

Two-jets of conformal fields along their zero sets

Research Article

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Abstract: The connected components of the zero set of any conformal vector field v , in a pseudo-Riemannian manifold (M, g) of arbitrary signature, are of two types, which may be called 'essential' and 'nonessential'. The former consist of points at which v is essential, that is, cannot be turned into a Killing field by a local conformal change of the metric. In a component of the latter type, points at which v is nonessential form a relatively-open dense subset that is at the same time a totally umbilical submanifold of (M, g) . An essential component is always a null totally geodesic submanifold of (M, g) , and so is the set of those points in a nonessential component at which v is essential (unless this set, consisting precisely of all the singular points of the component, is empty). Both kinds of null totally geodesic submanifolds arising here carry a 1-form, defined up to multiplications by functions without zeros, which satisfies a projective version of the Killing equation. The conformal-equivalence type of the 2-jet of v is locally constant along the nonessential submanifold of a nonessential component, and along an essential component on which the distinguished 1-form is nonzero. The characteristic polynomial of the 1-jet of v is always locally constant along the zero set.

MSC: 53B30**Keywords:** Conformal vector field • Fixed-point set • Two-jet

1. Introduction

A vector field v on a pseudo-Riemannian manifold (M, g) of dimension $n \geq 2$ is called *conformal* if $\mathcal{L}_v g$ equals a function times g , that is, if for some section A of $\mathfrak{so}(TM)$ and some function $\phi : M \rightarrow \mathbb{R}$,

$$2\nabla v = A + \phi \text{Id}, \quad \text{or, in coordinates, } v_{j,k} + v_{k,j} = \phi g_{jk}. \quad (1)$$

The covariant derivative ∇v is treated here as the bundle morphism $TM \rightarrow TM$ sending any vector field w to $\nabla_w v$, and sections of $\mathfrak{so}(TM)$ are endomorphisms of TM , skew-adjoint at every point; clearly, $\phi = (2/n) \text{div } v$.

If $n \geq 3$, such v is known to be uniquely determined by its 2-jet at any given point. Determining how the 2-jet of v may vary along the zero set Z of v is thus an obvious initial step towards understanding the dynamics of v near Z .

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Theorem 4.1 of this paper, which is an easy consequence of some facts proved in [4], deals with the 1-jet of v , establishing a restriction on its variability: the characteristic polynomial of ∇v must be locally constant on Z .

A point $x \in Z$ is called *nonessential* if some local conformal change of the metric at x turns v into a Killing field, and *essential* otherwise. A connected components of Z is either *essential* (if it consists of essential points only) or *nonessential* (when it contains some nonessential points, possibly along with essential ones).

The next main result, Theorem 5.1, explores structural properties of components of Z . Every essential component turns out to be a null totally geodesic submanifold, and so is, when nonempty, the possibly-disconnected set Σ of essential points in any given nonessential component N . At the same time, Σ coincides with the set of singular points of N , while $N \setminus \Sigma$ is a totally umbilical submanifold. The tangent spaces of these submanifolds at all points x are explicitly described in terms of ∇v_x and $d\phi_x$.

For N and Σ as above, let the same symbol Σ also stand for an essential component of Z . Section 6 discusses geometric structures on $N \setminus \Sigma$ and both types of Σ , naturally induced by the underlying conformal structure of (M, g) . They consist of a constant-rank, possibly-degenerate conformal structure on $N \setminus \Sigma$ along with its nullspace distribution, a projective structure on Σ , and a 1-form ξ on Σ defined only up to multiplications by functions without zeros. Their basic properties are listed in Proposition 6.1.

Finally, Section 10 addresses the question, mentioned above, of variability of the 2-jet of v along Z . The conformal-equivalence type of the 2-jet is proved to be locally constant in $N \setminus \Sigma$ and, generically, in Σ . The word 'generically' means here *in any component of Σ on which ξ is not identically zero*. Examples show that, in the case of Σ , some form of the 'generic' assumption is necessary. On the other hand, if $\Sigma \subset N$ is nonempty, the equivalence types at points of Σ are always different from those realized in $N \setminus \Sigma$.

2. Preliminaries

Manifolds *need not be connected*. However, their connected components must all have the same dimension. Submanifold are always endowed with the subset topology. All manifolds, mappings, bundles and their sections, including tensor fields and functions, are of class C^∞ . The symbol ∇ denotes both the Levi-Civita connection of a given pseudo-Riemannian metric g on a manifold M , and the g -gradient. Thus, for a vector field u and a function τ on M , we have $d_u\tau = g(u, \nabla\tau)$.

Given a submanifold K of a manifold M , we denote by $T_K M$ the restriction of TM to K . The *normal bundle* of

K is defined, as usual, to be the quotient vector bundle $T_K M / TK$. Any fixed torsion-free connection ∇ on M gives rise to the *second fundamental form* of K , which is a section b of $[T^*M]^{\odot 2} \otimes T_K M / TK$ (so that, at every $x \in K$, the mapping $b_x : T_x K \times T_x K \rightarrow T_x M / T_x K$ is bilinear and symmetric). We have

$$b(\dot{x}, w) = \pi \nabla_{\dot{x}} w \quad (2)$$

whenever $t \mapsto w(t)$ is a vector field tangent to K along a curve $t \mapsto x(t)$ in K , with $\pi : TM \rightarrow T_K M / TK$ denoting the quotient projection. When $b = 0$ identically, K is said to be *totally geodesic* relative to ∇ . If ∇ is the Levi-Civita connection of a pseudo-Riemannian metric g and $b = g_K \otimes u$ for some section u of $T_K M / TK$, where g_K is the restriction of g to K , one calls K *totally umbilical* in (M, g) . This last property is conformally invariant: changing g to $e^\tau g$ causes b to be replaced by $b - g_K \otimes \pi \nabla \tau / 2$.

As shown by Weyl [8, p. 100], two torsionfree connections on a manifold M are *projectively equivalent*, in the sense of having the same re-parametrized geodesics, if and only if their difference E can be written as $E = \theta \odot \text{Id}$ for some 1-form θ on M (in coordinates: $2E_{jk}^l = \theta_j \delta_k^l + \theta_k \delta_j^l$). On the other hand, given a pseudo-Riemannian metric g on M , with the Levi-Civita connection ∇ , and a function $\tau : M \rightarrow \mathbb{R}$, the conformally related metric $e^\tau g$ has the Levi-Civita connection $\nabla + E$, where $E = d\tau \odot \text{Id} - g \otimes \nabla \tau / 2$. Thus, for any null totally geodesic submanifold Σ of M , the connections on Σ induced by the Levi-Civita connections of g and $e^\tau g$ are projectively equivalent.

For every conformal vector field v on a pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$ and any vector fields u, w on M one has the well-known equalities of bundle morphisms $TM \rightarrow TM$ and functions $M \rightarrow \mathbb{R}$:

$$\begin{aligned} 2\nabla_u \nabla v &= 2R(v \wedge u) + d\phi \otimes u - g(u, \cdot) \otimes \nabla \phi + g(u, \nabla \phi) \text{Id}, \\ (1 - n/2)[\nabla d\phi](u, w) &= S(u, \nabla_w v) + S(w, \nabla_u v) + [\nabla_v S](u, w), \end{aligned} \quad (3)$$

cf. [4, formula (22)], where R and S are the curvature and Schouten tensors. In coordinates, (3) reads $2v^l{}_{,kj} = 2R_{pjk}{}^l v^p + \phi_{,k} \delta_j^l - \phi_{,j} g_{jk} + \phi_{,j} \delta_k^l$ and $(1 - n/2)\phi_{,jk} = S_{jp} v^p{}_{,k} + S_{kp} v^p{}_{,j} + S_{jk,p} v^p$.

Remark 2.1.

For a conformal vector field v on a pseudo-Riemannian manifold (M, g) and any function $\tau : M \rightarrow \mathbb{R}$, the conformally equivalent metric $e^\tau g$ satisfies, along with v , the analog of (1) in which the role of ϕ is played by $\phi + d_v \tau$. In fact, (1) is equivalent to $\mathcal{L}_v g = \phi g$, while $\mathcal{L}_v(e^\tau g) = e^\tau \mathcal{L}_v g + (d_v \tau) e^\tau g$. At a point x such that $v_x = 0$, switching from g to $e^\tau g$ thus results in replacing $d\phi_x$ by $d\phi_x + (d\tau_x) \nabla v_x$.

Remark 2.2.

A Killing field v and any vector field u on a pseudo-Riemannian manifold satisfy (3) with $\phi = 0$. Thus, ∇v is parallel along any curve to which v is tangent, such as an integral curve of v or a curve of zeros of v .

3. The zero set Z of a conformal field v

In addition to the function $\phi = (2/n) \operatorname{div} v : M \rightarrow \mathbb{R}$ appearing in (1), let us also consider

the zero set Z of a conformal vector field v on a pseudo-Riemannian manifold (M, g) , $\dim M = n \geq 3$. (4)

If $x \in Z$, the *simultaneous kernel* at x of the differential $d\phi$ and the bundle morphism $\nabla v : TM \rightarrow TM$ is the space

$$\mathcal{H}_x = \operatorname{Ker} \nabla v_x \cap \operatorname{Ker} d\phi_x. \quad (5)$$

When x is fixed, the symbol H may be used instead of \mathcal{H}_x .

As in [2], we call $x \in Z$ a *nonessential* zero of v if v restricted to a suitable neighborhood of x is a Killing field for some metric conformal to g . When no such neighborhood and metric exist, the zero of v at x is said to be *essential*.

By a *nonsingular point* of Z we mean any $x \in Z$ such that, for some neighborhood U of x in M , the intersection $Z \cap U$ is a submanifold of M . Points of Z not having a neighborhood with this property will be called *singular*.

For $(M, g), v, Z$ as above, a point $x \in Z$, and the exponential mapping \exp_x of g at x , we will repeatedly consider

any sufficiently small neighborhoods U of 0 in $T_x M$ and U' of x in M such that (6)
 U is a union of line segments emanating from 0 and \exp_x is a diffeomorphism $U \rightarrow U'$.

Theorem 3.1 (Kobayashi [6]).

For $(M, g), v, Z, U, U'$ and $H = \mathcal{H}_x$ as in (4) – (6), let v also be a Killing field. Then

$$Z \cap U' = \exp_x[H \cap U], \quad \text{with } H = \operatorname{Ker} \nabla v_x \text{ since } \phi = 0 \text{ in (1).}$$

Thus, the connected components of Z are totally geodesic submanifolds of even codimensions in (M, g) .

Theorem 3.2 (Beig [1, 3]).

Let Z be the zero set of a conformal vector field v on a pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$. A point $x \in Z$ is *nonessential* if and only if

$$\phi(x) = 0 \quad \text{and} \quad \nabla \phi_x \in \nabla v_x(T_x M). \quad (7)$$

for the function $\phi = (2/n) \operatorname{div} v : M \rightarrow \mathbb{R}$ appearing in (1). In other words, $x \in Z$ is *essential* if and only if

$$\text{either } \phi(x) \neq 0, \quad \text{or } \phi(x) = 0 \quad \text{and} \quad \nabla \phi_x \notin \nabla v_x(T_x M). \quad (8)$$

Theorem 3.3 (Derdzinski [4]).

Suppose that v is a conformal vector field on a pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$. If Z is the zero set of v , while $x \in Z$ satisfies (8), and $C = \{w \in T_x M : g_x(w, w) = 0\}$ stands for the null cone, then, with U, U' as in (6) and $H = \text{Ker } \nabla v_x \cap \text{Ker } d\phi_x$,

$$Z \cap U' = \exp_x[C \cap H \cap U]. \tag{9}$$

In addition, $\phi = (2/n) \text{div } v$ is constant along each connected component of Z .

Given a conformal vector field v on a pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$, and a parallel vector field $t \mapsto w(t) \in T_{y(t)}M$ along a geodesic $t \mapsto y(t)$ contained in the zero set Z of v , such that $g(\dot{y}, w) = 0$, replacing u in (3) with \dot{y} we obtain

$$(a) \quad 2\nabla_{\dot{y}}\nabla_w v = g(w, \nabla\phi)\dot{y}, \quad (b) \quad (1 - n/2)[g(w, \nabla\phi)]' = S(\dot{y}, \nabla_w v). \tag{10}$$

(The other terms vanish since $v = \nabla_{\dot{y}}v = 0$ at $y(t)$, while $g(\dot{y}, \nabla\phi) = 0$ due to the final clause of Theorem 3.3.)

Remark 3.1.

In view of Theorems 3.1 – 3.3, Z in (4) is always locally pathwise connected. Thus, the connected components of Z are pathwise connected, closed subsets of M .

Remark 3.2.

Away from singularities, the connected components of Z are totally umbilical submanifolds of (M, g) , and their codimensions are even unless the component is a null totally geodesic submanifold.

In fact, for the connected components of the set of nonsingular zeros of v , this easily follows from Theorems 3.1 – 3.3; see also [4, Theorem 1.1]. That the connected components of Z , with the singularities removed, are submanifolds as well (in other words, their own connected components all have the same dimension) is also immediate from Theorems 3.1 – 3.3: by (9), the set of singular points in $Z \cap U'$ coincides with $\exp_x[H \cap H^\perp \cap U]$ when (8) holds and the metric g_x restricted to H is not semidefinite, and is empty otherwise, while, in the former case, all components of $(C \setminus H^\perp) \cap H$ are clearly of dimension $\dim H - 1$.

4. The characteristic polynomial of ∇v

Given a torsion-free connection ∇ on an n -dimensional manifold M , and a vector field v on M , we denote by \mathcal{P}_n the space of real all polynomials in one variable of degrees not exceeding n , and by $\chi(\nabla v)$ the function $M \rightarrow \mathcal{P}_n$ assigning to each $x \in M$ the characteristic polynomial of the endomorphism $\nabla v_x : T_x M \rightarrow T_x M$.

Lemma 4.1 (Derdzinski [4], Lemma 12.2(b)–(iii)).

If a conformal vector field v on a pseudo-Riemannian manifold (M, g) is tangent to a null geodesic segment Γ , and ϕ appearing in (1) is constant along Γ , then $\chi(\nabla v)$ is constant along Γ as well.

Theorem 4.1.

Let Z be the zero set of a conformal vector field v on a pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$. Then $\chi(\nabla v) : M \rightarrow \mathcal{P}_n$ is constant on every connected component of Z and, consequently, so is $\phi = (2/n) \text{div } v : M \rightarrow \mathbb{R}$.

Proof. We fix $x \in Z$ and show that $\chi(\nabla v)$, at zeros of v near x , is the same as at x , cf. Remark 3.1.

First, if x is a nonessential zero of v , changing the metric conformally near x , we may assume that v is a Killing field. By Theorem 3.1, the nearby zeros of v then form a submanifold K of M , while, according to Remark 2.2, ∇v is parallel along K . This proves our assertion for nonessential zeros x .

Now let the zero of v at x be essential. Theorem 3.2 then gives (8). In view of Theorem 3.3, every nearby point of Z is joined to x by a null geodesic segment Γ contained in Z . Our claim about ϕ follows in turn from the final clause of Theorem 3.3. Constancy of $\chi(\nabla v)$ along Γ is therefore immediate from Lemma 4.1. \square

5. Essential and nonessential components of Z

By the *components* of the set Z appearing in (4) we mean its (pathwise) connected components, cf. Remark 3.1.

A component of Z will be called *essential* if all of its points are essential zeros of v , as defined in Section 3. Otherwise, the component is said to be *nonessential*.

This definition allows a nonessential component N to contain some essential zeros of v . However, as shown in Theorem 5.1(v) below, the set of nonessential zeros in N is relatively open and dense.

Unlike the components, submanifolds—such as Σ and $N \setminus \Sigma$ in case (b) of Theorem 5.1—need not be connected, although all of their components are required to have the same dimension.

Theorem 5.1.

Given a conformal vector field v on a pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$, suppose that

- (a) Σ is an essential component of Z , or
- (b) Σ is the set of essential points in a nonessential component N of Z ,

where Z is the zero set of v . Then, with $\mathcal{H}_x = \text{Ker } \nabla v_x \cap \text{Ker } d\phi_x$,

- (i) Σ , if nonempty, is a null totally geodesic submanifold of (M, g) , closed as a subset of M ,
- (ii) $T_x \Sigma = \mathcal{H}_x \cap \mathcal{H}_x^\perp$ at every point x of Σ ,
- (iii) for any $x \in \Sigma$ the metric g_x restricted to \mathcal{H}_x is semidefinite in case (a), non-semidefinite in case (b).

Finally, in case (b), with singular points and C, H defined as in Section 3 and Theorem 3.3,

- (iv) Σ consists of singular, $N \setminus \Sigma$ of nonsingular zeros of v in N ,
- (v) $N \setminus \Sigma$ is a totally umbilical submanifold of M , while the sign pattern of g restricted to $N \setminus \Sigma$, including its rank r , is the same at all points, and $\dim(N \setminus \Sigma) - \dim \Sigma = r + 1$,
- (vi) whenever $y \in N \setminus \Sigma$ one has $T_y(N \setminus \Sigma) = \text{Ker } \nabla v_y$, and $\text{rank } \nabla v_y = 2 + \text{rank } \nabla v_x$ if $x \in \Sigma$,
- (vii) for $x \in \Sigma$ and sufficiently small U, U' in (6), $\Sigma \cap U' = \exp_x[H \cap H^\perp \cap U]$ and $N \cap U' = \exp_x[C \cap H \cap U]$.

Proof. As a consequence of Theorem 3.2, for $x \in Z$ and $H = \mathcal{H}_x$ there are three possibilities:

(α) x is a nonessential zero of v , that is, (7) holds,

(β) x is essential and the metric g_x is semidefinite on H ,

(γ) x is essential, g_x restricted to H is not semidefinite, $\phi(x) = 0$ and $\nabla\phi_x \notin \nabla v_x(T_x M)$.

The assertions about $\phi = (2/n) \operatorname{div} v$ in (γ) follow from (8); note that, if $\phi(x)$ were nonzero, $\operatorname{Ker} \nabla v_x$ would be a null subspace of $T_x M$ (as an eigenspace of the skew-adjoint endomorphism A_x in (1) for a nonzero eigenvalue), and so g_x would be semidefinite on $H \subset \operatorname{Ker} \nabla v_x$. In view of Theorem 3.1 and [4, second paragraph on p. 22],

$$x \text{ is nonsingular in cases } (\alpha) \text{ and } (\beta), \text{ but singular in case } (\gamma). \quad (11)$$

If $Z \cap U'$ satisfy (γ) then, for U, U' as in (6), $\Sigma' = \exp_x[H \cap H^\perp \cap U]$ and $N' = \exp_x[C \cap H \cap U] \setminus \Sigma'$ are submanifolds of M such that

$$T_y N' = \operatorname{Ker} \nabla v_y \text{ and } \operatorname{rank} \nabla v_y = 2 + \operatorname{rank} \nabla v_x \text{ for every } y \in N'. \quad (12)$$

In fact, $H \cap H^\perp$ clearly is the set of singular points in $C \cap H$. (For more details, see [4, Remark 6.2(a)].) To verify (12) for sufficiently small U, U' , note that $T_y N' \subset \operatorname{Ker} \nabla v_y$ as $N' \subset Z$, while $\dim N' = \dim H - 1 = \dim \operatorname{Ker} \nabla v_x - 2$ due to the definition of $H = \mathcal{H}_x$ and the last relation in (γ). This yields $\dim \operatorname{Ker} \nabla v_x - 2 \leq \dim \operatorname{Ker} \nabla v_y$ or, equivalently, $\operatorname{rank} \nabla v_x \leq \operatorname{rank} \nabla v_y \leq 2 + \operatorname{rank} \nabla v_x$ (where, for y near x , we have also used semicontinuity of the rank). The two inequalities cannot be both strict, as both ranks are even: ∇v_x and ∇v_y are skew-adjoint by (1), with

$$\phi = 0 \text{ on } N' \cup \Sigma' \quad (13)$$

in view the final clause of Theorem 3.3 and (γ). If we now did not have $\operatorname{rank} \nabla v_y = 2 + \operatorname{rank} \nabla v_x$ for all $y \in N'$ close to x , there would be a sequence of points $y \in N'$ such that $\operatorname{rank} \nabla v_y = \operatorname{rank} \nabla v_x$, converging to x and, by continuity, (γ) with x replaced by y would hold for all but finitely many of its terms y . (They would be essential in view of (8), with $\phi(y) = 0$ by (13).) The result would be a contradiction, as y would then be singular by (11), yet at the same time nonsingular since, in view of Theorem 3.3 the submanifold N' of M , containing y , is a relatively open subset of Z .

Furthermore, for Σ', N' chosen as above, with sufficiently small U, U' ,

$$\text{points of } \Sigma' \text{ have property } (\gamma), \text{ while points of } N' \text{ satisfy } (\alpha). \quad (14)$$

The first claim is obvious from (11), since Σ' consists of singular points of Z , cf. [4, Remark 6.2(a)]. As for the second one, its failure would—again by (11)—amount to (β) for some points $y \in N'$, arbitrarily close to x . Combined with Theorem 3.3, this would imply that $T_y N' = \mathcal{H}_y \cap \mathcal{H}_y^\perp$. Since $T_y N' = \text{Ker } \nabla v_y$ by (12), both inclusions

$$\mathcal{H}_y \cap \mathcal{H}_y^\perp \subset \mathcal{H}_y = \text{Ker } \nabla v_y \cap \text{Ker } d\phi_y \quad \text{and} \quad \mathcal{H}_y = \text{Ker } \nabla v_y \cap \text{Ker } d\phi_y \subset \text{Ker } \nabla v_y$$

would be equalities. As $\phi(y) = 0$ by (13), the second inclusion-turned-equality would give $\nabla \phi_y \in \nabla v_y(T_y M)$, and so Theorem 3.2 would yield case (α) for y rather than (β) . The ensuing contradiction proves the second part of (14).

Let Π_α (or Π_β , or Π_γ) denote the subset of a given component N of Z formed by all points $x \in N$ with (α) (or (β) or, respectively, (γ)). According to [4, Remark 17.1], Π_α and Π_β are relatively open in N . So is, consequently, the set $N' \cup \Sigma' = \exp_x[C \cap H \cap U]$ appearing in (13) (by Theorem 3.3), as well as the union $\Pi_\alpha \cup \Pi_\gamma$ (in view of (14)). Thus, due to connectedness of N , either $N = \Pi_\beta$ (in which case N is essential, and we denote it by the symbol Σ), or $N = \Pi_\alpha \cup \Pi_\gamma$ is a nonessential component (and we let Σ stand for the set of its essential points, so that $\Sigma = \Pi_\gamma$). In other words, since Π_α , Π_β and Π_γ are pairwise disjoint, we have

$$(*) \quad \Sigma = \Pi_\beta \quad \text{in case (a),} \quad (**) \quad \Sigma = \Pi_\gamma \quad \text{and} \quad N \setminus \Sigma = \Pi_\alpha \quad \text{in case (b).} \quad (15)$$

Assertions (i) – (iii), in both cases (a) and (b), are now immediate (with one exception): for any $x \in \Sigma$, Theorem 3.3 and (14) imply that $\Sigma \cap U' = \Sigma'$, with Σ' as above and sufficiently small U, U' . The exception is the possibility, still to be excluded, that, in case (b), Σ might have connected components of different dimensions.

Assuming now case (b), we obtain (iv) as an obvious consequence of (11) and (15-**). Now (iv) and Remark 3.2 yield the first part of (v), while (vi) and (vii) follow from (12), (14) and (15-**).

To prove the remainder of (v), first note that the claim about the sign pattern is true locally: a local conformal change of the metric allows us to treat v as a Killing field and use the final clause of Theorem 3.1, which implies that the tangent spaces of $N \setminus \Sigma$ are invariant under parallel transports along $N \setminus \Sigma$. The corresponding global claim could fail only if some connected component of Σ would locally disconnect N , leading to different sign patterns on the resulting new components. This, however, cannot happen since, for any $x \in \Sigma$, any $\varepsilon \in (0, \infty)$, and any null geodesic $(-\varepsilon, \varepsilon) \ni t \mapsto y(t)$ with $y(0) = x$ which lies in $N \setminus \Sigma$ except at $t = 0$, the family of tangent spaces $T_{y(t)}(N \setminus \Sigma)$, for $t \neq 0$, is parallel along the geodesic. Namely, whenever $t \mapsto w(t) \in T_{y(t)}M$ is a parallel vector field and $g(\dot{y}, w) = 0$, relations (10) form a system of first-order linear homogeneous ordinary differential equations with the unknowns $\nabla_w v$ and $g(w, \nabla \phi)$. Therefore, if we choose a parallel field w satisfying at some fixed $t \neq 0$ the condition $w(t) \in T_{y(t)}(N \setminus \Sigma)$ (so that, by (vi) and (13) – (15), $\nabla_w v$ and $g(w, \nabla \phi)$ both vanish at t), uniqueness of solutions gives $\nabla_w v = 0$ and $g(w, \nabla \phi) = 0$ at every t . Now, by (vi), $w(t) \in T_{y(t)}(N \setminus \Sigma)$ for all $t \neq 0$, as required.

By (vii) the limit as $t \rightarrow 0$ of the above parallel family $t \mapsto T_{y(t)}(N \setminus \Sigma)$ is $u^\perp \cap H$, where $u = \dot{y}(0) \in T_x M$, so that $u \in (C \cap H) \setminus H^\perp$. Clearly, $\dim(N \setminus \Sigma) = \dim(u^\perp \cap H)$. Letting Σ temporarily stand for the connected component of Σ which contains x , we have, by (ii), $\dim \Sigma = \dim(H \cap H^\perp)$. Note that $H \cap H^\perp$ is the nullspace of the symmetric bilinear form $\langle \cdot, \cdot \rangle$ in H obtained by restricting the metric g , and r is the rank of the restriction of $\langle \cdot, \cdot \rangle$ to $u^\perp \cap H$. Since $u \in (C \cap H) \setminus H^\perp$, the sign pattern of the latter restriction arises from that of $\langle \cdot, \cdot \rangle$ in H by replacing a plus-minus pair with a zero. Consequently, $\langle \cdot, \cdot \rangle$ has the rank $r + 2$, and $\dim(N \setminus \Sigma) - \dim \Sigma = \dim(u^\perp \cap H) - \dim(H \cap H^\perp) = (\dim H - 1) - [\dim H - (r + 2)] = r + 1$. Now the dimension formula in (v) follows. This in turn shows that all connected components of Σ have the same dimension, completing the proof. \square

6. Induced structures on Σ and $N \setminus \Sigma$

Again, let Σ now be either an essential component, or the set of essential points in a nonessential component N of the zero set Z of a conformal vector field v on a pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$.

Both Σ and $N \setminus \Sigma$ carry geometric structures naturally induced by the underlying conformal structure of (M, g) . Fixing our metric g within the conformal structure allows us in turn to represent the induced structures by more concrete geometric objects, as explained below.

First, according to Theorem 5.1(v), g (or, the conformal structure), restricted to $N \setminus \Sigma$, is a symmetric 2-tensor field having the same sign pattern at all points (or, respectively, a class of such tensor fields, arising from one another via multiplications by functions without zeros). We refer to it as the *possibly-degenerate metric* (or, *possibly-degenerate conformal structure*) of $N \setminus \Sigma$. If $\Sigma \subset N$ is nonempty, the metric/structure must actually be degenerate due to the inequality in Theorem 5.1(v), which gives $r < \dim(N \setminus \Sigma)$, and also shows that this is the zero metric/structure (with $r = 0$) only in the case where $\dim \Sigma = \dim(N \setminus \Sigma) - 1$.

A further natural structure on $N \setminus \Sigma$ is the *nullspace distribution* \mathcal{P} of the restriction of the metric g (or conformal structure) to $N \setminus \Sigma$. The dimension of \mathcal{P} is positive when $\Sigma \subset N$ is nonempty: as we just saw, the restricted metric is degenerate. Instead of the argument in the last paragraph, we can also derive its degeneracy from the fact that, by the Gauss lemma, short null geodesic segments emanating from Σ into $N \setminus \Sigma$ are all tangent to \mathcal{P} . (The Gauss lemma and its standard proof in the Riemannian case [7, Lemma 10.5] remain valid for indefinite metrics.)

From now on Σ is assumed nonempty. In view of Theorem 5.1(i), g gives rise to an obvious torsion-free connection D on Σ , while the conformal structure of g induces on Σ a natural *projective structure*, that is, a class of torsion-free connections having the same family of nonparametrized geodesics. (See the third paragraph of Section 2.)

In addition, g naturally leads to a 1-form ξ on Σ . (Using the conformal structure instead of g , we obtain a 1-form ξ defined only up to multiplications by functions without zeros.) To describe ξ , we consider two cases, noting that $\phi = (2/n) \operatorname{div} v$ is constant on Σ and, in fact, on every component of Z , cf. the final clause of Theorem 3.3. Specifically, if $\phi = 0$ on Σ , then $\Sigma \ni x \mapsto \mathcal{H}_x = \operatorname{Ker} \nabla v_x \cap \operatorname{Ker} d\phi_x$ is, in both cases, a parallel subbundle of $T_\Sigma M$ contained in $\operatorname{Ker} \nabla v$ as a codimension-one subbundle [4, Lemma 13.1(b),(d)], and we set $\xi = g(u, \cdot)$, on Σ , for any section u of $\operatorname{Ker} \nabla v$ over Σ with $g(u, \nabla \phi) = 1$. If $\phi \neq 0$ on Σ , we declare that $\xi = 0$.

Proposition 6.1.

Under the assumptions made at the beginning of this section, for \mathcal{P}, D and ξ defined above,

- (i) \mathcal{P} is integrable and its leaves are null totally geodesic submanifolds of (M, g) ,
- (ii) if $\Gamma \subset \Sigma$ is a geodesic segment and $T_x \Gamma \subset \operatorname{Ker} \xi_x$ for some $x \in \Gamma$, then $T_x \Gamma \subset \operatorname{Ker} \xi_x$ for every $x \in \Gamma$,
- (iii) in the open subset $\Sigma' \subset \Sigma$ on which $\xi \neq 0$,

$$\operatorname{sym} D\xi = \mu \odot \xi, \quad \text{that is, } \xi_{j,k} + \xi_{k,j} = \mu_j \xi_k + \mu_k \xi_j \quad \text{for some 1-form } \mu \text{ on } \Sigma', \quad (16)$$

- (iv) ξ has the following unique continuation property: if $\xi = 0$ at all points of some codimension-one connected submanifold Δ of Σ , then $\xi = 0$ everywhere in the connected component of Σ containing Δ .

Proof. For any sections w, w' of \mathcal{P} and any curve $t \mapsto y(t)$ in the totally umbilical submanifold $K = N \setminus \Sigma$, (2) gives $\pi \nabla_{\dot{x}} w = 0$, that is, $\nabla_{\dot{x}} w$ is tangent to K . Hence so is $\nabla_w w'$ and, for any vector field u tangent to K we have $g(\nabla_w w', u) = -g(\nabla_w u, w') = 0$, as one sees applying (2), this time, to w' instead of w and an integral curve $t \mapsto y(t)$ of w . Thus, $\nabla_w w'$ is a section of \mathcal{P} , and (i) follows.

To prove (ii) – (iv), we may assume that $\phi = 0$ on Σ . For Γ and x as in (ii), let $t \mapsto w(t) \in T_{y(t)} M$ be a parallel vector field along a geodesic parametrization of $t \mapsto y(t)$ of Γ such that $y(0) = x$ and $\nabla_{w(0)} v = \dot{y}(0)$. (That $\dot{y}(0) \in \nabla v_x(T_x M)$ is clear: as $\dot{y}(0)$ lies in $T_x \Gamma \subset \operatorname{Ker} \xi_x \subset T_x \Sigma$, it is orthogonal not just to $\mathcal{H}_x = \operatorname{Ker} \nabla v_x \cap \operatorname{Ker} d\phi_x$, cf. Theorem 5.1(ii), but also to the whole space $\operatorname{Ker} \nabla v_x$, while $\nabla v_x(T_x M) = [\operatorname{Ker} \nabla v_x]^\perp$ as ∇v_x is skew-adjoint by (1) with $\phi(x) = 0$.) Choosing a function $t \mapsto \kappa(t)$ with $2\dot{\kappa} = g(w, \nabla \phi)$ and $\kappa(0) = 1$, then integrating (10-a), we get $\nabla_{w(t)} v = \kappa(t) \dot{y}(t)$, so that $\dot{y}(t) \in \nabla v_{y(t)}(T_x M) = [\operatorname{Ker} \nabla v_{y(t)}]^\perp$ for all t near 0, which yields (ii).

Assertion (iii) is in turn a consequence of (ii): if $x \in \Sigma$ and $w \in \operatorname{Ker} \xi_x$, setting $y(t) = \exp_x tw$ and differentiating the resulting equality $\xi(\dot{y}) = 0$, we obtain $[\nabla_{\dot{y}} \xi](\dot{y}) = 0$, so that $[\nabla_{ty} \xi](\dot{y}) = 0$, so that $[\nabla_w \xi](w) = 0$. Thus, $\operatorname{sym} D\xi_x$ treated as a polynomial function on $T_x \Sigma$ vanishes on the zero set of the linear function ξ_x , which is well-known to imply divisibility of the former by the latter, cf. [5, Lemma 17.1(i)]. This proves (iii).

Finally, (iv) follows from (ii) since under the hypothesis of (iv), ξ must vanish on an open set containing Δ , namely, the set of points at which an open set of tangent directions is realized by geodesics intersecting Δ . \square

Note that condition (16) involves D only through its underlying projective structure, and remain valid after ξ has been multiplied by a function without zeros.

7. One-jets of v along components of Z

As before, Z stands for the zero set of a conformal vector field v on a pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$. For $x \in Z$, the endomorphism ∇v_x of $T_x M$, obviously independent of the choice of the connection ∇ , is also known as the *linear part* (or *Jacobian*, or *derivative*, or *differential*) of v at the zero x . At the same time, ∇v_x is the infinitesimal generator of the local flow of v acting in $T_x M$.

Since $v_x = 0$, we may also identify ∇v_x with the 1-jet of v at x .

Given $x, y \in Z$, we say that the 1-jets of v at x and y are *conformally equivalent* if, for some vertical-arrow conformal isomorphism $T_x M \rightarrow T_y M$, the following diagram commutes:

$$\begin{array}{ccc}
 T_x M & \xrightarrow{\nabla v_x} & T_x M \\
 \downarrow & & \downarrow \\
 T_y M & \xrightarrow{\nabla v_y} & T_y M
 \end{array} \tag{17}$$

By *conformal isomorphisms* we mean here nonzero scalar multiples of linear isometries.

Proposition 7.1.

Under the assumptions of Theorem 5.1, with ξ defined in Section 6,

- (i) in case (b) of Theorem 5.1, for any connected component N' of $N \setminus \Sigma$, the 1-jets of v at all points of N' are conformally equivalent to one another, but not to the 1-jet of v at any point of Σ ,
- (ii) in both cases (a) – (b) of Theorem 5.1, if ξ is not identically zero on a connected component Σ' of Σ , then the 1-jets of v at any two points of Σ' are conformally equivalent.

Proof. Since some local conformal change of the metric near any $y \in N'$ turns v into a Killing field, (i) follows: ∇v then becomes parallel along a neighborhood of y in N' . The claim about Σ is obvious from Theorem 5.1(vi).

In (ii), the definition of ξ (see Section 6) implies that $\phi = 0$ on Σ' . Thus, by Theorem 3.2, $\nabla \phi_x \notin \nabla v_x(T_x M)$ at every point $x \in \Sigma'$, and so $\mathcal{H}_x^\perp = \nabla v_x(T_x M) \oplus \mathbb{R} \nabla \phi_x$. (Note that $\nabla v_x(T_x M) = [\text{Ker } \nabla v_x]^\perp$ by (1) with $\phi(x) = 0$, while (5) gives $\nabla \phi_x \in \mathcal{H}_x^\perp$.) Hence, in view of Theorem 5.1(ii), $T_x \Sigma' \subset \nabla v_x(T_x M) \oplus \mathbb{R} \nabla \phi_x$, while the vectors tangent to Σ' at x and lying in the summand $\nabla v_x(T_x M)$ form precisely the subspace $\text{Ker } \xi_x$, which has codimension one in $T_x \Sigma'$ for all points x of a dense open subset of Σ' (Proposition 6.1(iv)). We will now show that the conformal equivalence type of the 1-jets of v is constant along any geodesic segment Γ in Σ' with a parametrization $t \mapsto y(t)$ satisfying the condition $\dot{y}(t) \notin \nabla v_x(T_x M)$ at each $x = y(t)$. (As any two points of Σ' can be joined by piecewise

smooth curves made up from such geodesic segments, due to the denseness and openness property just mentioned, (ii) will then clearly follow.)

Specifically, our assumption about $\dot{y}(t)$ yields $\nabla\phi = \rho\dot{y} + \nabla_w v$ for some function $t \mapsto \rho(t)$ and a vector field $t \mapsto w(t) \in T_{y(t)}M$ along the geodesic; since $\text{rank } \nabla v$ is constant on Σ' by [4, Lemma 13.1(d)], w may be chosen differentiable.) From (3) we now obtain $2\nabla_{\dot{y}}\nabla v = g(\nabla\phi, \cdot) \otimes \dot{y} - g(\dot{y}, \cdot) \otimes \nabla\phi$, and one easily verifies that ∇v is D-parallel for the new metric connection D in $T_T M$ given by $2D_{\dot{y}} = 2\nabla_{\dot{y}} + g(w, \cdot) \otimes \dot{y} - g(\dot{y}, \cdot) \otimes w$. \square

8. The associated quintuples

The symbol $[\eta]$ stands for the *homothety class* of a pseudo-Euclidean inner product η on a finite-dimensional vector space \mathcal{T} , that is, the set of all nonzero scalar multiples of η . The underlying conformal structure $[g]$ of a pseudo-Riemannian manifold (M, g) may thus be identified with the assignment $M \ni x \mapsto [g_x]$. Let us consider quintuples

$$(\mathcal{T}, [\eta], B, \lambda, \delta) \tag{18}$$

formed by a pseudo-Euclidean vector space \mathcal{T} , the homothety class of its inner product η , a skew-adjoint endomorphism $B \in \mathfrak{so}(\mathcal{T})$, a real number λ , and a linear functional $\delta \in [\text{Ker}(B + \lambda)]^*$ on the subspace $\text{Ker}(B + \lambda)$ of \mathcal{T} (which, if nontrivial, is the eigenspace of B for the eigenvalue $-\lambda$).

We call (18) *algebraically equivalent* to another such quintuple $(\mathcal{T}', [\eta'], B', \lambda', \delta')$ if $\lambda' = \lambda$ and some linear isomorphism $\mathcal{T} \rightarrow \mathcal{T}'$ sends $[\eta], B, \delta$ to $[\eta'], B'$ and δ' .

Examples of quintuples (18) arise as follows. Given a conformal vector field v on a pseudo-Riemannian manifold (M, g) and a point $x \in M$ at which $v_x = 0$, we define the *quintuple associated with v and x* to be $(\mathcal{T}, [\eta], B, \lambda, \delta) = (T_x M, [g_x], A_x, \phi(x), \delta)$, where A and ϕ are determined by v as in (1), and δ is the restriction of $d\phi_x$ to the subspace $\text{Ker}(B + \lambda) = \text{Ker } \nabla v_x$. In other words, B equals twice the skew-adjoint part of $\nabla v_x : T_x M \rightarrow T_x M$ (the value at x of the morphism $\nabla v : TM \rightarrow TM$), and λ is $2/n$ times $\text{tr } \nabla v_x$, where $n = \dim M$.

The associated quintuple $(\mathcal{T}, [\eta], B, \lambda, \delta)$ depends, besides v and x , only on the underlying conformal structure $[g]$, rather than the metric g . This is obvious for $\mathcal{T}, [\eta], B$ and $\lambda = (2/n) \text{tr } \nabla v_x$, cf. the beginning of Section 7. Similarly, as $\text{Ker}(B + \lambda) = \text{Ker } \nabla v_x$, the last line in Remark 2.1 yields the claim about δ .

9. Conformal equivalence of two-jets

Let v and w be conformal vector fields on pseudo-Riemannian manifolds (M, g) and, respectively, (N, h) , such that v vanishes at a point $x \in M$, and w at $y \in N$. We say that the 2-jet of v at x is *conformally equivalent* to the 2-jet

of w at y if some diffeomorphism F between a neighborhood U of x in M and one of y in N , with $F(x) = y$, sends the former 2-jet to the latter, while, at the same time, for some function $\tau : U \rightarrow \mathbb{R}$, the metrics F^*h and $e^\tau g$ have the same 1-jet at x .

As v and w vanish at x and y , the above condition on F involves F only through its 2-jet at x .

Lemma 9.1.

For M, g, v, x and N, h, w, y as in the last two paragraphs, the 2-jets of v at x and of w at y are conformally equivalent if and only if the quintuple $(\mathcal{T}, [\eta], B, \lambda, \delta)$ associated with v and x is algebraically equivalent, in the sense of Section 8, to the analogous quintuple $(\mathcal{T}', [\eta'], B', \lambda', \delta')$ for w and y .

Proof. The ‘only if’ part of our claim is obvious from functoriality of the associated quintuple. To prove the ‘if’ part, we fix local coordinates x^j for M at x and y^a for N at y such that the corresponding Christoffel symbols of g , or h vanish at x , or y . We also set $v_j^k = \partial_j v^k$, $F_j^a = \partial_j F^a$, $F_{jk}^a = \partial_j \partial_k F^a$, $\tau_j = \partial_j \tau$, $w_c^a = \partial_c w^a$, where all the partial derivatives stand for their values at x or y (and those involving F or τ are treated as unknowns).

It now suffices to show that, if $(\mathcal{T}, [\eta], B, \lambda, \delta)$ and $(\mathcal{T}', [\eta'], B', \lambda', \delta')$ are equivalent, the system

$$\begin{aligned} \text{i)} \quad & w_c^a F_j^c = F_k^a v_j^k, \\ \text{ii)} \quad & F_l^a \partial_j \partial_k v^l + F_{jl}^a v_k^l + F_{kl}^a v_j^l = F_j^b F_k^c \partial_b \partial_c w^a + F_{jk}^c w_c^a, \\ \text{iii)} \quad & h_{ac} F_j^a F_k^c = e^\tau g_{jk}, \quad \text{iv)} \quad h_{ac} (F_j^a F_{kl}^c + F_{jl}^a F_k^c) = e^\tau \tau_l g_{jk}, \end{aligned} \tag{19}$$

where the values of g_{jk} , $\partial_j \partial_k v^l$, h_{ac} and $\partial_b \partial_c w^a$ are taken at x or y , has a solution consisting of a real number τ and some quantities F_j^a , F_{jk}^a , τ_j with $F_{jk}^a = F_{kj}^a$.

As the first part of such a solution we choose a matrix F_j^a which, when treated as a linear isomorphism $T_x M \rightarrow T_y N$ (with the aid of our fixed coordinates x^j and y^a), realizes the equivalence of the two quintuples. As $\lambda' = \lambda$ and the isomorphism in question sends $([\eta], B, \delta)$ to $([\eta'], B', \delta')$, we now clearly have (19-iii) for some $\tau \in \mathbb{R}$, (19-i), and there exists a 1-form $\sigma \in T_x^* M$ with $\phi_{,j} - \psi_{,a} F_j^a = 2v_j^k \sigma_k$, where ϕ is determined by v as in (1), ψ is its analog for w , and the components of $d\phi$ and $d\psi$ are evaluated at x or y . (That σ exists is obvious since our isomorphism sends δ to δ' , and so $u^j(\phi_{,j} - \psi_{,a} F_j^a) = 0$ whenever $u \in T_x M$ and $u^j v_j^k = 0$.) It follows that

$$\text{a)} \quad g^{jk} F_j^a F_k^c = e^\tau h^{ac}, \quad \text{b)} \quad v_k^j \sigma^k = \phi \sigma^j - g^{jl} v_l^k \sigma_k, \quad \text{where } \sigma^j = g^{jk} \sigma_k. \tag{20}$$

In fact, (19-iii) states that the matrix $e^{-\tau/2} F_j^a$, as a linear isomorphism $T_x M \rightarrow T_y N$, sends the metric g_x to h_y , and so the reciprocal metrics of g_x and h_y correspond to each other under the dual isomorphism $T_y^* N \rightarrow T_x^* M$. This amounts to (20-a), while (20-b) is a trivial consequence of (1).

Finally, let us set $F_{jk}^a = F_l^a \sigma^l g_{jk} - \sigma_j F_k^a - \sigma_k F_j^a$ and $\tau_j = -2\sigma_j$. Then (19-iii) implies (19-iv). Next, at x and y , our choice of coordinates and the coordinate form of (3) yield $\partial_j \partial_k v^l = \phi_{,k} \delta_j^l - \phi^l g_{jk} + \phi_{,j} \delta_k^l$ and, analogously,

$\partial_b \partial_c w^a = \psi_{,c} \delta_b^a - \psi^{,a} g_{bc} + \psi_{,b} \delta_c^a$. Now (19-iii) follows from (20-a), as one sees replacing $\phi_{,j}$ with $\psi_{,a} F_j^a + 2v_j^k \sigma_k$ and noting that, by (1), $v_j^l F_p^a \sigma^p g_{kl} + v_k^l F_p^a \sigma^p g_{jl} = \phi F_p^a \sigma^p g_{jk}$, while (19-i) and (20-b) give $w_c^a F_l^c \sigma^l = F_p^a v_l^p \sigma^l = \phi F_p^a \sigma^p - g^{pq} F_p^a v_q^l \sigma_l$. \square

10. Two-jets of v along components of Z

Proposition 7.1 remains true if the word 1-jet(s) is replaced everywhere with 2-jet(s).

For both assertions (i) and (ii), this is a direct consequence of Lemma 9.1. Specifically, in the case of (i), the invariant δ vanishes at every point of $N \setminus \Sigma$ by Theorem 3.2, while the remaining four objects in the associated quintuple (18) represent the 1-jet of v at the point in question. The algebraic-equivalence type of the quintuple is thus locally constant on $N \setminus \Sigma$ as a consequence of Proposition 7.1.

Similarly, for the new metric connection D in $T_T M$ used to prove Proposition 7.1(ii), the second formula in (3) shows that the restriction of $d\phi$ to $\text{Ker } \nabla v$ is D -parallel as well, as long as one chooses w with $g(w, \nabla \phi) = 0$. Since $\nabla \phi_x \notin \nabla v_x(T_x M) = [\text{Ker } \nabla v_x]^\perp$, whenever $x \in \Sigma'$, such a choice is always possible.

The following example shows that the assumption about ξ in Proposition 7.1(ii) cannot in general be removed.

On a pseudo-Euclidean space $(V, \langle \cdot, \cdot \rangle)$ of dimension n we may define a conformal vector field v by

$$v_x = w + Bx + cx + 2\langle u, x \rangle x - \langle x, x \rangle u, \quad (21)$$

using any fixed vectors $w, u \in V$, any skew-adjoint endomorphism B , and any scalar $c \in \mathbb{R}$. Let us now choose n to be even, $\langle \cdot, \cdot \rangle$ to have the neutral signature, B with two null n -dimensional eigenspaces for the eigenvalues $c, -c$, and u which does not lie in the $-c$ eigenspace, along with $w = 0$. Then $\dim \text{Ker } \nabla v_x$ is easily verified to decrease when one replaces $x = 0$ by any nearby vector x orthogonal to u and lying in the $-c$ eigenspace of B .

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References

- [1] Beig, R.: Conformal Killing vectors near a fixed point, Institut für Theoretische Physik, Universität Wien, unpublished manuscript, 1992
- [2] Belgun, F., Moroianu, A., Ornea, L., Essential points of conformal vector fields, *J. Geom. Phys.*, 2011, 61(3), 589–593
- [3] Capocci, M. S., Essential conformal vector fields, *Classical Quantum Gravity*, 1999, 16, 927–935
- [4] Derdzinski, A., Zeros of conformal fields in any metric signature, *Classical Quantum Gravity*, 2011, 28(7), #075011
- [5] Derdzinski, A., Maschler, G., A moduli curve for compact conformally-Einstein Kähler manifolds, *Compos. Math.*, 2005, 141(4), 1029–1080
- [6] Kobayashi S., Fixed points of isometries, *Nagoya Math. J.*, 1958, 13, 63–68
- [7] Milnor, J., *Morse Theory*, Ann. of Math. Stud., 51, Princeton University Press, Princeton, 1963
- [8] Weyl, H., *Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung*, *Göttinger Nachr.* 1921, 99–112.