Dong Du

MATH 132 PRACTICE PROBLEMS FOR MIDTERM 3

1. Determine if the following integral is convergent or divergent. If it is convergent, then find its value

(a) $\int_3^{\infty} \frac{1}{(2x-1)^3}\,dx$ (b) $\int_{-\infty}^{-2} \frac{1}{(x+1)^3}\,dx$ (c) $\int_0^{\infty} e^{-5x+17}\,dx$ (d) $\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

Ans: (a) $\frac{1}{100}$, (b) $-\frac{1}{2}$, (c) $\frac{e^{17}}{5}$, (d) $\frac{2}{e}$

2. Find (a) $\frac{\partial z}{\partial x}$ for the function $z=(x^2+y)e^{3x+4y}$. Ans: $(3x^2+2x+3y)e^{3x+4y}$

(b) $\frac{\partial z}{\partial y}$ for the function $z=\sqrt[3]{x^2+y^2}$. Ans: $\frac{2y}{3(x^2+y^2)^{\frac{2}{3}}}$

(c) $\frac{\partial h}{\partial x}$ for the function $h(x,y)=\frac{x^2+3xy+y^2}{\sqrt{x^2+y^2}}$. Ans: $\frac{x^3+xy^2+3y^3}{(x^2+y^2)^{\frac{3}{2}}}$

(d) $\frac{\partial h}{\partial y}$ for the function $h(x,y)=\ln(x^3+3xy^2+x^2y^3)e^{3x+5y+6xy}$

Ans: $\frac{6xy+3x^2y^2}{x^3+3xy^2+x^3y^3}e^{3x+5y+6xy}+(5+6x)\ln(x^3+3xy^2+x^2y^3)e^{3x+5y+6xy}$

3. Find the marginal cost $\frac{\partial C}{\partial y}$ with $x=20$ and $y=30$ for the function $C=7x+.03x^2+2xy+y^2+900$. Ans: 100

4. Find the marginal cost $\frac{\partial C}{\partial x}$ with $x=40$ and $y=60$ for the function $C=x\sqrt{x+y}+5000$. Ans: 12

5. The demand functions q_A, q_B for the products A and B are given by

$$q_A=\frac{500}{p_A\sqrt[3]{p_B}} \text{ and } q_B=\frac{7550}{(p_B)^2\sqrt[3]{p_A}}.$$

Determine whether the products A and B are competitive, complementary or neither. Ans: Complementary.

6. The demand functions q_A, q_B for the products A and B are given by $q_A=\frac{30\sqrt{p_B}}{\sqrt[3]{p_A^{\ 2}}}$ and $q_B=\frac{50p_A}{\sqrt[3]{p_B}}$.

(a) Determine whether the products A and B are competitive, complementary or neither.

(b) Find the values of the two marginal demands for product A, when $p_A=8$ and $p_B=64$.

(c) If p_B were reduced to 62 from 64, with p_A fixed at 8, use part (b) to estimate the corresponding change in demand for the product A.

Ans: (a) Competitive, (b) $\frac{\partial q_A}{\partial p_A}=-5$, $\frac{\partial q_A}{\partial p_B}=\frac{15}{32}$, (c) decrease the demand for A by $\frac{15}{16}$ Correct

7. (a) Use implicit differentiation to find $\frac{\partial z}{\partial x}$ from the equation

$$z^2x^3y^2 - 7x^2 + y^3 + 3xy = 11$$

Ans: $\frac{14x - 3y \ - 3x^2y^2z^2}{2x^3y^2z}$

(b) Use implicit differentiation to find $\frac{\partial z}{\partial y}$ from the equation

$$zxy - 7x^3 + y^3 = 9.$$

Ans: $-\frac{3y^2 + xz}{xy}$

8. Find f_{xx} and f_{xy} for the following functions:

(a) $f(x,y) = (x^2 + xy + y^2)(x^2 + xy + 1)$

Ans: $f_{xx} = 2(2x^2 + 2xy + y^2 + 1) + 2(2x + y)^2$, $f_{xy} = 6x^2 + 8xy + 3y^2 + 1$.

(b) $f(x,y\cancel{)} = xy^3z^2 + 11$. Ans: $f_{xx} = 0$, $f_{xy} = 3y^2z^2$.
,z)

9. Find the critical points of the following functions, and use the Second Derivative Test to determine whether it corresponds to a relative maximum, minimum or the test gives no information.

(a) $f(x,y) = x^2 + 4y^2 - 6x + 16y$. Ans: relative mininum at $(3,-2)$.

(b) $f(x,y) = x^2 + y^2 + xy - 9x + 1$. Ans: relative mininum at $(6,-3)$.

(c) $f(x,y) = \frac{1}{3}(x^3 + 8y^3) - 2(x^2 + y^2) + 1$.

Ans: rel. max. at $(0,0)$, saddle points at $(0,\frac{1}{2})$ and $(4,0)$, rel.min.at $(4,\frac{1}{2})$.

Second Derivative Test: If (x_0,y_0) is a critical point and D be the function defined by

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - \left[f_{xy}(x,y)\right]^2$$

Then (a) if $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) > 0$, then (x_0,y_0) is a point of rel. mininum.

(b) if $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) < 0$, then (x_0,y_0) is a point of rel. maximum.

(c) if $D(x_0,y_0) < 0$, then (x_0,y_0) is a saddle point.

(d) if $D(x_0,y_0) = 0$, then no conclusion can be drawn.

Math 132 Practice Problems For Midterm 3

1. (a) $$\int_3^\infty \frac{1}{(2x-1)^3}\,dx$$

$$= \lim_{r\to\infty} \int_3^r \frac{1}{(2x-1)^3}\,dx$$

Consider $\int \frac{1}{(2x-1)^3}\,dx$, let $u = 2x-1$,

$du = 2dx$, $dx = \frac{1}{2}du$, so

$$\int \frac{1}{(2x-1)^3}\,dx = \frac{1}{2}\int \frac{1}{u^3}\,du = \frac{1}{2}\,\frac{u^{-2}}{-2} + C$$

$$= -\frac{1}{4}\,\frac{1}{(2x-1)^2} + C$$

$$\int_3^r \frac{1}{(2x-1)^3}\,dx = -\frac{1}{4}\,\frac{1}{(2x-1)^2}\Big|_3^r = -\frac{1}{4}\,\frac{1}{(2r-1)^2} + \frac{1}{4}\,\frac{1}{(2\cdot 3-1)^2}$$

$$= -\frac{1}{4}\,\frac{1}{(2r-1)^2} + \frac{1}{100}$$

$$\lim_{r\to\infty} \int_3^r \frac{1}{(2x-1)^3}\,dx = \lim_{r\to\infty}\left(-\frac{1}{4}\,\frac{1}{(2r-1)^2} + \frac{1}{100}\right)$$

$$= \frac{1}{100}.$$

(b) $$\int_{-\infty}^{-2} \frac{1}{(x+1)^3}\,dx$$

$$\int \frac{1}{(x+1)^3}\,dx \overset{u=x+1}{=\!=\!=} \int \frac{1}{u^3}\,du = \frac{u^{-2}}{-2} + C$$

$$= -\frac{1}{2}\,\frac{1}{(x+1)^2} + C$$

$$\int_{-r}^{-2} \frac{1}{(x+1)^3}\,dx = -\frac{1}{2}\,\frac{1}{(x+1)^2}\Big|_{-r}^{-2} = -\frac{1}{2}\,\frac{1}{(-2+1)^2} + \frac{1}{2}\,\frac{1}{(-r+1)^2}$$

$$= -\frac{1}{2} + \frac{1}{2}\,\frac{1}{(-r+1)^2}$$

$$\int_{-\infty}^{-2} \frac{1}{(x+1)^3}\,dx = \lim_{r\to\infty} \int_{-r}^{-2} \frac{1}{(x+1)^3}\,dx$$

$$= \lim_{r\to\infty} \left(-\frac{1}{2} + \frac{1}{2}\frac{1}{(-r+1)^2} \right) = -\frac{1}{2}$$

(c) $\int_0^{\infty} e^{-5x+17}\,dx$

$$\int e^{-5x+17}\,dx \overset{u=-5x+17}{=\!=\!=} \int e^{u}\left(-\frac{1}{5}\right)du$$

$$= -\frac{1}{5}e^{u} + C$$

$$= -\frac{1}{5}e^{-5x+17} + C$$

$$\int_0^{\infty} e^{-5x+17}\,dx = \lim_{r\to\infty} \int_0^{r} e^{-5x+17}\,dx$$

$$= \lim_{r\to\infty} \left(-\frac{1}{5}e^{-5x+17}\right)\Big|_0^r$$

$$= \lim_{r\to\infty} \left(-\frac{1}{5}e^{-5r+17} + \frac{1}{5}e^{17}\right)$$

$$= \frac{e^{17}}{5}$$

(d) $\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

$$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx \overset{u=-\sqrt{x}}{=\!=\!=} \int e^{u}(-2)\,du$$

$$= -2e^{u} + C$$

$$= -2e^{-\sqrt{x}} + C$$

$$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{r\to\infty} \int_1^{r} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{r\to\infty} \left(-2e^{-\sqrt{x}}\right)\Big|_1^r$$

$$= \lim_{r\to\infty} \left(-2e^{-\sqrt{r}} + 2e^{-1}\right) = \frac{2}{e}$$

2. (a) $z = (x^2+y)e^{3x+4y}$

$$\frac{\partial z}{\partial x} = \frac{\partial(x^2+y)}{\partial x} e^{3x+4y} + (x^2+y)\frac{\partial e^{3x+4y}}{\partial x}$$

$$= 2x e^{3x+4y} + (x^2+y)\,3\,e^{3x+4y}$$

$$= [2x + (x^2+y)3]\,e^{3x+4y}$$

$$= (3x^2+2x+3y)\,e^{3x+4y}$$

(b) $z = \sqrt[3]{x^2+y^2} = (x^2+y^2)^{\frac{1}{3}}$

$$\frac{\partial z}{\partial y} = \frac{1}{3}(x^2+y^2)^{-\frac{2}{3}} \frac{\partial(x^2+y^2)}{\partial y}$$

$$= \frac{1}{3}(x^2+y^2)^{-\frac{2}{3}} \cdot 2y$$

$$= \frac{2y}{3(x^2+y^2)^{\frac{2}{3}}}$$

(c) $h(x,y) = \dfrac{x^2+3xy+y^2}{\sqrt{x^2+y^2}}$

$$\frac{\partial h}{\partial x} = \frac{\frac{\partial}{\partial x}(x^2+3xy+y^2)\cdot\sqrt{x^2+y^2} - (x^2+3xy+y^2)\frac{\partial}{\partial x}\sqrt{x^2+y^2}}{\left(\sqrt{x^2+y^2}\right)^2}$$

$$= \frac{(2x+3y)\sqrt{x^2+y^2} - (x^2+3xy+y^2)\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\cdot 2x}{x^2+y^2}$$

$$= \frac{(x^2+y^2)^{-\frac{1}{2}}\left[(2x+3y)(x^2+y^2) - x(x^2+3xy+y^2)\right]}{x^2+y^2}$$

$$= \frac{(2x+3y)(x^2+y^2) - x(x^2+3xy+y^2)}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= \frac{2x^3 + 3x^2y + 2xy^2 + 3y^3 - x^3 - 3x^2y - xy^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= \frac{x^3 + xy^2 + 3y^3}{(x^2+y^2)^{\frac{3}{2}}}$$

(d) $h(x,y) = \ln(x^3 + 3xy^2 + x^2y^3)\, e^{3x+5y+6xy}$

$$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(\ln(x^3+3xy^2+x^2y^3)\right)\cdot e^{3x+5y+6xy}$$
$$+ \ln(x^3+3xy^2+x^2y^3)\,\frac{\partial}{\partial y} e^{3x+5y+6xy}$$

$$= \frac{1}{x^3+3xy^2+x^2y^3}\cdot(6xy+3x^2y^2)\, e^{3x+5y+6xy}$$
$$+ \ln(x^3+3xy^2+x^2y^3)\cdot e^{3x+5y+6xy}\cdot(5+6x)$$

$$= \frac{6xy+3x^2y^2}{x^3+3xy^2+x^2y^3} e^{3x+5y+6xy} + (5+6x)\ln(x^3+3xy^2+x^2y^3)\, e^{3x+5y+6xy}$$

3. $C = 7x + 0.03x^2 + 2xy + y^2 + 900$

$$\frac{\partial C}{\partial y} = 2x + 2y$$

$$\left.\frac{\partial C}{\partial y}\right|_{(20,30)} = 2\cdot 20 + 2\cdot 30 = 100$$

4. $C = x\sqrt{x+y} + 5000$

$$\frac{\partial C}{\partial x} = \sqrt{x+y} + x\,\frac{1}{2\sqrt{x+y}}$$

$$\left.\frac{\partial C}{\partial x}\right|_{(40,60)} = \sqrt{40+60} + 40\,\frac{1}{2\sqrt{40+60}} = 10 + 40\cdot\frac{1}{2\cdot 10} = 12$$

5. For definitions, see page 758.

$$\frac{\partial q_A}{\partial p_B} = \frac{\partial}{\partial p_B}\left(\frac{500}{p_A} p_B^{-\frac{1}{3}}\right) = \frac{500}{p_A}\left(-\frac{1}{3}\right)p_B^{-\frac{4}{3}} < 0$$

$$\frac{\partial q_B}{\partial p_A} = \frac{\partial}{\partial p_A}\left(\frac{7550}{p_B^2} p_A^{-\frac{1}{3}}\right) = \frac{7550}{p_B^2}\left(-\frac{1}{3}\right) p_A^{-\frac{4}{3}} < 0$$

So A and B are complementary.

6. (a) $$\frac{\partial q_A}{\partial p_B} = \frac{\partial}{\partial p_B}\left(\frac{30}{\sqrt[3]{p_A^2}} \cdot p_B^{\frac{1}{2}}\right) = \frac{30}{\sqrt[3]{p_A^2}}\left(\frac{1}{2}\right)p_B^{-\frac{1}{2}} > 0$$

$$\frac{\partial q_B}{\partial p_A} = \frac{\partial}{\partial p_A}\left(\frac{50}{\sqrt[3]{p_B}} \cdot p_A\right) = \frac{50}{\sqrt[3]{p_B}} > 0$$

So A and B are competitive.

(b) $$\frac{\partial q_A}{\partial p_A} = \frac{\partial}{\partial p_A}\left(30\sqrt{p_B} \cdot p_A^{-\frac{2}{3}}\right) = 30\sqrt{p_B}\left(-\frac{2}{3}\right) p_A^{-\frac{5}{3}}$$

$$= 30\sqrt{64}\left(-\frac{2}{3}\right) 8^{-\frac{5}{3}}$$

$$= 30 \cdot 8 \cdot \left(-\frac{2}{3}\right) \cdot \frac{1}{32} = -5$$

$$\cancel{\frac{\partial q_B}{\partial p_B} = \frac{\partial}{\partial p_B}\left(50 p_A p_B^{-\frac{1}{3}}\right) = 50 p_A \left(-\frac{1}{3}\right) p_B^{-\frac{4}{3}}}$$

$$\cancel{= 50 \cdot 8 \cdot \left(-\frac{1}{3}\right) \cdot 64^{-\frac{4}{3}}}$$

$$\cancel{= 50 \cdot 8 \left(-\frac{1}{3}\right) \cdot \frac{1}{256}}$$

$$\frac{\partial q_A}{\partial p_B} = \frac{\partial}{\partial p_B}\left(30 p_A^{-\frac{2}{3}} \cdot p_B^{\frac{1}{2}}\right) = 30 p_A^{-\frac{2}{3}}\left(\frac{1}{2}\right) p_B^{-\frac{1}{2}}$$

$$= 30 \cdot 8^{-\frac{2}{3}} \cdot \frac{1}{2} \cdot 64^{-\frac{1}{2}} = 30 \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{8} = \frac{15}{32}$$

(c) $$\Delta q_A \approx \frac{\partial q_A}{\partial p_B} \cdot \Delta p_B$$

$$= \frac{15}{32} \cdot (-2)$$

$$= -\frac{15}{16}$$

So the demand for A deceased by $\frac{15}{16}$.

7. (a) $\frac{\partial}{\partial x}\left(z^2x^3y^2 - 7x^2 + y^3 + 3xy\right) = \frac{\partial}{\partial x}(11)$

where $z = z(x,y)$, so $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}$, $\frac{\partial x}{\partial x} = 1$, $\frac{\partial y}{\partial x} = 0$

$$2z\frac{\partial z}{\partial x}x^3y^2 + z^2 \cdot 3x^2 \cdot y^2 - 14x + 3y = 0$$

$$2x^3y^2z\frac{\partial z}{\partial x} + 3x^2y^2z^2 - 14x + 3y = 0$$

$$2x^3y^2z\frac{\partial z}{\partial x} = 14x - 3y - 3x^2y^2z^2$$

$$\frac{\partial z}{\partial x} = \frac{14x - 3y - 3x^2y^2z^2}{2x^3y^2z}$$

(b) $\frac{\partial}{\partial y}\left(zxy - 7x^3 + y^3\right) = \frac{\partial}{\partial y}(9)$

$$\frac{\partial z}{\partial y} \cdot xy + z \cdot x + 3y^2 = 0$$

$$xy\frac{\partial z}{\partial y} + xz + 3y^2 = 0$$

$$xy\frac{\partial z}{\partial y} = -(3y^2 + xz)$$

$$\frac{\partial z}{\partial y} = -\frac{3y^2 + xz}{xy}$$

8. (a) $f(x,y) = (x^2+xy+y^2)(x^2+xy+1)$

$f_x = (2x+y)(x^2+xy+1) + (x^2+xy+y^2)(2x+y)$

$f_{xx} = 2(x^2+xy+1) + (2x+y)(2x+y) + (2x+y)(2x+y) + (x^2+xy+y^2)\cdot 2$

$= \underbrace{2(2x^2+2xy+y^2+1)}_{\text{1st term + last term}} + 2(2x+y)^2$

$f_{xy} = (x^2+xy+1) + (2x+y)x + (x+2y)(2x+y) + (x^2+xy+y^2)$

$= x^2+xy+1 + 2x^2+xy + 2x^2+5xy+2y^2 + x^2+xy+y^2$

$= 6x^2 + 8xy + 3y^2 + 1$

(b) $f(x,y,z) = xy^3z^2 + 11$

$f_x = y^3z^2$

$\cancel{f_{xy} = 3y^2z^2}$ $f_{xx} = 0$

$\cancel{f_{xz} = 2y^3z}$ $f_{xy} = 3y^2z^2$

9. (a) $f(x,y) = x^2+4y^2-6x+16y$

$$\left.\begin{aligned} f_x &= 2x-6 = 0 \\ f_y &= 8y+16 = 0 \end{aligned}\right\} \Rightarrow \begin{cases} x_0 = 3 \\ y_0 = -2 \end{cases}$$

$f_{xx} = 2$

$f_{xy} = 0$

$f_{yy} = 8$

$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = 16$

So $D(x_0, y_0) = D(3,-2) = 16 > 0$
$f_{xx}(3,-2) = 2 > 0$

Therefore, $(3,-2)$ is a point of rel. minimum.

(b) $f(x,y) = x^2+y^2+xy-9x+1$

$$\left.\begin{aligned} f_x &= 2x+y-9 = 0 \\ f_y &= 2y+x = 0 \end{aligned}\right\} \Rightarrow \begin{cases} 2x+y=9 \\ x+2y=0 \end{cases} \Rightarrow \begin{cases} x_0 = 6 \\ y_0 = -3 \end{cases}$$

$f_{xx} = 2$, $f_{xy} = 1$, $f_{yy} = 2$

$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = 2 \cdot 2 - 1^2 = 3 > 0$

$f_{xx}(x,y) = 2 > 0$

So $(6,-3)$ is a point of rel. minimum.

(c) $f(x,y) = \frac{1}{3}(x^3+8y^3) - 2(x^2+y^2) + 1$

$$\left.\begin{aligned} f_x &= x^2 - 4x = 0 \\ f_y &= 8y^2 - 4y = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 0, 4 \\ y &= 0, 2 \end{aligned}$$

So $(x_0, y_0) = (0,0), (0,2), (4,0), (4,2)$

$f_{xx} = 2x - 4$, $f_{xy} = 0$, $f_{yy} = 16y - 4$

$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = (2x-4)(16y-4)$

Case 1 : $(0,0)$

$D(0,0) = (-4)(-4) > 0$

$f_{xx}(0,0) = -4 < 0$

So $(0,0)$ rel. maximum point.

Case 2 : $(0,2)$

$D(0,2) = (-4)(16 \cdot 2 - 4) = (-4) \cdot 28 < 0$

So $(0,2)$ is a saddle point.

Case 3 : $(4,0)$

$D(4,0) = (2 \cdot 4 - 4)(0-4) = 4 \cdot (-4) = -16 < 0$

So $(4,0)$ is also a saddle point.

Case 4 : $(4,2)$

$D(4,2) = (2 \cdot 4 - 4)(16 \cdot 2 - 4) = 4 \cdot 28 > 0$

$f_{xx}(4,2) = 2 \cdot 4 - 4 = 4 > 0$

So $(4,2)$ is a rel. minimum point.