BULLETIN DE LA S. M. F.

YUVAL FLICKER

On analytic models of degenerating abelian varieties

Bulletin de la S. M. F., tome 107 (1979), p. 283-293.

<http://www.numdam.org/item?id=BSMF_1979__107__283_0>

Bull. Soc. math. France,
107, 1979, p. 283-293.

ON ANALYTIC MODELS OF DEGENERATING ABELIAN VARIETIES

BY

YUVAL FLICKER (*)
[Institute for advanced study, Princeton]

RÉSUMÉ. — Nous définissons un objet arithmétique, le corps de stabilité, associé à certains ensembles de fonctions abéliennes sur un tore p-adique multiplicatif, et nous démontrons que ce corps de stabilité est une extension transcendante du corps des nombres rationnels.

ABSTRACT. — We associate an arithmetic object, the stability field, to a set of abelian functions on a multiplicative p-adic torus, and prove that the stability field is a transcendental extension of the rational numbers.

0. Introduction

Let k be a complete algebraically closed extension of the field Q_p of p-adic numbers, and denote by $(k^x)^d$ the product of $d (\geqslant 1)$ copies of the multiplicative group k^x of non-zero elements of k. Suppose L is a lattice in $(k^x)^d$. The geometric structure of the p-adic torus $(k^x)^d/L$ has been studied by many authors, especially GERRITZEN [8], MORIKAWA [10] and MUMFORD [11]. Indeed, in analogy with the classical theory of uniformization of abelian varieties by complex tori, it has been shown that when the Riemann's period relations are satisfied, there is a biholomorphic mapping $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_{d'})$, from the torus $(k^x)^d/L$ to an abelian variety A of dimension d in a projective d'-space $(d' \geqslant d)$. We shall follow [11] in naming A a "degenerating" abelian variety. Put $f_i = \theta_i/\theta_0$ $(1 \leqslant i \leqslant d')$. On applying a suitable normalization on $\boldsymbol{\theta}$ (which we shall discuss at the end of this section), there exist subfields K of k, for which the differential operators $D_i = z_i\, \partial/\partial z_i$ $(1 \leqslant i \leqslant d)$ map the algebra $K[f_1, \ldots, f_{d'}]$ into itself; here $z_1, \ldots, z_d$ are independent variables on k^d, and $\partial/\partial z_i$ denotes the partial derivative by z_i. The minima-field K with this property will here be called the stability field of the normal

(*) Supported in part by an NSF grant.

Texte reçu le 20 décembre 1978.

Yuval FLICKER, Institute for advanced study, Princeton, N.J. 08540 (États-Unis).

lized representation $\boldsymbol{\theta}$ of A. Now, once the geometric nature of the representation $\boldsymbol{\theta}$ is known one may study it from an arithmetic point of view, and consider its stability field in the above sense. This is our object here. We shall establish the following Theorem.

THEOREM. — *Let K be the stability field of the normalized representation $\boldsymbol{\theta}$ of a degenerating abelian variety A. Then the field K is a transcendental extension of the field $\mathbf{Q}$ of rational numbers.*

BERTRAND [2] was the first to obtain a result of this kind; he dealt with the case of a variety A of dimension $d = 1$, that is, an elliptic curve. In this case, the associated functions (normalized as in [2]) are the Jacobi-Tate elliptic function $P_q(z)$ and the function $DP_q(z)$, which is obtained by applying the operator $D = zd/dz$ to $P_q(z)$; here q denotes the generator of L whose valuation is less than 1. These functions satisfy the elliptic equation

$$(DP_q)^2 = 4\,P_q^3-(1/12)\,E_4(q)\,P_q+(1/216)\,E_6(q),$$

where $E_{2i}(q)$ ($i = 2,3$) are the values at q of the normalized Eisenstein series

$$E_{2i}(q) = 1+(-1)^i(4\,i/B_i)\sum_{n=1}^{\infty} s_{2i-1}(n)\,q^n;$$

here B_i is the ith Bernoulli number and $s_{2i-1}(n)$ denotes the sum of the $(2\,i-1)$th powers of the divisors of the integer n. The stability field of the elliptic curve is generated over $\mathbf{Q}$ by the values $E_4(q)$ and $E_6(q)$. Hence BERTRAND's work [2] implies that at least one of $E_4(q)$ and $E_6(q)$ is transcendental, for any non-zero q in the domain of convergence of E_4 and E_6.

The proof of our Theorem is based on arguments from the theory of transcendental numbers (*see* SCHNEIDER [12], chapter II, and LANG [9], chapter III). We shall derive a contradiction from the supposition that the stability field of the above representation is algebraic, by estimating a certain value of some well-chosen auxiliary function. Our work depends fundamentally on the interesting phenomenon, that the "orders" of the degenerating abelian functions f_i are arbitrarily small (*see* Proposition 2). More precisely we shall use a transcendende criterion (*see* Proposition 1) which unlike BOMBIERI's Theorem [5], will contradict the existence of only one algebraic point. There the one-dimensional multiplicative Schwarz lemma (Lemma 3; *see* [2], Lemma 2) will suffice for our needs. This contrasts with the methods employed in [4], [6] and [7], which involve a multi-dimensional additive Schwarz Lemma; also here we are dealing with global rather than local functions.

Finally we note that the same arguments which we employ in the non-archimedean case can furnish also a complex analogue for the Theorem. This will be discussed briefly in the last section.

We have now to specify the normalization of the representation $\boldsymbol{\theta}$. Signify by $|\ \ |$ the normalized valuation on the field k. The lattice L is a free subgroup of $(k^x)^d$ of maximal rank d, and we denote its generators by $\mathbf{g}_1, \ldots, \mathbf{g}_d$. Let $q(\mathbf{g}, \mathbf{g}')$ be a bimultiplicative function on L, and put $q_{ij} = q(\mathbf{g}_i, \mathbf{g}_j)\ (1 \leqslant i, j \leqslant d)$. Riemann's period relations are that the $d \times d$ matrix $(-\log |q_{ij}|)$ is symmetric and positive definite. As we mentioned above when these relations are satisfied for some form q, there is a biholomorphic mapping $\boldsymbol{\theta} = (\theta_0, \ldots, \theta_{d'})$ from the torus $(k^x)^d/L$ to a degenerating abelian variety A of dimension d in a projective d'-space $(d' \geqslant d)$. The components $\theta_i\ (0 \leqslant i \leqslant d')$ of $\boldsymbol{\theta}$ are holomorphic theta-functions and the quotients $f_i = \theta_i/\theta_0$ generate the field of meromorphic functions on $(k^x)^d$ with periods in L. This field of functions is of transcendence degree d over k. On applying a linear (projective) transformation to the functions θ_i we assume that $f_i(\mathbf{1}) = 0\ (1 \leqslant i \leqslant d')$; here $\mathbf{1}$ signifies the d-vector $(1, \ldots, 1)$. This is our first normalization. The second normalization is the following. On applying a transformation to the f_i's, by virtue of Noether's normalization Theorem we may assume that $f_1, \ldots, f_d$ are algebraically independent functions, and that $f_{d+1}, \ldots, f_{d'}$ are integral over the algebra $k[f_1, \ldots, f_d]$. It is now easy to verify that the differential operators $z_i \partial/\partial z_i\ (1 \leqslant i \leqslant d)$ map the algebra $k[f_1, \ldots, f_{d'}]$ into itself (*see* also [3]). Under this normalization, we define the stability field, as above and note that it is finitely generated over the rationals $\mathbf{Q}$. We are now ready to begin the proof of the Theorem.

1. Proof of the Theorem

We shall demonstrate the Theorem using the following generalization of the Schneider-Lang transcendence criterion. Let K be a number field in k, and suppose that $f_0, f_1, \ldots, f_{d'}\ (d' \geqslant d)$ are finitely many meromorphic functions in the vector variable $\mathbf{z} = (z_1, \ldots, z_d)$ on $(k^x)^d$, whose orders (*see* below) are arbitrarily small. As above, out $D_i = z\, \partial/\partial z_i\ (1 \leqslant i \leqslant d)$. Then we prove the following proposition.

PROPOSITION 1. — *If the algebra $K[f_0, \ldots, f_{d'}]$ is mapped into itself by all D_i, and it is of transcendence degree at least $d+1$ over K, then there is no point of $(k^x)^d$ at which all $f_i(z)$ obtain simultaneously algebraic values.*

Let h be a positive number. A meromorphic function on $(k^x)^d$ is said to be of order at most h if it can be expressed as a quotient of two analytic function f_1 and f_2 on $(k^x)^d$, which satisfy the relation

$$\log |f_i(\mathbf{z})| \leqslant c \max(|\mathbf{z}|^h, |\mathbf{z}|^{-h}) \qquad (i = 1, 2),$$

for all $\mathbf{z}$ in $(k^x)^d$; here we put

$$|\mathbf{z}| = \max(|z_1|, \ldots, |z_n|),$$

and c is a positive constant depending only on f_1, f_2 and h.

We shall establish Proposition 1 in section 3. Our proof of the Theorem consists of showing that Proposition 1 is applicable in the circumstances described by the Theorem. The main property which we shall need to verify is the following Proposition.

PROPOSITION 2. — *In the notations of the Theorem, each function f_i $(1 \leqslant i \leqslant d')$ has arbitrarily small order.*

The proof of this Proposition will be given in section 2.

We shall now deduce the Theorem from the Propositions. Suppose, contrary to the assertion of the Theorem that the stability field of the representation $\boldsymbol{\theta}$ of A is an algebraic number field K. In order to derive a contradiction which will establish the Theorem, we consider the algebra $K[f_0, f_1, \ldots, d_{d'}]$, where $f_0(\mathbf{z}) = z_j$, and j is a fixed integer between 1 and d. By the definition of the stability field, we deduce that this algebra is mapped into itself by the differential operators D_i; note that $D_i z_j = \delta_{ij} z_j$, where δ_{ij} denotes the Kronecker delta function. Further, by virtue of Proposition 2, each of the functions f_i $(1 \leqslant i \leqslant d')$ is of arbitrarily small order; obviously the function f_0 has the same property. Furthermore, we recall that since the dimension of the abelian variety A is equal to d and $f_1, \ldots, f_{d'}$ generate the field of meromorphic functions on A over K, the transcendence degree of the ring $K[f_1, \ldots, f_{d'}]$ over K is equal to d. But each of the functions f_i $(1 \leqslant i \leqslant d')$ is periodic on $(k^x)^d$, while the function f_0 is not periodic and non-constant. It follows that the algebra $K[f_0, \ldots, f_{d'}]$ is of transcendence degree $d+1$ over K. Finally, since all of the components of the origin $(f_1(\mathbf{1}), \ldots, f_{d'}(\mathbf{1}))$ of A lie in the number field K, and also $f_0(\mathbf{1}) = 1$, we obtain a contradiction to Proposition 1, by virtue of the first normalization. This is a contradiction to the assumption that K is algebraic which establishes the Theorem.

2. Proof of Proposition 2

The subsequent discussion is based on the description and results concerning the representation $\boldsymbol{\theta}$ which are given in [8], especially sections 1 and 4, and in [10].

Let $\mathbf{z} = (z_1, \ldots, z_d)$ denote again a variable vector on $(k^x)^d$, and suppose that L is a free subgroup of maximal rank in $(k^x)^d$, generated by $\mathbf{g}_1, \ldots, \mathbf{g}_d$. Let H be the group of characters $\tilde{z}$ on $(k^x)^d$ such that $\tilde{z}(\mathbf{z}) = z_1^{n_1} \ldots z_d^{n_d}$ for some integers $n_1, \ldots, n_d$. Such a character induces a homomorphism from L to k^x, mapping the element $\mathbf{g} = (g_1, \ldots, g_d)$ of L to $q(\mathbf{g}, \tilde{z}) = g_1^{n_1} \ldots g_d^{n_d}$ in k^x. We shall write $\mathbf{g}.\tilde{z}$ for $q(\mathbf{g}, \tilde{z})\,\tilde{z}$.

We shall now assume that the holomorphic torus $(k^x)^d/L$ is an abelian variety. According to GERRITEN's results ([8], § 3 and 4; *see* also [10]), this implies the existence of a homomorphism t from L to H, such that $q(\mathbf{g}, t(\mathbf{g}')) = q(t(\mathbf{g}), \mathbf{g}')$ for any $\mathbf{g}$, $\mathbf{g}'$ in L, and such that the symmetric bilinear from

$$(\mathbf{g}, \mathbf{g}') \mapsto -\log |q(\mathbf{g}, t(\mathbf{g}'))|$$

is positive definite (the Riemann relations).

Under these conditions (and since k is algebraically closed), there exists a symmetric bilinear form p, defined over k, satisfying

$$p(\mathbf{g}, \mathbf{g}')^2 = q(\mathbf{g}, t(\mathbf{g}')),$$

and a homomorphism m from L to k^x, such that, if $w_0, \ldots, w_{d'}$ denotes a complete set of representatives of the finite groupe $H/t(L)$, then the $d'+1$ functions

$$\theta_i = w_i \textstyle\sum_{\mathbf{g} \text{ in } L} m(\mathbf{g})\, p(\mathbf{g}, \mathbf{g})\, q(\mathbf{g}, w_i)\, t(\mathbf{g}) \qquad (0 \leqslant i \leqslant d'), \tag{1}$$

which satisfy the transformation formula

$$\theta_i(\mathbf{z}) = m(\mathbf{g})\, p(\mathbf{g}, \mathbf{g})\, t(\mathbf{g})(\mathbf{z})\, \theta_i(\mathbf{g}.\mathbf{z}), \tag{2}$$

define a holomorphic projective embedding of $(k^x)^d/L$. In other words, the field of meromorphic functions on $(k^x)^d/L$ (which, by definition of an abelian variety is of transcendence degree d) is generated over k by the functions $\{\theta_i/\theta_0\}$ $(1 \leqslant i \leqslant d')$. Since the normalizations of the representation $\boldsymbol{\theta}$ which we applied in the introduction were obtained by transformations which will not affect the estimations below, we may assume that $\boldsymbol{\theta}$ is the normalized representation, and that the degenerating abelian functions $f_i(\mathbf{z})$ are simply the quotients $\theta_i(\mathbf{z})/\theta_0(\mathbf{z})$.

It remains to estimate the orders of the functions f_i. Since the f_i are given as the quotients of the holomorphic functions θ_i, it suffices to deal with the latter functions only. For brevity, we put $\theta = \theta_i\,(0 \leqslant i \leqslant d')$, and proceed to show that the order of θ is arbitrarily small. Clearly

$$-\log|m(\mathbf{g})\,p(\mathbf{g},\mathbf{g})\,t(\mathbf{g})(\mathbf{z})| \tag{3}$$

is equal to

$$Q(\mathbf{n})-(\textstyle\sum_i n_i \log|m(\mathbf{g}_i)|+\sum_{ij} n_j t_{ij} \log|z_i|),$$

where

$$\mathbf{g} = \mathbf{g}_1^{n_1}\ldots\mathbf{g}_d^{n_d}, \qquad t(\mathbf{g}_i)(\mathbf{z}) = z_1^{t_{1j}}\ldots z_d^{t_{dj}},$$

and

$$Q(\mathbf{n}) = \textstyle\sum_{ij}(-\log|p(\mathbf{g}_i,\mathbf{g}_j)|)\,n_i n_j.$$

By virtue of the period relations $Q(\mathbf{n})$ is a positive definite quadratic form.

It follows that for any $\mathbf{g} \neq \mathbf{1}$ and for any $\mathbf{z}$ with $c_0 < |\mathbf{z}| < c_1$, we have that (3) is bounded by

$$\begin{aligned} c_2 \max\{n_i^2\} &< c_3 \log(\max(|\mathbf{g}|, |\mathbf{g}|^{-1})^2) \\ &< c_4 \log(\max(|\mathbf{g}.\mathbf{z}|, |\mathbf{g}.\mathbf{z}|^{-1})^2). \end{aligned}$$

Here $c_0, \ldots, c_9$ denote positive constants which depend only on θ.

Since $\theta(\mathbf{z})$ satisfies the functional equation (2), and is bounded on the set $\max(|\mathbf{z}|, |\mathbf{z}|^{-1}) \leqslant c_5$, we further deduce that

$$\begin{aligned} \log|\theta(\mathbf{g}.\mathbf{z})| &= \log(|\theta(\mathbf{z})|\,(m(\mathbf{g})\,p(\mathbf{g},\mathbf{g})\,t(\mathbf{g})(\mathbf{z}))^{-1}) \\ &< c_6 \log(\max(|\mathbf{g}.\mathbf{z}|, |\mathbf{g}.\mathbf{z}|^{-1})^2). \end{aligned}$$

Let R denote the maximum of the numbers $|\mathbf{g}_i|$ and $|\mathbf{g}_i|^{-1}$ $(1 \leqslant i \leqslant d)$. Since every point on $(k^\times)^d$ can be expressed in the form $\mathbf{g}.\mathbf{z}$, where $\mathbf{g}$ belongs to L and $\mathbf{z}$ in $(k^\times)^d$ satisfies $\max(|\mathbf{z}|, |\mathbf{z}|^{-1}) \leqslant R$, we finally deduce that for any $h > 0$ and for any $\mathbf{z}$ in $(k^\times)^d$ with $a = \max(|\mathbf{z}|, |\mathbf{z}|^{-1}) > c_7$, we have

$$\log|\theta(\mathbf{z})| < c_8 (\log a)^2 < c_9\, a^h,$$

as required.

3. Proof of Proposition 1

In the notations of section 1, we shall assume that, contrary to the assertion of the Proposition, there exists a vector $\mathbf{u} = (u_1, \ldots, u_d)$ in $(k^\times)^d$ such that all of the $f_i(\mathbf{u})$ are algebraic, and will eventually derive a contradiction.

For the construction of the auxiliary function in Lemma 1, below, we assume that all of the values $f_i(\mathbf{u})$ belong to K (taking a finite extension of K if necessary). Since the transcendence degree of $K[f_0, \ldots, f_{d'}]$ over K is at least $d+1$ we assume (upon reordering indeces if necessary) that the functions $f_0, \ldots, f_d$ are algebraically independent over K. We introduce the differential operator $D^{\mathbf{k}} = D_1^{k_1} \ldots D_d^{k_d}$ for any d-tuple $\mathbf{k}$ of non-negative integers k_i. Any such differential operator maps the algebra $K[f_0, \ldots, f_{d'}]$ into itself by hypothesis. Finally, we recall that the size of a non-zero algebraic number a is defined to be the maximum between $\| a \|$ and den a, where $\| a \|$ denotes the maximum of the archimedean absolute values of the conjugates of a, and den a is the least natural number such that (a den a) is an algebraic integer.

Now let S be an integer so large that the estimates below are valid, and put

$$M = [(S^d \log S)^{1/(d+1)}].$$

By the letters $c_1, \ldots, c_{15}$ below we mean positive numbers which can be effectively calculated in terms of $\mathbf{u}$, f_i, d and K; in particular, they will be independent of S (and M). Assuming that $S > c_1$, we prove the following Lemma.

LEMMA 1. — *There exist K-integers $a(\mathbf{j})$, which are indexed by $(d+1)$-tuples $\mathbf{j}$ of non-negative integers j_i with $0 \leqslant j_i \leqslant M$, not all 0, with sizes whose logarithms are $< c_2 S$, such that for any d-tuple $\mathbf{k}$ with $|\mathbf{k}| = k_1 + \ldots + k_d < S$. we have $(D^{\mathbf{k}} F)(\mathbf{u}) = 0$, where*

$$F(\mathbf{z}) = \textstyle\sum_{\mathbf{j}} a(\mathbf{j}) f_0(\mathbf{z})^{j_0} \ldots f_d(\mathbf{z})^{j_d}.$$

Proof. — The conditions of the Lemma give a system of at most S^d equations in $(M+1)^{d+1} \geqslant S^d \log S$ unknowns $a(\mathbf{j})$. The coefficients in the system are readily seen to be elements of K whose sizes are bounded by $S^{c_3 S}$.

The proof is now complete by virtue of Siegel's Lemma, on noting that the relevant exponent is $< c_4 (\log S)^{-1}$.

We note that the function $F(\mathbf{z})$ does not vanish identically, since the functions $f_0, \ldots, f_d$ are algebraically independent; we deduce that there exists a d-tuple $\mathbf{k}$ such that $(D^{\mathbf{k}} F)(\mathbf{u}) \neq 0$. On denoting by s the minimal integer for which there exists a d-tuple $\mathbf{t}$ of non-negative integers t_i with $|\mathbf{t}| = s$ and with $(D^{\mathbf{t}} F)(\mathbf{u}) \neq 0$, we deduce from Lemma 1 that $s \geqslant S$. By hypothesis each of the functions $f_0, \ldots, f_d$ is meromorphic with arbitrarily small order, hence there exist functions $\theta_0, \ldots, \theta_d$, with $\theta_i(\mathbf{u}) \neq 0$, such that θ_i and $\theta_i f_i$ are analytic functions with orders at most h, where h is any positive number. Then we can introduce the analytic function

$$G(\mathbf{z}) = (\theta_0(\mathbf{z}) \ldots \theta_d(\mathbf{z}))^M F(\mathbf{z})$$

on $(k^x)^d$. We shall obtain the desired final contradiction upon comparing an upper and a lower bound for the valuation of the number

$$\beta = (\mathbf{t}!)^{-1} (D^{\mathbf{t}} G)(\mathbf{u}),$$

where $\mathbf{t}! = t_1! \ldots t_d!$ By the minimality of s, we deduce that

$$\beta = (\mathbf{t}!)^{-1} (\theta_0(\mathbf{u}) \ldots \theta_d(\mathbf{u}))^M (D^{\mathbf{t}} F)(\mathbf{u}),$$

and hence that β is non-zero. A lower bound for $|\beta|$ can easily be obtained, as follows:

LEMMA 2. — *We have* $\log |\beta| > -c_5 s \log s$.

Proof. — Since $s \geqslant S$, estimations as in the proof of Lemma 1, together with the result of Lemma 1, show that the logarithm of the size of the algebraic number $\beta' = (\mathbf{t}!)^{-1} (D^{\mathbf{t}} F)(\mathbf{u})$ in K is $< c_6 s \log s$. But $\beta' \neq 0$, hence we can deduce from the product formula on K that

$$\log |\beta'| > -c_7 s \log s.$$

Since $\theta_i(\mathbf{u})$ are non-zero constants, and $s > M$, the Lemma follows for any $S > c_8$.

It remains for us to find a complementary upper bound for $\log |\beta|$. This is a little more difficult, and we need to apply some arguments from p-adic analysis. Thus for any $R > 1$, we denote by k_R the set of x in k^x with

$$\max(|x|, |x|^{-1}) \leqslant R.$$

For any analytic function $f(x) = \sum_{n=-\infty}^{\infty} a_n x^n$ on k_R, and for any number r in the valuation group of k^x, with $R^{-1} \leqslant r \leqslant R$, we define $|f|_r$ to be the maximum of the terms $|a_n| r^n$, over all of the integers n. We note that

$$\max_{|x|=r} |f(x)| = |f|_r,$$

since x is taken in the algebraically closed field k, and we deduce from the maximum modulus principle that

$$|f(u)| \leqslant \max(|f|_r, |f|_{r^{-1}}),$$

for any u in k_r. We shall need the following improvement upon the last inequality, which is a multiplicative variant of the Schwarz Lemma principle.

LEMMA 3. — *Let r and R be elements of the valuation group of k^x, with $1 < r < R$. Suppose that f is an analytic function on k_R, which admits $m \geqslant 0$ zeros in k_r, counted with their multiplicities. Then for any u in k_r we have*

$$|f(u)| \leqslant (r/R)^{(1-b)m/2} \max(|f|_R, |f|_{R^{-1}}),$$

where $b = \log r/\log R$.

Proof. — This is Lemma 2 of [2].

We shall emphasize here again that the transcendence criterion of Proposition 1, which applies for functions in several variables, is established here using merely a Schwarz Lemma in a single variable. The key Lemma is the next, where we use the fact that only a single point $\mathbf{u}$ is considered.

We can now establish the desired upper bound.

LEMMA 4. — *For any $h < c_9$, and for any $s > c_{10}$, we have*

$$\log|\beta| < -c_{11} h^{-1} s \log s.$$

Proof. — Since $|\mathbf{t}| = s$ there is some component t_i of $\mathbf{t}$ with $t_i \geqslant s/d$; for this i we put $t = t_i$ and $u = u_i$, and we denote by $\mathbf{t}'$ and $\mathbf{x}$ the vectors obtained from $\mathbf{t}$ and $\mathbf{u}$ by replacing t by 0 and u by x, respectively. We shall restrict our attention to the function $H(x) = E(\mathbf{x})$ in the single variable x, where we define

$$E(\mathbf{z}) = (\mathbf{t}'!)^{-1} (D^{\mathbf{t}'} G)(\mathbf{z}).$$

We note that

$$\beta = (t!)^{-1} (D^t H)(u),$$

where $D = xd/dx$; from the minimality of $\mathbf{t}$ we deduce that

$$(D^t H)(u) = (x^t d^t/dx^t H)(u).$$

It follows that

$$|\beta| = |(t!)^{-1} (D^t H)(u)| \leqslant |u|^t |(t!)^{-1} (d^t/dx^t H)(u)|.$$

Hence signifying by r the maximum between $|u|+1$ and $|u|^{-1}+1$, we can apply Cauchy's inequality in a disc of radius r^{-2} about u, to obtain

$$|\beta| \leqslant |u|^t r^{2t} \max\{|H(x)|; |x| = r \text{ or } r^{-1}\}.$$

Since $|u| \leqslant r$ and $t \leqslant s$, we deduce from Lemma 3 that

$$|\beta| \leqslant r^{3s}(r/R)^{(1-b)t/2} \max\{|H|_R, |H|_{R^{-1}}\}.$$

But we have $t \geqslant s/d$; hence it follows that

$$\log|\beta| \leqslant 3s\log r - (1-b)^2(s/2d)\log R + \max(\log|H|_R, \log|H|_{R^{-1}}).$$

Now, choosing $R = [s^{1/(d+1)h}]$, we see that the last term above is $< c_{12}\, s\,(\log s)^{1/(d+1)}$. The proof is now complete, noting that $r < c_{13}$ and that $b \log R = \log r$.

Finally it is clear that Lemmas 2 and 4 imply that

$$s\log s > c_{14}\, h^{-1}\, s\log s.$$

It follows that $h > c_{15}$ and we obtain a contradiction to the supposition that each of the functions $f_0, \ldots, f_{d'}$ is of an arbitrarily small order. This is the required contradiction which establishes the Proposition, and, as explained in sɘction 1, also the main Theorem.

It will be noted that the same arguments can furnisch also a complex analogue for the Theorem. Indeed to establish a complex analogue for Proposition 1, we merely have to replace Lemma 3 here by the Hadamard-Schwarz Lemma (*see* [1], Proposition 2). The proof of Proposition 2 here applies also when the valuation is archimedean and as we showed, the Theorem is a formal consequence of the two propositions for a suitably normalized set of generators of the field of meromorphic functions on A. The stability field will now be a subfield of the complex numbers **C**.

I thank here D. BERTRAND for reading an early draft of this work and making several helpful suggestions.

REFERENCES

[1] BERTRAND (D.). — Un théorème de Schneider-Lang sur certains domaines non simplement connexes, *Séminaire Delange-Pisot-Poitou : Théorie des nombres*, 16e année, 1974/1975, n° G 18, 13 p.

[2] BERTRAND (D.). — Séries d'Eisenstein et transcendance, *Bull. Soc. math. France*, t. 104, 1976, p. 309-21.

[3] BERTRAND (D.). — Fonctions abéliennes p-adiques : Définitions et conjectures, *Groupe d'étude d'Analyse ultramétrique*, 4e année, 1976/1977, n° 21, 13 p.

[4] BERTRAND (D.) and FLICKER (Y.). — Linear forms on abelian varieties over local fields, *Acta Arithm.*, Warszawa (to appear).

[5] Bombieri (E.). — Algebraic values of meromorphic maps, *Invent. Math.*, Berlin, t. 10, 1970, p. 267-287.

[6] Flicker (Y.). — *Linear forms on abelian varieties: A sharpening* (to appear).

[7] Flicker (Y.). — *Linear forms on arithmetic abelian varieties: Ineffective bounds* (to appear).

[8] Gerritzen (L.). — On non-archimedean representations of abelian varieties, *Math. Annalen.*, t. 96, 1972, p. 323-346.

[9] Lang (S.). — *Introduction to transcendental numbers.* — Reading, Addison-Wesley, 1966 (*Addison-Wesley Series in Mathematics*).

[10] Morikawa (H.). — Theta functions and abelian varieties over valuation fields of rank one, I, *Nagoya math. J.*, t. 20, 1962, p. 1-27.

[11] Mumford (D.). — An analytic construction of degenerating abelian varieties over complex rings, *Comp. Math.*, Groningen, t. 24, 1972, p. 239-272.

[12] Schneider (T.). — *Einführung in die transzendenten Zahlen.* — Berlin, Springer-Verlag, 1957.