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SOLUTIONS OF SELF-DIFFERENTIAL FUNCTIONAL EQUATIONS

Abstract

The system of functional differential equations (1) has a continuously differentiable solution for every value of the parameter a. The boundary values and a are related with $d(2-a) = c(2+a)$. When $a \in S$ where

$$S = \left\{2^{2n+1} : n = 1, 2, 3, \dots\right\},$$

the system (1) has infinitely many solutions with boundary values $c = 0$ and $d = 0$. For all other values of a, the system (1) has a unique solution.

$$\begin{cases} F'(x) = aF(2x) & \text{if } 0 \leq x \leq \dfrac{1}{2} \\ F'(x) = aF(2-2x) & \text{if } \dfrac{1}{2} \leq x \leq 1 \\ F(0) = c, F(1) = d. \end{cases} \tag{1}$$

1 Introduction.

A function $f : [a, b] \to \mathbb{R}$ is self-differential if $[a, b]$ can be subdivided into a finite number of sub-intervals, and on each sub-interval the derivative of f is equal to f by the graph transformed by an affine map. The case to be studied here is (1), where $[0, 1]$ is decomposed into $\left[0, \dfrac{1}{2}\right]$ and $\left[\dfrac{1}{2}, 1\right]$, and the affine transformed images of F are $aF(2x)$ and $aF(2-2x)$.

In [4] Fabius showed that the distribution function F_2 of the random variable $U = \sum_{n=1}^{\infty} 2^{-n} U_n$, where $U_1, U_2, \dots$ are independent random variables uniformly distributed on $[0, 1]$, is the solution of (1) for a value of the parameter $a = 2$ and boundary values $c = 0$ and $d = 1$. The function F_2 is infinitely

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differentiable and nowhere analytic on $[0,1]$. The derivatives of F_2 of order $2n$ are also solutions of (1) for $a = 2^{2n+1}$ and $c = d = 0$. In Theorem 2.1 (i), we show that (1) has a unique solution for all other values of a. We prove the existence of the solution using a modification of the method of the successive approximations on a sub-interval $\left[0, \dfrac{1}{4^n}\right]$ and extend it to $[0,1]$ with formulas (4.3). This method is used by Kato and McLeod [5] for the solution of the initial value functional differential equation $y(x) = \alpha y'(\lambda x) + \beta y(x)$, $y(0) = 1$. Similar initial value functional differential equations have been studied by De Bruijn [1]. A distinctive feature of the solution F of (1) is that if it is known on any sub-interval, then it can be extended to $[0,1]$ using only polynomials. This property of the solutions of boundary value self-differential equations is analogous to the notion of self-similarity for fractals (p. 135, Edgar [3]).

Definition 1.1. A differentiable function f is polynomially divided on the interval $[0,1]$ if for every $N \geq 0$ there exists an integer $n \geq N$ and polynomials $\{p_{n,i}(x)\}_{i=1}^{n-1}$ such that either $f\left(\dfrac{i}{n} + x\right) = f\left(\dfrac{i}{n} - x\right) + p_{n,i}(x)$ or $f\left(\dfrac{i}{n} + x\right) = f\left(\dfrac{i-1}{n} + x\right) + p_{n,i}(x)$ for $i = 1, 2, \ldots, n-1$ and $x \in \left[0, \dfrac{1}{n}\right]$.

This definition means that if $[0,1]$ is partitioned to n sub-intervals of equal length, the values of f on two neighboring intervals differ only by a polynomial. The solutions of (1) are polynomially divided by Lemma 3.1. In Section 2 we find a relation between the boundary values which allows us to decompose equations (1) to the simpler functional differential equations $(*)$ and $(**)$. The decomposition of the solutions is different, depending on whether or not a belongs to the set $S = \left\{2^{n+1} : n = 1, 2, 3, \ldots\right\}$. The main motivation to study self-differential equations is to generalize the exponential functions which have derivatives constant multiples of themselves. It is an interesting question to find self-differential equations which have practical applications.

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2 Basic Properties.

When $a = 0$, equations (1) have a solution $F(x) = c$ with boundary values $c = d$. For boundary conditions $c = d = 0$, equations (1) have a solution $F(x) = 0$ for all values of the parameter a. The Fabius function $dF_2(x)$ is a solution of (1) for $a = 2$, $c = 0$ and every value of $F(1) = d$. Other solutions are obtained from the derivatives of $F_2(x)$ of even order.

Lemma 2.1. *The function* $F_2^{(2n)}$ *is a solution of* (1) *with* $a = 2^{2n+1}$ *and boundary values* $c = d = 0$, *for all* $n = 1, 2, \ldots$

Proof. The Fabius function F_2 satisfies

$$\begin{cases} F_2'(x) = 2F_2(2x) & \text{if } 0 \le x \le \dfrac{1}{2}, \\ F_2'(x) = 2F_2(2-2x) & \text{if } \dfrac{1}{2} \le x \le 1. \end{cases}$$

By differentiating $2n$ times these equations, we obtain

$$\begin{cases} F_2^{(2n+1)}(x) = 2^{2n+1} F_2^{(2n)}(2x) & \text{if } 0 \le x \le \dfrac{1}{2}, \\ F_2^{(2n+1)}(x) = 2^{2n+1} F_2^{(2n)}(2-2x) & \text{if } \dfrac{1}{2} \le x \le 1. \end{cases}$$

From the first formula and $x = 0$ we obtain

$$F_2^{(2n+1)}(0) = F_2^{(2n)}(0) = \cdots = F_2'(0) = F_2(0) = 0,$$

and by the second formula and $x = 1$ we have that

$$F_2^{(2n+1)}(1) = 2^{2n+1} F_2^{(2n)}(0) = 0.$$

Therefore $F_2^{(2n)}$ satisfies (1) with parameter $a = 2^{2n+1}$ and boundary values $c = d = 0$. □

The graphs of the solutions of (1) for $a = 2, 8$ and 32 are given in Fig. 1. Every constant multiple function of $F_2^{(2n)}(x)$ is also a solution of (1). Therefore (1) has infinitely many solutions for every $a \in S$. In Section 3 we show that equations (1) have a unique solution for all other values of $a \notin S$. Now we want to find a relation between the boundary values and the parameter a. Suppose that the function $F(x)$ is a solution of the system of equations (1).

Proposition 2.1. *The solution* F *of* (1) *satisfies*

$$F\left(\frac{1}{2} + x\right) + F\left(\frac{1}{2} - x\right) = c + d \tag{2.1}$$

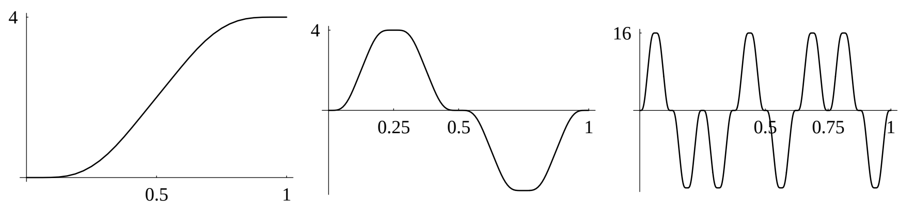

Fig. 1: The solutions $4F_2, F_2''$ and $F_2^{(4)}$ of (1) for $a = 2, 8$ and $a = 32$

PROOF.

$$\text{Let } G(x) = F\left(\frac{1}{2} + x\right) + F\left(\frac{1}{2} - x\right) \text{ for } 0 \le x \le \frac{1}{2}.$$
$$\text{Then } G'(x) = F'\left(\frac{1}{2} + x\right) + F'\left(\frac{1}{2} - x\right)$$
$$G'(x) = aF\left(2 - 2\left(\frac{1}{2} + x\right)\right) + aF\left(2\left(\frac{1}{2} - x\right)\right)$$
$$G'(x) = aF(1 - 2x) - aF(1 - 2x) = 0.$$

Therefore,

$$G(x) = G\left(\frac{1}{2}\right) = F(0) + F(1) = c + d.$$ □

From (2.1) and $x = \frac{1}{2}$, we obtain $F\left(\frac{1}{2}\right) = \frac{c+d}{2}$.

Lemma 2.2. *The boundary values of equations* (1) *satisfy* $d(2-a) = c(2+a)$.

PROOF. We evaluate $\int_0^1 F(x)\,dx$ in two ways:

$$\int_0^1 F(x)\,dx = \int_0^{\frac{1}{2}} F(x)\,dx + \int_{\frac{1}{2}}^1 F(x)\,dx = \int_0^{\frac{1}{2}} F(x)\,dx + \int_0^{\frac{1}{2}} F(1-u)\,du$$

and

$$\int_0^1 F(x)\,dx = \int_0^{\frac{1}{2}} [F(u) + F(1-u)]\,du.$$

By Proposition 2.1 we have that $F(u) + F(1-u) = c + d$. Then

$$\int_0^1 F(x)\,dx = \int_0^{\frac{1}{2}} [c+d]\,dx = \frac{c+d}{2}. \tag{2.2}$$

Let $x = 2y$.

$$\begin{aligned}\int_0^1 F(x)\,dx &= 2\int_0^{\frac{1}{2}} F(2y)\,dy = \frac{2}{a}\int_0^{\frac{1}{2}} F'(y)\,dy \\ &= \frac{2}{a}\left(F\left(\frac{1}{2}\right) - F(0)\right) = \frac{2}{a}\left(\frac{c+d}{2} - c\right) = \frac{d-c}{a}\end{aligned} \tag{2.3}$$

By (2.2) and (2.3) we obtain $\dfrac{d-c}{a} = \dfrac{c+d}{2}$. Therefore, $c(a+2) = d(2-a)$. □

In Lemma 2.2 we showed that if equations (1) have a solution, then the boundary values satisfy $c(a+2) = d(2-a)$. In this way we have the following four possibilities for the values of the parameters c and d depending on the values of a:

- $a \neq \pm 2$. The value of d is determined from c with $d = \dfrac{(a+2)c}{2-a}$.
- $a = 2$. Then $c = 0$ and d is an arbitrary real number.
- $a = -2$. Then $d = 0$ and c is an arbitrary real number. These relations of the boundary values lead to the following system of functional differential equations.

$$\begin{cases} f'(x) = af(2x) & \text{if } 0 \leq x \leq \dfrac{1}{2} \\ f'(x) = af(2-2x) & \text{if } \dfrac{1}{2} \leq x \leq 1 \\ f(0) = 2-a, f(1) = 2+a. \end{cases} \tag{$*$}$$

In Theorem 2.1 we show that $(*)$ has a unique solution when $a \notin S$.

- $c = d = 0$ and a is an arbitrary real number. With these values of the parameters equations (1) become

$$\begin{cases} f'(x) = af(2x) & \text{if } 0 \le x \le \dfrac{1}{2} \\ f'(x) = af(2-2x) & \text{if } \dfrac{1}{2} \le x \le 1 \\ f(0) = 0, f(1) = 0. \end{cases} \tag{$**$}$$

In Lemma 2.2 we have already found infinitely many nonzero solutions of $(**)$ for every $a \in S$. The solution of (1) is either a constant multiple of a solution of $(*)$ or it is a solution of $(**)$ if $c = d = 0$. These results are summarized in Theorem 2.1 and Corollary 2.1.

Theorem 2.1. *(i) Equations $(*)$ have a unique C^1 solution for all $a \notin S$ and have no solution if $a \in S$.*
(ii) Equations $(**)$ have infinitely many C^1 solutions $\left\{ rF_2^{(2n)} | r \in R \right\}$ for every $a = 2^{2n+1} \in S$ and have no nonzero solutions if $a \notin S$.

The proof of Theorem 2.1 is divided into lemmas. In Section 3 we show that a necessary condition for a solution of equations $(*)$ is that $a \notin S$ and we prove that all solutions of $(**)$ with parameter $a = 2^{2n+1}$ are constant multiples of $F_2^{(2n)}$. In Section 4 we construct the solution of $(*)$ using a modification of the method of the successive approximations. The solution of equations (1) is derived from the solution of $(*)$ in the following way.

Corollary 2.1. *Let $a \notin S$ and $f(x)$ be the unique solution of equations $(*)$. If $d(2-a) = c(a+2)$, then equations (1) have a unique C^1 solution $F(x)$ where $F(x) = \dfrac{cf(x)}{2-a}$ if $a \neq 2$. When $a = 2$ and $c = 0$ the solution is $F(x) = \dfrac{d}{4} f(x)$.*

3 Necessary Conditions.

In Section 2 we found infinitely many solutions of $(**)$ for every $a \in S$. In this Section we prove that these are the only solutions of $(**)$ and that $(*)$ has no solution if $a \in S$. Let $f(x)$ be a solution of $(*)$ or $(**)$ and denote $b_k = f\left(\dfrac{1}{2^k}\right)$. In Proposition 2.1 we showed that $f\left(\dfrac{1}{2} + x\right) + f\left(\dfrac{1}{2} - x\right) = c + d$. Now we show that similar property holds at each point $\dfrac{1}{2^n}$.

Lemma 3.1. *The function f satisfies*

$$f\left(\frac{1}{2^n} + x\right) + (-1)^{n+1} f\left(\frac{1}{2^n} - x\right) = p_n(x) \tag{3.1}$$

for $0 \le x \le \dfrac{1}{2^n}$*. The polynomials* p_n *are defined recursively by*

$$\begin{cases} p'_{2l-1}(x) = ap_{2l-2}(2x) \\ p_{2l-1}(0) = 2b_{2l-1} \end{cases} \tag{3.2}$$

and

$$\begin{cases} p'_{2l}(x) = ap_{2l-1}(2x) \\ p_{2l}(0) = 0 \end{cases} \tag{3.3}$$

where

$$p_1(x) = \begin{cases} 4 & \text{if } f \text{ is a solution of } (*) \\ 0 & \text{if } f \text{ is a solution of } (**). \end{cases}$$

Proof. We prove Lemma 3.1 by induction on n. When $n = 1$ equation (3.1) is satisfied by Proposition 2.1. Suppose that

$$f\left(\frac{1}{2^n} + x\right) + (-1)^{n+1} f\left(\frac{1}{2^n} - x\right) = p_n(x).$$

Put $x = 2t$ to obtain

$$f\left(\frac{1}{2^n} + 2t\right) + (-1)^{n+1} f\left(\frac{1}{2^n} - 2t\right) = p_n(2t).$$

From the second equations of $(*)$ and $(**)$ with $x = \dfrac{1}{2^n} + 2t$ and $x = \dfrac{1}{2^n} - 2t$ we have that

$$f\left(\frac{1}{2^n} - 2t\right) = \frac{1}{a} f'\left(\frac{1}{2^{n+1}} - t\right)$$

and

$$f\left(\frac{1}{2^n} + 2t\right) = \frac{1}{a} f'\left(\frac{1}{2^{n+1}} + t\right).$$

Then

$$f'\left(\frac{1}{2^{n+1}} + t\right) + (-1)^{n+1} f'\left(\frac{1}{2^{n+1}} - t\right) = ap_n(2t).$$

By integrating the above equation from 0 to x we obtain

$$\int_0^x \left[f'\left(\frac{1}{2^{n+1}} + t\right) + (-1)^{n+1} f'\left(\frac{1}{2^{n+1}} - t\right)\right] dt = \int_0^x ap_n(2t)\, dt$$

$$\left[f\left(\frac{1}{2^{n+1}} + t\right) + (-1)^{n+2} f\left(\frac{1}{2^{n+1}} - t\right)\right]_0^x = a\int_0^x p_n(2t)\, dt$$

$$f\left(\frac{1}{2^{n+1}} + x\right) + (-1)^{n+2} f\left(\frac{1}{2^{n+1}} - x\right) - \gamma f\left(\frac{1}{2^{n+1}}\right) = a\int_0^x p_n(2t)\, dt$$

where $\gamma = 1 + (-1)^{n+2}$.

Case1 n is even. Then $\gamma = 0$, and when $n = 2l$, we have that

$$f\left(\frac{1}{2^{2l+1}} + x\right) + f\left(\frac{1}{2^{2l+1}} - x\right) - 2f\left(\frac{1}{2^{2l+1}}\right) = a\int_0^x p_{2l}(2t)\,dt$$
$$f\left(\frac{1}{2^{2l+1}} + x\right) + f\left(\frac{1}{2^{2l+1}} - x\right) = 2b_{2l+1} + a\int_0^x p_{2l}(2t)\,dt.$$

Therefore, f satisfies (3.1) with $n = 2l + 1$ and

$$p_{2l+1}(x) = 2b_{2l+1} + a\int_0^x p_{2l}(2t)\,dt. \tag{3.4}$$

Case 2 n is odd. Then $\gamma = 0$, and when $n = 2l - 1$ we have that

$$\begin{aligned} f\left(\frac{1}{2^{2l}} + x\right) - f\left(\frac{1}{2^{2l}} - x\right) &= a\int_0^x p_{2l-1}(2t)\,dt \\ p_{2l}(x) &= a\int_0^x p_{2l-1}(2t)\,dt. \end{aligned} \tag{3.5}$$

We obtain equations (3.2) and (3.3) from (3.4) and (3.5) by differentiation with respect to x. □

The solution f of (1) is polynomially divided because it satisfies (3.1). If f is known on the interval $\left[0, \frac{1}{2^m}\right]$, then we can extend it to $\left[0, \frac{1}{2^{m-1}}\right]$ with formula (3.1) and $n = m$. By using the same procedure $m - 1$ times with $n = m-1, m-2, \ldots, 1$ we can reconstruct f on the interval $[0, 1]$. Even more, if the function f is known on an arbitrary sub-interval (a, b) of $[0, 1]$ where $a < \frac{s}{2^m} < \frac{s+1}{2^m} < b$, for some integers s and m, then using the reverse procedure we can find the values of f on $\left[0, \frac{1}{2^m}\right]$ and then extend f to the interval $[0, 1]$. In this way, we can reconstruct the solution of (1) from any sub-interval of $[0, 1]$ using only the polynomials p_n. Now we express the coefficients of $p_{2n}(x)$ with $b_1, b_3, \ldots, b_{2n-1}$. From (3.1) and $x = \frac{1}{2^n}$ we can find the values of p_n and $x = \frac{1}{2^n}$.

Corollary 3.1. *Suppose that f is a solution of $(*)$. Then*

$$p_{2n}\left(\frac{1}{2^{2n}}\right) = b_{2n-1} - 2 + a \text{ and } p_{2n+1}\left(\frac{1}{2^{2n+1}}\right) = b_{2n} + 2 - a.$$

Corollary 3.2. *Suppose that f is a solution of $(**)$. Then*

$$p_n\left(\frac{1}{2^n}\right) = b_{n-1}.$$

In the next lemma we find a formula for the polynomials p_{2n}.

Lemma 3.2. *The polynomials p_n with even index are given by*

$$p_{2n}(x) = \sum_{k=1}^{n} \frac{a^{2k-1}2^{2k^2-3k+2}b_{2n-2k+1}x^{2k-1}}{(2k-1)!}. \tag{3.6}$$

PROOF. By Lemma 3.1, p_{2n} is a polynomial of degree $2n-1$. The coefficients may be computed by Taylor's formula. Compute successive derivatives using Lemma 3.1:

$$\begin{aligned}
p'_{2n}(x) &= ap_{2n-1}(2x)\\
p''_{2n}(x) &= 2ap'_{2n-1}(2x) = 2a^2p_{2n-2}(4x)\\
p'''_{2n}(x) &= 8a^2p'_{2n-2}(4x) = 22^2a^3p_{2n-3}(8x) = 2^3a^3p_{2n-3}(8x)\\
p^{(4)}_{2n}(x) &= 64a^3p'_{2n-3}(8x) = 22^22^3a^4p_{2n-4}(16x) = 2^6a^4p_{2n-4}(16x),
\end{aligned}$$

and in general, by induction

$$p^{(l)}_{2n}(x) = 2^{1+2+3+\cdots+(l-1)}a^lp_{2n-l}(2^lx) = 2^{\frac{l(l-1)}{2}}a^lp_{2n-l}(2^lx).$$

But $p_{2n-2k}(0) = 0$ and $p_{2n-2k+1}(0) = 2b_{2n-2k+1}$; so

$$p_{2n}(x) = \sum_{k=1}^{2n-1} \frac{p^{(l)}_{2n}(0)x^l}{l!} = \sum_{k=1}^{n} \frac{a^{2k-1}2^{2k^2-3k+2}b_{2n-2k+1}x^{2k-1}}{(2k-1)!}. \qquad \square$$

In Lemma 2.1, we found infinitely many solutions

$$f(x) = rF_2^{(2n)}(x)$$

of equations $(**)$ for $a = 2^{2n+1} \in S$ and $r \in \mathbb{R}$. In the next corollary, we show that $a \in S$ is a necessary condition for a solution of $(**)$.

Corollary 3.3. *Let f be a nonzero solution of $(**)$ such that* $b_1 = \cdots = b_{2n-1} = 0$ *and* $b_{2n+1} \neq 0$. *Then* $a = 2^{2n+1}$.

PROOF. By Lemma 3.2,

$$p_{2n+2}(x) = \sum_{k=1}^{n+1} \frac{a^{2k-1} 2^{2k^2-3k+2} b_{2n-2k+3} x^{2k-1}}{(2k-1)!} = 2ab_{2n+1}x.$$

Then

$$f\left(\frac{1}{2^{2n+2}} + x\right) - f\left(\frac{1}{2^{2n+2}} - x\right) = 2ab_{2n+1}x.$$

Put $x = \dfrac{1}{2^{2n+2}}$ to obtain

$$f\left(\frac{1}{2^{2n+1}}\right) - f(0) = \frac{ab_{2n+1}}{2^{2n+1}} \text{ and } b_{2n+1} = \frac{ab_{2n+1}}{2^{2n+1}}.$$

Therefore, $a = 2^{2n+1}$ as required. □

Now we use Lemma 3.2 to find a formula which relates the numbers $b_1, b_3, \ldots, b_{2n-1}$.

Lemma 3.3. *The numbers* $\{b_{2n-1}\}_{n=0}^{\infty}$ *satisfy*

$$b_{2n-1}\left(2^{2n-1} - a\right) = 2^{2n-1}\left(2 - a + \sum_{k=2}^{n} \frac{a^{2k-1} 2^{2k^2-3k+2-2n(2k-1)} b_{2n-2k+1}}{(2k-1)!}\right). \tag{3.7}$$

PROOF. By Corollary 3.1,

$$p_{2n}\left(\frac{1}{2^{2n}}\right) = b_{2n-1} - 2 + a.$$

From (3.6) and $x = \dfrac{1}{2^{2n}}$ we have that

$$p_{2n}\left(\frac{1}{2^{2n}}\right) = \sum_{k=1}^{n} \frac{a^{2k-1} 2^{2k^2-3k+2-2n(2k-1)} b_{2n-2k+1}}{(2k-1)!}.$$

Then

$$b_{2n-1}\left(1 - \frac{a}{2^{2n-1}}\right) = 2 - a + \sum_{k=2}^{n} \frac{a^{2k-1} 2^{2k^2-3k+2-2n(2k-1)} b_{2n-2k+1}}{(2k-1)!}. \qquad \square$$

So far we have found formulas to compute the polynomials $p_{2n}(x)$ and the numbers b_{2n-1}.

Corollary 3.4. *Let f be a solution of $(*)$ or $(**)$. The numbers b_{2n} and the coefficients of p_{2n+1} are obtained from $\{b_{2k-1}\}_{k=1}^{n}$ by*

$$p_{2n+1}(x) = \sum_{k=0}^{n} \frac{a^{2k} 2^{2k^2-k+1} b_{2n-2k+1} x^{2k}}{(2k)!}$$

and

$$b_{2n} = \begin{cases} \sum_{k=0}^{n} \dfrac{a^{2k} 2^{2k^2-k+1-2k(2n+1)} b_{2n-2k+1}}{(2k)!} + a - 2 & \text{if } f \text{ satisfies } (*) \\ \sum_{k=0}^{n} \dfrac{a^{2k} 2^{2k^2-k+1-2k(2n+1)} b_{2n-2k+1}}{(2k)!} & \text{if } f \text{ satisfies } (**) \end{cases} \tag{3.8}$$

PROOF. We have that

$$p_{2n+1}(x) = 2b_{2n+1} + a \int_0^x p_{2n}(2t)\, dt.$$

From (3.6) and $x = 2t$,

$$p_{2n}(2t) = \sum_{k=1}^{n} \frac{a^{2k-1} 2^{2k^2-k+1} b_{2n-2k+1} t^{2k-1}}{(2k-1)!}.$$

Then

$$\begin{aligned} p_{2n+1}(x) &= 2b_{2n+1} + a \sum_{k=1}^{n} \int_0^x \frac{a^{2k-1} 2^{2k^2-k+1} b_{2n-2k+1} t^{2k-1}}{(2k-1)!}\, dt \\ &= \sum_{k=0}^{n} \frac{a^{2k} 2^{2k^2-k+1} b_{2n-2k+1} x^{2k}}{(2k)!}. \end{aligned}$$

If f is a solution of $(*)$, then $p_{2n+1}\left(\frac{1}{2^{2n+1}}\right) = b_{2n} + 2 - a$ and

$$b_{2n} = \sum_{k=0}^{n} \frac{a^{2k} 2^{2k^2-k+1-2k(2n+1)} b_{2n-2k+1}}{(2k)!} + a - 2.$$

If f is a solution of $(**)$, then $p_{2n+1}\left(\frac{1}{2^{2n+1}}\right) = b_{2n}$ and

$$b_{2n} = \sum_{k=0}^{n} \frac{a^{2k} 2^{2k^2-k+1-2k(2n+1)} b_{2n-2k+1}}{(2k)!}. \qquad \square$$

Remark 3.1. If $a \notin S$ and f is a solution of $(*)$, then the values $b_n = f\left(\frac{1}{2^n}\right)$ are computed with formulas (3.7) and (3.8). The values of f on the set

$$D = \left\{ \frac{k}{2^n} | k = 1, \ldots, 2^n, n \in N \right\}$$

are computed from $\{b_n\}$ with formula (3.1). The set D is dense in $[0,1]$ and the values of f are determined for every $x \in [0,1]$ from the values of f on D with $f(x) = \lim_{n\to\infty} f(d_n)$ where $d_n \in D$ and $\lim_{n\to\infty} d_n = x$. Therefore, $(*)$ has at most one solution for every $a \notin S$. We show that $(*)$ has a unique solution for all $a \notin S$ in Section 4.

Remark 3.2. If $a = 2^{2n+1}$ and f is a solution of $(**)$, then $b_1 = 0$ and $b_3, \ldots, b_{2n-1}$ are computed with (3.7), but the value of b_{2n+1} cannot be computed with (3.7). If we choose b_{2n+1} to be an arbitrary number, then $b_{2n+3}, b_{2n+5}, \ldots$ may also be computed with (3.7). The values of b_{2n} may be computed from the values of b_{2n-1} with formula (3.8). Similarly to Remark 3.1, the values of f on D may be computed with (3.1). Therefore, $(**)$ has at most one solution for every choice of b_{2n+1}. This solution is $\dfrac{b_{2n+1} F_2^{(2n)}(x)}{F_2^{(2n)}\left(\frac{1}{2^{2n+1}}\right)}$. Therefore, all solutions of $(**)$ are $\left\{ r F_2^{(2n)}(x) | r \in R \right\}$.

The graphs of f obtained by calculating the values of f on D for $a = -32, -4, 7.9, 8.1, 16, 50$ are given on Fig. 2. Now we show that $(*)$ has no solution when $a \in S$. Let $a = 2^{2m-1}$. With this value of a, equation (3.7)

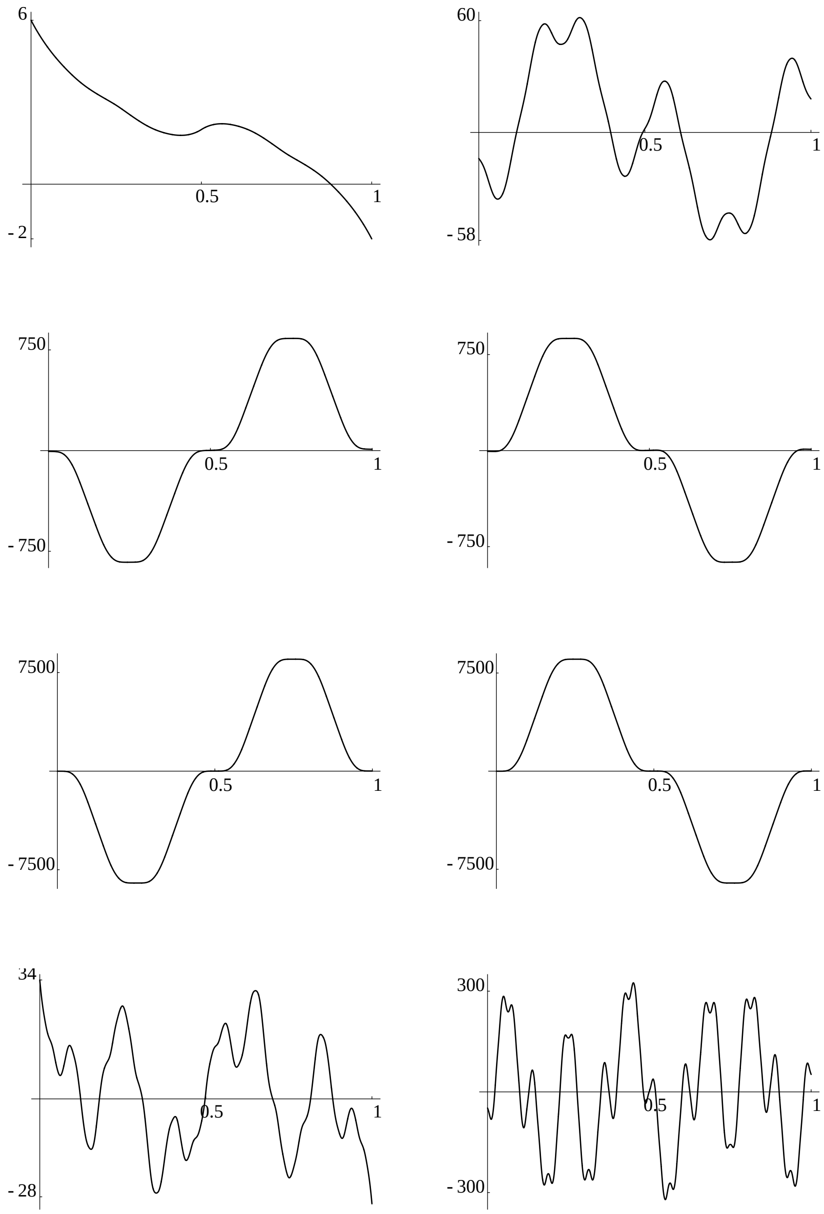

Fig. 2: The solutions of equations $(*)$ for $a = -4, 16$ (top), $a = 7.9, 8.1$ and $a = 7.99, 8.01$ (middle) and $a = -32, 50$ (bottom)

becomes

$$b_{2n-1}\left(2^{2n-1}-2^{2m-1}\right)=2^{2n-1}\Bigg(2-2^{2m-1}+$$
$$\sum_{k=2}^{n}\frac{2^{(2m-1)(2k-1)}2^{2k^2-3k+2-2n(2k-1)}b_{2n-2k+1}}{(2k-1)!}\Bigg)$$
$$b_{2n-1}\left(1-2^{2(m-n)}\right)=2-2^{2m-1}+ \tag{3.9}$$
$$\sum_{k=2}^{n}\frac{2^{2k^2-3k+2+(2m-2n-1)(2k-1)}b_{2n-2k+1}}{(2k-1)!}$$

$$b_{2n-1}=\frac{1}{1-4^{m-n}}\left(2\left(1-4^{m-1}\right)+\sum_{k=2}^{n}\frac{2^{2k^2-3k+2+(2m-2n-1)(2k-1)}b_{2n-2k+1}}{(2k-1)!}\right).$$

The numbers $b_1=2,b_3,\dots,b_{2m-3}$ are computed with the above formula. When $n=m$, formula (3.9) becomes

$$0=2(1-4^{m-1})+\sum_{k=2}^{m}\frac{2^{2k^2-5k+3}b_{2m-2k+1}}{(2k-1)!}$$
$$4^{m-1}-1=\sum_{k=2}^{m}\frac{2^{2k^2-5k+2}b_{2m-2k+1}}{(2k-1)!}. \tag{3.10}$$

Formula (3.10) gives a relation between $b_1,\dots,b_{2m-3}$ and is a necessary condition for existence of a solution of equations $(*)$ when $a=2^{2^{2m-1}}$. In Lemma 3.5 we show that (3.10) is not satisfied. In the proof of Lemma 3.5 we use Proposition 3.1 and Lemma 3.4.

Proposition 3.1. *Let e_k be the power of 3 in $(2k-1)!$. Then $e_k\le k-1$.*

PROOF.

$$e_k=\sum_{3^r\le 2k-1}\left\lfloor\frac{2k-1}{3^r}\right\rfloor\le\sum_{r=1}^{r_0}\frac{2k-1}{3^r}$$

where $r_0=\lfloor\log_3(2k-1)\rfloor$.

$$e_k\le\frac{2k-1}{3}\frac{1-\frac{1}{3^{r_0}}}{1-\frac{1}{3}}=\frac{2k-1}{2}\left(1-\frac{1}{3^{r_0}}\right)<\frac{2k-1}{2}<k.$$

Therefore, $e_k\le k-1$. □

The numbers $b_1, b_3, \ldots, b_{2m-3}$ are rational numbers in lowest terms. Let d_{2n-1} be the power of 3 in the denominator of b_{2n-1}. The sequence $\{d_{2n-1}\}_{n=1}^{m-1}$ is increasing and satisfies the following inequality.

Lemma 3.4.

$$d_{2n-1} - d_{2n-3} \geq 2 \tag{3.11}$$

PROOF. We prove (3.11) by induction on n. $b_1 = 2$ and $d_1 = 0$. By (3.7),

$$\begin{aligned} b_3 &= \frac{1}{1-4^{m-2}} \left(2(1-4^{m-1}) + \frac{2^{8-6+2+1+(2m-5)3}}{3!} \right) \\ &= \frac{6(1-4^{m-1}) + 2^{6m-11}}{3\,(1-4^{m-2})}. \end{aligned}$$

Let's denote by $d(k)$ the power of 3 in $4^k - 1$. Then $d(k) \geq 1$ because

$$4^k - 1 = 3(4^{k-1} + 4^{k-2} + \cdots + 1).$$

The numerator of b_3 is not divisible by 3. Therefore $d_3 = d(m-2) + 1 \geq 2$ and (3.11) is satisfied for $n = 2$. Suppose that $d_{2k-1} - d_{2k-3} \geq 2$ for every $k = 2, 3, \ldots, n-1$.

$$\begin{aligned} b_{2n-1} = \frac{1}{1-4^{m-n}} \Bigg(& 2(1-4^{m-1}) + \frac{64^{m-n} b_{2n-3}}{3} + \\ & \sum_{k=3}^{n} 2^{2k^2-3k+2+(2m-2n-1)(2k-1)} \frac{b_{2n-2k+1}}{(2k-1)!} \Bigg). \end{aligned}$$

By the induction assumption,

$$d_{2n-3} - d_{2n-2k+1} = \sum_{l=2}^{k-1} [d_{2n-2l+1} - d_{2n-2l-1}] \geq 2(k-2) = 2k-4$$

$$d_{2n-2k+1} + 2k - 4 \leq d_{2n-3}.$$

The power of 3 in the denominator of $\dfrac{b_{2n-2k+1}}{(2k-1)!}$ is $d_{2n-2k+1} + e_k$. From Proposition 3.1, we have that

$$d_{2n-2k+1} + e_k \leq d_{2n-2k+1} + k - 1 \leq d_{2n-2k+1} + 2k - 4 \leq d_{2n-3} < d_{2n-3} + 1$$

for $3 \leq k \leq n$. The power of 3 in the denominator of $\dfrac{b_{2n-3}}{3!}$ is $d_{2n-3}+1$. Therefore, the power of 3 in the denominator of $\dfrac{b_{2n-3}}{3!}$ is greater than the power of 3 in the denominator of $\dfrac{b_{2n-2k+1}}{(2k-1)!}$ for every $k=3,4,\dots,n$. Then the power of 3 in the denominator of

$$\frac{64^{m-n}b_{2n-3}}{3}+\sum_{k=2}^{n}2^{2k^2-3k+2-(2m-2n-1)(2k-1)}\frac{b_{2n-2k+1}}{(2k-1)!}$$

is exactly $d_{2n-3}+1$, and so the power of 3 in the denominator of

$$\frac{1}{1-4^{m-n}}\left(\frac{64^{m-n}b_{2n-3}}{3}+\sum_{k=3}^{n}2^{2k^2-3k+2-(2m-2n-1)(2k-1)}\frac{b_{2n-2k+1}}{(2k-1)!}\right)$$

is equal to $d_{2n-3}+d(m-n)+1$. Then

$$d_{2n-1}=d_{2n-3}+d(m-n)+1 \geq d_{2n-3}+2. \qquad \square$$

In Lemma 3.5, we prove that the necessary condition (3.10) for a solution of $(*)$ is not satisfied.

Lemma 3.5.

$$4^{m-1}-1 \neq \sum_{k=2}^{n}\frac{2^{2k^2-5k+2}b_{2m-2k+1}}{(2k-1)!}.$$

PROOF. We want to show that the denominator of

$$\sum_{k=2}^{n}2^{2k^2-5k+2}\frac{b_{2m-2k+1}}{(2k-1)!}$$

is divisible by 3.

$$\sum_{k=2}^{n}2^{2k^2-5k+2}\frac{b_{2m-2k+1}}{(2k-1)!}=\frac{b_{2m-3}}{3!}+\sum_{k=3}^{n}2^{2k^2-5k+2}\frac{b_{2m-2k+1}}{(2k-1)!}$$

The power of 3 in the denominator of

$$2^{2k^2-5k+2}\frac{b_{2m-2k+1}}{(2k-1)!}$$

is $d_{2m-2k+1} + e_k$. By Proposition 3.1 and Lemma 3.4, we have that $d_{2m-2k+1} + e_k \le d_{2m-2k+1} + k - 1 \le d_{2m-2k+1} + 2k - 4 \le d_{2m-3} < d_{2m-3} + 1$ for $3 \le k \le n$. The power of 3 in the denominator of $\dfrac{b_{2m-3}}{3!}$ is $d_{2m-3} + 1$. Therefore, the power of 3 in the denominator of

$$\sum_{k=2}^{n} 2^{2k^2-5k+2} \frac{b_{2m-2k+1}}{(2k-1)!}$$

is $d_{2m-3} + 1$. Then the denominator of

$$\sum_{k=2}^{n} 2^{2k^2-5k+2} \frac{b_{2m-2k+1}}{(2k-1)!}$$

is divisible by 3. Hence,

$$4^{m-1} - 1 \ne \sum_{k=2}^{n} \frac{2^{2k^2-5k+2} b_{2m-2k+1}}{(2k-1)!}$$

as required. □

From Lemma 3.5 it follows that equations $(*)$ have no solution when $a \in S$ because the necessary condition (3.10) is not satisfied.

4 Successive Approximations to the Solution.

In Section 2 and Section 3, we proved part (ii) of Theorem 2.1. We also showed that if $a \in S$, equations $(*)$ have no solution. In Section 4, we show that equations $(*)$ have a solution for every $a \notin S$. According to the following remark, this solution is unique. Remark 3.1. Let f be a solution of $(*)$ and $x \in \left[0, \frac{1}{2}\right]$. Then $f(x) = f(0) + \int_0^x f'(t)\,dt = 2 - a + \int_0^x af(2t)\,dt = 2 - a + \frac{a}{2}\int_0^{2x} f(t)\,dt$. Let n be the smallest integer such that $2\,|a| < 4^n$. Now, we use the above equation to define a sequence of functions $\{h_k\}_{k=0}^{\infty}$ which approximates f on the interval $\left[0, \frac{1}{4^{n-1}}\right]$ by

$$\begin{cases} h_0(x) &= x \\ h_k(x) &= 2 - a + \dfrac{a}{2}\displaystyle\int_0^{2x} h_{k-1}(t)\,dt \text{ for } 0 \le x \le \dfrac{2}{4^n} \\ h_k\left(\dfrac{2}{4^n} + x\right) &= p_{2n-1}(x) - h_k\left(\dfrac{2}{4^n} - x\right) \text{ for } 0 < x \le \dfrac{2}{4^n}. \end{cases} \tag{4.1}$$

Proposition 4.1. *The functions h_k are continuous for all $k \geq 2$.*

PROOF. The functions h_k are continuous on the intervals $\left[0, \frac{2}{4^n}\right]$ and $\left(\frac{2}{4^n}, \frac{1}{4^{n-1}}\right]$. Now, we show that h_k is continuous at $\frac{2}{4^n}$. From the first equation of (4.1) and $x = \frac{2}{4^n}$, we have that

$$\begin{aligned}
h_k\left(\frac{2}{4^n}\right) &= 2 - a + \frac{a}{2}\int_0^{\frac{1}{4^{n-1}}} h_{k-1}(t)\,dt \\
&= 2 - a + \frac{a}{2}\int_0^{\frac{2}{4^n}} h_{k-1}(t)\,dt + \frac{a}{2}\int_{\frac{2}{4^n}}^{\frac{1}{4^{n-1}}} h_{k-1}(t)\,dt \\
&= 2 - a + \frac{a}{2}\int_0^{\frac{2}{4^n}} h_{k-1}(t)\,dt + \frac{a}{2}\int_0^{\frac{2}{4^n}} h_{k-1}\left(\frac{2}{4^n} + u\right)\,du \\
&= 2 - a + \frac{a}{2}\int_0^{\frac{2}{4^n}} h_{k-1}(t)\,dt + \frac{a}{2}\int_0^{\frac{2}{4^n}} \left[p_{2n-1}(u) - h_{k-1}\left(\frac{2}{4^n} - u\right)\right]\,du \\
h_k\left(\frac{2}{4^n}\right) &= 2 - a + \frac{a}{2}\int_0^{\frac{2}{4^n}} p_{2n-1}(u)\,du.
\end{aligned}$$

From (3.2), we have that $ap_{2n-1}(u) = p'_{2n}\left(\frac{u}{2}\right)$. Then

$$\begin{aligned}
h_k\left(\frac{2}{4^n}\right) &= 2 - a + \frac{1}{2}\int_0^{\frac{2}{4^n}} p'_{2n}\left(\frac{u}{2}\right)\,du = 2 - a + \int_0^{\frac{1}{4^n}} p'_{2n}(u)\,du \\
h_k\left(\frac{2}{4^n}\right) &= 2 - a + p_{2n}\left(\frac{1}{4^n}\right) - p_{2n}(0) = 2 - a + b_{2n-1} - 2 + a = b_{2n-1}.
\end{aligned}$$

From the second equation of (4.1), we obtain

$$\begin{aligned}
h_k\left(\frac{2}{4^n}+\right) &= \lim_{x\to 0^+} f\left(\frac{2}{4^n} + x\right) = \lim_{x\to 0^+}\left[p_{2n-1}(x) - h_k\left(\frac{2}{4^n} - x\right)\right] \\
h_k\left(\frac{2}{4^n}+\right) &= p_{2n-1}(0) - h_k\left(\frac{2}{4^n}-\right) = 2b_{2n-1} - b_{2n-1} = b_{2n-1}.
\end{aligned}$$

Therefore,

$$h_k\left(\frac{2}{4^n}\right) = h_k\left(\frac{2}{4^n}+\right) = b_{n-1}.$$

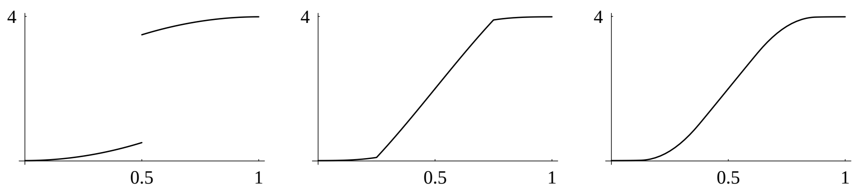

Fig. 3: The approximations h_1, h_2 and h_3 to the solution of $(*)$ for $a = 1.99$

The function h_k is right continuous at $x = \frac{2}{4^n}$ and so it is continuous on the interval $\left[0, \frac{1}{4^{n-1}}\right]$. □

When $a = 1.99$, then $n = 1$, because $2|a| < 4$. The sequence of functions $\{h_k\}_{k=0}^{\infty}$ approximates the solution of $(*)$ on the interval $[0,1]$. The graphs of the first three approximations h_1, h_2 and h_3 are given on Fig. 3. The solution of equations $(*)$ for $a = 2$ is $4F_2$ (Fig. 2). The solutions of $(*)$ for $a = 1.99$ and $a = 2$ differ by less than 0.015. Now we define a system of functional differential equations on the interval $\left[0, \frac{1}{4^{k-1}}\right]$. Let k be a positive integer, and denote by $Eqns[k]$ the following functional differential equations.

$$\begin{cases} f'(x) & = af(2x) \text{ for } 0 \le x \le \frac{2}{4^k} \\ f\left(\frac{2}{4^k} + x\right) & = p_{2k-1}(x) - f\left(\frac{2}{4^k} - x\right) \text{ for } 0 < x \le \frac{2}{4^k} \\ f(0) & = 2 - a \end{cases} \qquad (Eqns[k])$$

Proposition 4.2. *Let f be a solution of $Eqns[k]$. Then*

$$f\left(\frac{1}{4^{k-1}}\right) = b_{2k-2} \quad \text{and} \quad f'\left(\frac{1}{4^{k-1}}\right) = ab_{2k-3}.$$

PROOF. From the second equation of $Eqns[k]$ and $x = \frac{2}{4^k}$, we have that

$$f\left(\frac{1}{4^{k-1}}\right) = p_{2k-1}\left(\frac{2}{4^k}\right) - f(0) = p_{2k-1}\left(\frac{2}{4^k}\right) - 2 + a = b_{2k-2}.$$

By differentiating the second equation of $Eqns[k]$, we obtain

$$f'\left(\frac{2}{4^k}+x\right)=p'_{2k-1}(x)+f'\left(\frac{2}{4^k}-x\right).$$

Put $x=\dfrac{2}{4^k}$ in the above equation,

$$f'\left(\frac{1}{4^{k-1}}\right)=p'_{2k-1}\left(\frac{2}{4^k}\right)+f'(0)=ap_{2k-2}\left(\frac{1}{4^{k-1}}\right)+af(0)$$
$$f'\left(\frac{1}{4^{k-1}}\right)=a(b_{2k-3}-2+a)+a(2-a)=ab_{2k-3}. \qquad \square$$

By Proposition 4.2, equations $(*)$ are the same as $Eqns[1]$.

Lemma 4.1. *(i) The sequence of functions $h_k(x)$ converges uniformly.*
(ii) The limit function $f_n(x)=\lim_{k\to\infty}h_k(x)$ is continuously differentiable and satisfies $Eqns[n]$.

PROOF. *(i)* Let $M_k=\sup_{x\in[0,\frac{1}{4^{n-1}}]}|h_k(x)-h_{k-1}(x)|$. From the second equation of (4.1), we have that

$$M_k=\sup_{x\in[0,\frac{2}{4^n}]}|h_k(x)-h_{k-1}(x)|.$$

From the first equation of (4.1),

$$h_k(x)-h_{k-1}(x)=\frac{a}{2}\int_0^{2x}h_{k-1}(u)\,du-\frac{a}{2}\int_0^{2x}h_{k-2}(u)\,du$$
$$|h_k(x)-h_{k-1}(x)|\le\frac{a}{2}\int_0^{2x}|h_{k-1}(u)-h_{k-2}(u)|\,du.$$

Therefore,

$$M_k\le\frac{a}{2}\int_0^{2x}M_{k-1}\,du=\frac{|a|}{2}2xM_{k-1}\le\frac{2\,|a|}{4^n}M_{k-1}.$$

Let $t = \dfrac{2\,|a|}{4^n}$. Then $0 < t < 1$, because n is chosen such that $2\,|a| < 4^n$. Therefore, $M_k \leq tM_{k-1}$. By induction, we obtain $M_k \leq t^{k-1}M_1$. Then

$$\begin{aligned} \sup_{x\in[0,\frac{1}{4^{n-1}}]} |h_s(x) - h_r(x)| &\leq \sum_{k=r+1}^{s} \sup_{x\in[0,\frac{1}{4^{n-1}}]} |h_k(x) - h_{k-1}(x)| \\ &\leq \sum_{k=r+1}^{s} M_k \leq M_1 \sum_{k=r+1}^{s} t^{k-1} \leq M_1 \sum_{k=r}^{\infty} t^k \leq \frac{M_1 t^r}{1-t} \end{aligned}$$

for $s > r$. Therefore, h_k is a Cauchy sequence, and so, it converges uniformly.

(ii) The function f_n is continuous because it is a uniform limit of continuous functions. In the proof of Proposition 4.1, we showed that $h_k\left(\dfrac{2}{4^n}\right) = b_{2n-1}$ for $k \geq 2$. Then $f_n\left(\dfrac{2}{4^n}\right) = b_{2n-1}$. By letting $k \to \infty$ in (4.1), we obtain

$$\begin{cases} f_n(x) = 2 - a + \dfrac{a}{2}\displaystyle\int_0^{2x} f_n(t)\,dt & \text{for } 0 \leq x \leq \dfrac{2}{4^n} \\ f_n\left(\dfrac{2}{4^n} + x\right) = p_{2n-1}(x) - f_n\left(\dfrac{2}{4^n} - x\right) & \text{for } 0 < x \leq \dfrac{2}{4^n}. \end{cases} \tag{4.2}$$

Form the first and second equations of (4.2), the function f_n is continuously differentiable on the intervals $\left[0, \dfrac{2}{4^n}\right]$ and $\left(\dfrac{2}{4^n}, \dfrac{1}{4^{n-1}}\right]$. Now, we show that f_n is differentiable at $x = \dfrac{2}{4^n}$. Let $f'(x-)$ and $f'(x+)$ denote

$$f'(x+) = \lim_{t\to x^+} \frac{f(t) - f(x)}{t - x} \text{ and } f'(x-) = \lim_{t\to x^-} \frac{f(t) - f(x)}{t - x}.$$

Then

$$\begin{aligned} f_n'\left(\frac{2}{4^n}+\right) &= \lim_{x\to 0^+} \frac{f_n\left(\frac{2}{4^n} + x\right) - f_n\left(\frac{2}{4^n}\right)}{x} \\ &= \lim_{x\to 0^+} \frac{p_{2n-1}(x) - f_n\left(\frac{2}{4^n} - x\right) - f_n\left(\frac{2}{4^n}\right)}{x} \\ &= \lim_{x\to 0^+} \frac{p_{2n-1}(x) - p_{2n-1}(0) - f_n\left(\frac{2}{4^n} - x\right) + f_n\left(\frac{2}{4^n}\right)}{x} \end{aligned}$$

$\left(\text{because } p_{2n-1}(0) - f_n\left(\frac{2}{4^n}\right) = 2b_{2n-1} - b_{2n-1} = f_n\left(\frac{2}{4^n}\right)\right)$

$$\begin{aligned}
&= \lim_{x\to 0^+} \frac{p_{2n-1}(x) - p_{2n-1}(0)}{x} + \lim_{x\to 0^+} \frac{f_n\left(\frac{2}{4^n} - x\right) - f_n\left(\frac{2}{4^n}\right)}{-x} \\
&= p'_{2n-1}(0) + f'_n\left(\frac{2}{4^n}-\right) = ap_{2n-2}(0) + f'_n\left(\frac{2}{4^n}-\right) \\
&= f'_n\left(\frac{2}{4^n}-\right)
\end{aligned}$$

Therefore, f_n is differentiable at $x = \frac{2}{4^n}$. From the second equation of (4.3), we have that

$$f'_n\left(\frac{2}{4^n} + x\right) = p'_{2n-1}(x) + f'_n\left(\frac{2}{4^n} - x\right).$$

Then

$$\lim_{x\to 0^+} f'_n\left(\frac{2}{4^n} + x\right) = p'_{2n-1}(0) + \lim_{x\to 0^+} f'_n\left(\frac{2}{4^n} - x\right) = f'_n\left(\frac{2}{4^n}-\right).$$

Therefore, f_n is continuously differentiable at $x = \frac{2}{4^n}$, and so f_n is continuously differentiable at each point of the interval $\left[0, \frac{1}{4^{n-1}}\right]$. From the first equation of (4.2), we have that $f_n(0) = 2 - a$ and $f'_n(x) = af_n(2x)$ for $0 \le x \le \frac{2}{4^n}$. Therefore, f_n satisfies $Eqns[n]$ as required. □

Lemma 4.2. *Suppose that $g_1(x)$ is a continuously differentiable function which satisfies $Eqns[k+1]$. Let $g_2(x)$ be an extension of $g_1(x)$ defined by*

$$\begin{cases}
g_2(x) & = g_1(x) & \text{if } 0 \le x \le \frac{1}{4^k} \\
g_2\left(\frac{1}{4^k} + x\right) & = p_{2k}(x) + g_2\left(\frac{1}{4^k} - x\right) & \text{if } 0 < x \le \frac{1}{4^k} \\
g_2\left(\frac{2}{4^k} + x\right) & = p_{2k-1}(x) - g_2\left(\frac{2}{4^k} - x\right) & \text{if } 0 < x \le \frac{2}{4^k}
\end{cases} \tag{4.3}$$

Then $g_2(x)$ is continuously differentiable on $\left[0, \frac{1}{4^{k-1}}\right]$, and satisfies $Eqns[k]$.

Proof. The function g_2 is continuously differentiable on the intervals $\left[0, \frac{1}{4^k}\right]$, $\left(\frac{1}{4^k}, \frac{2}{4^k}\right]$ and $\left(\frac{2}{4^k}, \frac{1}{4^{k-1}}\right]$. Now, we show that g_2 is differentiable at $x = \frac{1}{4^k}$.

$$\begin{aligned}
g_2'\left(\frac{1}{4^k}+\right) &= \lim_{x\to 0^+} \frac{g_2\left(\frac{1}{4^k}+x\right) - g_2\left(\frac{1}{4^k}\right)}{x} \\
&= \lim_{x\to 0^+} \frac{p_{2k}(x) + g_2\left(\frac{1}{4^k}-x\right) - g_2\left(\frac{1}{4^k}\right)}{x} \\
&= \lim_{x\to 0^+} \frac{p_{2k}(x) - p_{2k}(0)}{x} - \lim_{x\to 0^+} \frac{g_2\left(\frac{1}{4^k}-x\right) - g_2\left(\frac{1}{4^k}\right)}{-x} \\
&= p_{2k}'(0) - g_2'\left(\frac{1}{4^k}-\right)
\end{aligned}$$

By Proposition 4.2, $g_2'\left(\frac{1}{4^k}-\right) = ab_{2k-1}$. Then

$$g_2'\left(\frac{1}{4^k}+\right) = ap_{2k-1}(0) - ab_{2k-1} = 2ab_{2k-1} - ab_{2k-1} = g_2'\left(\frac{1}{4^k}-\right).$$

Therefore, g_2 is differentiable at $x = \frac{1}{4^k}$, and $g_2'\left(\frac{1}{4^k}\right) = ab_{2k-1}$. Now we show that g_2' is continuous at $x = \frac{1}{4^k}$. It is enough to show that

$$\lim_{x\to 0^+} g_2'\left(\frac{2}{4^k}+x\right) = g_2'\left(\frac{2}{4^k}\right)$$

because g_2' is continuous on $\left[0, \frac{1}{4^k}\right]$. From the second equation of (4.3), we have that

$$g_2'\left(\frac{1}{4^k}+x\right) = p_{2k}'(x) - g_2'\left(\frac{1}{4^k}-x\right).$$

Then

$$\begin{aligned}\lim_{x\to 0^+} g_2'\left(\frac{1}{4^k}+x\right) &= \lim_{x\to 0^+}\left[p_{2k}'(x) - g_2'\left(\frac{1}{4^k}-x\right)\right]\\ &= p_{2k}'(0) - g_2'\left(\frac{1}{4^k}-\right) = 2ab_{2k-1} - ab_{2k-1}\\ &= ab_{2k-1} = g_2'\left(\frac{1}{4^k}\right).\end{aligned}$$

Therefore, g_2 is continuously differentiable at $x = \dfrac{1}{4^k}$. The proof that g_2 is continuously differentiable at $x = \dfrac{2}{4^k}$ is similar. Now we show that g_2 satisfies the conditions of $Eqns[k]$. By the definition of g_2, we have that $g_2(0) = 0$ and

$$g_2\left(\frac{2}{4^k}+x\right) = p_{2k-1}(x) - g_2\left(\frac{2}{4^k}-x\right)$$

for $0 < x \le \dfrac{2}{4^k}$. Also, $g_2'(x) = ag_2(2x)$ for $0 < x \le \dfrac{2}{4^{k+1}}$ because $g_2(x) = g_1(x)$ on the interval $\left[0, \dfrac{1}{4^k}\right]$ and g_1 satisfies $Eqns[k+1]$. By the third equation of $Eqns[k+1]$, we have that

$$g_2\left(\frac{2}{4^{k+1}}+x\right) = p_{2k+1}(x) - g_2\left(\frac{2}{4^{k+1}}-x\right)$$

for $0 < x \le \dfrac{2}{4^{k+1}}$. By differentiating the above equation, we obtain

$$\begin{aligned}g_2'\left(\frac{2}{4^{k+1}}+x\right) &= p_{2k+1}'(x) + g_2'\left(\frac{2}{4^{k+1}}-x\right)\\ g_2'\left(\frac{2}{4^{k+1}}+x\right) &= ap_{2k}(2x) + ag_2\left(\frac{1}{4^k}-2x\right) = ag_2\left(\frac{1}{4^k}+2x\right).\end{aligned}$$

By the second equation of (4.3),

$$g_2\left(\frac{1}{4^k}+x\right) = p_{2k}(x) + g_2\left(\frac{1}{4^k}-x\right)$$

for $0 < x \le \dfrac{1}{4^k}$. Then

$$\begin{aligned}g_2'\left(\frac{1}{4^k}+x\right) &= p_{2k}'(x) - g_2'\left(\frac{1}{4^k}-x\right)\\ g_2'\left(\frac{1}{4^k}+x\right) &= ap_{2k-1}(2x) - ag_2\left(\frac{2}{4^k}-2x\right) = ag_2\left(\frac{2}{4^k}+2x\right).\end{aligned}$$

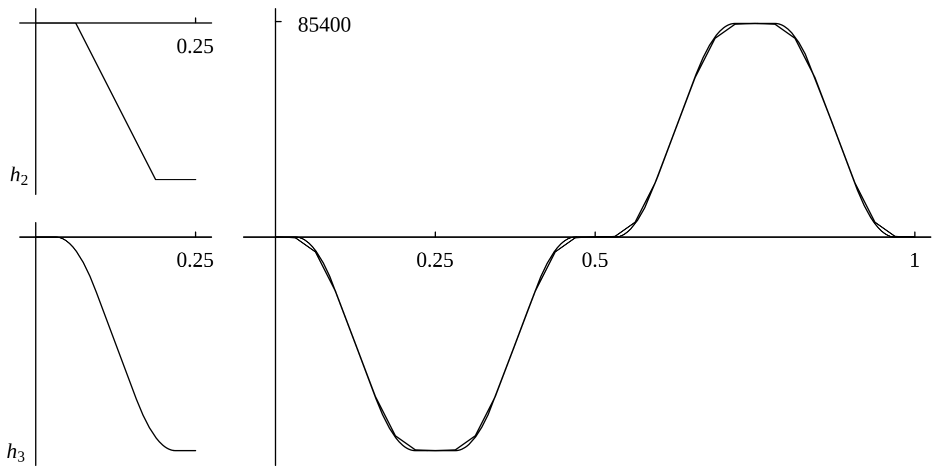

Fig. 4: The graphs of h_2 and h_3 (left) and $\bar{f}_3$ and f on the same axis (right) for $a = 7.999$

Therefore, $g_2'(x) = ag_2(2x)$ for all $0 < x \leq \dfrac{2}{4^k}$. Then g_2 is a continuously differentiable function on $\left[0, \dfrac{1}{4^{k-1}}\right]$ which satisfies $Eqns[k]$. □

Let $n = \lfloor \log_4 2a \rfloor + 1$. In Lemma 4.1, we showed that f_n satisfies $Eqns[n]$. Let $\{\tilde{f}_k\}_{k=1}^n$ be a sequence of functions where $\tilde{f}_n = f_n$, and $\tilde{f}_k$ is the extension of $\tilde{f}_{k+1}$ to the interval $\left[0, \dfrac{1}{4^{k-1}}\right]$ with formulas (4.3) for $k = 1, 2, \ldots, n-1$. By Lemma 4.2, the functions $\tilde{f}_k$ satisfy $Eqns[k]$ for all $k = 1, 2, \ldots, n$.

Corollary 4.1. *The function $\tilde{f}_1$ is continuously differentiable and satisfies equations* $(*)$.

PROOF. The function $\tilde{f}_1$ satisfies $Eqns[1]$ and by Proposition 4.2: $\tilde{f}_1(1) = 2 + a$. Therefore $\tilde{f}_1$ satisfies $(*)$. □

When $a = 7.999$, the sequence $\{h_k\}_{k=0}^{\infty}$ defined with (4.1) and $h_0(x) = x$ converges to the solution of $(*)$ on the interval $\left[0, \dfrac{1}{4}\right]$. The graphs of h_2 and h_3 are given on Figure 4 (left). Let $\bar{f}_3$ be the extension of h_3 with formulas (4.3) where $g_1 = h_3$ and $g_2 = \bar{f}_3$. The function $\bar{f}_3$ is an approximation to the solution f of $(*)$ on $[0,1]$. Although the values of f are in the interval $[-85400, 85400]$, the graph of the third approximation $\bar{f}_3$ already resembles the graph of f (Fig. 4). When $a = 2$, the solution of $(*)$ is $4F_2$ and the Fabius function F_2 is infinitely differentiable and nowhere analytic in $[0,1]$. Now we show that this is the only infinitely differentiable solution of $(*)$.

Corollary 4.2. *Let f be the solution of $(*)$ where $a \neq 2$ and $a \notin S$. Then $f''(x)$ is discontinuous at $\frac{1}{2}$.*

PROOF. By differentiating the first two equations of $(*)$ we obtain

$$\begin{cases} f''(x) = 2a^2 f(4x) & \text{if} \quad 0 \leq x \leq \frac{1}{4}, \\ f''(x) = 2a^2 f(2-4x) & \text{if} \quad \frac{1}{4} < x \leq \frac{1}{2}, \\ f''(x) = -2a^2 f(4x-2) & \text{if} \quad \frac{1}{2} < x \leq \frac{3}{4}, \\ f''(x) = -2a^2 f(4-4x) & \text{if} \quad \frac{3}{4} < x \leq 1. \end{cases} \tag{4.4}$$

From the second and the third equations of (4.4) and $x = \frac{1}{2}$, we obtain

$$f''\left(\frac{1}{2}-\right) = 2a^2(2-a)$$

and

$$f''\left(\frac{1}{2}+\right) = -2a^2(2-a).$$

Therefore, f'' is discontinuous at $x = \frac{1}{2}$. □

References

[1] N. G. De Bruijn, *The Difference-Differential Equation* $F'(x) = e^{\alpha x+\beta}F(x-1)$, Indagationes Math., **15** (1953), 449–464.

[2] Y. Dimitrov, *Polynomially-divided Solutions of Bipartite Self-differential Functional Equations*, Ph.D. Thesis, (2006).

[3] G. Edgar, *Integral, Probability and Fractal Measure*, Springer-Verlag, New York, 1998.

[4] J. Fabius, *A Probabilistic Example of a Nowhere Analytic C^∞-Function*, Z. Wahrsch. Verw. Gebiete, **5** (1966), 173–174.

[5] T. Kato and J. B. McLeod, *The Functional-Differential Equation* $y'(x) = ay(\lambda x) + by(x)$, Bull. Amer. Math. Soc., **77** (1971) 891–937.

[6] S. Wolfram, *The Mathematica Book*, 4th ed., Wolfram Media/Cambridge University Press, 1999.