

# *A Fractal Dimension Estimate for a Graph-directed Iterated Function System of Non-similarities*

G. A. EDGAR & JEFFREY GOLDS

ABSTRACT. Suppose a graph-directed iterated function system consists of maps  $f_e$  with upper estimates of the form  $d(f_e(x), f_e(y)) \leq r_e d(x, y)$ . Then the fractal dimension of the attractor  $K_v$  of the IFS is bounded above by the dimension associated to the Mauldin–Williams graph with ratios  $r_e$ . Suppose the maps  $f_e$  also have lower estimates of the form  $d(f_e(x), f_e(y)) \geq r'_e d(x, y)$  and that the IFS also satisfies the strong open set condition. Then the fractal dimension of the attractor  $K_v$  of the IFS is bounded below by the dimension associated to the Mauldin–Williams graph with ratios  $r'_e$ . When  $r_e = r'_e$ , then the maps are similarities and this reduces to the dimension computation of Mauldin and Williams for that case.

**0. Introduction.** Fractal sets may be constructed in many different ways. Barnsley [3] singled out the “iterated function system” method: The fractal set  $K$  is made up of parts, each of which is a shrunken copy of the whole set. Mauldin and Williams [13] provided a more general setting, where several sets  $K_v$  are involved, each of them is made up of parts, and each part is a shrunken copy of one of the parts (the same one or a different one). The combinatorics of the way in which the parts fit together is described by a directed multigraph. This is described in detail below (Definition 1.5). In addition to Mauldin and Williams, compare “recurrent iterated function system” [3, Ch. X], “Markov self-similar sets” [20], and “mixed self-similar systems” [1]; see also [5], [19].

In the text [6, Section 6.4] there is an exposition of the Mauldin–Williams computation of the dimension of the attractors for a graph-directed iterated function system consisting of similarities. “Similarities” are functions  $f: S \rightarrow T$  between metric spaces that satisfy equations of the form

$$(0.1) \quad d(f(x), f(y)) = r d(x, y) \quad \text{for all } x, y \in S.$$

Here we have used  $d$  for the (possibly different) metrics in the two metric spaces. This dimension computation involves the spectral radius of some nonnegative matrices. Next in the text are two “Exercises” [6, (6.4.9) and (6.4.10)] asking for appropriate generalizations when the equalities (0.1) are replaced by inequalities of the form

$$d(f(x), f(y)) \leq r d(x, y) \quad \text{for all } x, y \in S$$

or of the form

$$d(f(x), f(y)) \geq r d(x, y) \quad \text{for all } x, y \in S.$$

In this paper we provide some possible solutions for those exercises. The non-graph case (or, in our language, a graph with only one node and several loops) may be found in [9, Theorems 9.6, 9.7].

We also take this opportunity to make a generalization beyond the setting used in [6, Section 6.4]: we allow complete metric spaces other than Euclidean space. (Even when Euclidean space is the main interest, it is useful to apply the results in ultrametric “model” spaces consisting of strings or sequences.) This means that for the lower bound estimates we must use the “strong open set condition” rather than the original “open set condition” (this has been discovered since the publication of [6]: for example [2], [16], [17]).

The paper is arranged as follows. Section 1 reviews the notation for graph-directed iterated function systems. Section 2 reviews the Perron–Frobenius theory to be used. Section 3 contains the proofs. Section 4 describes an example.

This paper is based in part on a portion of the dissertation [11] of the second author, written at The Ohio State University under the direction of the first author.

## 1. THE SETTING

We will describe here: the “fractal dimensions” that will be used, the “graph-directed iterated function systems” and the fractal sets  $K_u$  that they define, and the “string models”  $E_u^{(\omega)}$  that will be used in the investigation. The notation will mostly follow [6], especially Section 4.3 on graph self-similarity.

**Fractal dimensions.** The first dimension we consider is the box-counting dimension. See [9] for a more complete discussion.

Let  $S$  be a metric space, and let  $K \subseteq S$  be a totally bounded set. For each  $\delta > 0$ , define  $N_\delta(K)$  to be the smallest number of sets of diameter  $\leq \delta$  that can cover  $E$ . Since we have postulated that  $K$  is totally bounded,  $N_\delta(K)$  is a finite natural number for every  $\delta > 0$ . Unless  $K$  is a finite set,  $N_\delta(K)$  increases to  $\infty$  as  $\delta$  decreases to 0; the rate at which  $N_\delta(K)$  increases will tell us something about the size of the set  $K$ .

**1.1. Definition.** The *upper box-counting dimension* or *Bouligand dimension* of  $K$  is

$$\overline{\dim}_B K = \limsup_{\delta \rightarrow 0} \frac{\log N_\delta(K)}{-\log \delta}.$$

The *lower box-counting dimension* of  $K$  is

$$\underline{\dim}_B K = \liminf_{\delta \rightarrow 0} \frac{\log N_\delta(K)}{-\log \delta}.$$

Next we will review the definition of the Hausdorff dimension. See for example [6, p. 149], [9, p. 28].

**1.2. Definition.** Let  $S$  be a metric space, let  $K \subseteq S$ , and let  $\delta > 0$  be a positive number. A collection  $\mathcal{C}$  of subsets of  $S$  is  $\delta$ -*fine* if and only if  $\text{diam } A \leq \delta$  for all  $A \in \mathcal{C}$ . The collection  $\mathcal{C}$  is a *cover* of  $K$  if and only if  $K \subseteq \bigcup_{A \in \mathcal{C}} A$ .

**1.3. Definition.** Let  $S$  be a metric space, let  $K \subseteq S$ , and let  $s > 0$ . For  $\delta > 0$ , define

$$\mathcal{H}_\delta^s(K) = \inf_{A \in \mathcal{C}} \sum (\text{diam } A)^s,$$

where the infimum is over all countable  $\delta$ -fine covers  $\mathcal{C}$  of  $K$ . Define

$$\mathcal{H}^s(K) = \sup_{\delta > 0} \mathcal{H}_\delta^s(K) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(K).$$

The *Hausdorff dimension* of the set  $K$  is

$$\dim_H K = \sup \{ s : \mathcal{H}^s(K) = \infty \} = \inf \{ s : \mathcal{H}^s(K) = 0 \}.$$

Another fractal dimension (not defined here) is the packing dimension  $\dim_P$ . See [7, Section 1.2] for a discussion. In [6] the packing measure is mentioned only as an afterthought, but in [7] its role is nearly as important as the Hausdorff measure.

Many other fractal dimensions may be found in the literature.

It is known ([7, Corollary 1.3.5] and [9, Lemma 3.7]) that all of the dimensions mentioned here (and many other fractal dimensions, as well) are between the Hausdorff dimension and the upper box dimension:

$$\dim_H K \leq \dim K \leq \overline{\dim}_B K,$$

where “ $\dim$ ” is any of  $\dim_H$ ,  $\dim_P$ ,  $\underline{\dim}_B$ ,  $\overline{\dim}_B$ .

**Graphs and Iterated Function Systems.** First, a directed multigraph  $(V, E)$  should be fixed. The elements  $v \in V$  are the *vertices* or *nodes* of the graph; the elements  $e \in E$  are the *edges* of the graph. For  $u, v \in V$ , there is a subset  $E_{uv}$  of  $E$ , known as the edges from  $u$  to  $v$ . Each edge belongs to exactly one of these subsets. We will sometimes write  $E_u = \bigcup_v E_{uv}$ , the set of all edges leaving the vertex  $u$ .

We will often think of the set  $E$  as a set of “letters” that label the edges of the graph, so we will talk about “words” or “strings” made up of these letters. A *path* in the graph is a finite string  $\alpha = e_1 e_2 \cdots e_k$  of edges, such that the terminal vertex of each edge  $e_i$  is the initial vertex of the next edge  $e_{i+1}$ . The initial vertex of  $\alpha$  is the initial vertex of the first letter  $e_1$  and the terminal vertex of  $\alpha$  is the terminal vertex of the last letter  $e_k$ . We write  $E_{uv}^{(k)}$  for the set of all paths of length  $k$  that begin at  $u$  and end at  $v$ ; and  $E_u^{(k)}$  for the set of all paths of length  $k$  that begin at  $u$ ; and  $E_u^{(*)}$  for the set of all finite paths of any length that begin at  $u$ ; and  $E^{(*)}$  for the set of all finite paths. By convention,  $E_u^{(0)}$  consists of a single “empty path”  $\Lambda_u$  of length zero from node  $u$  to itself.

If  $\alpha \in E_{uv}^{(k)}$  and  $\beta \in E_{vw}^{(n)}$ , then we write  $\alpha\beta$  for the path made by concatenation of the two strings, so that  $\alpha\beta \in E_{uw}^{(k+n)}$ .

A partial order may be defined on  $E^{(*)}$  as follows: Write  $\alpha \leq \beta$  if and only if  $\alpha$  is a prefix of  $\beta$ : that is,  $\beta = \alpha\gamma$  for some path  $\gamma$ . With this ordering, each set  $E_u^{(*)}$  becomes a tree. The root of the tree  $E_u^{(*)}$  is the empty path  $\Lambda_u$ . The entire set  $E^{(*)}$  is a disjoint union of trees (a forest). Two strings are *incomparable* if and only if neither is a prefix of the other.

A nonempty path that begins and ends at the same node is called a *cycle*.

We will assume that the graph  $(V, E)$  is *strongly connected*, that is, there is a path from any vertex to any other, along the edges of the graph (taken in the proper directions). We will also assume that there are at least two edges leaving each node. (In [8] may be found a method to eliminate this assumption.)

**1.4. Definition.** An *iterated function system* (or IFS) directed by the graph  $(V, E)$  consists of: one complete metric space  $S_v$  for each vertex  $v \in V$  and one function  $f_e: S_v \rightarrow S_u$  for each edge  $e \in E_{uv}$ .

To avoid trivialities, we will normally assume that each metric space  $S_v$  has at least two points. For this sort of graph-directed arrangement see [13], [6], and references in both. Note that in some places, such as [19], the direction of the edges of the graph are the reverse of the convention used in this paper; this may lead to consideration of the transpose of our matrix, but of course that does not affect the spectral radius.

**1.5. Definition.** An *invariant set list* for the iterated function system  $(f_e)$  consists of one nonempty set  $K_v \subseteq S_v$  for each  $v \in V$  such that

$$K_u = \bigcup_{v \in V} \bigcup_{e \in E_{uv}} f_e[K_v]$$

for all  $u \in V$ .

If the functions  $f_e$  satisfy appropriate conditions, then it may be proved that there is a unique invariant set list of nonempty compact sets. For example [9, Theorem 9.1], [6, Theorem 4.3.5].

**The models.** We will use some “string models” in our investigation of these invariant sets. Write  $E_u^{(\omega)}$  for the set of all *infinite* strings, using symbols from  $E$ , where the initial vertex of the first edge is  $u$  and the terminal vertex of each edge is the initial vertex of the next edge. These sets are naturally compact metric spaces. For each finite string  $\alpha \in E^{(*)}$ , the *cylinder*  $[\alpha]$  is the set of all infinite strings  $\sigma \in E^{(\omega)}$  that begin with  $\alpha$ . Then the set  $\{[\alpha] : \alpha \in E_u^{(*)}\}$  is the base for the topology on  $E_u^{(\omega)}$ . For  $\sigma \in E_u^{(\omega)}$  and a positive integer  $k$ , the *restriction*  $\sigma \upharpoonright k$  is the finite string made up of the first  $k$  letters of  $\sigma$ . The same notation  $\alpha \upharpoonright k$  is used for finite strings  $\alpha$  when  $k$  is less than the length of  $\alpha$ . As a special case of this: if  $\alpha$  has length  $k$ , then the *parent* of  $\alpha$  is  $\alpha^- = \alpha \upharpoonright (k - 1)$ , obtained by omitting the last letter of  $\alpha$ .

Suppose  $(E, V)$  directs an iterated function system  $(f_e)$  consisting of contractions. Then we may define an “action” of  $V$  starting from the maps  $f_e$ . If  $\alpha = e_1 e_2 \cdots e_k \in E_{uv}^{(k)}$ , then the composition  $f_\alpha = f_{e_1} \circ f_{e_2} \circ \cdots \circ f_{e_k}$  is a function from  $S_v$  to  $S_u$ . We will usually write simply  $\alpha(x)$  for  $f_\alpha(x)$ .

There is a *model map*  $h_u : E_u^{(\omega)} \rightarrow S_u$  for each  $u$ , defined so that  $h_u(\sigma)$  is the unique element of the set

$$\bigcap_{k=1}^{\infty} (\sigma \upharpoonright k) K_{v_k}.$$

(Of course, here  $v_k$  is the terminal vertex for the  $k$ th letter of  $\sigma$ .) Then clearly  $K_u = h_u[E_u^{(\omega)}]$ , since the sets  $h_u[E_u^{(\omega)}]$  satisfy the defining conditions for an invariant set list of the IFS  $(f_e)$ . If  $h_u(\sigma) = x$ , then we say that the string  $\sigma$  is an *address* of the point  $x$ .

**Mauldin–Williams graphs.** The computations to be done here can be stated in terms of a combinatorial structure that assigns a positive number to each edge of the graph.

**1.6. Definition.** A *Mauldin–Williams Graph* is a directed multigraph  $(V, E)$  together with a positive number  $r_e$  for each edge  $e \in E$ .

If  $\alpha = e_1 e_2 \cdots e_k \in E^{(k)}$  is a path, we define  $r(\alpha) = \prod_{i=1}^k r_{e_i}$ ; thus  $r(\alpha\beta) = r(\alpha)r(\beta)$  and  $r(\Lambda_u) = 1$ .

**1.7. Definition.** The system  $(r_e)$  is *strictly contracting* if and only if  $r_e < 1$  for all  $e$ .

## 2. PERRON–FROBENIUS THEORY

The computations will use some information from the Perron–Frobenius theory of nonnegative matrices. This material may be found in [10], [15], or [18]. We will state here the information that is used below.

Let  $A$  be an  $n \times n$  matrix. We say that  $A$  is nonnegative, and write  $A \geq 0$ , if all entries of  $A$  are nonnegative. When  $A$  and  $B$  are both  $n \times n$  matrices, we write  $A \geq B$  if  $A - B \geq 0$ . Similarly, if  $\mathbf{x}$  is an  $n \times 1$  column vector we write  $\mathbf{x} \geq 0$  if all entries are nonnegative, and we write  $\mathbf{x} > 0$  if all entries are positive.

The *spectral radius* of an  $n \times n$  matrix  $A$  is the maximum absolute value of the complex eigenvalues of  $A$ . Write  $\rho(A)$  for the spectral radius of  $A$ .

An  $n \times n$  matrix  $A$  is *reducible* if and only if the index set  $\{1, 2, \dots, n\}$  can be partitioned into two nonempty sets  $I, J$  such that for all  $i \in I, j \in J$ , the entry of  $A$  in row  $i$  column  $j$  is zero. The matrix  $A$  is *irreducible* if and only if it is not reducible.

We cite here a few results from the Perron–Frobenius theory of nonnegative matrices:

- (2.1) If  $A \geq 0$  is irreducible, then the spectral radius  $\lambda = \rho(A)$  is an eigenvalue of  $A$  and there is a positive eigenvector  $\mathbf{x} > 0$  with  $A\mathbf{x} = \lambda\mathbf{x}$ . [10, Theorem 2, p. 53], [15, Theorem 4.1, p. 11], [18, Theorem 1.5, p. 20].
- (2.2) If  $A \geq 0$ ,  $A$  irreducible,  $\mathbf{x} \geq 0, \mathbf{x} \neq 0, A\mathbf{x} = \lambda\mathbf{x}$ , then  $\lambda$  is the spectral radius  $\rho(A)$ . [10, Remark 3, p. 63], [15, Theorem 4.4, p. 16], [18, Theorem 1.6, p. 20].
- (2.3) If  $A \geq B \geq 0$ , where  $A$  is irreducible, then  $\rho(A) \geq \rho(B)$ . [10, Lemma 2, p. 57], [15, Corollary 2.2, p. 38], [18, Exercise 1.9, p. 25].

**Dimension of a graph.** The “dimension” for a Mauldin–Williams graph is computed as follows (see [13], [6]).

**2.1. Definition.** Let  $(V, E, (r_e)_{e \in E})$  be a strictly contracting Mauldin–Williams graph. For each positive real number  $s$ , let  $M(s)$  be the matrix (with rows and columns indexed by  $V$ ) with entry

$$M_{uv}(s) = \sum_{e \in E_{uv}} r_e^s$$

in row  $u$  column  $v$ . Let  $\Phi(s) = \rho(M(s))$  be the spectral radius of  $M(s)$ . Then  $\Phi$  is continuous, strictly decreasing,  $\Phi(0) \geq 1, \lim_{s \rightarrow \infty} \Phi(s) = 0$ . The unique solution  $s_1 \geq 0$  of  $\Phi(s_1) = 1$  is the *dimension* associated to the Mauldin–Williams graph  $(V, E, (r_e)_{e \in E})$ .

Note that, as a consequence of the rule for matrix multiplication, if  $k \in \mathbb{N}$ , then the  $k$ th power  $M(s)^k$  has entry

$$\sum_{\alpha \in E_{uv}^{(k)}} r(\alpha)^s$$

in row  $u$  column  $v$ . Because of our conventions about empty strings, this is true even for  $k = 0$ .

Suppose that  $(V, E)$  is a strongly connected graph. Since  $M(s_1)$  has non-zero entry in row  $u$  column  $v$  whenever there is an edge in  $E_{uv}$ , it follows that the matrix is irreducible. But 1 is an eigenvalue, so by (2.1) there is a positive eigenvector:

$$(2.2) \quad \lambda_u > 0, \quad \lambda_u = \sum_{v \in V} \sum_{e \in E_{uv}} r_e^{s_1} \lambda_v, \quad \sum_{u \in V} \lambda_u = 1.$$

(For convenience, we will write  $\lambda_{\min} = \min_v \lambda_v$  and  $\lambda_{\max} = \max_v \lambda_v$ . Thus  $0 < \lambda_{\min} \leq \lambda_{\max} \leq 1$ .)

This summation property shows that if we define

$$(2.3) \quad \mu_u([\alpha]) = r(\alpha)^{s_1} \lambda_v \quad \text{for all } \alpha \in E_{uv}^{(*)},$$

then we have the consistency property

$$\mu_u([\alpha]) = \sum_{e \in E_v} \mu_u([\alpha e])$$

when  $\alpha \in E_{uv}^{(*)}$ . Thus the  $\mu_u$  extend to Borel measures on the spaces  $E_u^{(\omega)}$ .

### 3. THE PROOFS

Let  $(V, E)$  be a strongly connected directed multigraph, let  $(S_v)_{v \in V}$  be a family of nonempty complete metric spaces, and let  $(f_e)_{e \in E}$  be a family of functions  $f_e: S_v \rightarrow S_u$  if  $e \in E_{uv}$ . Suppose each of these functions  $f_e$  satisfies a Lipschitz condition

$$(3.1) \quad d(f_e(x), f_e(y)) \leq r_e d(x, y)$$

for  $x, y \in S_v, e \in E_{uv}$ . We assume all  $r_e < 1$  (although [6, Exercise 4.3.9] it is actually enough to assume that the system of ratios  $(r_e)$  is “contracting” in

the sense that all cycles have ratio  $< 1$ ). It follows that there is a unique list of nonempty compact sets  $K_v \subseteq S_v$  satisfying the invariance condition

$$K_u = \bigcup_{\substack{v \in V \\ e \in E_{uv}}} f_e [K_v]$$

for all  $u \in V$ . For future use, we define

$$d_{\min} = \min \{ \text{diam } K_u : u \in V \}, \quad d_{\max} = \max \{ \text{diam } K_u : u \in V \}.$$

Following Definition 2.4, let  $s_1$  be the dimension associated with the Mauldin-Williams graph  $(V, E, (r_e)_{e \in E})$ . Write  $M(s)$  and  $\Phi(s)$  as in that definition.

We will show (Theorem 3.5) that the upper box dimensions satisfy

$$\overline{\dim}_B K_v \leq s_1$$

for all  $v$ . (See, for example, [9, Section 9.3] for a proof of this in the non-graph case and in Euclidean space.)

Now suppose there are also lower bounds of the form

$$(3.2) \quad d(f_e(x), f_e(y)) \geq r'_e d(x, y)$$

for  $x, y \in S_v, e \in E_{uv}$ . Let  $s_2$  be the dimension associated with the Mauldin-Williams graph  $(V, E, (r'_e)_{e \in E})$ . Write  $M'(s)$  for the matrix and  $\Phi'(s)$  for the spectral radius in Definition 2.4.

Under the assumption of the “strong open set condition”, we will show (Theorem 3.14) that the Hausdorff dimensions satisfy

$$\dim_H K_v \geq s_2$$

for all  $v$ . (See, for example, [9, Section 9.3] for a proof of this in the non-graph case, in Euclidean space, and in the disjoint case.)

Certainly  $r'_e \leq r_e$  (since each  $S_v$  has at least two points, as we have assumed). So  $M'(s) \leq M(s)$  and by (2.3)  $\Phi'(s) \leq \Phi(s)$  so that  $s_2 \leq s_1$ . The two estimates together will give us

$$s_2 \leq \dim K_v \leq s_1$$

where “dim” is any of the fractal dimensions mentioned above.

**Upper bounds.** For the proof of the upper bound, we need to estimate the counts  $N_\delta(K_u)$ . We will use “cross-cuts” of our forest of finite strings for this.

**3.3. Definition.** A cross-cut is a finite set  $T \subseteq E^{(*)}$  such that, for every infinite string  $\sigma \in E^{(\omega)}$  there is exactly one  $n$  with the restriction  $\sigma|n \in T$ .

If  $T$  is a cross-cut, we will write  $T_u = T \cap E_u^{(*)}$  and  $T_{uv} = T \cap E_{uv}^{(*)}$  for  $u, v \in V$ . Then, according to the definition, for each  $u \in V$ , the cylinders  $\{[\alpha] : \alpha \in T_u\}$  constitute a partition of  $E_u^{(\omega)}$ .

Let  $\mu_u$  be the measures on the spaces  $E_u^{(*)}$  as in (2.6). If  $T$  is a cross-cut we have

$$(3.4) \quad \lambda_u = \mu_u(E_u^{(\omega)}) = \sum_{\alpha \in T_u} \mu([\alpha]) = \sum_{v \in V} \sum_{\alpha \in T_{uv}} r(\alpha)^{s_1} \lambda_v.$$

**3.5. Theorem.** Let  $f_e$  satisfy upper bounds (3.1). Let  $s_1$  be the dimension for the Mauldin–Williams graph  $(V, E, (r_e)_{e \in E})$ . Then the attractors  $K_v$  satisfy  $\dim_B K_u \leq s_1$ .

*Proof.* Let  $\delta > 0$  be fixed. Then

$$(3.6) \quad T = \{ \alpha : u, v \in V, \alpha \in E_{uv}^{(*)}, r(\alpha) < \delta \leq r(\alpha^-) \}$$

is a cross-cut (called “first time less than  $\delta$ ”). For each  $\alpha \in T$ , we have  $\delta r_{\min} \leq r(\alpha) < \delta$ . We may estimate the cardinalities of the sets  $T_u$ :

$$\lambda_u = \sum_{v \in V} \sum_{\alpha \in T_{uv}} r(\alpha)^{s_1} \lambda_v \geq (\delta r_{\min})^{s_1} \lambda_{\min} \#T_u$$

and therefore, for every  $u \in V$ , we have

$$\#T_u \leq (\delta r_{\min})^{-s_1} \frac{\lambda_{\max}}{\lambda_{\min}}.$$

But of course the set  $K_u$  is covered as follows:

$$K_u \subseteq \bigcup_{\alpha \in T_u} \alpha K_v,$$

and the covering sets have diameters

$$\text{diam } \alpha K_v \leq r(\alpha) \text{diam } K_v \leq \delta d_{\max}.$$

Therefore we have

$$N_{\delta d_{\max}}(K_u) \leq (\delta r_{\min})^{-s_1} \frac{\lambda_{\max}}{\lambda_{\min}}$$

or (write  $\eta = \delta d_{\max}$ )

$$N_{\eta}(K_u) \leq \left( \frac{\eta r_{\min}}{d_{\max}} \right)^{-s_1} \frac{\lambda_{\max}}{\lambda_{\min}}.$$

Taking logarithms, dividing by the positive number  $-\log \eta$ , and letting  $\eta \rightarrow 0$ , we get

$$\overline{\dim}_B K_u = \limsup_{\eta \rightarrow 0} \frac{\log N_{\eta}(K_u)}{-\log \eta} \leq s_1. \quad \square$$

**Lower bound: Disjoint case.** Now we will consider the lower bound estimate on the dimension of the sets  $K_v$  that we get from the inequalities (3.2). Note that we continue to assume upper bounds (3.1) as before, so that the  $f_e$  are continuous, the images  $\alpha K_u$  are compact, and so on. (Thus  $r'_e < 1$  for all  $e$ .)

Now as before, 1 is an eigenvalue of the matrix  $M'(s_2)$ , so by (2.1) there is an eigenvector with positive entries:

$$(3.7) \quad \lambda'_u > 0, \quad \lambda'_u = \sum_{v \in V} \sum_{e \in E_{uv}} r_e'^{s_1} \lambda'_v, \quad \sum_{u \in V} \lambda'_u = 1.$$

(As usual, we will write  $\lambda'_{\min} = \min_v \lambda'_v$  and  $\lambda'_{\max} = \max_v \lambda'_v$ .) If  $\mu'_u$  are defined by  $\mu'_u([\alpha]) = r'(\alpha)^{s_2} \lambda'_v$  for all  $\alpha \in E_{uv}^{(*)}$ , then they extend to Borel measures on  $E_u^{(\omega)}$ .

**3.8. Definition.** We will say that the graph-directed iterated function system  $(f_e)$  with attracting sets  $(K_v)$  falls in the *disjoint case* if, for any  $u, v, v' \in V$ ,  $e \in E_{uv}$ ,  $e' \in E_{uv'}$ ,  $e \neq e'$ , we have  $f_e[K_v] \cap f_{e'}[K_{v'}] = \emptyset$ .

**3.9. Theorem.** *Let  $f_e$  satisfy lower bounds (3.2). Let  $s_2$  be the dimension for the Mauldin–Williams graph  $(V, E, (r'_e)_{e \in E})$ . Suppose the IFS falls in the disjoint case. Then the Hausdorff dimensions satisfy  $\dim_{\mathbb{H}} K_u \geq s_2$ .*

*Proof.* Because of the disjointness, and because of the compactness of the sets  $K_v$ , there is  $\eta > 0$  so that  $\text{dist}(f_e[K_v], f_{e'}[K_{v'}]) > \eta$ , for all  $u, v, v', e, e'$  as in the definition. We claim that for any incomparable  $\alpha, \alpha' \in E_u^{(*)}$ ,

$$\text{dist}(\alpha K_v, \alpha' K_{v'}) > r'(\alpha^-) \eta.$$

To see this, let  $\gamma$  be the longest common prefix of  $\alpha, \alpha'$ . Because they are incomparable, they are both strictly longer than  $\gamma$ . (Thus  $\gamma \leq \alpha^-$ , so  $r'(\gamma) \geq r'(\alpha^-)$ .) There are distinct letters  $e, e'$  so that  $\alpha = \gamma e \beta, \alpha' = \gamma e' \beta'$ . Say  $\beta \in E_{wv}^{(*)}, \beta' \in E_{w'v'}^{(*)}$ ; these strings may be empty. Then  $\beta K_v \subseteq K_w$  and  $\beta' K_{v'} \subseteq K_{w'}$ ; since  $e \neq e'$  we have  $\text{dist}(e\beta K_v, e'\beta' K_{v'}) \geq \text{dist}(eK_w, e'K_{w'}) > \eta$ ; so finally  $\text{dist}(\gamma e \beta K_v, \gamma e' \beta' K_{v'}) \geq r'(\gamma)\eta \geq r'(\alpha^-)\eta$ .

Let  $B \subseteq S_u$  be a Borel set. Let  $\delta = \text{diam } B/\eta$  and let  $T$  be the cross-cut “first time less than  $\delta$ ”

$$(3.10) \quad T = \{ \alpha : u \in V, v \in V, \alpha \in E_{uv}^{(*)}, r'(\alpha) < \delta \leq r'(\alpha^-) \}.$$

The elements  $\alpha \in T_u$  are all incomparable, and  $\text{diam } B = \eta\delta \leq \eta r'(\alpha^-)$ , so at most one set  $\alpha K_v$  with  $\alpha \in T_u$  can meet  $B$ . This means  $h^{-1}[B] \subseteq [\alpha]$  for some  $\alpha \in T$ , and thus  $\mu'_u(h^{-1}[B]) \leq \mu'_u([\alpha]) = r'(\alpha)^{s_2} \lambda'_v \leq \lambda'_{\max} \eta^{-s_2} (\text{diam } B)^{s_2}$ .

Now if  $\mathcal{C}$  is a countable cover of  $K_u$  by Borel sets, we have

$$\sum_{B \in \mathcal{C}} (\text{diam } B)^{s_2} \geq \left( \frac{\eta^{s_2}}{\lambda'_{\max}} \right) \sum \mu'_u(h^{-1}[B]) \geq \left( \frac{\eta^{s_2}}{\lambda'_{\max}} \right) \mu'_u(K_u).$$

It follows that  $\mathcal{H}^{s_2}(K_u) \geq (\eta^{s_2}/\lambda'_{\max})\mu'_u(K_u) > 0$ . Therefore  $\dim_{\mathbb{H}} K_u \geq s_2$ . □

**Lower bound: Strong open set condition.** To prove a lower bound for the dimension of an IFS we must of course limit the overlap. This is often done with an “open set condition”. In general complete metric spaces (not necessarily subsets of Euclidean space) we require the “strong open set condition”. For the strong open set condition, and its application in this calculation, we follow [2], [16], [17].

**3.11. Definition.** Let  $(V, E)$  be a directed multigraph, let  $(S_v)_{v \in V}$  be complete metric spaces, let  $f_e : S_v \rightarrow S_u$  for  $e \in E_{uv}$  be a strictly contracting graph-directed IFS, and let  $(K_v)_{v \in V}$  be the attractor list. We say that the IFS  $(f_e)$  satisfies the *open set condition* if and only if, for each  $v \in V$  there is an open set  $U_v \subseteq S_v$  such that:

- (a) For all  $u, v \in V, e \in E_{uv}$ , we have  $f_e[U_v] \subseteq U_u$ .
- (b) For all  $u, v, v' \in V, e \in E_{uv}, e' \in E_{uv'}, e \neq e'$ , we have  $f_e[U_v] \cap f_{e'}[U_{v'}] = \emptyset$ .

We say that the IFS  $(f_e)$  satisfies the *strong open set condition* if and only if there exist open sets  $U_v$  satisfying (a), (b), and

- (c) For all  $v \in V$ , we have  $U_v \cap K_v \neq \emptyset$ .

Unlike the case where each  $S_v$  is Euclidean space, it now no longer follows that the images  $f_e[U_v]$  are open sets. Even if  $f_e$  is a homeomorphism of  $U_v$  onto  $f_e[U_v]$ , it need not follow that  $f_e[U_v]$  is open.

Notice the following consequence of the open set condition. (Of course, (3.2) implies that  $f_e$  is injective.)

**3.12. Lemma.** *Let  $f_e$  satisfy the open set condition with open sets  $U_v$ . Suppose the maps  $f_e$  are injective. If  $\alpha \in E_{uv}^{(*)}$ ,  $\beta \in E_{u'v'}^{(*)}$  are incomparable, then  $f_\alpha[U_v] \cap f_{\alpha'}[U_{v'}] = \emptyset$ .*

*Proof.* Let  $\gamma$  be the longest common prefix of  $\alpha, \alpha'$ . Because they are incomparable, they are both strictly longer than  $\gamma$ . There are distinct letters  $e, e'$  so that  $\alpha = \gamma e \beta$ ,  $\alpha' = \gamma e' \beta'$ . Say  $\beta \in E_{wv}^{(*)}$ ,  $\beta' \in E_{w'v'}^{(*)}$ ; these strings may be empty. Now by (a), we have  $\beta U_v \subseteq U_w$  and  $\beta' U_{v'} \subseteq U_{w'}$ . Next,  $e \beta U_v \subseteq f_e[U_w]$  and  $e' \beta' U_{v'} \subseteq f_{e'}[U_{w'}]$  and by (b) these are disjoint. Since the maps  $f_e$  are injective, so are their compositions, so we conclude that  $\alpha U_v = \gamma e \beta U_v$  and  $\alpha' U_{v'} = \gamma e' \beta' U_{v'}$  are also disjoint. □

From the uniqueness properties of the invariant sets  $K_v$ , it is easy to see that  $K_v \subseteq \overline{U_v}$ . But  $K_v \subseteq U_v$  need not be true. The strong open set condition provides a substitute for this.

**3.13. Lemma.** *Let  $(V, E)$  be a strongly connected directed multigraph. Suppose the IFS  $(f_e)$  satisfies the strong open set condition with open sets  $U_v$ . For each  $v \in V$ , there is a cycle  $\zeta \in E_{vv}^{(*)}$  such that  $\zeta K_v \subseteq U_v$ .*

*Proof.* By (c), there is a point  $x \in K_v \cap U_v$ . Let  $\sigma \in E_v^{(\omega)}$  be an address of  $x$ . Then the singleton  $\{x\}$  is the decreasing intersection of compact sets of the form  $(\sigma \upharpoonright n) K_{u_n}$ . Now  $U_v$  is a neighborhood of the point  $x$ , so we conclude by compactness that there is  $n$  so that  $(\sigma \upharpoonright n) K_{u_n} \subseteq U_v$ . Because the graph  $(V, E)$  is strongly connected, there is a path  $\gamma \in E_{u_n v}^{(*)}$ . Let  $\zeta = (\sigma \upharpoonright n) \gamma$ . Then we have  $\gamma K_v \subseteq K_{u_n}$  so that  $\zeta K_v = (\sigma \upharpoonright n) \gamma K_v \subseteq (\sigma \upharpoonright n) K_{u_n} \subseteq U_v$ . □

**3.14. Theorem.** *Let  $f_e$  satisfy lower bounds (3.2). Let  $s_2$  be the dimension for the Mauldin–Williams graph  $(V, E, (r'_e)_{e \in E})$ . Suppose  $(f_e)$  satisfies the strong open set condition. Then the attractors  $K_v$  satisfy  $\dim_{\mathbb{H}} K_v \geq s_2$ .*

*Proof.* Let  $s < s_2$ . We will show that  $\dim_{\mathbb{H}} K_v \geq s$ . Now  $\Phi'(s) > 1$ , so the spectral radius  $t = \rho(M'(s))$  is  $> 1$ .

Let  $U_v$  be open sets for the strong open set condition. For each  $v \in V$ , choose a cycle  $\zeta_v \in E_{vv}^{(*)}$  so that  $\zeta_v K_v \subseteq U_v$ . Then

$$c = \min_{v \in V} r'(\zeta_v)^s$$

is a positive number. Let  $n \in \mathbb{N}$  be such that  $ct^n > 1$ .

Now we will define a new IFS. First we define a new directed multigraph  $(V, \tilde{E})$ . The vertices of the new graph will be  $V$  as before. The edges from vertex  $u$  to vertex  $v$  will be

$$\tilde{E}_{uv} = \{ \alpha\zeta_v : \alpha \in E_{uv}^{(n)} \}.$$

The corresponding maps are  $\tilde{f}_{\alpha\zeta_v} = f_{\alpha\zeta_v}$ . Note that  $\alpha\zeta_v[K_v] \subseteq K_u$ , so the new attractors  $\tilde{K}_v$  are subsets of the old attractors  $K_v$ .

We claim that that the new IFS falls in the disjoint case. Indeed, let  $\alpha\zeta_v \in \tilde{E}_{uv}$  and  $\alpha'\zeta_{v'} \in \tilde{E}_{uv'}$  be distinct edges. Then  $\zeta_v K_v \subseteq U_v$  and  $\zeta_{v'} K_{v'} \subseteq U_{v'}$ . But  $\alpha$  and  $\alpha'$  have the same length and are unequal, so they are not comparable, and thus by Lemma 3.12 we have  $\alpha\zeta_v K_v \cap \alpha'\zeta_{v'} K_{v'} \subseteq \alpha U_v \cap \alpha' U_{v'} = \emptyset$ .

Now the matrix  $\tilde{M}'(s)$  for the new IFS has entry

$$\sum_{\alpha \in E_{uv}^{(n)}} r'(\alpha\zeta_v)^s$$

in row  $u$  column  $v$ . But this entry is

$$\geq c \sum_{\alpha \in E_{uv}^{(n)}} r'(\alpha)^s,$$

which is the corresponding entry in the matrix  $cM'(s)^n$ . Now by (2.3),

$$\rho(\tilde{M}'(s)) \geq \rho(cM'(s)^n) = ct^n > 1.$$

This shows (by Theorem 3.9) that  $\dim_{\mathbb{H}} \tilde{K}_v > s$ , so also  $\dim_{\mathbb{H}} K_v > s$ . □

#### 4. EXAMPLE: A JULIA SET

We will consider an example showing how the theorem may be applied to estimate the fractal dimension of a set. This example is the Julia set for the transformation  $z^2 - 1/2$  of the complex plane  $\mathbb{C}$ . See, for example, [9, Chapter 14] for an explanation of Julia sets in simple cases like this. McMullen [14] has provided an algorithm that allows computation of Hausdorff dimension up to any desired accuracy of conformal expanding open maps (so having Markov partition) which include this example.

The Julia set we will consider here is shown in Figure 4.1. See [11] for more details of the following computation.

We divide the set into four parts  $A, B, C, D$ , using the four quadrants of the plane. Then under the two inverse maps  $\pm\sqrt{w+1/2}$  each of these parts is made up of the images of two of the parts.  $A$  is made up of images of  $A$  and  $B$ ;

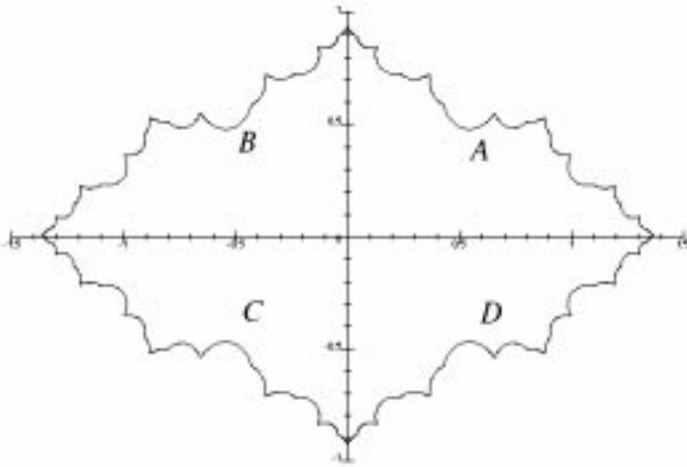


FIGURE 4.1. A Julia set

$B$  is made up of images of  $C$  and  $D$ ;  $C$  is made up of images of  $A$  and  $B$ ;  $D$  is made up of images of  $C$  and  $D$ . Thus our directed graph will have four vertices and eight edges. See Figure 4.2.

For the upper and lower bounds, we will use a simple estimate based on the derivatives.

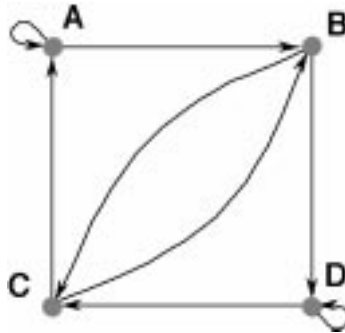


FIGURE 4.2. The graph

**4.3. Lemma.** *Suppose  $\varphi(z) = z^2 - 1/2$  maps convex set  $U \subseteq \mathbb{C}$  bijectively onto convex set  $W$ , and  $0 \notin W$ . Write  $f(w) = \sqrt{w + 1/2}$  for the branch of the inverse that maps  $W$  onto  $U$ . Let*

$$m = \inf \{ |z| : z \in U \}; \quad M = \sup \{ |z| : z \in U \}.$$

Then, for any  $w_1, w_2 \in W$ ,

$$\frac{1}{2M} |w_2 - w_1| \leq |f(w_2) - f(w_1)| \leq \frac{1}{2m} |w_2 - w_1|.$$

*Proof.* Note that  $\varphi'(z) = 2z$ , so  $|\varphi'(z)| \leq 2M$  for all  $z \in U$ . Also  $f'(w) = 1/(2f(w))$  and if  $w \in W$  then  $z = f(w) \in U$ , so  $|f'(w)| \leq 1/(2m)$  for all  $w \in W$ .

Now let  $w_1, w_2 \in W$ , and write  $z_1 = f(w_1)$ ,  $z_2 = f(w_2)$ . Since  $U$  is convex we may integrate along the line segment joining  $z_1$  to  $z_2$ :

$$\begin{aligned} |w_2 - w_1| &= |\varphi(z_2) - \varphi(z_1)| = \left| \int_{z_1}^{z_2} \varphi'(z) dz \right| \\ &\leq (\sup_{z \in U} |\varphi'(z)|) |z_2 - z_1| = 2M |z_2 - z_1|. \end{aligned}$$

Consequently we have  $|f(w_2) - f(w_1)| \geq (1/(2M))|w_2 - w_1|$ . Similarly,

$$\begin{aligned} |f(w_2) - f(w_1)| &= \left| \int_{w_1}^{w_2} f'(w) dw \right| \\ &\leq (\sup_{w \in W} |f'(w)|) |w_2 - w_1| = \frac{1}{2m} |w_2 - w_1|. \quad \square \end{aligned}$$

For the convex sets required here, we begin with the smallest circle that surrounds our set, a certain parallelogram inside, and segments on the imaginary axis. We get four three-sided regions, each containing the portion of the curve in one quadrant. See Figure 4.4.

The parallelogram is chosen so that for each of the four regions bounded by our curves, the images under  $\pm\sqrt{w + 1/2}$  are inside the original regions. (See Figure 4.5.)

Now we are in a position to estimate the constants for the inequalities. The outer circle has radius  $(1 + \sqrt{3})/2$ , so  $M = (1 + \sqrt{3})/2$  and our lower bounds are all  $r'(e) = 1/(1 + \sqrt{3})$ . Then

$$M'(s) = \begin{bmatrix} a & a & 0 & 0 \\ 0 & 0 & a & a \\ a & a & 0 & 0 \\ 0 & 0 & a & a \end{bmatrix}$$

where  $a = (1 + \sqrt{3})^{-s}$ . The spectral radius is 1 when  $s = s_2 = \log 2 / \log(1 + \sqrt{3}) > 0.689$ .

Similarly, for the closest point we get  $m = \sqrt{9 + 3\sqrt{3}}/6$ , and  $r(e) = 1/(2m) \approx 0.7962$ . Computing the spectral radius as before, we get  $s_1 < 3.042$ .

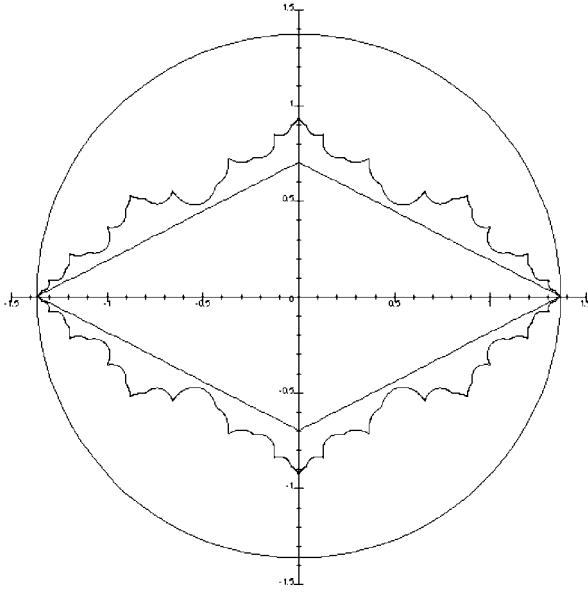


FIGURE 4.4. Inner and outer bounds

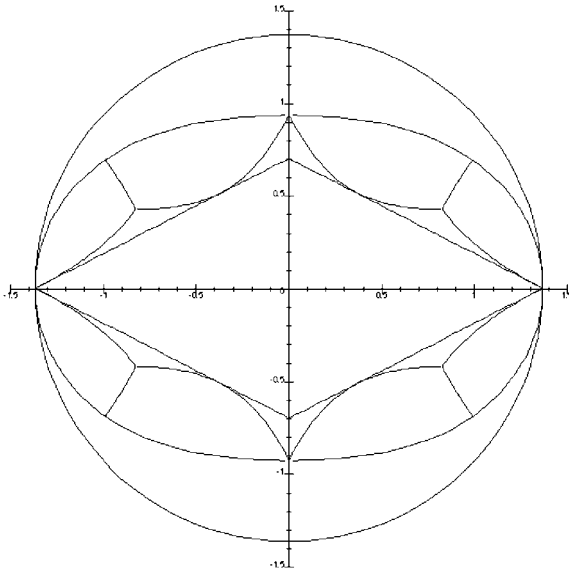


FIGURE 4.5. Images are inside the regions

The (rather poor) estimates are

$$0.689 < \dim K < 3.042.$$

But that need not be the end of our work. Consider the eight small regions in Figure 4.5. Their images, in turn, fall as shown in Figure 4.6. This graph-directed iterated function system has 8 nodes and 16 edges. We may work with an  $8 \times 8$  matrix. [Actually, because of symmetry, we may reduce this to a  $2 \times 2$  matrix, but such questions of efficiency are not dealt with in this paper.] Using the upper and lower estimates for these regions, we obtain

$$0.735 < \dim K < 1.758.$$

A considerable improvement, but still not very good.

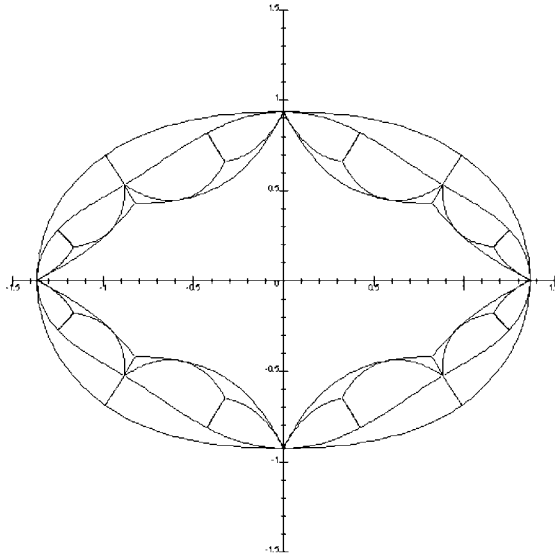


FIGURE 4.6. Eight regions and their images

If we have a computer to help us, we may continue to get better approximations. When we reach a matrix of size  $2^{12} \times 2^{12}$  our estimate is

$$1.069 < \dim K < 1.077.$$

The size of the computation is not as bad as the size of the matrix may suggest: each row has only two nonzero entries.

McMullen's method [14] computes the dimension for this Julia set as 1.07336 (according to the computer program on his web site). This agreement provides evidence that our methods (and our computer program) are correct.

Other Julia sets for the quadratic maps  $z^2 + c$  are considered in [11]. For  $c$  outside the Mandelbrot set, and for  $c$  in the main cardioid of the Mandelbrot set, the process required to carry out the inner and outer approximations was automated. It is hoped that details of this will be published elsewhere.

Note that the functions  $z^2 + c$  used in this example are "conformal" in the sense used in geometry (except at  $z = 0$ ). So this example is a "self-conformal" set. See [12] and [19] for more information on self-conformal fractals.

For a map  $f: S \rightarrow T$  of metric spaces, one might say that  $f$  is *conformal* if and only if, for every  $x_0 \in S$ , there is a positive constant  $c(x_0)$  such that for every  $\varepsilon > 0$  there is  $\delta > 0$  such that

$$\left| \frac{d(f(x), f(x_0))}{d(x, x_0)} - c(x_0) \right| < \varepsilon$$

whenever  $d(x, x_0) < \delta$ . But as far as we know, the only cases that have been considered (aside from similarities in general metric spaces) are differentiable manifolds.

#### REFERENCES

- [1] C. BANDT, *Self-similar sets 1: Markov shifts and mixed self-similar sets*, Math. Nachr. **142** (1989), 107–123.
- [2] C. BANDT AND S. GRAF, *Self-similar sets 7: A characterization of self-similar fractals with positive Hausdorff measure*, Proc. Amer. Math. Soc. **114** (1992), 995–1001.
- [3] M. F. BARNESLEY, *Fractals Everywhere*, Second Edition, Academic Press, 1993.
- [4] R. CAWLEY AND R. D. MAULDIN, *Multifractal decompositions of Moran fractals*, Advances in Math. **92** (1992), 196–236.
- [5] S. DROBOT AND J. TURNER, *Hausdorff dimension and Perron–Frobenius theory*, Illinois J. Math. **33** (1989), 1–9.
- [6] G. A. EDGAR, *Measure, Topology, and Fractal Geometry*, Springer-Verlag New York, 1990.
- [7] G. A. EDGAR, *Integral, Probability, and Fractal Measures*, Springer-Verlag New York, 1998.
- [8] G. A. EDGAR AND R. D. MAULDIN, *Multifractal decompositions of digraph recursive fractals*, Proc. London Math. Soc. **65** (1992), 604–628.
- [9] K. J. FALCONER, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley and Sons, 1990.
- [10] F. R. GANTMACHER, *The Theory of Matrices*, Volume 2, Chelsea, 1959, Chapter XIII: Matrices with non-negative elements.
- [11] JEFFREY A. GOLDS, *Estimation of the fractal dimension of selected classes of Julia sets using spectral radius calculations*, Ph. D. Dissertation, The Ohio State University, 1998.

- [12] R. D. MAULDIN, *Infinite iterated function systems: Theory and applications*, Progress in Prob. **37** (1995), 91–110.
- [13] R. D. MAULDIN AND S. C. WILLIAMS, *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc. **309** (1988), 811–829.
- [14] C. T. MCMULLEN, *Hausdorff dimension and conformal dynamics III: Computation of dimension*, Amer. J. Math. **120** (1998), 691–721.
- [15] H. MINC, *Nonnegative Matrices*, Wiley, 1988.
- [16] N. PATZSCHKE, *The strong open set condition in the random case*, Proc. Amer. Math. Soc. **125** (1997), 2119–2125.
- [17] A. SCHIEF, *Self-similar sets in complete metric spaces*, Proc. Amer. Math. Soc. **124** (1996), 481–490.
- [18] E. SENETA, *Non-Negative Matrices*, Wiley, 1973.
- [19] R. STRICHARTZ, *Self-similar measures and their Fourier transforms III*, Indiana Univ. Math. J. **42** (1993), 123–157.
- [20] Y. TSUJI, *Markov self-similar sets*, Hiroshima Math. J. **21** (1991), 491–519.

Department of Mathematics  
The Ohio State University  
Columbus, Ohio 43210

ALSO:  
Silicon Graphics  
1600 Amphitheatre Parkway  
Mountain View, California 94043  
E-MAIL: edgar@math.ohio-state.edu (G. A. Edgar)  
E-MAIL: golds@csd.sgi.com (Jeffrey Golds)

*Submitted: June 29th, 1998; revised: December 4th, 1998.*