{L}$ which, restricted to each fibre of $\mathcal{L}$, equals a times the radial (identity) vector field on the fibre, one easily verifies that $d_v = Q\,d/d\tau$, both sides being viewed as operators acting on C^∞ functions of τ. Consequently, v equals the g-gradient $\nabla\tau$ of τ. Note that $g(v,v) = Q$.

From now on the symbols w, w' will stand both for any two C^∞ vector fields in Σ and, simultaneously, for their horizontal lifts to $\mathcal{L}$ (which themselves are just the π-projectable horizontal vector fields on $\mathcal{L}$). We also define a vector field u on $\mathcal{L}$ by $u = iv$ (multiplication by i in each fibre), so that, for our J, and w as above, $Jv = u$, while Jw has the same meaning in $\mathcal{L}$ as in Σ. With ∇ and D denoting the Levi-Civita connections of g and h, one has, on a dense open subset of M',

$$
\begin{aligned}
&\nabla_v v = -\nabla_u u = \psi v, \quad \nabla_v u = \nabla_u v = \psi u,\\
&\nabla_v w = \nabla_w v = \phi w, \quad \nabla_u w = \nabla_w u = \phi J w,\\
&Q\nabla_w w' = Q\mathrm{D}_w w' - \phi[g(w,w')v + g(Jw,w')u]\\
&\qquad + (\tau_* - \gamma)^{-1}(\tau - \tau_*)\phi[h(\mathrm{D}\gamma, w)w' + h(\mathrm{D}\gamma, w')w - h(w,w')\mathrm{D}\gamma]
\end{aligned}
\tag{8}
$$

for $\psi, \phi : M' \to \mathbb{R}$ given by $2\psi = dQ/d\tau$ and $2\phi = (\tau - \gamma)^{-1}Q$. The dense open set in question is the union of the π-preimages of two subsets in Σ, which are: the γ-preimage of $\mathbb{R} = \mathbb{R}\mathrm{P}^1 \smallsetminus \{\infty\}$, cf. Remark 2.5; and the interior of the γ-preimage of ∞. On the former set, $\mathrm{D}\gamma$ denotes the h-gradient of γ treated as a real-valued function; on the latter, we set $\mathrm{D}\gamma = 0$.

In fact, the connection ∇ *defined* by (8) is clearly compatible with g and torsion-free, since v, u commute both with each other and with the horizontal lifts w, w', while the vertical component of $[w, w']$ is $a^{-1}\Omega(w, w')u$, cf. [4, formula (3.6)].

Also, J commutes with ∇_v, ∇_u, all ∇_w, and ∇v. These commutation relations are obvious from (8), possibly except $[J, \nabla_w]w' = 0$, which follows, as (8) yields

$$[J, \nabla_w]w' = [(\tau_* - \gamma)Q]^{-1}(\tau - \tau_*)\phi\,[\Xi(Jw, w', J\mathrm{D}\gamma) - \Xi(w, w', \mathrm{D}\gamma)],$$

with $\Xi(w, w', w'') = h(Jw, w')w'' + h(Jw', w'')w + h(Jw'', w)w'$. Skew-symmetry of Ξ and two-dimensionality of Σ now give $\Xi(w, w', w'') = 0$.

The conclusions of the last paragraph amount to $\nabla J = 0$ and $[J, \nabla v] = 0$. The former equality means that g is a Kähler metric; the latter states that $v = \nabla\tau$ is real-holomorphic, which makes τ a (nonconstant) Killing potential on the Kähler manifold (M, g), cf. Section 3. Also, τ has a geodesic gradient in view of the first line in (8). Note that $\Delta\tau = \mathrm{tr}\,\nabla v = 2\phi + 2\psi$, and so

$$\Delta\tau \;=\; (\tau - \gamma)^{-1}Q \;+\; dQ/d\tau. \tag{9}$$

***Remark* 5.1.** By (9) and Remark 3.3, our τ is a special Kähler-Ricci potential on (M, g) if and only if γ is constant. When γ is constant, our construction becomes that of [5, §5] for $m = 2, \tau_0 = \tau_{\min}$, and either $\varepsilon = 0$ with an undefined constant c (when $\gamma = \infty$), or $\varepsilon = \pm 1$ with $c \in \mathbb{R} = \mathbb{R}\mathrm{P}^1 \smallsetminus \{\infty\}$ equal to the value of γ (if $\gamma \neq \infty$); in the latter case, our h is $2|\tau_* - c|$ times the metric denoted by h in [5].

***Remark* 5.2.** The "base point" τ_* is not a geometric invariant of the triple (M, g, τ) constructed above, and one may choose it to be a different constant, or even a *function* $\widetilde{\tau}_* : \Sigma \to \mathbb{R}$, as long as $\tau \neq \gamma \neq \widetilde{\tau}_*$ everywhere in M, so that the definition of g makes sense. (Again, we treat τ, γ and $\widetilde{\tau}_*$ as functions $M \to \mathbb{R}$.) The resulting metric g will then remain unchanged, provided that we replace h with $\widetilde{h}$, equal to $(\tau_* - \gamma)^{-1}(\widetilde{\tau}_* - \gamma)h$ on the subset of Σ on which $\gamma \neq \infty$, and to h on its complement. (Condition (vii) for $\widetilde{\tau}_*$ and $\widetilde{h}$ will still hold, with the same $\mathcal{H}$ and Ω.)

More generally, we can relax conditions (iii) – (v), while keeping (ii), (vi) and (vii), so that Σ need not be compact, Q is defined and positive on an open interval, and $\gamma, \tau_* : \Sigma \to \mathbb{R}\mathrm{P}^1$. The construction then yields a triple (M, g, τ) with the same properties, except compactness of M, where M now is any connected component of the open set in $\mathcal{L} \smallsetminus \Sigma$ defined by requiring that $\tau \neq \gamma \neq \tau_*$ and that the values of the norm function r lie in the resulting new range.

The following result provides a classification of compact Kähler surfaces with nonconstant Killing potentials, which have geodesic gradients, yet are not special Kähler-Ricci potentials.

***Theorem* 5.3.** *Let τ be a nonconstant Killing potential with a geodesic gradient on a compact Kähler surface (M,g). If τ is not a special Kähler-Ricci potential on (M,g), then, up to a biholomorphic isometry, the triple (M,g,τ) arises from the above construction applied to some data* (7) *satisfying* (i) – (vii), *such that $\gamma : \Sigma \to \mathrm{I\!RP}^1 \smallsetminus \mathbf{I}$ is nonconstant.*

A proof of Theorem 5.3 is given in Sections 11 and 12. First, however, Theorem 5.3 will be used, in Section 6, to derive a more general conclusion.

6. A MORE GENERAL CLASSIFICATION RESULT

Compact Kähler manifolds of all complex dimensions $m \geq 2$, admitting special Kähler-Ricci potentials, have been completely described in [5, Theorem 16.3]. They form two types, referred to in [5] as Class 1 and Class 2, the universal models of which were constructed in [5, §5] and, respectively, [5, §6].

Class 1, for $m = 2$, is obtained as in Section 5, from data (7) with (i) – (vii), such that the mapping $\gamma : \Sigma \to \mathrm{I\!RP}^1 \smallsetminus \mathbf{I}$ is constant. For details, see Remark 5.1.

Class 2, in complex dimension 2, arises from a modified version of the construction in Section 5, in which γ is still constant, but instead of lying in $\mathrm{I\!RP}^1 \smallsetminus \mathbf{I}$, it is assumed to one of the endpoints of $\mathbf{I}$, and $a \in \mathrm{I\!R}$, rather than being positive, is nonzero, with the sign such that $(\tau_* - \gamma)a > 0$. The choice of other data in (7) is less general than in Section 5: (Σ, h) and $\mathcal{L}, (\,,\,), \mathcal{H}$ are, respectively, $\mathbb{C}\mathrm{P}^2$ with $2(\tau_* - \gamma)/a$ times the Fubini-Study metric, and the tautological bundle over $\mathbb{C}\mathrm{P}^2$ with its standard fibre metric and connection (both invariant under the full isometry group). The metric g on $M' = \mathcal{L} \smallsetminus \Sigma$, defined as before, still has a C^∞ extension to a tensor field on the total space $\mathcal{L}$ and, further, on the projective compactification of $\mathcal{L}$. Along the zero section $\Sigma \subset \mathcal{L}$, however, this extension fails to be positive definite, as $\tau = \gamma$ on Σ. One removes the resulting degeneracy of g by blowing down the zero section, which results in replacing the projective compactification of $\mathcal{L}$ with $\mathbb{C}\mathrm{P}^2$.

Using the result of [5], one can now generalize Theorem 5.3 a follows.

***Theorem* 6.1.** *Up to biholomorphic isometries, every triple (M,g,τ) formed by a compact Kähler surface (M,g) and a nonconstant Killing potential τ with a geodesic gradient on (M,g,τ) is either obtained as in Section* 5 *from some data* (7) *with* (i) – (vii), *or arises from the above construction of Class 2 special Kähler-Ricci potentials on $M = \mathbb{C}\mathrm{P}^2$.*

Proof. If τ is (or, is not) a special Kähler-Ricci potential, our claim follows from Theorem 5.3 or, respectively, [5, Theorem 16.3]. ☐

7. ONE-JETS OF GEODESIC VECTOR FIELDS AT THEIR ZEROS

As a first step toward the proof of Theorem 5.3, we now proceed to establish one general property of geodesic vector fields, defined in Section 4.

***Remark* 7.1.** If $\varepsilon > 0$ and a curve $[0,\varepsilon] \ni t \mapsto v(t) \in V$ in a normed vector space V with $\dim V < \infty$ is differentiable at $t = 0$, while $v(0) = 0 \neq w$, where $w = \dot{v}(0)$ and $\dot{v} = dv/dt$, then $v(t)/|v(t)| \to w/|w|$ as $t \to 0^+$. (In fact, $v(t)/t \to \dot{v}(0) = w$ as $t \to 0^+$. Thus, $|v(t)|/t \to |w|$ and $v(t)/|v(t)| = [v(t)/t][|v(t)|/t]^{-1} \to w/|w|$.)

***Lemma* 7.2.** *Let v be a geodesic vector field on a manifold M with a fixed connection ∇. If $y \in M$ and $v_y = 0$, then, for $E = [\nabla v]_y : T_yM \to T_yM$ and some $a \in \mathbb{R}$, we have $E^2 = aE$, that is, one of the following two cases occurs:*

- (i) *E is diagonalizable, and either it is a multiple of the identity, or it has exactly two distinct eigenvalues, one of which is zero.*
- (ii) *E is not diagonalizable and $E^2 = 0$.*

Proof. We may assume that $E \neq 0$ and identify a neighborhood of y in M with a neighborhood U of 0 in a vector space V, so that y corresponds to 0. This turns ∇ into a connection in TU. As $v = 0$ at the point 0, the operator E is now the differential at 0 of v viewed as a mapping $U \to V$. We also fix a vector subspace $V' \subset V$ of dimension $\operatorname{rank} E$ such that E maps V' isomorphically onto the image $E(V)$, and choose a linear projection $P : V \to E(V)$. In view of the inverse mapping theorem, there exists a neighborhood U' of 0 in V' such that $U' \subset U$ and $\Pi = P \circ v : U' \to U''$ is a diffeomorphism onto a neighborhood U'' of 0 in $E(V)$. Thus, $\Pi(0) = 0$ and $d\Pi_0$ equals E restricted to V'.

Given any nonzero vector $w \in E(V)$, let $\varepsilon > 0$ be such that $tw \in U''$ for all $t \in [0, \varepsilon]$. We set $x(t) = \Pi^{-1}(tw)$ if $t \in [0, \varepsilon]$. Thus, $v(x(t)) \neq 0$ for $t \in (0, \varepsilon]$, as $Pv(x(t)) = \Pi(x(t)) = tw \neq 0$. We may now set $u(t) = v(x(t))/|v(x(t))|$, if $0 < t \leq \varepsilon$, using a fixed norm $|\;|$ in V, so that $u(t) \to w/|w|$ as $t \to 0^+$ according to Remark 7.1, and an equality of the form (5) holds at each $x(t)$, $t \in (0, \varepsilon]$, with some function ψ (defined only at points where $v \neq 0$). Dividing both sides of that equality by $|v(x(t))|$ and setting $a(t) = \psi(x(t))$, we obtain $[\nabla_{u(t)} v]_{x(t)} = a(t)u(t)$. Consequently, $a(t)$ has a limit a_w as $t \to 0^+$ and, taking the limits of both sides of the last relation, we get $[\nabla_w v]_0 = a_w w$, that is, $Ew = a_w w$. Every $w \in E(V) \smallsetminus \{0\}$ is thus an eigenvector of E for some eigenvalue a_w, which is only possible if $a = a_w$ does not depend on w. Hence $E(V) \subset \operatorname{Ker}(E - a)$ or, equivalently, $E^2 - aE = (E - a)E = 0$. If $a \neq 0$, the subspaces $\operatorname{Ker} E$ and $\operatorname{Ker}(E - a)$ must, for dimensional reasons, be the summands in a direct-sum decomposition of V. This leads to case (i). Hence, if E is not diagonalizable, we have $a = 0$, and (ii) follows. $\square$

8. MORSE-BOTT FUNCTIONS WITH GEODESIC GRADIENTS

A *Morse-Bott function* on a manifold M is a C^∞ function $\tau : M \to \mathbb{R}$ such that the connected components of the set of critical points of τ are mutually isolated submanifolds of M (called the *critical manifolds* of τ), and the rank of the Hessian of τ at every critical point x is the codimension of the critical manifold containing x.

***Example* 8.1.** All Killing potentials are Morse-Bott functions, and their critical manifolds are totally geodesic complex submanifolds of the ambient Kähler manifold. This is a well-known consequence of Remark 3.1(b) and Kobayashi's result [7] mentioned in Remark 2.1. Cf. also [5, Example 11.1 and Remark 2.3(iii-c,d)].

***Remark* 8.2.** The standard examples of Morse-Bott functions are provided by homogeneous quadratic polynomials on finite-dimensional real vector spaces. The conclusion about the squared-norm function in Remark 2.2 now implies that τ of Example 4.2(d) is a Morse-Bott function.

The next remark and lemma use the symbols $\mathrm{Exp}^\perp$ and $N^\varepsilon\Sigma$ defined in Section 2.

***Remark* 8.3.** Given a critical manifold Σ of a Morse-Bott function τ on a manifold M and a point $y \in \Sigma$, there exist a neighborhood Σ' of y in Σ and $\varepsilon \in (0,\infty)$ such that the domain of $\mathrm{Exp}^\perp$ contains $N^\varepsilon\Sigma'$ and $\mathrm{Exp}^\perp$ maps $N^\varepsilon\Sigma'$ diffeomorphically onto a neighborhood U of y in M, while $\nabla\tau \neq 0$ everywhere in $U \smallsetminus \Sigma'$.

This is immediate from the inverse mapping theorem applied to $\mathrm{Exp}^\perp$ and the definition of a critical manifold.

***Lemma* 8.4.** *Let $y \in \Sigma$, for a critical manifold Σ of a nonconstant Morse-Bott function τ with a geodesic gradient on a Riemannian manifold (M, g).*

- (i) *The Hessian $\nabla d\tau$ at y has exactly one nonzero eigenvalue a.*
- (ii) *The eigenspace corresponding to a in* (i) *is the normal space $N_y\Sigma$ of Σ at y.*
- (iii) *For every sufficiently small $\varepsilon \in (0,\infty)$ there exists a neighborhood U of y in M such that the underlying one-dimensional manifolds of the maximal integral curves of the restriction of $v = \nabla\tau$ to $U \smallsetminus \Sigma$ coincide with the length ε open geodesic segments emanating from $\Sigma \cap U$ and normal to Σ.*
- (iv) *The gradient $v = \nabla\tau$ is tangent to every nonconstant geodesic $[0,b) \ni t \mapsto x(t)$ with $x(0) = y$ and $\dot{x}(0) \in N_y\Sigma$, where $b \in (0,\infty]$, and the set of $t \in [0,b)$ for which $v_{x(t)} = 0$ is discrete.*

Proof. Case (ii) in Lemma 7.2 for $v = \nabla\tau$ is excluded by self-adjointness of $B = [\nabla v]_y$. Now (i) and (ii) are immediate from Lemma 7.2(i) and the rank condition in the definition of a Morse-Bott function. Note that $B \neq 0$, for otherwise Σ would be both a submanifold of codimension 0 and a closed subset of M, which is not possible as $\Sigma \neq M$.

Assertion (iii) is a trivial consequence of Remark 8.3, since (ii) and [5, Lemma 8.2] imply that $\nabla\tau$ is tangent to all sufficiently short geodesic segments normal to Σ.

For b and $x(t)$ as in (iv), let t_{sup} be the supremum of $t' \in (0,b)$ such that v is tangent to the geodesic segment $[0,t'] \ni t \mapsto x(t)$ and the set of $t \in [0,t']$ with $v_{x(t)} = 0$ is finite. By (iii), $t_{\mathrm{sup}} > 0$.

Suppose now that $t_{\mathrm{sup}} < b$. The word 'supremum' then can be replaced with 'maximum' since, whether $v \neq 0$ or $v = 0$ at the point $x(t_{\mathrm{sup}})$, the parameter values $t \in [0,t_{\mathrm{sup}})$ with $v_{x(t)} = 0$ cannot form a strictly increasing sequence that converges to t_{sup}. (In the former case this follows from continuity of v, in the latter from (iii) applied to $y' = x(t_{\mathrm{sup}})$ and the critical manifold containing y', rather than y and Σ.) Next, maximality of t_{sup} gives $v = 0$ at y'. Applying (iii), again, to y' instead of y, we see that v is tangent to some segment $[0,t'] \ni t \mapsto x(t)$ with $t' > t_{\mathrm{sup}}$. The resulting contradiction shows that $t_{\mathrm{sup}} = b$, completing the proof. $\square$

***Remark* 8.5.** For $(M,g), \tau, \Sigma$, and y satisfying the hypotheses of Lemma 8.4, and any unit-speed geodesic $t \mapsto x(t)$ such that $x(0) = y$ and $\dot{x}(0) \in N_y\Sigma$, writing $\dot\tau(t) = d[\tau(x(t))]/dt$, we get, from Lemma 8.4(ii),

$$\dot\tau(0) = 0 \neq \ddot\tau(0) = a, \quad \text{with } a \text{ as in Lemma 8.4(i).} \tag{10}$$

9. AN IRP^1-VALUED INVARIANT

Any nonconstant Killing potential with a geodesic gradient on a Kähler surface (M,g) naturally gives rise to a C^∞ mapping $\gamma : M \to \mathrm{IRP}^1$, described in Lemma 9.1 below. We begin by introducing some notations.

In the remainder of the paper, except Section 10, τ is always assumed to be a nonconstant Killing potential with a geodesic gradient on a Kähler manifold (M,g) of complex dimension $m \geq 2$. We write

$$v = \nabla\tau, \qquad u = Jv, \qquad Q = g(v,v). \tag{11}$$

The open set $M' \subset M$ on which $v \neq 0$ is connected and dense in M, cf. [5, Remark 2.3(ii)]. On M' one has the distributions $\mathcal{V} = \mathrm{Span}(v,u)$ and $\mathcal{H} = \mathcal{V}^\perp$. At any point of M', nonzero vectors in $\mathcal{V}$ are eigenvectors of ∇v for the eigenvalue function ψ appearing in (5). Furthermore,

$$\begin{array}{l} \mathrm{a)}\ 2\psi = dQ/d\tau, \quad \mathrm{b)}\ d_v\tau = Q, \quad \mathrm{c)}\ d_vQ = 2\psi Q, \\ \mathrm{d)}\ g(v,v) = g(u,u) = Q, \quad g(v,u) = 0, \end{array} \tag{12}$$

where (12.a) makes sense in view of the line following (6). In fact, (11) yields (12.b) and (12.d), while (2), (5) and (11) give $dQ = 2\psi\, d\tau$, so that (12.a) and (12.c) follow.

If $m = 2$, nonzero vectors in $\mathcal{H}$ are also eigenvectors of ∇v, for the eigenvalue function ϕ given by $2\phi = \Delta\tau - 2\psi$. Thus,

$$\mathrm{i)}\ \Delta\tau = 2(\psi + \phi), \qquad \mathrm{ii)}\ |\nabla v|^2 = 2(\psi^2 + \phi^2). \tag{13}$$

(The vector-bundle morphism $\nabla v : TM \to TM$ is complex-linear and Hermitian at every point; see Section 3.) Since $\Delta\tau = \mathrm{div}\, v$, (3) combined with (5) implies, whenever $m \geq 2$, that $d_v\Delta\tau = 2(d_v\psi + \psi\Delta\tau - |\nabla v|^2)$. Consequently, by (13),

$$d_v\phi = 2(\psi - \phi)\phi \qquad \text{if} \quad m = 2. \tag{14}$$

***Lemma* 9.1.** *For any nonconstant Killing potential τ with a geodesic gradient on a Kähler surface (M,g), there exists a unique C^∞ mapping $\gamma : M \to \mathrm{IRP}^1$ such that, with the conventions of Remark* 2.5, *$\gamma = \tau - Q/(\Delta\tau - 2\psi)$ on M'. In addition,*

- (a) *At every point $x \in M$, the vectors v_x and u_x lie in* $\mathrm{Ker}\, d\gamma_x$.
- (b) *γ is constant along every geodesic issuing from a critical manifold Σ of τ in a direction normal to Σ, cf. Example* 8.1.
- (c) *γ is constant on M if and only if τ is a special Kähler-Ricci potential.*

Proof. We begin by establishing (a) and (c) for M' rather than M. Clearly, (a) holds if x lies in the interior of the set on which $\gamma = \infty$. On the set where $\gamma \neq \infty$, treating $\gamma = \tau - Q/(2\phi)$ as a real-valued function, we clearly have $d_u\gamma = 0$ since u is a Killing field and $d_u\tau = 0$ by (11) combined with (12.d), while $d_v\gamma = 0$ due to (12.b), (12.c) and (14). As the union of the two open sets is dense in M', (a) on M' follows.

To prove (c) on M', assume first that $\gamma : M' \to \mathrm{IRP}^1$ is constant. Both when $\gamma = \infty$ (and so $\Delta\tau = 2\psi$), and when $\gamma \neq \infty$, this implies that $\Delta\tau$ is, locally in M', a function of τ, since so are Q and ψ by (6) and (12.a). In view of Remark 3.3, τ then is a special Kähler-Ricci potential. On the other hand, if τ is a special Kähler-Ricci potential, we either have $\phi = 0$ identically on M', or $\phi \neq 0$ everywhere in M' [4, Lemma 12.5]. As $\gamma = \tau - Q/(2\phi)$, in the former case $\gamma = \infty$, and in the latter γ is a real constant [4, Lemma 12.5], which yields (c).

We now show that $\gamma : M' \to \mathrm{I\!RP}^1$ has a C^∞ extension to M. To this end, let $\varSigma$ be the critical manifold of τ containing a given point $y \in M \smallsetminus M'$, cf. Example 8.1. For $\varSigma', \varepsilon$ and U chosen as in Remark 8.3, $U \smallsetminus \varSigma'$ is a bundle over $\varSigma'$ with fibres which are even-dimensional (Example 8.1), and hence connected, punctured balls. By (a) for v along with Lemma 8.4(iv), the C^∞ mapping $\gamma : U \smallsetminus \varSigma' \to \mathrm{I\!RP}^1$ is constant on each fibre, so that it has an obvious C^∞ extension to U, as required.

Finally, Lemma 8.4(iv) and (a) for v imply (b). □

For (M, g) and τ constructed in Section 5, γ used in the construction, when viewed as a mapping $M \to \mathrm{I\!RP}^1$, coincides with γ defined in Lemma 9.1. This is clear from (8), (9) and (12.a).

We will show later (Lemma 11.3) that, if M in Lemma 9.1 is compact, the values of γ lie in $\mathrm{I\!RP}^1 \smallsetminus \mathbf{I}^\circ$, where $\mathbf{I}^\circ = (\tau_{\min}, \tau_{\max})$. Identifying $\mathrm{I\!RP}^1 \smallsetminus \mathbf{I}^\circ$ with an interval in $\mathrm{I\!R}$, we may then treat γ as real-valued invariant. However, such an adjustment is not possible in general, since $\gamma : M \to \mathrm{I\!RP}^1$ is *surjective* for some nonconstant Killing potentials τ with geodesic gradients on (noncompact) Kähler surfaces (M, g). An example arises when one modifies the construction in Section 5, as described in the second paragraph of Remark 5.2. Specifically, let $\varSigma = \mathbb{C}$, and so $\varSigma = U_+ \cup U_-$, where the open set $U_\pm$ is defined by the condition $\pm \mathrm{Re}\, z < 1$ imposed on $z \in \mathbb{C}$. We choose $\gamma : \mathbb{C} \to \mathrm{I\!RP}^1$ to be a surjective mapping such that $\gamma = \infty$ on the closure K of $U_+ \cap U_-$, while γ restricted to $\mathbb{C} \smallsetminus K$ is real-valued and has no critical points, and, finally, neither $\gamma : U_+ \to \mathrm{I\!RP}^1$ nor $\gamma : U_- \to \mathrm{I\!RP}^1$ is surjective. (For instance, γ with the above properties may be a function of $\mathrm{Re}\, z$.) We now select base points $\tau_*^\pm \in \mathrm{I\!R} \smallsetminus \gamma(U_\pm)$, any metric h on $\varSigma = \mathbb{C}$, and any $a \in (0, \infty)$. The 2-form $\varOmega$ on $\varSigma$ equal to $-a(\tau_*^\pm - \gamma)^{-1}\omega^{(h)}$ on $U_\pm$ is well defined, since both expressions yield $\varOmega = 0$ on $U_+ \cap U_-$. Being closed, $\varOmega$ is exact, and so it the curvature form of a Hermitian connection in the trivial complex line bundle $\mathcal{L}$ over $\varSigma$, with the bundle projection still denoted by $\pi : \mathcal{L} \to \varSigma$. We now define a metric g on an open subset $M^\pm$ of the line bundle $\mathcal{L}^\pm = \pi^{-1}(U_\pm)$ over $U_\pm$ as in Remark 5.2, using $\tau_*^\pm$ and the same function Q of the variable τ in both cases. As the two metrics agree on the intersection $\pi^{-1}(U_+ \cap U_-)$, they together form a metric g on $M = M^+ \cup M^-$, thus giving rise to a triple (M, g, τ) for which $\gamma : M \to \mathrm{I\!RP}^1$ is surjective.

***Remark* 9.2.** For later reference, note that, under the hypotheses made in the lines preceding (11), if $m = 2$, the $\mathcal{V}$ component $[w, w']^{\mathcal{V}}$, relative to the decomposition $TM' = \mathcal{H} \oplus \mathcal{V}$, of the Lie bracket of any two sections w, w' of $\mathcal{H}$ is given by

$$Q[w, w']^{\mathcal{V}} = -2\phi g(Jw, w')u. \tag{15}$$

If, in addition, w, w' commute with both v and u, then

$$d_v[\phi g(w, w')/Q] = d_u[\phi g(w, w')/Q] = 0. \tag{16}$$

Both equalities follow since ϕ is the eigenvalue function of ∇v in $\mathcal{H}$, and so

$$g(\nabla_w v, w') = \phi g(w, w'), \quad g(\nabla_w u, w') = g(J\nabla_w v, w') = \phi g(Jw, w') \tag{17}$$

for sections w, w' of $\mathcal{H}$. Hence, as $g(v, \nabla_w w') = -g(\nabla_w v, w')$ and $g(u, \nabla_w w') = -g(\nabla_w u, w')$, we have $g(v, \nabla_w w') = -\phi g(w, w')$ and $g(u, \nabla_w w') = -\phi g(Jw, w')$. Skew-symmetrized in w, w', this gives (15) due to symmetry of $g(w, w')$ and skew-symmetry of $g(Jw, w')$ in w, w'. For the same reasons of (skew)-symmetry, assuming that w, w' commute with v, u, we obtain $d_v[g(w, w')] = 2\phi g(w, w')$ and $d_u[g(w, w')] = 0$ in view of (17) and the Leibniz rule. Now (12.c) and (14) yield (16).

***Remark* 9.3.** Let τ be a nonconstant Killing potential with a geodesic gradient on a Kähler surface (M, g). If Σ is a critical manifold of τ, cf. Example 8.1, and $y \in \Sigma$, then the covariant derivative $[\nabla u]_y : T_y M \to T_y M$ of the Killing field $u = J(\nabla\tau)$ at y has the kernel $T_y\Sigma$, and acts as the operator aJ_y in the normal space $N_y\Sigma$, where a is the unique nonzero eigenvalue of $\nabla d\tau$ at y, cf. Lemma 8.4(i)–(ii).

In fact, $\nabla d\tau$ corresponds via g to ∇v, for $v = \nabla\tau$, while $\nabla u = J \circ \nabla v$.

10. MORSE-BOTT FUNCTIONS ON COMPACT MANIFOLDS

We now consider Morse-Bott functions τ with geodesic gradients such that

(18) all critical manifolds of τ are of codimensions greater than 1.

In view of Example 8.1, given a function τ on a Kähler manifold (M, g),

(19) condition (18) holds whenever τ is a nonconstant Killing potential.

***Lemma* 10.1.** *If the Hessian of a Morse-Bott function τ on a compact manifold is semi-definite at every critical point, and all critical manifolds are of codimensions $k > 1$, then*

- (a) *τ has exactly two critical manifolds, which are its maximum and minimum levels,*
- (b) *all levels of τ are connected.*

Proof. See [5, Proposition 11.4]. ☐

***Theorem* 10.2.** *Suppose that τ is a Morse-Bott function with a geodesic gradient on a compact Riemannian manifold (M, g) and all critical manifolds of τ have codimensions greater than* 1. *Let us also set $\mathbf{I} = [\tau_{\min}, \tau_{\max}]$ and $\mathbf{I}^\circ = (\tau_{\min}, \tau_{\max})$. Then*

- (i) *$Q = g(\nabla\tau, \nabla\tau)$ is a C^∞ function of τ, in the sense of Remark* 3.2,
- (ii) *for y, a as in Lemma* 8.4(i), *and $\tau \mapsto Q$ as in* (i), *$dQ/d\tau$ at y equals $2a$,*
- (iii) *for the function $\tau \mapsto Q$ in* (i), *the integral λ of $Q^{-1/2}$ over $\mathbf{I}$ is finite,*
- (iv) *λ in* (iii) *is the distance between the minimum and maximum levels of τ,*
- (v) *the assignment $\tau \mapsto s$, characterized by $ds/d\tau = Q^{-1/2}$ and $s = 0$ at $\tau = \tau_{\min}$, is a homeomorphism $\mathbf{I} \to [0, \lambda]$ which maps $\mathbf{I}^\circ$ diffeomorphically onto $(0, \lambda)$,*
- (vi) *s in* (v) *equals the distance from the minimum level of τ, when treated, due to its dependence on τ, as a function $s : M \to \mathbb{R}$.*

Proof. Let Σ and Σ^* be the minimum and maximum levels of τ.

By (6), Q restricted to the open set M' where $d\tau \neq 0$ is, locally, a C^∞ function of τ. The word 'locally' can be dropped in view of Lemma 10.1(b). The resulting C^∞ function $\mathbf{I}^\circ \ni \tau \mapsto Q$ has a continuous extension to $\mathbf{I}$, equal to 0 at the endpoints.

Next, let us fix a parametrization $[0, \delta] \ni t \mapsto x(t)$ of a shortest geodesic segment Γ joining Σ to Σ^*, with $x(0) \in \Sigma$. By (10), the infimum t' of those $t \in (0, \delta)$

for which $\dot{\tau}(t) = 0$ lies in $(0, \delta]$. As $v = \nabla\tau$ is tangent to Γ (Lemma 8.4(iv)), and $\dot{\tau} = g(v, \dot{x})$ vanishes at $t = t'$, at $x(t')$ we must also have $v = 0$, and hence $\tau = \tau_{\max}$. (The fact that $\tau(x(t))$ is an increasing function of $t \in (0, t')$ excludes the only other possibility left open by Lemma 10.1(a), namely, $\tau = \tau_{\min}$.) The distance-minimizing property of Γ now implies that $t' = \delta$, and so $v_{x(t)} \neq 0$ whenever $t \in (0, \delta)$, that is, the open-interval restriction $(0, \delta) \ni t \mapsto x(t)$ is a reparametrized integral curve of the gradient $v = \nabla\tau$. Thus, λ in (iii) is finite, as it equals the length of Γ (see Remark 4.6), which proves (iii) and (iv). Assertion (v) is in turn obvious from (iii). Finally, let us fix $x \in M'$. According to Remark 4.6 and (iii), the length of the maximal integral curve of v through x is finite, and so its underlying one-dimensional manifold C has limit endpoints $y_{\min}$ and $y_{\max}$ (Remark 2.4), at which $\tau = \tau_{\min}$ and $\tau = \tau_{\max}$ due to maximality of C and Lemma 10.1(a). By Remark 4.6, the length of C is λ. Hence, in view of (iv), $\Gamma = C \cup \{y_{\min}, y_{\max}\}$ is a distance-minimizing geodesic segment. Consequently, the same is true of the subsegment Γ' of Γ joining $y_{\min}$ to x, which is also the shortest geodesic segment joining Σ to x. The distance between Σ and x is therefore given by the length formula in Remark 4.6, applied to Γ', and (vi) follows.

For $(-\varepsilon, \varepsilon) \ni t \mapsto x(t)$ as in Remark 8.5, with $\varepsilon \in (0, \infty)$ chosen sufficiently small, $|t|$ equals $\mathrm{dist}(\Sigma, x(t))$ (or, $\mathrm{dist}(\Sigma^*, x(t))$), cf. Remark 2.2 and Lemma 10.1(a). Thus, by (vi), $|t|$ is the value of $s : M \to \mathrm{I\!R}$ or, respectively, $\lambda - s : M \to \mathrm{I\!R}$, at $x(t)$. (Note that replacing τ by $\tau_* - \tau$, where τ_* is the midpoint of $\mathbf{I}$, causes $\tau_{\min}$ to be switched with $\tau_{\max}$, and s with $\lambda - s$.) The homeomorphic correspondence between s and τ in (v) now implies that $\tau(x(t))$ is an even C^∞ function of t, and, due to the already-established dependence of Q on τ, the same is true of $Q(x(t))$. Evenness of both functions and the relation $\dot{\tau}(0) = 0 \neq \ddot{\tau}(0)$ (cf. (10)) are well-known to imply that Q restricted to some neighborhood of $\tau_{\min}$ (or, $\tau_{\max}$) in $\mathbf{I}$ is a C^∞ function of τ. See, for instance, [5, the last nine lines in §9]. Thus, the extension of Q from $\mathbf{I}^\circ$ to $\mathbf{I}$ is of class C^∞, which proves (i).

Finally, $dQ/d\tau = 2\psi$ on $\mathbf{I}^\circ$, and, consequently, on $\mathbf{I}$, since $dQ = 2\psi\, d\tau$ by (2) and (5). Again, let us choose a geodesic $t \mapsto x(t)$ as in Remark 8.5. Then v is tangent to it (Lemma 8.4(iv)) and so, by (5), $\dot{x}$ is, at every t, an eigenvector of $\nabla d\tau$ (that is, of ∇v) for the eigenvalue $\psi = [\nabla d\tau](\dot{x}, \dot{x}) = \ddot{\tau}$. Now (10) implies (ii). □

The next lemma uses the notations of Remark 2.2 and λ defined in Theorem 10.2.

***Lemma* 10.3.** *Let Σ and Σ^* be the minimum and maximum levels of a nonconstant Morse-Bott function τ with a geodesic gradient and* (18) *on a compact Riemannian manifold (M, g). Then* $\mathrm{Exp}^\perp$ *maps $N^\lambda\Sigma$ diffeomorphically onto $B_\lambda(\Sigma)$, and $B_\lambda(\Sigma) = M \smallsetminus \Sigma^*$.*

Proof. That $B_\lambda(\Sigma) = M \smallsetminus \Sigma^*$ is obvious from assertions (v) and (vi) in Theorem 10.2.

Let $M' \subset M$ be the open set given by $v \neq 0$, where $v = \nabla\tau$. If $x \in M'$, the geodesic segment $[0, 1] \ni t \mapsto x(t)$ of length $\mathrm{dist}(\Sigma, x)$, such that $x(0) = x$ and $\dot{x}(0)$ is a negative multiple of v_x, is also a shortest segment connecting x to Σ. In fact, choosing a shortest segment Γ connecting x to Σ, we see that it is normal to Σ, and so v is tangent to it (Lemma 8.4(iv)); as the diffeomorphism $\mathbf{I}^\circ \to (0, \lambda)$ in Theorem 10.2(v) is strictly increasing, on $\Gamma \smallsetminus \Sigma$ the gradient $v = \nabla\tau$ must, by

Theorem 10.2(vi), point away from Σ and toward x. Thus, both geodesic segments satisfy the same initial conditions at x.

Let the mapping $H : M' \to TM$ send any $x \in M'$ to the vector $-\dot{x}(1)$ tangent to M at $x(1)$, for $t \mapsto x(t)$ associated with x as in the last paragraph. Since $x(1) \in \Sigma$ and $\dot{x}(1)$ is normal to Σ (see above), H takes values in the subset $N^\lambda\Sigma \smallsetminus \Sigma$ of TM. Our claim now follows, since $H \circ \mathrm{Exp}^\perp$ and $\mathrm{Exp}^\perp \circ H$ are easily seen to be the identity mappings of $N^\lambda\Sigma \smallsetminus \Sigma$ and $M' = B_\lambda(\Sigma) \smallsetminus \Sigma$, while, if $\varepsilon \in (0,\infty)$ is sufficiently small, $\mathrm{Exp}^\perp : N^\varepsilon\Sigma \to B_\varepsilon(\Sigma)$ is a diffeomorphism (Remark 2.2). □

11. PROOF OF THEOREM 5.3, FIRST PART

In this section we construct the required data (7) for any triple (M, g, τ) satisfying the assumptions of Theorem 5.3, and verify conditions (i) – (vi) in Section 5.

***Lemma* 11.1.** *Let a nonconstant Killing potential τ on a complete Kähler manifold (M, g) have a geodesic gradient. Then*

(i) *at every critical point of τ, the Hessian $\nabla d\tau$ has exactly one nonzero eigenvalue, the absolute value of which is the same for all critical points,*
(ii) *if the set of critical points of τ is nonempty, the flow of the Killing vector field $u = J(\nabla\tau)$ is periodic.*

Proof. Obvious from Lemma 8.4(i) (cf. Example 8.1) and [5, Corollary 10.3]. □

***Lemma* 11.2.** *If τ is a nonconstant Killing potential with a geodesic gradient on a compact Kähler manifold (M, g), then, for some $a \in (0,\infty)$,*

(a) *$\tau_{\max}$ and $\tau_{\min}$ are the only critical values of τ,*
(b) *the τ-preimages of $\tau_{\max}$ and $\tau_{\min}$ are compact complex submanifolds of M,*
(c) *$Q = g(\nabla\tau, \nabla\tau)$ is a C^∞ function of τ, as defined in Remark* 3.2,
(d) *the values of $dQ/d\tau$ at $\tau = \tau_{\min}$ and $\tau = \tau_{\max}$ are $2a$ and $-2a$.*

Proof. Assertions (a) and (b) are immediate consequences of Lemma 10.1 combined with Example 8.1 and (19); (c) and (d) similarly follow from Theorem 10.2(i)–(ii) and the absolute-value clause in Lemma 11.1(i). □

***Lemma* 11.3.** *Given a nonconstant Killing potential τ with a geodesic gradient on a compact Kähler surface (M, g), let us set $\mathbf{I} = [\tau_{\min}, \tau_{\max}]$ and $\mathbf{I}^\circ = (\tau_{\min}, \tau_{\max})$.*

(i) *All values of $\gamma : M \to \mathrm{I\!RP}^1$, defined in Lemma* 9.1, *lie in $\mathrm{I\!RP}^1 \smallsetminus \mathbf{I}^\circ$.*
(ii) *If τ is not a special Kähler-Ricci potential, then*
    (a) *the maximum and minimum levels of τ both have complex dimension* 1,
    (b) *the values of γ all lie in $\mathrm{I\!RP}^1 \smallsetminus \mathbf{I}$.*

Proof. First, let $\gamma(y) \in \mathbf{I}^\circ$ at some $y \in M$. By Theorem 10.2(v)–(vi), which can be used here in view of Example 8.1 and (19), $\mathrm{dist}(\Sigma, y) \le \lambda$, for the minimum level Σ of τ. Hence y lies on a geodesic segment Γ of length λ emanating from Σ and normal to Σ. Due to injectivity of $\mathrm{Exp}^\perp$ on $N^\lambda\Sigma$ (Lemma 10.3), Γ also provides a shortest connection between Σ and any point of Γ. Therefore, the function s of Theorem 10.2(vi), restricted to Γ, serves as an arc-length parameter for Γ. Theorem 10.2(v) (or, Lemma 9.1(b)) implies now that the τ-image of Γ is $\mathbf{I}$

(or, respectively, that γ is constant on $\mathbf{I}$). Thus, $\varGamma$ contains a point x at which $\gamma(x) = \tau(x) \in \mathbf{I}^\circ$ and, consequently, $Q(x) > 0$ (cf. Lemma 11.2(a) and (11)). The equality $\gamma(x) = \tau(x)$ contradicts in turn the definition of γ, proving (i).

Next, if some critical manifold of τ (cf. Example 8.1) consisted of a single point, the Hopf-Rinow theorem and Lemma 9.1(b) would imply that γ is constant on M, thus making τ a special Kähler-Ricci potential (Lemma 9.1(c)). This implies (ii–a).

Finally, if $\gamma(y) = \tau_{\min}$ or $\gamma(y) = \tau_{\max}$ at some $y \in M$, we may assume that y is a critical point of τ and $\gamma(y) = \tau(y)$, which is achieved by choosing $\varGamma$ as above and replacing y with an endpoint of $\varGamma$. In view of (i), $\tau \neq \gamma$ everywhere in the open set $M' \subset M$ on which $d\tau \neq 0$. A fixed geodesic $t \mapsto x(t)$ having the properties listed in Remark 8.5, for our y, and the equality $2\phi = Q/(\tau - \gamma)$ on M' (immediate from the definition of γ in Lemma 9.1) now allow us to evaluate $2\phi(y)$ via l'Hospital's rule, with Q and $\tau - \gamma$ both vanishing at $y = x(0)$ due to (11). Consequently, $2\phi(y)$ is the limit, as $t \to 0$, of $\dot{Q}/(\dot{\tau} - \dot{\gamma}) = (d_v Q)/(d_v\tau - d_v\gamma)$, where we have used the 'dot' notation of Remark 8.5 and the fact that, since $v = \nabla\tau$ is tangent to the geodesic (Lemma 8.4(iv)) and nonzero at $x(t)$ for $t \neq 0$ close to 0 (Remark 8.3), d/dt equals a specific function of the variable $t \neq 0$ times d_v. From (12.b), (12.c) and Lemma 9.1(a) we now obtain $2\phi(y) = 2\psi(y)$. The two eigenvalues of the Hessian $\nabla d\tau$ at y thus coincide, and so, according to Lemma 8.4(i)–(ii), T_yM is the normal space at y of the critical manifold Σ of τ containing y. Hence $\Sigma = \{y\}$ and, by (a), τ is a special Kähler-Ricci potential, which yields (ii–b). □

For (M, g, τ) as in Theorem 5.3, we now define the data (7) by choosing: a and $\mathbf{I} \ni \tau \mapsto Q$, where $\mathbf{I} = [\tau_{\min}, \tau_{\max}]$, as in Lemma 11.2(c)–(d); Σ to be the minimum level of τ, with $\gamma : \Sigma \to \mathbb{R}\mathrm{P}^1$ obtained by restricting to Σ the mapping γ introduced in Lemma 9.1, and with the metric h on Σ given by

$$h = (\tau_{\min} - \gamma)^{-1}(\tau_* - \gamma)g, \tag{20}$$

$\tau_* \in \mathbf{I}$ being the midpoint; the normal bundle $\mathcal{L}$ of Σ with the Hermitian fibre metric $(\,,\,)$, the real part of which is g (that is, g restricted to $\mathcal{L}$); and, finally, the horizontal distribution $\mathcal{H}$ of the normal connection in $\mathcal{L}$. Lemmas 11.2(c)–(d) and 11.3(ii) state that these objects satisfy conditions (i) – (vii) in Section 5 except for the equality $\varOmega = -a(\tau_* - \gamma)^{-1}\omega^{(h)}$, which will be established in the next section.

12. PROOF OF THEOREM 5.3, SECOND PART

Using the data (7) just constructed for the given triple (M, g, τ), we also choose, as in Section 5, a C^∞ diffeomorphism $(\tau_{\min}, \tau_{\max}) \ni \tau \mapsto r \in (0, \infty)$ with $dr/d\tau = ar/Q$. Its inverse now gives rise to the composite $r \mapsto \tau \mapsto s$, for $\tau \mapsto s$ as in Theorem 10.2(v), allowing us to treat s as a function of r and write $s = \sigma(r)$, so that $r \mapsto \sigma(r)$ is a diffeomorphism $(0, \infty) \to (0, \lambda)$. This in turn leads to a fibre-preserving diffeomorphism $\theta : N\Sigma \smallsetminus \Sigma \to N^\lambda\Sigma \smallsetminus \Sigma$ of punctured-disk bundles, which sends a vector $w \neq 0$ normal to Σ at any point to $\sigma(r)w/r$, where $r = |w|$ is the g-norm of w. For later reference, note that, according to Theorem 10.2(v),

$$d[\sigma(r)]/dr = (ar)^{-1}Q^{1/2} \quad \text{and} \quad \sigma(0) = 0, \quad \text{while} \quad s = \sigma(r). \tag{21}$$

By Lemma 10.3, Example 8.1 and (19), $F = \mathrm{Exp}^\perp \circ \theta$ maps $N\Sigma \smallsetminus \Sigma$ diffeomorphically onto the open submanifold $M' \subset M$ on which $d\tau \neq 0$.

We now show that F is a biholomorphic isometry of $N\Sigma \smallsetminus \Sigma \subset N\Sigma = \mathcal{L}$, with the complex structure and metric obtained as in Section 5 from the data (7), onto our (M', g), and that it sends the Killing potential with a geodesic gradient, described in Section 5, onto our τ. The proof, split into three lemmas, closely follows the argument in [5, §§15–16].

To minimize confusion, the hatted symbols $\hat{M}, \hat{M}', \hat{\mathcal{V}}, \hat{\mathcal{H}}, \hat{g}, \hat{J}, \hat{v}, \hat{u}$ stand for for the objects constructed in Section 5 from our data (and from $\tau \mapsto r$ chosen above), which in Section 5 appeared as $M, M', \mathcal{V}, \mathcal{H}, g, J, v, u$. For $M, M', \mathcal{V}, \mathcal{H}, g, v, J, u$, the meaning is now the same as in Section 9: they are associated with (M, g) and the function $\tau : M \to \mathrm{I\!R}$. However, τ, r and s, in their original form, are used not only for the independent variables ranging over $\mathbf{I}^\circ, (0, \infty)$ and $(0, \lambda)$, but, along with Q and γ, also denote mappings defined on both manifolds M' and $\hat{M}'$. Similarly, Σ is treated as a submanifold both of M (the minimum level of τ) and of $\mathcal{L} = N\Sigma$ (the zero section). Again, $\pi : \mathcal{L} \to \Sigma$ is the bundle projection.

***Lemma* 12.1.** *The diffeomorphism $F : \hat{M}' \to M'$ sends the functions s, τ, Q and the mapping γ defined on $\hat{M}'$ to their analogs on M', and the vector field $\hat{v}$ to v.*

Proof. In the case of γ this is clear from Lemma 9.1(b), since F restricted to Σ is the identity mapping.

Because of how we defined $\hat{g}$ on $\hat{\mathcal{V}}$ in Section 5, given $y \in \Sigma$, (21) implies that a line segment of g_y-length r emanating from 0 in the normal space $N_y\Sigma$ has the $\hat{g}$-length $\sigma(r)$, which is at the same time the g_y-length of the segment's image under θ. That image is also a segment in $N_y\Sigma$ issuing from 0, and so $\mathrm{Exp}^\perp$ sends it to a geodesic segment of g-length $\sigma(r)$ in (M, g), normal to Σ at y. Since Theorem 10.2(vi) applies to both (M, g, τ) and $(\hat{M}, \hat{g}, \tau)$, our claim about s follows from the distance-minimizing clause of Remark 2.2.

As the homeomorphic correspondence $\mathbf{I} \to [0, \lambda]$ of Theorem 10.2(v) holds in both (M, g, τ) and $(\hat{M}, \hat{g}, \tau)$, the same now follows for τ and Q. Finally, we just saw that F sends line segments emanating from 0 in the normal spaces of Σ to normal g-geodesics issuing from Σ. Since $\hat{v}$ is tangent to the former (by definition), and $v = \nabla\tau$ to the latter (cf. Example 8.1 and Lemma 8.4(iv)), the F-image of $\hat{v}$ is the product of a function and v. That the function in question equals 1 is in turn obvious from the normalizing condition (12.b), valid in both (M, g, τ) and $(\hat{M}, \hat{g}, \tau)$, along with our assertion, already established for τ and Q. □

***Lemma* 12.2.** *The F-images of $\hat{u}$ and $\hat{\mathcal{V}}$ are, respectively, u and $\mathcal{V}$, while $\hat{g}$ and $\hat{J}$ restricted to $\hat{\mathcal{V}}$ correspond under F to g and J on $\mathcal{V}$.*

Proof. Obviously, θ preserves $\hat{u}$, that is, the θ-image of $\hat{u}$ is the restriction of $\hat{u}$ to $N^\lambda\Sigma \smallsetminus \Sigma$. As u is a Killing field, Remarks 2.1 and 9.3 combined with the definition of $\hat{u}$ (cf. Section 5) imply in turn that $\mathrm{Exp}^\perp$ sends $\hat{u}$ to u. Hence so does F.

The rest of our assertion is now obvious from Lemma 12.1, since in both (M, g, τ) and $(\hat{M}, \hat{g}, \tau)$ we have the relations (12.d) and $\mathcal{V} = \mathrm{Span}(v, u)$ or, respectively, their hatted versions. $\square$

***Lemma* 12.3.** *The assertion of Lemma* 12.2 *remains true also when* $\hat{\mathcal{V}}$ *and* $\mathcal{V}$ *are replaced by* $\hat{\mathcal{H}}$ *and* $\mathcal{H}$*, while the data* (7) *constructed in Section* 11 *satisfy condition* (vii) *of Section* 5*.*

Proof. Let us fix a g-unit vector field $t \mapsto w(t) \in N_{y(t)}\Sigma$, normal to Σ, defined along a curve $t \mapsto y(t) \in \Sigma$, and parallel relative to the normal connection in $\mathcal{L} = N\Sigma$. Since Σ is totally geodesic in (M, g) (see Example 8.1), the last condition reads $\nabla_{\dot{y}} w = 0$, where ∇ is the Levi-Civita connection of g. The variable t ranges over some given open interval (b, c). For any $t \in (b, c)$ and $s \in (0, \lambda)$, we define $x(t, s) \in M$ to be the F-image of $rw(t)$ treated as an element of $\hat{M}'$, for the unique $r \in (0, \infty)$ with $s = \sigma(r)$. Thus, by the definition of F, we obtain a mapping

$$(b, c) \times (0, \lambda) \ni (t, s) \;\mapsto\; x(t, s) \;=\; \exp_{y(t)} s w(t) \in M. \tag{22}$$

We will use subscripts for its partial derivatives x_t, x_s, and their partial covariant derivatives x_{ts}, x_{ss}, etc. All such derivatives are sections of the pullback of TM under the mapping (22). The subscript-style partial (or, partial covariant) derivatives also make sense for functions (or, respectively, vector fields) on M, which amounts to differentiating the latter objects along each of the curves given by (22) with fixed s or fixed t. More details can be found in [5, §14].

Writing $\langle\,,\rangle$ instead of $g(\;,\;)$, and denoting by $|\;|$ the g-norm, we now have

(a) $x_s = Q^{-1/2} v, \quad |v| = |u| = Q^{1/2}$,
(b) $\langle u, x_t \rangle_s = 2\langle u, x_{st} \rangle$,
(c) $\langle u, x_t \rangle_s = 2\langle u, x_t \rangle \psi Q^{-1/2}$,
(d) $Q_s = 2\psi Q^{1/2}$.

Although equalities (a) – (d) all appear in [5, p. 101], they have to be established here independently, as [5] makes a stronger assumption about τ. However, the argument is the same as in [5].

First, (12.d) implies the second part of (a), and the first part then follows: by (22) and Lemma 12.1, v equals a positive function times x_s, and $|x_s| = 1$. Furthermore, u is a Killing field, so that $\langle u_t, x_s \rangle = \langle [\nabla u] x_t, x_s \rangle = -\langle u_s, x_t \rangle$, while $\langle u, x_{st} \rangle = -\langle u_t, x_s \rangle$, as (a) and (12.d) give $\langle u, x_s \rangle = 0$. Consequently, $\langle u, x_t \rangle_s = \langle u, x_{st} \rangle + \langle u, x_{ts} \rangle$, which yields (b), since ∇ is torsion-free, and so $x_{ts} = x_{st}$. The relations just established and (a) also show that $\langle u, x_t \rangle_s / 2 = \langle u, x_{st} \rangle = -\langle u_t, x_s \rangle = \langle u_s, x_t \rangle = \langle [\nabla u] x_s, x_t \rangle = Q^{-1/2} \langle \nabla_v u, x_t \rangle$, which proves (c), as $\nabla_v u = \nabla_v (Jv) = J\nabla_v v = \psi J v = \psi u$ by (11) and (5). Finally, (d) is obvious from (12.c) and (a).

By (c) and (d), $[\langle u, x_t \rangle / Q]_s = 0$. Hence $\langle u, x_t \rangle / Q$ is constant as a function of s. To show that $\langle u, x_t \rangle / Q = 0$, we take the limit of $\langle u, x_t \rangle / Q$ as $s \to 0$, that is (cf. Theorem 10.2(v)), as $\tau \to \tau_{\mathrm{min}}$. We may use l'Hospital's rule, since the numerator and denominator both vanish at $\tau = \tau_{\mathrm{min}}$ due to (11) and the fact that (22) has an obvious C^∞ extension to $(b, c) \times [0, \lambda)$. Now (b), (d) and (a) give $\langle u, x_t \rangle_s / Q_s = \langle u, x_{st} \rangle Q^{-1/2} / \psi = \langle u / |u|, x_{st} \rangle / \psi$. The last expression tends to 0 as $s \to 0$ since

$\psi = a \neq 0$ at $\tau = \tau_{\min}$ due to Lemma 11.2(d) and (12.a), while x_{st} at $s = 0$ equals $\nabla_{\dot{y}} w$, and so $x_{st} \to 0$ as $s \to 0$.

Consequently, $\langle u, x_t \rangle = 0$, while $\langle v, x_t \rangle = 0$ in view of (a) and the generalized Gauss lemma [6, p. 26]. Therefore, $\mathcal{H}$ is the F-image of $\hat{\mathcal{H}}$.

Combined with the assertion about u in Lemma 12.2 and (15), this yields the formula for Ω required by condition (vii) of Section 5, since, given sections $\hat{w}, \hat{w}'$ of $\hat{\mathcal{H}}$, the $\hat{\mathcal{V}}$ component of $[\hat{w}, \hat{w}']$ is $a^{-1}\Omega(\hat{w}, \hat{w}')\hat{u}$, cf. [4, formula (3.6)].

For fixed $t \in (b, c)$, let $\hat{w}$ be the $\hat{\mathcal{H}}$-horizontal lift to $\pi^{-1}(\Sigma') \smallsetminus \Sigma'$ of a vector field on a neighborhood Σ' of $y(t)$ in Σ, having the value $\dot{y}(t)$ at $y(t)$. As we just showed, the F-image of $\hat{w}$ is a section w of $\mathcal{H}$, defined on $F(\pi^{-1}(\Sigma') \smallsetminus \Sigma')$. Since $\hat{w}$ obviously commutes with $\hat{v}$ and $\hat{u}$, Lemmas 12.1 and 12.2 imply that w commutes with v and u, while, by (22), $w_{x(t,s)} = x_t(t,s)$ for all $s \in (0, \lambda)$ and our fixed t. Therefore (a) and (16) give $[\phi g(x_t, x_t)/Q]_s = 0$. Thus, since $Q/(2\phi) = \tau - \gamma$ (see Lemma 9.1), $\langle x_t, x_t \rangle / (\tau - \gamma)$ is constant as a function of s, that is, equal to its value at $s = 0$. In other words, writing $y, \dot{y}, \gamma, \tau$ instead of $y(t), \dot{y}(t), \gamma(y(t))$ and $\tau(x(t,s))$, we have $\langle x_t, x_t \rangle = (\tau_{\min} - \gamma)^{-1}(\tau - \gamma)\langle \dot{y}, \dot{y} \rangle$, both if $\gamma(y(t)) \neq \infty$, and when $\gamma(y(t)) = \infty$ (provided that, in the latter case, one lets $(\tau_{\min} - \gamma)^{-1}(\tau - \gamma)$ stand for 1). In view of (20), with g now denoted by $\langle\, , \rangle$, the definition of $\hat{g}$ in Section 5 thus shows that $\langle x_t, x_t \rangle$ at (t, s) equals $\hat{g}(\hat{w}, \hat{w})$ at $F^{-1}(x(t,s))$, proving our claim about $\hat{g}$ and g.

Finally, since $\dim_{\mathbb{R}} \Sigma = 2$, both $\hat{g}$ and g, restricted to $\hat{\mathcal{H}}$ and $\mathcal{H}$, determine $\hat{J}$ on $\hat{\mathcal{H}}$ and J on $\mathcal{H}$ uniquely up to a sign. Hence F sends $\hat{J}$ on $\hat{\mathcal{H}}$ to J on $\mathcal{H}$, with the plus sign due to the fact that $F = \mathrm{Id}$ on Σ (which is tangent to both $\hat{\mathcal{H}}$ and $\mathcal{H}$). $\square$

According to Lemmas 12.1 – 12.3, F is a biholomorphic isometry of $(\hat{M}', \hat{g})$ onto (M', g), sending the Killing potential τ on $(\hat{M}', \hat{g})$ to τ on (M', g). Lemma 2.3 now implies that F has an extension $M' \to M$, which proves Theorem 5.3.

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APPENDIX. KILLING FIELDS WITH ZEROS

The following fact, although not needed for our argument, helps explain how Killing potentials fit into the general landscape of Kähler geometry.

Kobayashi [8, p. 95, Corollary 4.5] observed that, due to a result of Lichnerowicz [9], gradients of Killing potentials on a compact Kähler manifold are in a one-to-one correspondence, provided by J, with Killing fields which have zeros:

***Theorem* A.1.** *A vector field v on a compact Kähler manifold (M, g) is the gradient of a Killing potential if and only if Jv is a Killing field with zeros.*

Consequently, for any Killing field u on a compact Kähler manifold with non-zero Euler characteristic, Ju is the gradient of a Killing potential. On the other hand, nontrivial parallel vector fields on flat complex tori are examples of Killing fields u such that Ju is a geodesic vector field, but not a gradient.

For the reader's convenience, a proof of Theorem A.1 is given below.

First, for any vector fields u and w on a Kähler manifold (M,g),

$$\text{(23)}\qquad \begin{aligned} &\text{i)}\quad \operatorname{tr} J[R(u,w)] \,=\, 2\operatorname{Ric}(u,Jw),\\ &\text{ii)}\quad |\pounds_u J|^2 \,=\, \operatorname{tr}(A+A^*)A \,+\, d_u \operatorname{tr} A \,+\, \operatorname{div}[(JA^*J-A)u], \end{aligned}$$

where A^* is the (pointwise) adjoint of $A = \nabla u : TM \to TM$, cf. the lines following (1), $|B|^2 = \operatorname{tr} BB^*/2$ for endomorphisms B of TM, and the left-hand side of (23.i) denotes the (real) trace of the composition of $J : TM \to TM$ with the curvature operator $R(u,w) : TM \to TM$, sending any vector field v to $R(u,w)v$.

In fact, as $\nabla J = 0$, the Levi-Civita connection ∇ is a connection in the *complex* vector bundle TM. For any vector fields u,w on M, the vector-bundle morphism $R(u,w) : TM \to TM$ is thus complex-linear (commutes with J). At every point, the commuting morphisms $R(u,w)$ and J are skew-adjoint, and so their composite is self-adjoint. Hence $R_{klsp}J_q^p = R_{klqp}J_s^p$, which, contracted against g^{ks}, gives $R_{lk}J_q^k = R_{kql}{}^sJ_s^k$. However, due to skew-adjointness of J and the first Bianchi identity, $2R_{kql}{}^sJ_s^k - R_{lqk}{}^sJ_s^k = (R_{kql}{}^s - R^s{}_{qlk} - R_{lqk}{}^s)J_s^k = (R_{kql}{}^s + R_{lkq}{}^s + R_{qlk}{}^s)J_s^k = 0$. Now (23.i) follows if one transvects the last two equalities with w^qu^l.

Next, $\pounds_u J = [J,A]$ for $A = \nabla u$ (see Section 3), and so $|\pounds_u J|^2 = \operatorname{tr}[J,A][J,A]^*/2 = \operatorname{tr} JAJA^* + \operatorname{tr} AA^*$. Subtracting from this the right-hand side in (23.ii), we obtain the sum of two expressions: $\operatorname{tr} JAJA^* - \operatorname{div}(JA^*Ju)$ and $\operatorname{div} Au - \operatorname{tr} A^2 - d_u \operatorname{tr} A$. The first expression, $J_q^pu^q{}_{,k}J_s^ku_{p,}{}^s - (J_q^pu^qJ_s^ku_{p,}{}^s)_{,k}$, clearly equals $-J_q^pu^qJ_s^ku_{p,}{}^s{}_k$ which, due to skew-adjointness of J and the Ricci identity, is the same as one-half of $J_q^pu^q$ times $J_s^k(u_{p,k}{}^s - u_{p,}{}^s{}_k) = J_s^kR^s{}_{kpl}u^l$. The second expression is $(u^ju^k{}_{,j})_{,k} - u^j{}_{,k}u^k{}_{,j} - u^ju^k{}_{,kj} = u^j(u^k{}_{,jk} - u^k{}_{,kj}) = \operatorname{Ric}(u,u)$, since $u^k{}_{,jk} - u^k{}_{,kj} = R_{jk}u^k$ by (1.b). Now (23.i) with $w = -Ju$ yields (23.ii).

Secondly, given a Killing field u on a compact Riemannian manifold (M,g) and a vector field w on M, suppose that the 1-form $\alpha = \langle w,\,\cdot\,\rangle$ is harmonic. Then

$$\text{(24)}\qquad \begin{aligned} &\text{a)}\quad \text{the function } \langle u,w\rangle \text{ is constant,}\qquad \text{b)}\quad \text{the 1-form } \langle Jw,\,\cdot\,\rangle \text{ is harmonic,}\\ &\text{c)}\quad \text{the vector field } u \text{ is real-holomorphic,} \end{aligned}$$

where $\langle\,,\rangle$ stands for $g(\;,\;)$ and, in (24.b-c), (M,g) is a (compact) Kähler manifold.

In fact, the flow of u acts trivially in cohomology and preserves harmonicity of 1-forms, so that $\pounds_u\alpha = 0$ or, equivalently, $[u,w] = 0$. For any vector field v, the Leibniz rule, skew-adjointness of ∇u, and self-adjointness of ∇w give $d_v\langle u,w\rangle = \langle\nabla_v u,w\rangle + \langle u,\nabla_v w\rangle = -\langle\nabla_w u,v\rangle + \langle v,\nabla_u w\rangle = \langle v,[u,w]\rangle = 0$, proving (24.a).

As for (24.b), it follows since the Hodge Laplacian $\operatorname{div}\circ d + d\circ\operatorname{div}$ acting on 1-forms equals $\Delta - \operatorname{Ric}$ (and so it commutes with J, as both Δ and Ric do). Here Δ is the rough Laplacian, and the coordinate form of this equality, $(\beta_{j,k} - \beta_{k,j})^{,k} + \beta_{k,}{}^k{}_j = \beta_{j,k}{}^k - R_j^k\beta_k$, with β denoting any 1-form, amounts to $R_j^k\beta_k = \beta_{k,j}{}^k - \beta_{k,}{}^k{}_j$, that is, the Bochner identity (1.b) for $\beta = \langle v,\,\cdot\,\rangle$. Finally, integration over M, applied to (23.ii) with $A^* = -A$ and $\operatorname{tr} A = 0$, yields (24.c).

Proof of Theorem A.1. The 'only if' part is provided by Remark 3.1(b). Conversely, suppose that $u = Jv$ is a Killing field with zeros. By (24.c), J and $\nabla u = J\nabla v$ are commuting skew-adjoint morphisms $TM \to TM$, so that their composite $J\nabla u = -\nabla v$ is self-adjoint. In other words, the 1-form $\beta = \langle v, \cdot\, \rangle$ is closed. The harmonic-exact Hodge decomposition of β amounts to an L^2-orthogonal decomposition $v = w + \nabla\tau$ with some function τ and some vector field w such that the 1-form $\alpha = \langle w, \cdot\, \rangle$ is harmonic. As u has zeros, (24.a-b) give $\langle v, w\rangle = -\langle Ju, w\rangle = \langle u, Jw\rangle = 0$. Thus, w is L^2-orthogonal to $\nabla\tau$, to $v = w + \nabla\tau$, and hence also to itself, so that $w = 0$ and $v = \nabla\tau$, as required. ☐

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