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Explicit realization of a higher metaplectic representation

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0. Let $F\neq\mathbb{C}$ be a local field of characteristic $\neq 2$, and n an integer ≥ 1. Denote by $p:S_{n+1}\to SL(n+1,F)$ the unique non-trivial topological double covering group of $SL(n+1,F)$. Choose a section $\underset{\sim}{s}:SL(n+1,F)\to S_{n+1}$ corresponding to a choice of a two-cocycle $\beta':S_{n+1}\times S_{n+1}\to\ker p$ which defines the group law on S_{n+1}. Put $G_n'=p^{-1}(\iota(\bar{G}_n))$, where ι is the embedding $\bar{G}_n=GL(n,F)\to SL(n+1,F)$, by

$$g\mapsto\begin{pmatrix} g & 0\\ 0 & \det g^{-1}\end{pmatrix}.$$

Let $(.,.):F^\times\times F^\times\to\{\pm 1\}$ be the Hilbert symbol. Identify $\ker p$ with $\{\pm 1\}$. Put $\beta(g,g')=\beta'(g,g')(\det g,\det g')$ $(g,g'\in\bar{G}_n)$. Denote by G_n the group which is equal to G_n' as a set, whose product rule is given by $\underset{\sim}{s}(g)\zeta\underset{\sim}{s}(g')\zeta'=\underset{\sim}{s}(gg')\zeta\zeta'\beta(g,g')$. Let $\bar{A}$ and $\bar{B}$ be the groups of diagonal and upper-triangular matrices in $\bar{G}_n$, and A and B their preimages in G_n. The section $\underset{\sim}{s}:\bar{G}_n\to G_n$ is a homomorphism on the group $\bar{N}$ of upper-triangular unipotent matrices. Put $N=\underset{\sim}{s}(\bar{N})$. Let $\bar{Z}$ be the center of $\bar{G}_n$, and Z the center of G_n. Put $A^2=p^{-1}(\bar{A}^2)$, where $\bar{A}^2$ is the group of squares in $\bar{A}$. Then ZA^2 is the center of A. Put $\underset{\sim}{z}=\underset{\sim}{s}(z)$ for z in $\bar{Z}\simeq F^\times$, and $\underset{\sim}{a}=\underset{\sim}{s}(\bar{a})$ for $\bar{a}=\operatorname{diag}(a_1,\dots,a_n)$ in $\bar{A}$. Note that

$$\underset{\sim}{z}\underset{\sim}{a}=\underset{\sim}{s}(z\bar{a})(z,\prod_{i=1}^{n-1}a_{i+1}^i),\quad \underset{\sim}{a}\underset{\sim}{z}=\underset{\sim}{s}(\bar{a}z)(\prod_{i=1}^{n-1}a_i^{n-i},z).$$

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Then $Z=p^{-1}(\bar{Z}^2)=A^2\cap p^{-1}(\bar{Z})$ when n is even. When n is odd then $Z=p^{-1}(\bar{Z})$.

Define a character $\bar{\delta}=\bar{\delta}_n:\bar{A}\to\mathbb{C}^\times$ by $\bar{\delta}(\mathrm{diag}(a_i))=\prod_{1\le i\le n}|a_i|^{i-(n+1)/2}$. Given a non-trivial additive character $\psi:F\to\mathbb{C}^\times$, define the function $\gamma=\gamma_\psi:F^\times\to\mathbb{C}^\times$ by

$$\gamma(a)=|a|^{1/2}\int\psi(-ax^2/2)dx/\int\psi(-x^2/2)dx;$$

dx is a Haar measure on F. Then γ is trivial on $F^{\times 2}$, and satisfies $\gamma(a)\gamma(b)=\gamma(ab)(a,b)$ (see [W], p. 176). The function $Z\to\mathbb{C}^\times$, $\zeta\underline{s}(z)\mapsto\zeta\gamma(z^{n(n-1)/2})$ (ζ in $\ker p$, z in $F^\times\simeq\bar{Z}$), is a character. Define the function $\delta=\delta_{\psi,n}:ZA^2\to\mathbb{C}^\times$ by

$$\delta(\zeta\underline{s}(za^2))=\zeta\gamma(z^{(n-1)n^2/2})\bar{\delta}(a)\quad(\zeta\in\ker p, z\in\bar{Z}\simeq F^\times, a\in\bar{A}).$$

There exists a unique (up to isomorphism) irreducible representation $\varrho=\varrho_{\psi,n}$ of A whose restriction to ZA^2 is δ. Extend ϱ to a representation of B trivial on N. Let $(\pi, V^{\mathrm{ind}})=(\pi_{\psi,n}, V^{\mathrm{ind}}_{\psi,n})$ be the G_n-module normalizedly (see [BZ2], (1.8)) induced from ϱ. Then (π, V^{ind}) has a unique irreducible subrepresentation (see [KP1], p. 72), denoted by $(\Theta, V^{\mathrm{sub}})=(\Theta_{\psi,n}, V^{\mathrm{sub}}_{\psi,n})$. This Θ, which is sometimes called exceptional, or unipotent, corresponds to the trivial $\bar{G}$-module $\mathbb{1}$ by the metaplectic correspondence of [FK] (when $n=3$; for $n>3$, the statement $\Theta_n\to\mathbb{1}_n$ follows from a certain conjecture concerning orbital integrals, see [FK], p. 67, [KP2] and also Hales [H] and Waldspurger [Wa]). By [KP1], Thm II.2.1, p. 118, this Θ is unitarizable.

The representation Θ of the *two*-fold covering group G_n of $\bar{G}_n=GL(n,F)$ is probably the most natural generalization of the Weil representation [W] of the two-fold covering of the symplectic group $Sp(n)$. Indeed, it has recently been used (by Patterson and Piatetski-Shapiro [PS] when $n=3$, and then for general n by D. Ginzburg (his proof was later simplified by Flicker-Rallis)) to construct an integral presentation of the symmetric square L-function attached to a cuspidal representation of $GL(n)$. Analogous representations of higher fold covering groups of $GL(n)$ have not yet been found to afford such meaningful applications.

The purpose of this paper is to construct an explicit model of $\Theta=\Theta_n$ *for all* $n\ge 3$, *and determine the unique unitary structure of* Θ, *thus generalizing the Theorem of* [FKS], *using the methods of* [FKS], *from the context of* $n=3$ *to that of any* $n\ge 3$.

In fact, as in [FKS] we construct a model of the extension of Θ to the semi-direct product $G^\#=G\rtimes\langle\sigma\rangle$, where σ is an involution of G defined as follows. Let $w=w(n)$ be the anti-diagonal matrix $((-1)^{i+1}\delta_{i,n+1-j})$ in $\bar{G}$, considered as an element of $SL(n+1,F)$ via ι. Denote by $\bar{\sigma}$ the involution $\bar{\sigma}(g)=w^{-1}\,{}^tg^{-1}w$ of $SL(n+1,F)$. The Steinberg group $St(n+1,F)$ is generated by elementary matrices (see [M], p. 39). Since $\bar{\sigma}$ maps elementary matrices to elementary matrices, and it preserves the relations which define $St(n+1,F)$, it lifts to an involution of $St(n+1,F)$, hence to an involution $\tilde{\sigma}$ of G. Since

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & a^{-1} \\ -a & 0 \end{pmatrix} = u(-1)d(1)u(-1)u(a^{-1})d(-a)u(a^{-1}),$$

where

$$u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad d(x) = {}^t u(x),$$

it is easy to check that

$$\tilde{\sigma}(\underline{s}(\operatorname{diag}(a_j))) = \underline{s}(\operatorname{diag}(a_{n+1-j}^{-1})) \cdot \prod_{i=1}^{n-1} \Big(\prod_{j=i+1}^{n} a_j, a_i \Big).$$

Hence

$$\tilde{\sigma}(\underline{s}(z)) = \underline{s}(z^{-1})(-1, z)^{n(n-1)/2} \text{ for } z \in F^\times \simeq \bar{Z}.$$

Put $\sigma(g) = (-1, \det p(g))^{(n-1)/2} \tilde{\sigma}(g)$. Then $\sigma \circ \underline{s} = \underline{s} \circ \bar{\sigma}$ on $\bar{Z}\bar{A}^2$, hence $\delta_{\psi,n} \circ \sigma = \delta_{\psi,n}$ on ZA^2. Consequently $\varrho_{\psi,n} \circ \sigma \simeq \varrho_{\psi,n}$ on A, and $\pi_{\psi,n} \circ \sigma \simeq \pi_{\psi,n}$ on G. We conclude that $\Theta_{\psi,n} \circ \sigma \simeq \Theta_{\psi,n}$, namely there exists a non-zero operator $I: V^{\text{sub}} \to V^{\text{sub}}$ such that $\Theta(g)I = I\Theta(\sigma g)$ for all g in G. Since Θ is irreducible, I^2 is a scalar by Schur's lemma. Multiplying I by a scalar we may assume that $I^2 = Id$. This determines I uniquely up to a sign. The choice $\Theta(\sigma) = I$ determines an extension of Θ to the semi-direct product $G^\# = G \rtimes \langle \sigma \rangle$. It is this extension of Θ to $G^\#$ whose model we construct.

1. To state the Theorem we need more notations. Consider $\bar{G}_j = GL(j, F)$, for $1 \le j \le n$, as a subgroup of $\bar{G}_n$, via

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & I_{n-j} \end{pmatrix}.$$

Then $G_j = p^{-1}(\bar{G}_j)$ is a subgroup of G_n, and G_1 is the direct product of $F^\times$ and $\ker p$. Put $H_j = G_j \underline{s}(\bar{Z}_n)$.

A *genuine* representation ϱ of a subgroup H of $G = G_n$ is one which satisfies $\varrho(h\zeta) = \zeta\varrho(h)$ for ζ in $\ker p, h$ in H. Let (Θ_1, V_1) be the genuine representation of G_1 which is trivial on $F^\times$.

Let $\bar{P}_j$ $(2 \le j \le n)$ be the upper-triangular parabolic subgroup of $\bar{G}_j$ of type $(j-1, 1)$. Let $\bar{U}_j$ be the unipotent radical of $\bar{P}_j$. Put $P_j = p^{-1}(\bar{P}_j \bar{Z}_n)$ and $U_j = \underline{s}(\bar{U}_j)$. Then $P_j = H_j U_j$. Consider the surjection $pr_j: P_j \to F^{j-1} - \{\underline{0}\}$, $(p_{ab}) \to (p_{j-1,b};\ 1 \le b < j)$. It yields an isomorphism $P_{j-1}U_j \backslash P_j \simeq F^{j-1} - \{\underline{0}\}$. Denote by e_k the row vector of length j whose only non zero entry is 1, at the kth place. Define a section $s_j: F^{j-1} - \{\underline{0}\} \to P_j$ by $s_j(x_1^{(j)}, \ldots, x_{j-1}^{(j)}) = \underline{s}(A)$, where if $x_1^{(j)} = \cdots = x_i^{(j)} = 0$, $x_{i+1}^{(j)} \neq 0$, then A is the j by j matrix whose rows, from top to bottom, are $e_1, \ldots, e_i, e_{i+2}, \ldots, e_{j-1}$, $(0, \ldots, 0, x_{i+1}^{(j)}, \ldots, x_{j-1}^{(j)}, 0), e_j$.

As in [BZ1] we denote by Ind the functor of (unnormalized) induction, and by ind the functor of induction with compact supports. Denote by I and i normalized (as in [BZ2]) induction; thus

$$i_{GH}\varrho = i(\varrho; G, H) = \operatorname{ind}((\delta_H/\delta_G)^{1/2}\varrho; G, H),$$

where ϱ is an H-module, and $\Delta_H = \delta_H^{-1}$, $\Delta_G = \delta_G^{-1}$ are defined in [BZ2], p. 444.

Our definition of the model (Θ_n, V_n) of the G_n-module $(\Theta_n, V_n^{\text{sub}})$ is inductive. Given any model $(\Theta_{n-2}, V_{n-2}^a)$ of Θ_{n-2}, let V_{n-1}^b be the space of smooth genuine functions $f_0: G_{n-1} \to V_{n-2}^a$ which satisfy

$$f_0(\zeta \underline{s}(z) g_{n-2} u g_{n-1}) = \zeta |z|^{-(n-1)/4} \Theta_{n-2}(g_{n-2}) f_0(g_{n-1})$$

where

$$g_i \in G_i,\ u \in U_{n-1},\ z \in p(Z_{n-1}),\ \zeta \in \ker p.$$

The homogeneous space $U_{n-1}G_{n-2}Z_{n-1} \setminus G_{n-1}$ is compact. Thus, putting $\nu(t) = |t|$, we have

$$\begin{aligned} V_{n-1}^b &= \operatorname{ind}(V_{n-2}^a \times \nu^{-(n-1)/4}; G_{n-1}, U_{n-1}Z_{n-1}G_{n-2}) \\ &= i(V_{n-2}^a \otimes \nu^{-1/2} \times \nu^{(n-3)/4}; G_{n-1}, U_{n-1}Z_{n-1}G_{n-2}); \end{aligned}$$

here i is the normalized induction, while ind is not normalized.

Let G_{n-1} act on V_{n-1}^b by $\varrho(g) f_0(h) = |\det p(g)|^{1/2} f_0(hg)$. Then the G_{n-1}-module (ϱ, V_{n-1}^b) is isomorphic to

$$\nu^{1/2} \otimes \operatorname{ind}(V_{n-2}^a \times \nu^{-(n-1)/4}) = \nu^{1/4} \otimes i(V_{n-2}^a \otimes \nu^{-1/4} \times \nu^{(n-2)/4}).$$

LEMMA 1. *The G_{n-1}-module $i(\Theta_{n-2} \otimes \nu^{-1/4} \times \nu^{(n-2)/4})$ has a unique irreducible submodule.*

PROOF. The G_{n-2}-module Θ_{n-2} is, by definition, the unique irreducible submodule of π_{n-2}. Since the functor i is exact, $i(\Theta_{n-2} \otimes \nu^{-1/4} \times \nu^{(n-2)/4})$ is a submodule of $\pi_{n-1} = i(\pi_{n-2} \otimes \nu^{-1/4}, \nu^{(n-1)/4})$. Since Θ_{n-1} is the unique irreducible submodule of π_{n-1}, the lemma follows.

Denote by $(\Theta_{n-1} \otimes \nu^{1/4}, V_{n-1}^c)$ the unique irreducible submodule of (ϱ, V_{n-1}^b). Denote the element

$$\begin{pmatrix} 1 & & 0 & u_1 \\ & \ddots & & \vdots \\ & & 1 & u_{n-1} \\ 0 & & & 1 \end{pmatrix}$$

of U_n by $u(u_1, \ldots, u_{n-1})$. Let V_n^d be the space of smooth genuine functions $f: P_n \to \bar{V}_{n-2}^a$ which satisfy

$$(\underline{0})\qquad f(\zeta \underline{s}(z) g_{n-2} u' u p_n) = \zeta \psi(u_{n-1}) \gamma(z^{n(n-1)/2}) \Theta_{n-2}(g_{n-2}) f(p_n),$$

where

$$p_n \in P_n,\ g_i \in G_i,\ u' \in U_{n-1},\ z \in p(Z_n),\ \zeta \in \ker p,\ u = u(u_1, \ldots, u_{n-1}) \in U_n,$$

and are compactly supported on $U_n P_{n-1} \setminus P_n$. Let V_n^e be the space of f in V_n^d such that there exist $A_f > 0$ and f_0 in V_{n-1}^c, with $f(p) = f_0(p)$ on the $p = s_n(x_1, \ldots, x_{n-1})$ in P_n which satisfy $\max(|x_i|; 1 \le i < n) \le A_f$. Let $\delta_P: P \to \mathbb{R}_{>0}^\times$ be the character which maps $p \in P$ to the absolute value of the Jacobian of $u \mapsto pup^{-1}$, $U \to U$. Then V_n^e is a genuine P_n-module under the action

$$(\underline{1})\qquad \Theta_n(\zeta g) f(p) = \zeta \delta_P(g)^{1/2} f(pg) \quad (p \in P_n, g \in P_{n-1}, \zeta \in \ker p).$$

In particular

$$(2) \qquad \Theta_n(u)f(p)=\psi\Big(\sum_{1\le i<n} u_i x_i\Big)f(p)$$

for all

$$u=u(u_1,\dots,u_{n-1})\in U_n,\ pr_n(p)=(x_i;\ 1\le i<n),$$

and

$$\Theta_n(\underline{s}(z))f(x_1^{(n)},\dots)=(x_1^{(n)},z)^{n-1}\gamma(z^{n(n-1)/2})f(x_1^{(n)},\dots)\quad (z\in F^\times\simeq\bar{Z}).$$

We will define V_n as a space of smooth genuine (in each variable) functions

$$f: P_n\times\cdots\times P_{n-2j}\times\cdots\to\mathbb{C}$$

which satisfy

$$f(p_n,\dots,qu'up_{n-2j},p_{n-2j-2},\dots)$$
$$=\psi(u_{n-2j-1})\delta_{P_{n-2j-2}}(q)^{1/2}f(p_n,\dots,p_{n-2j},p_{n-2j-2}q,\dots),$$

for any $0\le j\le(n-2)/2$ and

$$q\in P_{n-2j-2},\ u=u(u_1,\dots,u_{n-2j-1})\in U_{n-2j},\ u'\in U_{n-2j-1},\ p_i\in P_i.$$

Moreover, for every j $(1\le j\le(n-2)/2)$ there is an operator $k_j: V_n\to V_n$ such that

$$(k_j f)(p_n,\dots,p_{n-2j+2},p_{n-2j},\dots)$$
$$=\int f(p_n,\dots,p_{n-2j+2},q_{n-2j},\dots)K_j(q_{n-2j},\dots;p_{n-2j},\dots)\prod_{i\ge j} dq_{n-2i}$$

for some kernel $K_j:(P_{n-2j}\times\dots)\times(P_{n-2j}\times\dots)\to\mathbb{C}$, with the following property: For every f in V_n, we have

$$f(p_n,\dots,\underline{s}(w(n-2j))p_{n-2j},p_{n-2j-2},\dots)$$
$$=(k_j f)(p_n,\dots,p_{n-2j+2},p_{n-2j},\dots).$$

Since G_{n-2j} is generated by P_{n-2j} and $\underline{s}(w(n-2j))$, such a function is completely determined by its values on

$$(3)\qquad \begin{cases} f(x_1^{(n)},\dots,x_{n-1}^{(n)};\dots;x_1^{(n-2j)},\dots,x_{n-2j-1}^{(n-2j)};\dots)\\ \quad =f(s_n(x_1^{(n)},\dots,x_{n-1}^{(n)});\dots;s_{n-2j}(x_1^{(n-2j)},\dots,x_{n-2j-1}^{(n-2j)});\dots).\end{cases}$$

Note that (Θ_1,V_1) is always taken to be the genuine G_1-module which is trivial on $\underline{s}(\bar{G}_1)$, and (Θ_0,V_0) is the genuine representation of $G_0=\ker p$. Having defined the G_{n-2}-module (Θ_{n-2},V_{n-2}) $(n>1)$, using $V_{n-2}^a=V_{n-2}$ we obtain the G_{n-1}-module $(\Theta_{n-1}\otimes\nu^{1/4},V_{n-1}^c)$ and the P_n-module (Θ_n,V_n^e).

Put $V_n=V_n^e$. Define an operator J on V_n by

$$(4)\qquad \begin{cases}(Jf)(\dots;x_1^{(n-2j)},\dots,x_{n-2j-1}^{(n-2j)};\dots)\\ \quad =\big[\prod_{0\le j\le n/2-1}(|x_1^{(n-2j)}|^{j+1-n/2}/\gamma(x_1^{(n-2j)}))\big]\cdot(J_N f)(\dots),\end{cases}$$

where

$$(J_N f)(\dots) = \int f(\dots; -x_1^{(n-2j)}, y_1^{(n-2j)}, \dots, y_{n-2j-2}^{(n-2j)}; \dots)$$
$$\cdot \psi[\sum_{0 \le j \le n/2-1} \sum_{1 \le i \le n-2j-2} (-1)^{i-1} y_i^{(n-2j)} x_{n-2j-i}^{(n-2j)} / x_1^{(n-2j)}]$$
$$\cdot \prod_{i,j} dy_i^{(n-2j)}.$$

Note that when $n \ge 3$ the group $G_n^{\#} = G_n \rtimes \langle \sigma \rangle$ is generated by P_n and σ. It suffices to show that V_n is a $G_n^{\#}$-module, isomorphic to V_n^{sub}. For then V_n is a G_n-module, isomorphic to V_n^{sub}, and we obtain an inductive process, beginning with (Θ_1, V_1), or (Θ_2, V_2), to define an explicit realization of $(\Theta_3, V_3), (\Theta_5, V_5), \dots$, or $(\Theta_4, V_4), (\Theta_6, V_6), \dots$, and finally (Θ_n, V_n), by means of unitary operators.

THEOREM. (i) *The space V_n is isomorphic to V_n^{sub}.*

(ii) *There exists $c \neq 0$, unique up to a sign, and a representation (denoted Θ_n) of $G_n^{\#}$ on V_n, given by (1)-(2) on P_n and with $\Theta_n(\sigma) = cJ$.*

(iii) *The $G_n^{\#}$-module (Θ_n, V_n) is isomorphic to $(\Theta_n, V_n^{\text{sub}})$.*

(iv) *By (3), the space V_n can be regarded as a subspace of $L^2(F^{n-1} \times \dots \times F^2)$. Then up to a scalar there exists a unique Hermitian scalar product on the unitarizable G_n-module (Θ_n, V_n). It is given by L^2-product.*

REMARK. (i) When $n=2$ the Theorem has been worked out in [FM]. For $n=3$ the Theorem coincides with the Theorem of [FKS]; our proofs generalize those of [FKS], and put the example of [FKS] in a general framework. It is likely to have applications in harmonic analysis as in [FKS], but these will not be discussed here.

(ii) By (iv), the unitary completion of (Θ_n, V_n) is $(\Theta_n, L^2(F^{n-1} \times \dots \times F^2))$, where Θ_n acts by (1)-(2) on P_n, and by $\Theta_n(\sigma) = cJ$. When $n=3$ and $F = \mathbb{R}$, the unitary G_3-module $(\Theta_3, L^2(\mathbb{R}^2))$, or at least its restriction to $p^{-1}(SL(3, \mathbb{R}))$, was first constructed by Torasso [T].

(iii) The explicit realization (Θ_n, V_n) of the G_n-module Θ_n is analogous to the realization of the representation of the two-fold covering group $\tilde{Sp}$ of the symplectic group Sp in Weil [W]. See the example below for the case of $n=2$. As noted in Section 0, our Θ_n is the most useful (to the theory of L-functions) analogue of the representation of [W], in the context of covering groups of $GL(n)$. Some experts are more attracted to the analogue for the n-fold cover of $GL(n)$.

(iv) Our proofs are inductive, passing from n to $n+2$. Hence the study of Θ_n for odd n is completely independent of the study of Θ_n for even n, and vice versa.

EXAMPLE. As noted in [FM], when $n=2$, our model is easily obtained from the well-known model of the genuine $p^{-1}(SL(2, F))$-model ϱ constructed in Weil [W]. To see this, recall that there is a choice $\gamma(-1)^{-1/2}$ of a square-root of $\gamma(-1)^{-1}$, such that ϱ of [W] acts on the space of even ($\varphi(-t)=\varphi(t)$) smooth compactly-supported complex-valued functions φ on F, by

$$\varrho\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\varphi(t)=\psi(bt^2/2)\varphi(t), \qquad \varrho\left(\underline{s}\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\right)\varphi(t)=\gamma(a)|a|^{1/2}\varphi(at),$$

$$\varrho\left(\underline{s}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)\varphi(t)=\gamma(-1)^{-1/2}\check{\varphi}(-t)=\gamma(-1)^{-1/2}\int\varphi(y)\psi(-yt)dy$$

$(b\in F, a\in F^{\times})$.

Since $p^{-1}(\bar{Z})$ is the center of $p^{-1}(\bar{Z}\cdot SL(2,F))$, this ϱ extends to a $p^{-1}(\bar{Z}\cdot SL(2,F))$-module by $\varrho(\underline{s}(z))\varphi=\gamma(z)\varphi$ ($z\in\bar{Z}\simeq F^{\times}$); note that φ is assumed to be even. Then Θ_2 is $\mathrm{ind}(\varrho\otimes\nu^{1/2}; G_2, p^{-1}(\bar{Z}\cdot SL(2,F)))$. Choosing the section

$$x\mapsto\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}$$

to the isomorphism $p^{-1}(SL(2,F))\setminus G_2\to F^{\times}$, $g\mapsto\det p(g)$, the space of Θ_2 consists of $f: F^{\times}\times F\to\mathbb{C}$ with $f(x,t)=|t|^{1/2}f(xt^2,1)$ (note that f is even in t). Putting $f(x)=f(x,1)$, the group G_2 acts as follows:

$$\Theta\left(\underline{s}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)f(x)=|a|^{1/2}f(ax), \qquad \Theta\left(\underline{s}\begin{pmatrix}z & 0\\ 0 & z\end{pmatrix}\right)f(x)=(x,z)\gamma(z)f(x),$$

$$\Theta\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}f(x)=\psi(bx/2)f(x),$$

$$\Theta\left(\underline{s}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)f(x)=\gamma(-1)^{-1/2}\gamma(x)|x|^{1/2}\int_F|y|^{1/2}f(xy^2)\psi(xy)dy.$$

Since $\varphi\in\varrho$ is locally constant on F, in particular it is constant in some neighborhood of 0 in F. Hence $f(x,t)$ is constant near $t=0$ for each fixed x, and for every $f\in\Theta$ there is A_f, and $f_0: F^{\times}\to\mathbb{C}^{\times}$ satisfying $f_0(xa^2)=|a|^{-1/2}f_0(x)$ ($x, a\in F^{\times}$), such that $f(x)=f_0(x)$ for $|x|\leq A_f$. This determines the space of Θ, and the action of G. It is easy to check that

$$(\Theta(\sigma)f)(x)=\gamma(-1)^{1/2}\gamma(x)^{-1}f(-x)$$

defines an extension of Θ from G to $G^{\#}=G\rtimes\langle\sigma\rangle$, unique up to a sign, which is consistent with (4̲).

Of course the unitary structure on $\Theta=\{f\}$ is given by

$$\langle f,f'\rangle=\int_F f(x)\bar{f}'(x)dx.$$

2. In the proof we use the functor r of coinvariants. Let $\psi_U: U\to\mathbb{C}^{\times}$ be a character (possibly degenerate) of the unipotent radical U of a parabolic subgroup P of G. Let V be a smooth P-module, and put

$$V_{U,\psi_U}=V/\langle\pi(u)v-\psi_U(u)v;\ v\in V, u\in U\rangle.$$

It is a $\mathrm{Stab}_{\psi_U}(P)$-module. Put $r_{U,\psi_U}V = \delta_P^{-1/2} \otimes V_{U,\psi_U}$. Then r_{U,ψ_U} is the normalized functor of coinvariants, from the category $\mathbb{M}(P)$ of smooth P-modules, to the category $\mathbb{M}(\mathrm{Stab}_{\psi_U}(P))$ of smooth $\mathrm{Stab}_{\psi_U}(P)$-modules. Of course, when ψ_U is trivial, U acts trivially on $V_U = V_{U,\psi_U}$ and r_{U,ψ_U}, which we now denote by r_U, maps $\mathbb{M}(P)$ to $\mathbb{M}(P/U)$.

The proof of the Theorem is based on a study of the restriction $\tau = \Theta|P$ of the G_n-module $(\Theta_n, V_n^{\mathrm{sub}})$ to its subgroup $P = P_n$. The space of τ is $V = V_n^{\mathrm{sub}}$. Put $U = U_n$, and denote by $\psi = \psi_U$ the character $u(u_1, \ldots, u_{n-1}) \mapsto \psi(u_{n-1})$ of U; here $\psi : F \to \mathbb{C}^\times$ is the non-trivial character of F fixed in the definition of $\Theta = \Theta_\psi$. In particular we have the (normalized) functors

$$r_U : \mathbb{M}(P_n) \to \mathbb{M}(G_{n-1}), \qquad r_{U,\psi} : \mathbb{M}(P_n) \to \mathbb{M}(P_{n-1}),$$

of coinvariants, and

$$i_U : \mathbb{M}(G_{n-1}) \to \mathbb{M}(P_n), \qquad i_{U,\psi}, I_{U,\psi} : \mathbb{M}(P_{n-1}) \to \mathbb{M}(P_n)$$

of induction, as in [BZ2], § 3. The k-th derivative $\tau^{(k)} \in \mathbb{M}(G_{n-k})$ (here $1 \le k < n$) of τ is defined to be $r_U \circ r_{U,\psi}^{k-1}\tau$. The P_n-module τ has a composition series (see [BZ2], (3.5))

$$\tau_{n+1} = 0 \subset \tau_n \subset \ldots \subset \tau_1 = \tau, \text{ where } \tau_k = i_{U,\psi}^{k-1} \circ r_{U,\psi}^{k-1}(\tau) \in \mathbb{M}(P_n).$$

The composition factors are

$$\tau_k/\tau_{k+1} = i_{U,\psi}^{k-1} \circ i_U(\tau^{(k)}) = i_{U,\psi}^{k-1} \circ i_U \circ r_U \circ r_{U,\psi}^{k-1}(\tau) \in \mathbb{M}(P_n).$$

For any k $(1 \le k < n)$, let $r_{(n-k,k)}$ denote the normalized functor of coinvariants with respect to the standard (containing B) parabolic subgroup $P_{n-k,k}$ of G_n of type $(n-k,k)$. It maps $\mathbb{M}(G_n)$ to $\mathbb{M}(G_{n-k} \times G_k)$, and $\mathbb{M}(P_n)$ to $\mathbb{M}(G_{n-k} \times P_k)$. Similarly introduce $\delta_{(n-k,k)}$.

LEMMA 2. (i) *For each k $(1 \le k < n)$ we have*

$$r_{(n-k,k)}\Theta_n = \nu^{-k/4} \otimes \Theta_{n-k} \times \nu^{(n-k)/4} \otimes \Theta_k.$$

(ii) *The dimension of $r_{U,\psi}^{k-1}\Theta_k$ is one if $k=2$ and zero if $k>2$.*

PROOF. (i) The functor $r_N : \mathbb{M}(G_n) \to \mathbb{M}(A_n)$, where as usual N is the unipotent radical of $B = B_n$, yields an equivalence of the category $\mathbb{M}(A_n)$ with the subcategory $\mathbb{M}_{A_n}(G_n)$ of $\mathbb{M}(G_n)$ consisting of the G_n-modules whose irreducible constituents are all subquotients of G_n-modules of the form $i(\varrho; G_n, A_nN)$, where $\varrho \in \mathbb{M}(A_n)$ is extended trivially on N. Similarly the functor $r_k = r_{N_{n-k}} \times r_{N_k}$ establishes an equivalence of $\mathbb{M}_{A_{n-k}}(G_{n-k}) \times \mathbb{M}_{A_n}(G_n)$ with $\mathbb{M}(A_n)$. Since the functor r is transitive, we have $r_k \circ r_{(n-k,k)} = r_N$. The A_n-module $r_N\Theta_n$ is computed in [KP1], Thm I.2.9(e); it is irreducible. Using [KP1], Thm I.2.9(e), it is easy to see that $r_k[r_{(n-k,k)}(\Theta_n)]$ is equivalent to $r_k[\nu^{-k/4} \otimes \Theta_{n-k} \times \nu^{(n-k)/4} \otimes \Theta_k]$, and so (i) follows.

(ii) According to [KP1], p. 74, the space $r_{U,\psi}^{k-1}\Theta_k$ is dual to the space $Wh(\Theta_k)$ of Whittaker functionals on Θ_k, and by [KP1], Cor. I.3.6, dim $Wh(\Theta_k)$ is 1 if $k=2$ and 0 if $k>2$, as asserted.

REMARK. Cor. I.3.6 of [KP1] is claimed only for F with $|2|=1$, but the proof there extends also to F with $|2|\neq 1$ once it is shown that Θ_k corresponds to the trivial $\bar{G}_k$-module via the metaplectic correspondence. This correspondence is reduced (for $k>3$) in [KP2] and [FK] to a certain conjecture concerning non-metaplectic orbital integrals. Progress towards a proof of this conjecture has recently been announced by Hales [H] and Waldspurger [Wa].

PROPOSITION 1. *There is an exact sequence* $0\to V_0\to V\to V_U\to 0$ *of* $P=P_n$-*modules. The P-module* V_U *is isomorphic to* $\nu^{1/4}\otimes\Theta_{n-1}$ *as a* G_{n-1}-*module; it is irreducible. The P-module* V_0 *is irreducible; it is equivalent to*

$$(5)\quad \begin{cases} V_0=\delta_P^{1/2} \\ \qquad\otimes\operatorname{ind}\left[\zeta\underline{s}(z)\begin{pmatrix} g & * & * \\ 0 & 1 & x \\ 0 & 0 & 1\end{pmatrix}\mapsto\zeta\gamma(z^{n(n-1)/2})\psi(x)\Theta_{n-2}(g);\, P,\, P'U\right];\end{cases}$$

here $P'=P_{n-1}$.

PROOF. By Lemma 2(i), $r_U V$ is $\nu^{-1/4}\otimes\Theta_{n-1}$ as a G_{n-1}-module, and $\delta_P^{1/2}=\nu^{1/2}\otimes I_{n-1}$ as a G_{n-1}-module. Hence $V_U=\nu^{1/4}\otimes\Theta_{n-1}$.

Proposition 3.2(e) of [BZ2] asserts that the kernel V_0 of the P-module morphism $V\to V_U$ is τ_2. Since the functor r is transitive we have (by Lemma 2(i))

$$r_U\circ r_{U,\psi}^{k-1}\tau=\nu^{-k/4}\otimes\Theta_{n-k}\times\nu^{(n-k)/4}\otimes r_{U,\psi}^{k-1}\Theta_k.$$

Lemma 2(ii) asserts that $r_{U,\psi}^{k-1}\Theta_k$ is zero for $k\geq 3$. Hence $\tau_k=0$ for $k\geq 3$, and $\tau_2=\tau_2/\tau_3$. Consequently

$$\begin{aligned} V_0=\tau_2/\tau_3&=i_{U,\psi}\circ i_U\circ r_U\circ r_{U,\psi}(\tau)\\ &=\operatorname{ind}(\delta_{(n-2,2)}^{1/2}\otimes[\nu^{-1/2}\Theta_{n-2}\times r_{U,\psi}(\nu^{(n-2)/4}\otimes\Theta_2)];\, P,\, P_{(n-2,2)}).\end{aligned}$$

As

$$\delta_{(n-2,2)}^{1/2}=\nu\otimes I_{n-2}\times\nu^{-(n-2)/2}\otimes I_2,$$

and

$$\delta_P^{1/2}=\delta_{(n-1,1)}^{1/2}=\nu^{1/2}\otimes I_{n-1}\times\nu^{-(n-2)/4},$$

we have

$$\begin{aligned} V_0&=\operatorname{ind}\left[\zeta\underline{s}(z)\begin{pmatrix} g & * & * \\ 0 & 1 & x \\ 0 & 0 & 1\end{pmatrix}\mapsto\zeta\gamma(z^{n(n-1)/2})\psi(x)(\nu^{1/2}\otimes\Theta_{n-2})(g);\, P,\, P'U\right]\\ &=\delta_P^{1/2}\otimes\operatorname{ind}(\psi\otimes\Theta_{n-2};\, P,\, P'U).\end{aligned}$$

Since the stabilizer in P of the character

$$\begin{pmatrix} I_{n-1} & & & * \\ & & & x \\ 0 & \dots & 0 & 1\end{pmatrix}\mapsto\psi(x)$$

of U is $P'U$, and $\psi\otimes\Theta_{n-2}$ is an irreducible $P'U$-module, V_0 is irreducible by Mackey's Theorem 4.2(i) of [FKS], as required.

3. As in [FKS], (4.1), given a group H and a smooth H-module $V=V(H)$, let $V'(H)$ be the Hermitian dual of V, namely the smooth H-module obtained on conjugating the complex structure of the smooth dual of V. Write V' for $V'(H)$ when H is specified. Note that an H-invariant Hermitian form on V is equivalent to an H-invariant map from V to V'.

In our case, since (Θ, V) is unitarizable we obtain a sequence

$$V_0 \to V \to V' \to V_0'$$

of P-modules. Here $V'=V'(P)$, $V_0'=V_0'(P)$. Mackey's Theorem 4.2(iv) of [FKS] implies that

$$[\delta_P^{1/2}\otimes \operatorname{ind}(\psi\otimes\Theta_{n-2}; P, P'U)]'$$

$$=\delta_P^{-1/2}\otimes \operatorname{Ind}[(\delta_{P'U}/\delta_P)\otimes(\bar\psi^{-1}\otimes\Theta_{n-2}'); P, P'U]$$

as P-modules. Since $\Theta_{n-2}'=\Theta_{n-2}$ (as G_{n-2}-modules), and $\psi^{-1}=\bar\psi$, and $\delta_{P'U}/\delta_P=\delta_{P'}=\delta_P$ on $P'U$, we conclude that

$$V_0'=\delta_P^{1/2}\otimes \operatorname{Ind}(\psi\otimes\Theta_{n-2}; P, P'U). \tag{6}$$

We shall now show that V is a P-submodule of V_0' and later characterize V in V_0'.

PROPOSITION 2. (i) *The composition* $\varphi: V\to V'\to V_0'$ *is an embedding. Moreover, the map* $V'\to V_0'$ *is also an embedding.*

(ii) *We have* $\operatorname{Hom}_P(V_0, V_0')=\mathbb{C}$. *In particular the restriction of* φ *to* V_0 *is a multiple of the natural inclusion*

$$\delta_P^{1/2}\otimes \operatorname{ind}(\psi\otimes\Theta_{n-2}; P, P'U)\hookrightarrow \delta_P^{1/2}\otimes \operatorname{Ind}(\psi\otimes\Theta_{n-2}; P, P'U).$$

PROOF. (i) The kernel of φ consists of all $v\in V$ such that $\langle v, v_0\rangle=0$ for all $v_0\in V_0$. Since $V_0=\ker(V\to V_U)$ is spanned by the vectors $v-\Theta(u)v$, $v\in V$, $u\in U$, the space $\ker\varphi$ consists of vectors fixed under the action of U. The claim then follows from the following variant of a result of Howe-Moore [HM], Prop. 5.5, p. 85.

LEMMA 3. *Let G be a covering group of $GL(n, F)$, and V a non-trivial irreducible unitarizable G-module. Then the only vector in V fixed by a one-parameter additive subgroup of G is the zero vector.*

The injectivity of $V'\to V_0'$ follows analogously.

(ii) By (6) and Frobenius reciprocity (see [BZ2], (1.9(b)), p. 445), we have

$$\operatorname{Hom}_P(V_0, V_0')=\operatorname{Hom}_{P'U}((V_0)_{U,\psi_U}, \delta_{P'}^{1/2}\otimes(\psi\otimes\Theta_{n-2})).$$

Since the functor of coinvariants is exact we have $(V_0)_{U,\psi_U} = V_{U,\psi_U}$. As in the proof of Proposition 1, we have $r_{U,\psi_U}V = i_U \circ r_U \circ r_{U,\psi_U}V$. By Lemma 2(i), we have

$$r_{(n-2,2)}\Theta_n = \nu^{-1/2} \otimes \Theta_{n-2} \times \nu^{(n-2)/4} \otimes \Theta_2.$$

Since i_U is simply multiplication by $\delta_{P'}^{1/2}$, and $r_{U,\psi_U}\Theta_2 = \psi_U$, we conclude that $r_{U,\psi_U}V = \Theta_{n-2} \otimes \psi_U$. Hence $V_{U,\psi_U} = \delta_P^{1/2} \otimes r_{U,\psi_U}V = \delta_{P'}^{1/2} \otimes (\psi \otimes \Theta_{n-2})$. Consequently

$$\mathrm{Hom}_P(V_0, V_0') = \mathrm{Hom}_{P'U}(V_{U,\psi_U}, V_{U,\psi_U}),$$

and this is one-dimensional since Θ_{n-2} is irreducible. Hence (ii) follows.

We can now describe V as a P-submodule of V_0'.

PROPOSITION 3. *The space of V consists of all f in the P-module $\delta_P^{1/2} \otimes \mathrm{Ind}(\psi \otimes \Theta_{n-2}; P, P'U)$ for which there is $A_f > 0$, and f_0 in the unique irreducible subspace $\Theta_{n-1} \otimes \nu^{1/4}$ (see Lemma 1) of*

$$i(\Theta_{n-2} \otimes \nu^{-1/4} \times \nu^{(n-2)/4}) \otimes \nu^{1/4},$$

such that $f = f_0$ on the $p = s_n(x_1, \dots, x_{n-1})$ in P with $\max(|x_i|; 1 \le i < n) \le A_f$.

PROOF. The space V is a subspace of $V_0' = \delta_P^{1/2} \otimes \mathrm{Ind}(\psi \otimes \Theta_{n-2}; P, P'U)$ which contains $V_0 = \delta_P^{1/2} \otimes \mathrm{ind}(\psi \otimes \Theta_{n-2}; P, P'U)$. Write $\bar{f}$ for the class of $f \in V_0'$ modulo V_0. According to Proposition 1, V is the space of f in V_0' such that $\bar{f}$ lies in $V_U = \nu^{1/4} \otimes \Theta_{n-1}$. Hence for any f in V we have that

$$|t|^{-(n-1)/2}\Theta_n(\mathrm{diag}(t^2, \dots, t^2, 1))\bar{f} = \bar{f} \text{ in } \nu^{1/4} \otimes \Theta_{n-1} = V/V_0.$$

Consequently

$$|t|^{-(n-1)/2}\Theta_n(\mathrm{diag}(t^2, \dots, t^2, 1))f - f \text{ lies in } V_0 \quad (t \text{ in } F^\times).$$

Then there is $A_f > 0$, and $c (0 < c < 1/2)$, such that $|t|^{(n-1)/2} f(p(t^2)) = f(p(1))$ for $p(t^2) = s_n(t^2x_1, \dots, t^2x_{n-1})$ in P with $\max(|x_i|; 1 \le i < n) \le A_f$ and $c \le |t| \le 1$ (since f is locally constant and the domain of t is compact). But then this relation holds for all t with $0 < |t| \le 1$. Define f_0 by $f_0(p(1)) = |t|^{(n-1)/2} f(p(t^2))$ for t such that $\max(|t^2x_i|; 1 \le i < n) \le A_f$. It follows that given an $f \in V$ there is $A_f > 0$ and f_0 in the space

$$\begin{aligned}\mathrm{ind}(\Theta_{n-2} \times \nu^{(n-1)/4}) &= \mathrm{ind}(\delta_{P'}^{1/2} \otimes [\nu^{-1/2} \otimes \Theta_{n-2} \times \nu^{(n-3)/4}]) \\ &= i(\nu^{-1/2} \otimes \Theta_{n-2} \times \nu^{(n-3)/4}) = \nu^{-1/4} \otimes i(\nu^{-1/4} \otimes \Theta_{n-2} \times \nu^{(n-2)/4})\end{aligned}$$

[thus $f_0 : G_{n-1} \to \Theta_{n-2}$ satisfies $f_0(g_{n-2}ug_{n-1}t^2) = |t|^{-(n-1)}\Theta_{n-2}(g_{n-2})f_0(g_{n-1})$ $(g_i \in G_i, u \in U_{n-1})$], such that $f(s_n(x_1, \dots, x_{n-1})) = f_0(s_n(x_1, \dots, x_{n-1}))$ for $\max(|x_i|; 1 \le i < n) \le A_f$. Note that G_{n-1} acts on f_0 by $\varrho(g)f_0(p) = |\det p(g)|^{1/2} f_0(pg)$. Hence f_0 lies in the G_{n-1}*-module*

$$\mathrm{ind}(\Theta_{n-2} \times \nu^{(n-1)/4}) \otimes \nu^{1/2} = i(\Theta_{n-2} \otimes \nu^{-1/4} \times \nu^{(n-2)/4}),$$

which according to Lemma 1 has a unique irreducible submodule $\Theta_{n-1}\otimes\nu^{1/4}$. But Proposition 1 then asserts that f_0 lies in this submodule $\Theta_{n-1}\otimes\nu^{1/4}$, and the proposition follows.

4. Proposition 3 determines V as a P-submodule of the induced P-module $V_0'=\delta_P^{1/2}\otimes\operatorname{Ind}(\psi\otimes\Theta_{n-2}; P, P'U)$. It remains to extend the action of P on V to an action of G, or in fact $G^{\#}=G\rtimes\langle\sigma\rangle$, on V. Since $G^{\#}$ is generated by P and σ when $n\geq 3$ (as we now assume), it remains to describe the action of σ. To do that we construct below an irreducible B-submodule $W_0=W_{n,0}$ of the P-module V_0, and define (the restriction of) σ on W_0 by a formula which extends to V. The first step in this plan is to find an irreducible P''-submodule W of V_0, where $P''=p^{-1}(\bar{P}'')$ is the pullback of the intersection $\bar{P}''$ of $\bar{P}$ with $\bar{\sigma}(\bar{P})$.

Let w' be the transposition $(1,n-1)$ in the Weyl group of G. Namely it is the image under $\underline{s}$ of the matrix in $\bar{G}$ whose non-zero entries are 1, located at $(i,j)=(1,n-1)$, $(n-1,1)$, (k,k) ($k=n$ or $1<k<n-1$). We have the disjoint union decomposition

$$P=P'UP''\cup P'Uw'B=P'UP''\cup P'Uw'P''.$$

The subset $P^*=P_n^*=P'Uw'B=P'Uw'P''$ of P consists of all $p\in P$ with $x_1\neq 0$, where $pr_n(p)=(x_i;\ 1\leq i<n)$. The space W of the elements f of V_0 which are supported on P^* is then a P''-module. It is

$$W=\delta_P^{1/2}\otimes\operatorname{ind}[(\psi\otimes\Theta_{n-2})^{w'}; P'', w'P'Uw'\cap P''].$$

Since the $w'P'Uw'\cap P''$-module

$$(\psi\otimes\Theta_{n-2})^{w'}:\underline{s}(z)\begin{pmatrix}1 & 0 & \dots & 0 & x\\ 0 & & g & & \vdots\\ 0 & \dots & & 0 & 1\end{pmatrix}\mapsto\psi(x)\gamma(z^{n(n-1)/2})\Theta_{n-2}(g)$$

is irreducible, W is an irreducible P''-module by Mackey's Theorem (4.2(ii)) of [FKS]. Let $W'=W'(P'')$ be the Hermitian dual of the P''-module W. By Mackey's Theorem [FKS], (4.2(iv)), we have

$$W'=\delta_P^{1/2}\otimes\operatorname{Ind}[(\psi\otimes\Theta_{n-2})^{w'}; P'', w'P'Uw'\cap P''].$$

This W' consists of all functions $f: P^*\to\Theta_{n-2}$ smooth under the action ($\underline{1}$), ($\underline{2}$) of P''. Hence we have the following inclusions of P''-modules:

$$W\subset V_0\subset V\subset V'=V'(P)\subset V_0'=V_0'(P)\subset W'=W'(P'').$$

PROPOSITION 4. *Let $J: W'\to W'$ be a P''-module morphism such that $J(W)\subset W$, $J^2=Id$, and $J\Theta(p'')=\Theta(\sigma p'')J$ for all $p''\in P''$. Then $J(V)\subset V$ and $J|V$ is equal to I (up to a sign).*

PROOF. For any $m\in F$ let $u(m)$ be the matrix in $\bar{N}$ whose only non-zero entry above the diagonal is m, located at $(i,j)=(1,n)$. The subgroup $N_{1,n}=\{\underline{s}(u(m));$

$m\in F\}$ of N acts on W', according to (2), by

$$\Theta_n(u(m))f(x_1,\dots,x_{n-1})=f(s_n(x_1,\dots,x_{n-1})u(m))$$
$$=f(u(mx_1)s_n(x_1,\dots,x_{n-1}))=\psi(mx_1)f(x_1,\dots,x_{n-1}).$$

Hence the only $N_{1,n}$-fixed vector in W' is the zero vector (W' consists of the f on $x_1\neq 0$). Moreover, for every $f\in W'$ and $m\in F$ it is clear that $\Theta_n(u(m))f-f$ lies in W. Namely $\Theta_n(u(m))f=f$ in W'/W, and $N_{1,n}$ acts trivially on W'/W. Since $W\subset V\subset W'$ it follows that

$$\mathrm{Hom}_B(V/W, W')=0, \qquad \mathrm{Hom}_B((V/W)', W')=0.$$

Since

$$\mathrm{Hom}_B(W, V/W)\hookrightarrow \mathrm{Hom}_B((V/W)', W'),$$

we further have that $\mathrm{Hom}_B(W, V/W)=0$.

We conclude that $I\colon V\to V$ maps W to W. If not, the operator I induces a non-zero P''-module morphism $W\to V/W$. But this is impossible since $\mathrm{Hom}_B(W, V/W)=0$.

Since W is irreducible, any P''-module morphism $J\colon W\to W$ with $J^2=Id$ and $J\Theta(p'')=\Theta(\sigma p'')J$ for all $p''\in P''$ has to be equal to $I|W$ up to a sign.

Finally we claim that the restriction $J|V$ to V of J of the proposition is equal to I, up to a sign. Indeed, if $J|W=I|W$ then the P''-module morphism $J|V-I\colon V/W\to W'$ is well-defined. Since $\mathrm{Hom}_B(V/W, W')=0$, the proposition follows.

5. Put $B''=B\cap w'P'Uw'$. Since $P=P'UP''\cup P'Uw'B$, and $P'U\setminus P'Uw'B\simeq B''\setminus B$, as a B-module the restriction of W to B is

$$W|B=\delta_P^{1/2}\otimes \mathrm{ind}((\psi\otimes\Theta_{n-2})^{w'}|B''; B, B'').$$

Then $W|B$ is reducible since $\Theta_{n-2}|B_{n-2}$ is reducible. We shall construct an irreducible B-module $W_0=W_{n,0}$ of $W=W_n$ by induction, as follows. By induction on n, we have inclusions of B''-modules:

$$W_{n-2,0}\subset W_{n-2}\subset V_{n-2,0}\subset V_{n-2}\subset V'_{n-2}\subset V'_{n-2,0}\subset\dots.$$

Here $W_{n-2,0}$ is an irreducible B_{n-2}-submodule of the irreducible P''_{n-2}-module W_{n-2}. This W_{n-2} consists of the restrictions $f|P^*_{n-2}$ (namely restrictions to the subvariety of $F^{n-3}-\{\bar{0}\}$ determined by $x_1^{(n-2)}\neq 0$) of the $f\in V_{n-2,0}$. In turn $V_{n-2,0}$ is the unique proper (necessarily irreducible) P_{n-2}-submodule of $V_{n-2}=\Theta_{n-2}|B_{n-2}$.

DEFINITION. Put $W_0=W_{n,0}=\delta_P^{1/2}\otimes\mathrm{ind}(\psi\otimes W_{n-2,0}; B, B'')$.

Then W_0 is an irreducible B-module by Mackey's Theorem (4.2(iv)) of [FKS], since $W_{n-2,0}$ is irreducible. This W_0 is the desired irreducible B-submodule of

$$W = W_n = \delta_P^{1/2}$$

$$\otimes \operatorname{ind}\left[\zeta \underline{s}(z)\begin{pmatrix}1 & 0 & \dots & 0 & x\\ 0 & & & & \\ \vdots & & b & & \vdots\\ 0 & \dots & & 0 & 1\end{pmatrix} \mapsto \zeta\psi(x)\gamma(z^{n(n-1)/2})\Theta_{n-2}(b);\ B,\ B''\right].$$

Note that W_0 consists of the $f: P_n \times \dots \times P_{n-2j} \times \dots \to \mathbb{C}$ in W which are supported on $P_n^* \times \dots \times P_{n-2j}^* \times \dots$. If the elements f of W are regarded as functions of $(F^{n-1} - \{\underline{0}\}) \times \dots \times (F^{n-2j-1} - \{\underline{0}\}) \times \dots$ as in ($\underline{3}$), then W_0 consists of the restrictions of the $f \in W$ to the subvariety determined by

$$x_1^{(n)} \neq 0, \dots, x_1^{(n-2j)} \neq 0, \dots .$$

Let $W_0' = W_0'(B)$ be the Hermitian dual of the irreducible B-module W_0. As usual, we have inclusions of B-modules:

$$W_0 \subset W \subset V \subset W_0'.$$

PROPOSITION 5. *Let $J: W_0' \to W_0'$ be a B-module morphism such that $J(W_0) \subset W_0$, $J^2 = Id$, and $J\Theta(b) = \Theta(\sigma b)J$ for all $b \in B$. Then $J(V) \subset V$ and $J \mid V$ is equal to I (up to a sign).*

PROOF. The proof follows that of Proposition 4. For any $m \in F$ let $u'(m)$ be the matrix in $\bar{N}$ whose only non-zero entry above the diagonal is m, located at $(i, j) = (2, n-1)$. The subgroup $N_{2,n-1} = \{\underline{s}(u'(m));\ m \in F\}$ of N acts on W_0', according to ($\underline{2}$), by

$$\Theta_n(u'(m))f(x_1^{(n)}, \dots, x_{n-1}^{(n)}; x_1^{(n-2)}, \dots) = f(s_n(x_1^{(n)}, \dots, x_{n-1}^{(n)})u'(m))(x_1^{(n-2)}, \dots)$$

$$= \Theta_{n-2}(u(m)) \cdot f(x_1^{(n)}, \dots, x_{n-1}^{(n)}; x_1^{(n-2)}, \dots) = \psi(m x_1^{(n-2)}) \cdot f(x_1^{(n)}, \dots).$$

Since W_0' consists of f on the variety determined in particular by $x_1^{(n-2)} \neq 0$, the only $N_{2,n-1}$-fixed vector in W_0' is the zero vector.

Moreover, for every $f \in W_0'$ and $m \in F$ it is clear that $\Theta_n(u'(m))f - f$ lies in W_0. Namely $\Theta_n(u'(m))f = f$ in W_0'/W_0, and $N_{2,n-1}$ acts trivially on W_0'/W_0. Since $W_0 \subset V \subset W_0'$ we conclude that

$$\operatorname{Hom}_B(W_0, V/W_0)(\subset \operatorname{Hom}_B((V/W_0)', W_0')) = 0, \quad \operatorname{Hom}_B(V/W_0, W_0') = 0.$$

It follows that $I: V \to V$ maps W_0 to W_0. Otherwise the operator I induces a non-zero B-module morphism $W_0 \to V/W_0$. But this is impossible since $\operatorname{Hom}_B(W_0, V/W_0) = 0$.

Since W_0 is irreducible, any B-module morphism $J: W_0 \to W_0$ with $J^2 = Id$ and $J\Theta(b) = \Theta(\sigma b)J$ for all $b \in B$ has to be equal to I up to a sign.

Finally we claim that the restriction $J \mid V$ to V of J of the proposition is equal to I, up to a sign. Indeed, if $J \mid V = I$ on W_0 then the B-module morphism $J \mid V - I: V/W_0 \to W_0'$ is well-defined. Since $\operatorname{Hom}_B(V/W_0, W_0') = 0$, the proposition follows.

6. To complete the construction of the model of Θ, it remains to write down an explicit expression for J of Proposition 5. We shall use the notations of (3), and claim that up to a scalar, which is unique up to a sign, J is given by (4). To check that one needs to write explicitly the action of $B=AN$ on $f\in W_0'$. For that, given $u\in F$ and $1\le i'<j'\le n$, denote by $u(u;i',j')$ the unipotent matrix whose only non-zero entry above the diagonal is u at (i',j'). Then

$$\Theta(u(u;i',j'))f(x_1^{(n)},\dots)$$
$$=\psi(ux_{i'+j'-n}^{(2j'-n)})f(\dots,x_{j'}^{(n)}+ux_{i'}^{(n)},\dots;\dots,x_{j'-j}^{(n-2j)}+ux_{i'-j}^{(n-2j)},\dots;\dots);$$

in the last expression, only variables affected by the action of $u(u;i',j')$ are written out. Since

$$\sigma(u(u;i',j'))=u((-1)^{1+j'-i'}u;n+1-j',n+1-i'),$$

we also have

$$\Theta(\sigma(u(u;i',j'))f(x_1^{(n)},\dots)=\psi((-1)^{1+i'+j'}ux_{n+2-i'-j'}^{(n+2-2i')})$$
$$\cdot f(\dots,x_{n+1-i'}^{(n)}-(-1)^{i'+j'}ux_{n+1-j'}^{(n)},\dots;$$
$$\dots,x_{n+1-i'-j}^{(n-2j)}-(-1)^{i'+j'}ux_{n+1-j'-j}^{(n-2j)},\dots;\dots).$$

It is easy to see that

$$(J_Nf)(\dots)=\int f(\dots;-x_1^{(n-2j)},y_1^{(n-2j)},\dots,y_{n-2j-2}^{(n-2j)};\dots)$$
$$\cdot\psi[\sum_{0\le j\le n/2-1}\ \sum_{1\le i\le n-2j-2}(-1)^{i-1}y_i^{(n-2j)}x_{n-2j-i}^{(n-2j)}/x_1^{(n-2j)}]\cdot\prod_{i,j}dy_i^{(n-2j)}$$

satisfies $J_N(\Theta(u(u;i',j'))f)=\Theta(\sigma(u(u;i',j')))J_Nf$, for all $i'<j'$ and $u\in F$.

Using the decomposition

$$X(x_1,\dots,x_{n-1})=\begin{pmatrix}0,1,0,\dots & ,0\\ \cdot & \\ 0,\dots,0,\dot{1},0 & \\ x_1,\dots,x_{n-1},0 & \\ 0,\dots\quad 0,1 & \end{pmatrix}$$
$$=\begin{pmatrix}1,0,\dots & & 0\\ \cdot & & \\ 0,\dots,\dot{1}, & 0, & 0\\ 0,\dots & x_1, & 0\\ 0,\dots & 0, & 1\end{pmatrix}\begin{pmatrix}0,1,0,\dots,0\\ \cdot\\ 0,\dots,0,\dot{1},0\\ 1,0,\dots,0,0\\ 0,\dots\quad 0,1\end{pmatrix}\begin{pmatrix}1,x_2/x_1,\dots & & x_{n-1}/x_1 & \\ \cdot & & & \\ 0,\dots,\dot{0}, & 1, & 0, & 0\\ 0,\dots & 0, & 1, & 0\\ 0,\dots & & 0, & 1\end{pmatrix}$$

and the multiplication law in G (see [KP1]), it is easy to verify that

$$\underline{s}(X(x_1,\dots,x_{n-1}))\cdot\underline{s}(\mathrm{diag}(a_1,\dots,a_n))$$
$$=\underline{s}(a_n)\cdot\underline{s}(\mathrm{diag}(a_2/a_n,\dots,a_{n-1}/a_n,1,1))$$
$$\underline{s}(X(a_1x_1/a_n,\dots,a_{n-1}x_{n-1}/a_n)\cdot((-1)^{n-1}x_1,a_2\dots a_n)$$
$$\cdot(a_n,\prod_{1\le j<n}(-a_j)^{j-1})\cdot(a_n,-1)^{n-1}.$$

Applying induction on j, and the recurrence relations

$$f(\dots, qp_{n-2j+2}, p_{n-2j}, \dots)$$
$$= \delta_{P_{n-2j}}(q)^{1/2} f(\dots, p_{n-2j+2}, p_{n-2j}q, \dots), \quad q \in P_{n-2j},$$

it is easy to verify that

$$\Theta(\underline{s}(\operatorname{diag}(a_1, \dots, a_n))) f(\dots, x_i^{(n-2j)}, \dots)$$
$$= f(a_1 x_1^{(n)}/a_n, \dots, a_{n-1} x_{n-1}^{(n)}/a_n; \dots; a_{j+1} x_1^{(n-2j)}/a_{n-j}, \dots,$$
$$a_{n-2j-1} x_{n-2j-1}^{(n-2j)}/a_{n-j}; \dots) \cdot \prod_{1 \le j \le n/2-1} [\Theta_{n-2j}(\underline{s}(a_{n-j}/a_{n-j+1}))$$
$$\cdot ((-1)^{n-1} x_1^{(n-2j)}, \prod_{j+1<k\le n-j} (a_k/a_{n+1-j})) \cdot (a_{n-j}/a_{n-j+1},$$
$$(-1)^{n-1} \cdot \prod_{1<k<n-2j} (-a_{k+j}/a_{n+1-j})^{k-1}) \cdot \prod_{j<i<n-j} |a_i/a_{n-j}|^{1/2}].$$

Here $\Theta_m(\underline{s}(a))$ is multiplication by the scalar $\gamma(a^{m(m-1)/2})$.
Recall that $\sigma(g) = (-1, \det p(g))^{(n-1)n^2/2} \sigma(g)$ and that

$$\tilde{\sigma}(\underline{s}(\operatorname{diag}(a_j))) = \underline{s}(\operatorname{diag}(a_{n+1-j}^{-1})) \cdot \prod_{i=1}^{n-1} (\prod_{j=i+1}^{n} a_j, a_i).$$

Hence

$$\Theta(\sigma(\underline{s}(\operatorname{diag}(a_1, \dots, a_n)))) f(\dots, x_i^{(n-2j)}, \dots) = (-1, a_1 a_2 \dots a_n)^{(n-1)n^2/2}$$
$$\cdot \prod_{i=1}^{n-1} (\prod_{j=i+1}^{n} a_j, a_i) \cdot \prod_{j=0}^{n/2-1} [\Theta_{n-2j}(\underline{s}(a_j/a_{j+1}))$$
$$\cdot ((-1)^{n-1} x_1^{(n-2j)}, \prod_{k=j+1}^{n-1-j} (a_k/a_j)) \cdot (a_j/a_{j+1}, (-1)^{n-1}$$
$$\cdot \prod_{1<k<n-2j} (-a_j/a_{n+1-k-j})^{k-1}) \cdot \prod_{j<i<n-j} |a_{j+1}/a_{n+1-i}|^{1/2}]$$
$$\cdot f(a_1 x_1^{(n)}/a_n, \dots, a_1 x_{n-1}^{(n)}/a_2; \dots; a_{j+1} x_1^{(n-2j)}/a_{n-j}, \dots, a_{j+1} x_{n-2j+1}^{(n-2j)}/a_{2j+2}; \dots).$$

One can then check that

$$(Jf)(\dots, x_i^{(n-2j)}, \dots) = [\prod_{0 \le j \le n/2-1} (|x_1^{(n-2j)}|^{j+1-n/2}/\gamma(x_1^{(n-2j)}))]$$
$$\cdot (J_N f)(\dots, x_i^{(n-2j)}, \dots),$$

satisfies

$$J(\Theta(\operatorname{diag}(a_i))f) = (\Theta(\sigma(\operatorname{diag}(a_i)))J)f.$$

Hence J^2 is a scalar, and the product of J with some constant c satisfies $(cJ)^2 = Id$. This completes the proof of (i)–(iii) in the Theorem.

7. It remains to prove (iv) in the Theorem. By Proposition 2(ii) we have $\dim \operatorname{Hom}_P(V_0, V_0') = 1$. By Proposition 2(i), we have $V' \hookrightarrow V_0'$. Hence the space

$\mathrm{Hom}_P(V_0, V')$ is a subspace of $\mathrm{Hom}_P(V_0, V_0')$, necessarily one-dimensional. Consider the map $\mathrm{Hom}_P(V, V') \to \mathrm{Hom}_P(V_0, V')$, obtained by restriction from V to V_0. Its kernel is $\mathrm{Hom}_P(V/V_0, V')$. Now $V/V_0 \simeq V_U$, and U acts trivially on V_U. On the other hand, the only vector in W', and in particular in its subspace V', which is fixed by U, is the zero vector. Hence $\mathrm{Hom}_P(V, V')$ injects in $\mathrm{Hom}_P(V_0, V')$, and it is one-dimensional. The L^2-product on V yields a P-invariant Hermitian form on V, hence a non-zero P-module morphism $i: V \to V'$. The unitary structure on V yields a non-zero morphism $j: V \to V'$ of G-modules. In particular j is a P-module morphism. Since $\dim \mathrm{Hom}_P(V, V') = 1$, j is a multiple of i, as required.

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