HAUSDORFF DIMENSION, ANALYTIC SETS AND TRANSCENDENCE

G. A. EDGAR AND CHRIS MILLER

ABSTRACT. Every analytic real closed proper subfield of $\mathbb{R}$ has Hausdorff dimension zero. Equivalently, every analytic set of real numbers having positive Hausdorff dimension contains a transcendence base for $\mathbb{R}$.

How large can a proper subfield of the real numbers be? Of course, before attempting to answer this question, we must make sense of what we should mean by "large"—cardinality, measure, Baire category, and so on—but there is another implicit question: What kinds of subfields of $\mathbb{R}$ should we consider? Set-theoretic independence issues quickly arise if we do not narrow the focus. We show in this note that proper subfields of $\mathbb{R}$ that are well behaved in certain algebraic and descriptive set-theoretic senses (to be made precise below) are quite small when viewed measure theoretically, although every uncountable such field has the cardinality of the continuum.

Given $E \subseteq \mathbb{R}$ and $n \in \mathbb{N}$, we write E^n for the n-fold Cartesian product $E \times E \times \cdots \times E$.

A subset of $\mathbb{R}^n$ is **analytic** (also called **Souslin** or **Suslin** in the literature) if it is the continuous image of a Borel subset of $\mathbb{R}$. (There are several equivalent definitions; this one is perhaps the easiest to state.) The collection of all analytic subsets of $\mathbb{R}^n$ properly contains the collection of all Borel subsets of $\mathbb{R}^n$. Analytic sets are Lebesgue measurable and have the property of Baire. Every uncountable analytic set contains a nonempty perfect set and thus has the cardinality of the continuum. Basic information on analytic sets can be found in Cohn [2, Ch. 8] or Rogers [11]; see Kechris [8] for an extensive modern development.

Suppose that G is a proper additive subgroup of $\mathbb{R}$. It follows immediately from elementary results (see e.g. Oxtoby [10, 4.8]) that if G is Lebesgue measurable then it has measure zero, and if G has the property of Baire then it is of Baire first category. Hence, proper analytic subgroups of $(\mathbb{R}, +)$ are small if we restrict our attention to Lebesgue measure and Baire category. On the other hand, any uncountable analytic subgroup of $\mathbb{R}$ has the cardinality of $\mathbb{R}$. Finer tools are needed in order to understand the situation better.

For $s \geq 0$, the **Hausdorff s-measure** of $A \subseteq \mathbb{R}^n$, denoted by $\mathcal{H}^s(A)$, is defined as follows: For $\delta > 0$, let $\mathcal{H}^s_\delta(A) \in [0, \infty]$ be the infimum of all sums of the form $\sum_{j\in\mathbb{N}}[\text{diameter}(A_j)]^s$ where $\{A_j\}_{j\in\mathbb{N}}$ is a collection of subsets of $\mathbb{R}^n$ with $A \subseteq \bigcup_{j\in\mathbb{N}} A_j$ and $\text{diameter}(A_j) \leq \delta$ for all $j \in \mathbb{N}$. Now put $\mathcal{H}^s(A) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(A)$. The **Hausdorff dimension** of $A \subseteq \mathbb{R}^n$, denoted by $\dim_{\mathcal{H}} A$, is the infimum of all $s \geq 0$ such that $\mathcal{H}^s(A) = 0$. (See any of Edgar [5], Falconer [7], Mattila [9] or [11] for details).

From now on, "dimension" means "Hausdorff dimension".

An ordered field K is **real closed** if every positive element has a square root in K and every one-variable odd degree polynomial function with coefficients from K has a root in K (equivalently, if the ring $K[X]/(X^2 + 1)$ is an algebraically closed field). Real closed

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fields play a role in the theory of ordered fields analogous to that of algebraically closed fields in the theory of fields of characteristic zero.

Here is the main result of this note:

Theorem. *Every analytic real closed proper subfield of* $\mathbb{R}$ *has dimension* 0*.*

Actually, we shall prove that no analytic set $E \subseteq \mathbb{R}$ with $\dim_{\mathcal{H}} E > 0$ is contained in any proper real closed subfield of $\mathbb{R}$. (Another way of saying this is that E contains a **transcendence base** for $\mathbb{R}$, that is, a maximal algebraically independent subset of $\mathbb{R}$.) The converse does not hold: There are dimension 0 analytic subsets of $\mathbb{R}$ that are not contained in any proper real closed subfield of $\mathbb{R}$. Indeed, there are dimension 0 compact sets $C \subseteq \mathbb{R}$ such that the sum set $\{\, x+y : x, y \in C \,\}$ has interior, so C is not even contained in any proper additive subgroup of $\mathbb{R}$. (For example, put $C = E \cup F$ where E and F are as in [7, Example 7.8]).

We came to the theorem while considering an open question about the possible Hausdorff dimensions of subrings of $\mathbb{R}$ that are Borel. It is well known that every proper additive subgroup of $\mathbb{R}$ is either cyclic (that is, of the form $r\mathbb{Z}$ for some $r \in \mathbb{R}$) or is dense and co-dense in $\mathbb{R}$. By what we noted earlier, if such a subgroup is Borel (as a subset of $\mathbb{R}$) then it has Lebesgue measure zero and is of Baire first category. What are the possible values for its Hausdorff dimension? It turns out that the situation is quite wild: For each $d \in [0, 1]$ there is a subgroup G of $(\mathbb{R}, +)$ such that G is Borel and $\dim_H G = d$; see Erdős and Volkmann [6] or [7, §12.4].

A natural variant of the question is this: What are the possible Hausdorff dimensions for subrings of $\mathbb{R}$ that are Borel? Only a partial answer is known at present: The dimension of such a ring is either equal to 1 or at most 1/2. But no examples are known other than dimensions 0 and 1. A similar question for subfields of $\mathbb{R}$ that are Borel is also open. The same questions are open even when the subring or subfield is not required to be Borel but merely analytic. (See e.g. [9, pp. 166–7] for more information.)

Thus, we come to the question answered in this note: A real closed subfield of $\mathbb{R}$ that is a Borel set (or even an analytic set) has Hausdorff dimension either 0 or 1. Moreover, dimension 1 occurs only for $\mathbb{R}$ itself.

The theorem is immediate from the following four lemmas (each of independent interest).

Lemma 1. *Let* $E \subseteq \mathbb{R}$ *be compact,* $\dim_{\mathcal{H}} E > 0$*. Then there exist* $n \in \mathbb{N}$ *and an* $\mathbb{R}$*-linear function* $T \colon \mathbb{R}^n \to \mathbb{R}$ *such that* $T(E^n)$ *has interior (in* $\mathbb{R}$*).*

Proof. Put $\dim_{\mathcal{H}} E = d > 0$. Choose $k \in \mathbb{N}$ such that $kd > 1$; then $\dim_{\mathcal{H}}(E^k) \geq kd > 1$ (see [9, 8.10]). Hence, there is an orthogonal projection $\pi \colon \mathbb{R}^k \to \mathbb{R}$ such that the image πE has positive Lebesgue measure [9, Ch. 9]. (In $\mathbb{R}$, Hausdorff 1-measure is the same as Lebesgue measure). Then the difference set $\{\, a - b : a, b \in \pi E \,\}$ has interior; see [10, 4.8]. Put $n = 2k$ and define $T \colon \mathbb{R}^n \to \mathbb{R}$ by $T(x_1, \ldots, x_n) = \pi(x_1, \ldots, x_k) - \pi(x_{k+1}, \ldots, x_n)$. ☐

Lemma 2. *Let* $E \subseteq \mathbb{R}$ *be analytic,* $\dim_{\mathcal{H}} E > 0$*. Then there exist* $n \in \mathbb{N}$ *and an* $\mathbb{R}$*-linear function* $T \colon \mathbb{R}^n \to \mathbb{R}$ *such that* $T(E^n)$ *has interior.*

Proof. Since E is analytic, it contains a compact set of positive dimension [5, (1.7.11)]. Apply the previous lemma. ☐

Remark. Suppose, in Lemma 2, that E is moreover an additive subgroup of $\mathbb{R}$. Then $T(E^n)$ is an additive subgroup of $\mathbb{R}$ that has interior in $\mathbb{R}$. Hence, $T(E^n) = \mathbb{R}$, that is, there exist

$s_1, \ldots, s_n \in \mathbb{R}$ such that $\mathbb{R}$ is equal to the set all sums $\sum_{i=1}^n s_i e_i$ with $e_1, \ldots, e_n \in E$. (In particular, if E is also a subring of $\mathbb{R}$, then $\mathbb{R}$ is finitely generated as an E-module.) This shows that proper analytic subgroups of $\mathbb{R}$ having positive dimension are, in an algebraic sense, almost all of $\mathbb{R}$.

Before proceeding further, we need some definitions and basic facts from real algebraic geometry. (See Bochnak *et al* [1, Chs. 1,2] or van den Dries [4, Chs. 2,3] for facts used below.)

A **semialgebraic** set $S \subseteq \mathbb{R}^n$ is a finite union of sets of the form

$$\{\, x \in \mathbb{R}^n : p(x) = 0, q_1(x) < 0, \ldots, q_l(x) < 0 \,\}$$

where $p, q_1, \ldots, q_l \colon \mathbb{R}^n \to \mathbb{R}$ are real polynomial functions. We say that S is **defined over** a subfield $K \subseteq \mathbb{R}$ if every coefficient occuring in the description of S belongs to K (so "semialgebraic and defined over $\mathbb{R}$" means the same as "semialgebraic"). If $A \subseteq \mathbb{R}^m$ then a map $f \colon A \to \mathbb{R}^n$ is said to be semialgebraic (and defined over K) if its graph $\{\, (x, f(x)) : x \in A \,\}$ is a semialgebraic (and defined over K) subset of $\mathbb{R}^{m+n}$.

Semialgebraic sets play a role in real algebraic geometry analogous to that of constructible sets in complex algebraic geometry.

Lemma 3. *Let $E \subseteq \mathbb{R}$ be analytic and K be the smallest real closed subfield of $\mathbb{R}$ containing E. Then K is analytic.*

Proof. The field K is equal to the union of all sets of the form $f(E^n)$ where $n \in \mathbb{N}$ and $f \colon \mathbb{R}^n \to \mathbb{R}$ is semialgebraic and defined over $\mathbb{Q}$. (Aside: There are more explicit ways of producing K from E but we do not need them here.) Let $n \in \mathbb{N}$ and $f \colon \mathbb{R}^n \to \mathbb{R}$ be semialgebraic. By the cell decomposition theorem, there is a finite partition of $\mathbb{R}^n$ into locally closed sets $C_1, \ldots, C_k$ such that each restriction $f|C_i \colon C_i \to \mathbb{R}$ is continuous. Intersections of analytic sets are again analytic, so each $C_i \cap E^n$ is analytic. Continuous images of analytic sets are analytic, so each $f(C_i \cap E^n)$ is analytic. For each $n \in \mathbb{N}$ there are only countably many semialgebraic functions $\mathbb{R}^n \to \mathbb{R}$ that are defined over $\mathbb{Q}$. Countable unions of analytic sets are analytic, so K is analytic. $\square$

Remark. Lemma 3 fails with "Borel" in place of "analytic" even if E is also a subring of $\mathbb{R}$. However, if E is a Borel subfield of $\mathbb{R}$, then K is Borel. (These results were produced by R. Dougherty in personal communication with the second author.)

Remark to model theorists. An easy modification of the proof of Lemma 3 shows that if $\mathfrak{R}$ is an o-minimal expansion of the structure $(\mathbb{R}, <, +, 1)$ in a countable language and $E \subseteq \mathbb{R}$ is analytic then the definable closure of E—taken with respect to $\mathrm{Th}(\mathfrak{R})$—is again analytic.

Finally, we state without proof a special case of a model-theoretic result; see [3, Lemma 4.1].

Lemma 4. *Let K and L be real closed subfields of $\mathbb{R}$ such that K is properly contained in L. Let $n \in \mathbb{N}$ and $f \colon \mathbb{R}^n \to \mathbb{R}$ be semialgebraic and defined over L. Then $f(K^n)$ has empty interior in L.*

The point is that coefficients from L are allowed in the description of f. The result is rather trivial if f is defined over K, for then $f(K^n) \subseteq K$, and K has no interior in L.

Proof of the Theorem. Let $E \subseteq \mathbb{R}$ be analytic with $\dim_{\mathcal{H}} E > 0$. Let K be the smallest real closed subfield of $\mathbb{R}$ containing E; then K is analytic (by Lemma 3) and $\dim_{\mathcal{H}} K > 0$.

By Lemma 2, there exist $n \in \mathbb{N}$ and an $\mathbb{R}$-linear (hence semialgebraic) function $T\colon \mathbb{R}^n \to \mathbb{R}$ such that $T(K^n)$ has interior. Hence (by Lemma 4) we have $K = \mathbb{R}$. □

We don't know how optimal is the theorem. Lemma 4 says that all proper real closed subfields of $\mathbb{R}$ are small in a model-theoretic sense. Lemma 3 can be regarded more generally as saying that the smallest real closed subfield of $\mathbb{R}$ containing a given $E \subseteq \mathbb{R}$ is neither much larger nor much more complicated than is E. But the tools needed in order to obtain Lemma 2 do not extend in general to non-analytic sets.

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Department of Mathematics, The Ohio State University, Columbus, Ohio 43210
E-mail address: `edgar@math.ohio-state.edu`
URL: `http://www.math.ohio-state.edu/~edgar`

Department of Mathematics, The Ohio State University, Columbus, Ohio 43210
E-mail address: `miller@math.ohio-state.edu`
URL: `http://www.math.ohio-state.edu/~miller`