TWISTED CHARACTER OF A SMALL REPRESENTATION OF GL(4)

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Abstract. We compute by a purely local method the (elliptic) θ-twisted character χ_{π_Y} of the representation $\pi_Y = I_{(3,1)}(1_3 \times \chi_Y)$ of $G = \mathrm{GL}(4, F)$, where F is a p-adic field, $p \neq 2$, and Y is an unramified quadratic extension of F; χ_Y is the nontrivial character of $F^\times / N_{Y/F} Y^\times$.

The representation π_Y is normalizedly induced from $\begin{smallmatrix} m_3 & * \\ 0 & m_1 \end{smallmatrix} \mapsto \chi_Y(m_1)$, $m_i \in \mathrm{GL}(i, F)$, on the maximal parabolic subgroup of type $(3, 1)$; θ is the "transpose-inverse" involution of G.

We show that the twisted character χ_{π_Y} of π_Y is an unstable function: its value at a twisted regular elliptic conjugacy class with norm in $C_Y = \mathbf{C}_Y(F)$="$(\mathrm{GL}(2, Y)/F^\times)_F$" is minus its value at the other class within the twisted stable conjugacy class. It is 0 at the classes without norm in C_Y. Moreover π_Y is the endoscopic lift of the trivial representation of C_Y.

We deal only with unramified Y/F, as globally this case occurs almost everywhere. The case of ramified Y/F would require another paper.

Our $\mathbf{C}_Y$ ="$(\mathrm{R}_{Y/F}\,\mathrm{GL}(2)/\,\mathrm{GL}(1))_F$" has Y-points $\mathbf{C}_Y(Y) = \{(g, g') \in \mathrm{GL}(2, Y) \times \mathrm{GL}(2, Y);$ $\det(g) = \det(g')\}/Y^\times$ ($Y^\times$ embeds diagonally); $\sigma(\neq 1)$ in $\mathrm{Gal}(Y/F)$ acts by $\sigma(g, g') = (\sigma g', \sigma g)$. It is a θ-twisted elliptic endoscopic group of GL(4).

Naturally this computation plays a role in the theory of lifting of C_Y and GSp(2) to GL(4) using the trace formula, to be discussed elsewhere.

Our work extends – to the context of nontrivial central characters – the work of [FZ4], where representations of $\mathrm{PGL}(4, F)$ are studied. In [FZ4] a 4-dimensional analogue of the model of the small representation of $\mathrm{PGL}(3, F)$ introduced with Kazhdan in [FK] in a 3-dimensional case is developed, and the local method of computation introduced in [FZ3] is extended. As in [FZ4] we use here the classification of twisted (stable) regular conjugacy classes in $\mathrm{GL}(4, F)$ of [F], motivated by Weissauer [W].

Introduction

Let π be an admissible representation (see Bernstein-Zelevinsky [BZ], 2.1) of a p-adic reductive group G. Its character χ_π is a complex valued function defined by $\mathrm{tr}\,\pi(fdg) = \int_G \chi_\pi(g) f(g) dg$ for all complex valued smooth compactly supported measures fdg ([BZ], 2.17). It is smooth on the regular set of the group G. The character is important since it

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characterizes the representation up to equivalence. A fundamental result of Harish-Chandra [H] establishes that the character is a locally integrable function in characteristic zero.

Let θ be an automorphism of finite order of the group G. Define ${}^\theta\pi$ by ${}^\theta\pi(g) = \pi(\theta(g))$. When π is invariant under the action of θ (thus ${}^\theta\pi$ is equivalent to π), Shintani and others introduced an extension of π to the semidirect product $G \rtimes \langle\theta\rangle$. The twisted character $\chi_\pi(g \times \theta)$ is defined by $\operatorname{tr}\pi(fdg \times \theta) = \int_G \chi_\pi(g \times \theta)f(g)dg$ for all fdg. It depends only on the θ-conjugacy class $\{hg\theta(h)^{-1}; h \in G\}$ of g. It is again smooth on the θ-regular set, and characterizes the θ-invariant irreducible π up to isomorphism. Moreover, it is locally integrable (see Clozel [C]) in characteristic zero.

Characters provide a very precise tool to express a relation of representations of different groups, called lifting, initiated by Shintani and studied extensively in the case of base change, and also in non base change situations such as twisting by characters (Kazhdan [K], Waldspurger [Wa]), and the symmetric square lifting from SL(2) to PGL(3) ([Fsym], [FK]). In this last case twisted characters of θ-invariant representations of PGL(3) are related to packets of representations of SL(2), and θ is the involution sending g to its transpose-inverse.

The aim of the present work is to compute the twisted (*by* θ) *character of a specific representation* $\pi = \pi_Y = I_{(3,1)}(1_3 \times \chi_Y)$, *of the group* $G = \mathrm{GL}(4, F)$, F *a p-adic field, p odd.* Here Y/F is an unramified quadratic extension and χ_Y is the quadratic character of $F^\times$ which is trivial on the group $N_{Y/F}Y^\times$, where $N_{Y/F}$ is the norm map from Y to F. This π is normalizedly induced from the representation $\left(\begin{smallmatrix} m_3 & * \\ 0 & m_1 \end{smallmatrix}\right) \mapsto \chi_Y(m_1)$, $m_i \in \mathrm{GL}(i, F)$, of the standard (upper triangular) maximal parabolic subgroup P of type $(3, 1)$. It is invariant under the involution $\theta(g) = J^{-1}{}^tg^{-1}J$, where $J = (a_i\delta_{i,5-j})$, $a_1 = a_2 = 1$, $a_3 = a_4 = -1$.

A natural setting for the statement of our result is the theory of liftings to the group $\mathbf{G} = \mathrm{GL}(4)$ from its θ-twisted endoscopic (see Kottwitz-Shelstad [KS]) F-group

$$\mathbf{C}_Y = \{(g, g') \in \mathrm{GL}(2) \times \mathrm{GL}(2); \det g = \det g'\}/\mathbb{G}_m,$$

where the multiplicative group $\mathbb{G}_m = \mathrm{GL}(1)$ embeds as $z \mapsto (zI_2, zI_2)$, I_2 is the identity 2×2 matrix, with $\mathrm{Gal}(\overline{F}/F)$-action which is a composition of the usual Galois action on each of the two factors GL(2) with the transposition $(g, g') \mapsto (g', g)$ if $\sigma \in \mathrm{Gal}(\overline{F}/F)$ has nontrivial restriction to Y. Here $\overline{F}$ is a separable algebraic closure of F containing Y.

The corresponding map λ_Y of dual groups is simply the natural embedding in $\hat{G} = \mathrm{GL}(4, \mathbb{C})$ of the non connected $\hat{C}_Y = Z_{\hat{G}}(\hat{s}\hat{\theta})$

$$= \left\{g \in \hat{G} = \mathrm{GL}(4, \mathbb{C});\ g\hat{s}J{}^tg = \hat{s}J = \begin{pmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{pmatrix}\right\} = \mathrm{O}\left(\begin{pmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{pmatrix}, \mathbb{C}\right)$$

$$= \left\langle \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix}, \iota = \begin{pmatrix} 1 & & & 0 \\ & 0 & 1 & \\ & 1 & 0 & \\ 0 & & & 1 \end{pmatrix};\ \left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B\right) \in \left\{\mathrm{GL}(2, \mathbb{C})^2; \det A \cdot \det B = 1\right\}/\mathbb{C}^\times \right\rangle,$$

where $z \in \mathbb{C}^\times$ embeds as the central element (z, z^{-1}), and where $\hat{s} = \mathrm{diag}(-1, 1, -1, 1)$ and $\omega = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus $\hat{C}_Y$ is the $\hat{\theta}$-centralizer in $\hat{G}$ of the semisimple element $\hat{s}$ (and $\hat{\theta}$ is defined on $\hat{G}$ by the same formula that defines θ on G), and $\mathrm{Gal}(Y/F)$ acts via conjugation by ι.

Indeed, our result can be viewed as asserting that the θ-invariant representation π of $G = \mathbf{G}(F)$, whose central character is $\chi_Y \neq 1$ of order two, is the endoscopic lift of the trivial representation of $C_Y = \mathbf{C}_Y(F) = (\mathrm{GL}(2,Y)/F^\times)_F$. The subscript F here indicates that $(\mathrm{GL}(2,Y)/F^\times)_F$ consists of g in $\mathrm{GL}(2,Y)/F^\times$ with $\det(g)$ in $F^\times/F^{\times 2}$.

To state this we note that the embedding $\lambda_Y : \hat{C}_Y \to \hat{G}$ defines a norm map. This norm map relates the stable θ-conjugacy classes in G with stable conjugacy classes in C_Y, where "stable" means the elements in G of an orbit in the points of $\mathbf{G}$ in a separable algebraic closure $\overline{F}$ of F. The crucial case is that of θ-elliptic elements. A stable θ-conjugacy class consists of several θ-conjugacy classes. The stable θ-conjugacy classes of elements in G, and the θ-conjugacy classes within the stable θ-classes, have been described recently in [F], in analogy with the description of the (stable) conjugacy classes in the group of symplectic similitudes $\mathrm{GSp}(2,F)$ of Weissauer [W]. In fact in [FZ4] the simpler case of $\mathrm{PGL}(4,F)$ is used, but here, as in [F], we deal with θ-classes in $\mathrm{GL}(4,F)$. We give here full details of the description in our case.

There are four types of θ-elliptic elements of G, named in [F] and here I, II, III, IV, depending on their splitting behaviour. As in [F], our work relies on an explicit presentation of representatives of the θ-conjugacy classes within the stable such classes in G. We present here the same set of representatives as in [FZ4].

The norm map, which we describe explicitly here, relates θ-conjugacy classes of types II and IV to conjugacy classes in C_Y. It does not relate classes of types I, III to classes in C_Y. Our "quadratic" case behaves then in a complementary fashion to that of [FZ4], where θ-conjugacy classes of types I, III are related to conjugacy classes in the group $C = \mathrm{SO}(4)$ of [FZ4], but θ-conjugacy classes of types II, IV are not related to conjugacy classes in C.

The stable θ-conjugacy classes of types II and IV come associated with a quadratic extension E/E_3, where $Y = E_3$ is a quadratic extension of F. The two θ-conjugacy classes g_r within the stable θ-classes are parametrized by r in $E_3^\times/N_{E/E_3}E^\times$. We prove

Theorem. *The value of the θ-character $\chi_\pi(g \times \theta)$ of $\pi = \pi_Y$ at the θ-regular element $g = g_r$ of type II or IV, multiplied by a suitable Jacobian $\frac{\Delta(g_r\theta)}{\Delta_C(Ng)}$, is $2\kappa(r)$; here κ is the $\neq 1$ character of $E_3^\times/N_{E/E_3}E^\times$. At any θ-regular element g of type I or III, $\chi_\pi(g \times \theta) = 0$.*

In particular the character $\chi_\pi(g \times \theta)$ is an unstable function, namely its value at one θ-conjugacy class within a stable θ-conjugacy class of type II or IV is negative its value at the other θ-conjugacy class.

We deal only with unramified Y/F, as globally this case occurs almost everywhere. The case of ramified Y/F would require another paper.

Our result is a special case of the lifting with respect to λ_Y to the group $G = \mathrm{GL}(4,F)$ of representations of the group $C_Y = (\mathrm{GL}(2,Y)/F^\times)_F$.

Our work develops the method of [FZ4] to the context of representations with nontrivial central characters. We use a model of our representation $\pi_Y = I_{(3,1)}(1_3 \times \chi_Y)$, different from the standard model of a parabolically induced representation. It is a twist of the four dimensional analogue of [FZ4], dealing with $\pi_4 = I_{(3,1)}(1_3 \times 1)$, of a (three dimensional) model introduced and used with Kazhdan in [FK] to compute the twisted (by transpose-inverse) character of the representation $\pi_3 = I_{(2,1)}(1_2)$ of $\mathrm{PGL}(3,F)$ normalizedly induced from the trivial representation of the maximal parabolic subgroup. We do not use our results to prove the fundamental lemma since in our case, as well as that of [FZ4], the fundamental lemma is already established in [F]. In the case of the symmetric square lifting from $\mathrm{SL}(2,F)$ to $\mathrm{PGL}(3,F)$, an analogous purely local and simple proof of the fundamental lemma was given in [Fsym. Unit elements].

The work of [FK] uses local arguments to compute the twisted character of π_3 on one of the two twisted conjugacy classes within the stable one (where the quadratic form is anisotropic), and global arguments to reduce the computation on the other class (where the quadratic form is isotropic) to that computed by local means. A purely local computation for the second class is given in [FZ3]. In [FZ4] this local computation is developed in a four dimensional projective case. A global type of argument as in [FK] is harder to apply as there are not enough anisotropic quadratic forms in the four dimensional case. Anyway, a simpler, local proof, is better. Here we extend the work of [FZ4] to θ-invariant representations of $\mathrm{GL}(4,F)$ whose central character is nontrivial, necessarily quadratic. Our work is parallel to – but entirely independent of – the work of [FZ4].

Conjugacy classes

Let F be a local nonarchimedean field, and R its ring of integers. Put $\mathbf{G}=\mathrm{GL}(4)$, $G=\mathbf{G}(F)$ and $K=\mathbf{G}(R)$. Put $\mathbf{C}_Y=\{(g_1,g_2)\in\mathrm{GL}(2)\times\mathrm{GL}(2);\det(g_1)=\det(g_2)\}/\mathbb{G}_m$ ($\mathbb{G}_m$ embeds diagonally), viewed as a group over F with Galois action $\tau(g,g')=(\tau g,\tau g')$ unless $\tau\in\mathrm{Gal}(\overline{F}/F)$ has nontrivial restriction to Y, in which case $\tau(g,g')=(\tau g',\tau g)$, where $\tau(g_{ij})=(\tau g_{ij})$. It is a form of the group $\mathbf{C}$ of [FZ4], and in particular $\mathbf{C}_Y(Y)=\mathbf{C}(Y)$, but the $\mathrm{Gal}(\overline{F}/F)$-action is different: $\tau\in\mathrm{Gal}(\overline{F}/F)$ takes (g,g') of $\mathbf{C}(\overline{F})$ to $(\tau g,\tau g')$. Then $C_Y=\mathbf{C}_Y(F)=\{g\in GL(2,Y)/F^\times;\det(g)\in F^\times\}$ and $K_{C_Y}=\mathbf{C}_Y(R)$. Set $\theta(\delta)=J^{-1t}\delta^{-1}J$ for δ in G. Here J is $\begin{pmatrix}0&w\\-w&0\end{pmatrix}$, where $w=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Fix a separable algebraic closure $\overline{F}$ of F. The elements δ, δ' of G are called (*stably*) *θ-conjugate* if there is g in G (resp. $\mathrm{GL}(4,\overline{F})$) with $\delta'=g^{-1}\delta\theta(g)$.

Results of [F] concerning (stable) θ-twisted regular conjugacy classes are recalled in [FZ4], pp. 337-338. There are four types of θ-elliptic classes, but the norm map N from G to C_Y relates only the twisted classes in G of type II and IV to conjugacy classes in C_Y. We should then expect the twisted character of the representation considered here to vanish on the twisted classes of type I and III.

Norm map

The norm map $N:\mathbf{G}\to\mathbf{C}_Y$ is defined on the diagonal torus $\mathbf{T}^*$ of $\mathbf{G}$ by

$$N(\mathrm{diag}(a,b,c,d))=(\mathrm{diag}(ab,cd),\mathrm{diag}(ac,bd)).$$

Since both components have determinant $abcd$, the image of N is indeed in $\mathbf{C}_Y$.

In type II we have $a\in E_1^\times$, $E_1=E^\tau=F(\sqrt{D})$, $b\in E_2^\times$, $E_2=E^{\sigma\tau}=F(\sqrt{AD})$, and the norm map becomes

$$N(\mathrm{diag}(a,b,\tau b,\sigma a))=(\mathrm{diag}(ab,\tau(b)\sigma(a)),\mathrm{diag}(a\tau(b),b\sigma(a))).$$

The two components on the right are mapped to each other by τ, while the pairs of eigenvalues ($\{ab, \tau(b)\sigma(a)\}$ and $\{a\tau(b), b\sigma(a)\}$) are permuted by σ. Hence the right side defines a conjugacy class in $\mathrm{GL}(2, E_3)_F$ (the determinant $ab \cdot \tau(b)\sigma(a) = a\tau(b) \cdot b\sigma(a)$ lies in $F^\times$, and E_3 is the fixed field of σ in E). We choose Y to be the quadratic extension E_3 of F. The image of this torus is the torus (up to conjugacy) in $C_Y(F)$ which splits over the biquadratic extension E of F.

In type IV we have $\alpha \in E^\times$, $E = E_3(\sqrt{D})$, $\alpha\sigma^2\alpha \in E_3^\times$, $E_3 = F(\sqrt{A})$, and the norm map becomes

$$N(\mathrm{diag}(\alpha, \sigma\alpha, \sigma^3\alpha, \sigma^2\alpha)) = (\mathrm{diag}(\alpha\sigma\alpha, \sigma^2\alpha\sigma^3\alpha), \mathrm{diag}(\alpha\sigma^3\alpha, \sigma\alpha\sigma^2\alpha)).$$

Here σ^3 permutes the two diagonal matrices on the right, and σ^2 permutes each pair of eigenvalues. Since both components of $N(*)$ have equal determinants in $F^\times$, $\mathrm{diag}(*, *)$ defines a conjugacy class in $\mathrm{GL}(2, E_3)_F$. Hence the norm map defines a conjugacy class in $C_Y = \mathbf{C}_Y(F)$ for each θ-stable conjugacy class of type IV in $G = \mathbf{G}(F)$, where we take Y to be E_3.

In types I and III the image of the map N does not correspond to any conjugacy class in C_Y, for any quadratic extension Y of F.

Jacobians

The character relation that we study relates the product of the value at t of the twisted character of our representation $\pi_Y = I_{(3,1)}(1_3 \times \chi_Y)$ by a factor $\Delta(t \times \theta)$, with the product by a factor $\Delta_C(Nt)$ of the value at Nt of the character of the (trivial) representation $\mathbf{1}_{C_Y}$ of C_Y which lifts to π_Y.

The factors $\Delta(t \times \theta)$ and $\Delta_C(Nt)$ are defined and computed in [FZ4], pp. 339-340. We have

$$\frac{\Delta(t\theta)}{\Delta_C(Nt)} = \left| \frac{(a-d)^2}{ad} \cdot \frac{(b-c)^2}{bc} \right|^{1/2}.$$

Then in case II if $t = \mathrm{diag}(a, b, \tau b, \sigma a)$, $a = a_1 + a_2\sqrt{D} \in E_1^\times$, $b = b_1 + b_2\sqrt{AD} \in E_2^\times$, we get

$$\frac{\Delta(t\theta)}{\Delta_C(Nt)} = \left| \frac{(a-\sigma a)^2}{a\sigma a} \cdot \frac{(b-\sigma b)^2}{b\sigma b} \right|^{1/2} = \left| \frac{(2a_2\sqrt{D})^2}{a_1^2 - a_2^2 D} \cdot \frac{(2b_2\sqrt{AD})^2}{b_1^2 - b_2^2 AD} \right|^{1/2}.$$

In case IV, if $t = \mathrm{diag}(\alpha, \sigma\alpha, \sigma^3\alpha, \sigma^2\alpha)$, $\alpha = a + b\sqrt{D}$, $a = a_1 + a_2\sqrt{A}$, $b = b_1 + b_2\sqrt{A}$, $\sigma\alpha = \sigma a + \sigma b\sqrt{\sigma D}$, $\sigma^3\alpha = \sigma a - \sigma b\sqrt{\sigma D}$, $\alpha - \sigma^2\alpha = 2b\sqrt{D}$, $\sigma(\alpha - \sigma^2\alpha) = 2\sigma b\sqrt{\sigma D}$, and

$$\frac{\Delta(t\theta)}{\Delta_C(Nt)} = \left| \frac{(\alpha-\sigma^2\alpha)^2}{\alpha\sigma^2\alpha} \cdot \frac{\sigma(\alpha-\sigma^2\alpha)^2}{\sigma\alpha\sigma^3\alpha} \right|^{1/2} = \left| \frac{(4b\sigma b)^2 D\sigma D}{(a^2-b^2D)\sigma(a^2-b^2D)} \right|^{1/2}.$$

Characters

Denote by f (resp. f_{C_Y}) a complex-valued compactly-supported smooth (thus locally-constant since F is nonarchimedean) function on G (resp. C_Y). Fix Haar measures on G and on C_Y.

By a G-module π (resp. C_Y-module π_{C_Y}) we mean an admissible representation ([BZ]) of G (resp. C_Y) in a complex space. An irreducible G-module π is called *θ-invariant* if it is equivalent to the G-module ${}^\theta\pi$, defined by ${}^\theta\pi(g) = \pi(\theta(g))$. In this case there is an intertwining operator A on the space of π with $\pi(g)A = A\pi(\theta(g))$ for all g. Since $\theta^2 = 1$ we have $\pi(g)A^2 = A^2\pi(g)$ for all g, and since π is irreducible A^2 is a scalar by Schur's lemma. We choose A with $A^2 = 1$. This determines A up to a sign. When π has a Whittaker model, which happens for all components of cuspidal automorphic representations of the adele group $\mathrm{GL}(4, \mathbb{A})$, we specify a normalization of A which is compatible with a global normalization, as follows, and then put $\pi(g \times \theta) = \pi(g) \times A$.

Fix a nontrivial character $\boldsymbol{\psi}$ of F in $\mathbb{C}^\times$, and a character $\psi(u) = \boldsymbol{\psi}(a_{1,2} + a_{2,3} - a_{3,4})$ of $u = (u_{i,j})$ in the upper triangular subgroup U of G. Note that $\psi(\theta(u)) = \psi(u)$. Assume that π is a *non degenerate* G-module, namely it embeds in the space of "Whittaker" functions W on G, which satisfy – by definition – $W(ugk) = \psi(u)W(g)$ for all $g \in G$, $u \in U$, k in a compact open subgroup K_W of K, as a G-module under right shifts: $(\pi(g)W)(h) = W(hg)$. Then ${}^\theta\pi$ is non degenerate and can be realized in the space of functions ${}^\theta W(g) = W(\theta(g))$, W in the space of π. We take A to be the operator on the space of π which maps W to ${}^\theta W$.

A G-module π is called *unramified* if the space of π contains a nonzero K-fixed vector. The dimension of the space of K-fixed vectors is bounded by one if π is irreducible. If π is θ-invariant and unramified, and $v_0 \neq 0$ is a K-fixed vector in the space of π, then Av_0 is a multiple of v_0 (since $\theta K = K$), namely $Av_0 = cv_0$, with $c = \pm 1$. Replace A by cA to have $Av_0 = v_0$, and put $\pi(\theta) = A$.

When π is (irreducible) unramified and has a Whittaker model, both normalizations of the intertwining operator are equal. In this case $\boldsymbol{\psi}$ is unramified (trivial on R but not on $\boldsymbol{\pi}^{-1}R$, where $\boldsymbol{\pi}$ is a generator of the maximal ideal of R), and there exists a unique Whittaker function W_0 in the space of π with respect to ψ with $W_0 = 1$ on K. It is mapped by $\pi(\theta) = A$ to ${}^\theta W_0$, which satisfies ${}^\theta W_0(k) = 1$ for all k in K since K is θ-invariant. Namely A maps the unique normalized (by $W_0(K) = 1$) K-fixed vector W_0 in the space of π to the unique normalized K-fixed vector ${}^\theta W_0$ in the space of ${}^\theta\pi$, and we have ${}^\theta W_0 = W_0$.

For any (admissible) π and (smooth) f the convolution operator $\pi(fdg) = \int_G f(g)\pi(g)dg$ has finite rank. If π is θ-invariant put $\pi(fdg \times \theta) = \int_G f(g)\pi(g)\pi(\theta)dg$. Denote by tr $\pi(fdg \times \theta)$ the trace of the operator $\pi(fdg \times \theta)$. It depends on the choice of the Haar measure dg, but the (*twisted*) *character* χ_π of π does not; χ_π is a locally-integrable complex-valued function on $G \times \theta$ (see [C], [H]) which is θ-conjugacy invariant and locally-constant on the θ-regular set, with tr $\pi(fdg \times \theta) = \int_G f(g)\chi_\pi(g \times \theta)dg$ for all f.

Local integrability is not used in this work; rather it is recovered for our twisted character.

Small representation

To describe the G-module of interest in this paper, take P to be the upper triangular parabolic subgroup of type (3,1), and fix its Levi factor to be $M = \{m = \mathrm{diag}(m_3, m_1);\ m_3 \in \mathrm{GL}(3,F),\ m_1 \in F^\times\}$. It is isomorphic to $\mathrm{GL}(3,F) \times F^\times$. Let δ denote (as above) the character $\delta(p) = |\operatorname{Ad}(p)|\operatorname{Lie} N|$ of P; it is trivial on the unipotent radical N $(= F^3)$ of P. Then the value of δ at $p = mn$ is $|m_1^{-3}\det m_3|$. Denote by $I(\pi_1)$ the G-module $\pi = \mathrm{Ind}(\delta^{1/2}\pi_1; P, G)$ normalizedly induced from π_1 on P to G. It is clear from [BZ] that when π_1 is self-contragredient and $I(\pi_1)$ is irreducible then $I(\pi_1)$ is θ-invariant, and it is unramified if and only if π_1 is unramified.

Our aim in this work is to compute the θ-twisted character χ_{π_Y} of the $\mathrm{GL}(4,F)$*-module* $\pi_Y = I_{(3,1)}(1_3 \times \chi_Y)$*, where* $1_3 \times \chi_Y$ *is the* P*-module* $\left(\begin{smallmatrix} m_3 & * \\ 0 & m_1 \end{smallmatrix}\right) \mapsto \chi_Y(m_1)$*,* χ_Y *is a quadratic character of* $F^\times$*,* $m_i \in \mathrm{GL}(i,F)$*, by purely local means.*

We begin by describing a useful model of our representation, in analogy with the models of [FK] and [FZ4] of analogous representations $I_{(2,1)}(1_2)$ of $\mathrm{PGL}(3,F)$ and $I_{(3,1)}(1_3)$ of $\mathrm{PGL}(4,F)$. Indeed we shall express π_Y as an integral operator in a convenient model, and integrate the kernel over the diagonal to compute the character of π_Y.

Denote by $\mu = \mu_s$ the character $\mu(x) = |x|^{(s+1)/2}$ of $F^\times$, and by χ_Y a quadratic character of $F^\times$. This pair (μ, χ_Y) defines a character $\mu_P = \mu_{s,Y,P}$ of P, trivial on N, by $\mu_P(p) = \mu((\det m_3)/{m_1}^3)\chi_Y(m_1)$ if $p = mn$ and $m = \begin{pmatrix} m_3 & 0 \\ 0 & m_1 \end{pmatrix}$ with m_3 in $\mathrm{GL}(3,F)$, m_1 in $\mathrm{GL}(1,F)$. If $s = 0$, then $\mu_P = \delta^{1/2}\chi_Y$, where viewed as a character on P, χ_Y takes the value $\chi_Y(m_1)$ at p. Let $W_s = W_s^Y$ be the space of complex-valued smooth functions ψ on G with $\psi(pg) = \mu_P(p)\psi(g)$ for all p in P and g in G. The group G acts on W_s by right translation: $(\pi_s(g)\psi)(h) = \psi(hg)$. By definition, $I_{(3,1)}(1_3 \times \chi_Y)$ is the G-module W_s with $s = 0$. The parameter s is introduced for purposes of analytic continuation.

We prefer to work in another model $V_s = V_s^Y$ of the G-module W_s. Let V denote the space of column 4-vectors over F. Let V_s be the space of smooth complex-valued functions ϕ on $V - \{0\}$ with $\phi(\lambda\mathbf{v}) = \mu(\lambda)^{-4}\chi_Y(\lambda)\phi(\mathbf{v})$. The group G acts on V_s by $(\tau_s(g)\phi)(\mathbf{v}) = \mu(\det g)\phi({}^t g\mathbf{v})$. Let $\mathbf{v}_0 \neq 0$ be a vector of V such that the line $\{\lambda\mathbf{v}_0; \lambda$ in $F\}$ is fixed under the action of ${}^tP$. Explicitly, we take $\mathbf{v}_0 = {}^t(0,0,0,1)$. It is clear that the map $V_s \to W_s$, $\phi \mapsto \psi = \psi_\phi$, where $\psi(g) = (\tau_s(g)\phi)(\mathbf{v}_0) = \mu(\det g)\phi({}^t g\mathbf{v}_0)$, is a G-module isomorphism (check that $\psi_{\tau_s(g)\phi} = \pi_s(g)\psi_\phi$), with inverse $\psi \mapsto \phi = \phi_\psi$, $\phi(\mathbf{v}) = \mu(\det g)^{-1}\psi(g)$ if $\mathbf{v} = {}^t g\mathbf{v}_0$ (G acts transitively on $V - \{0\}$).

For $\mathbf{v} = {}^t(x,y,z,t)$ in V put $\|\mathbf{v}\| = \max(|x|, |y|, |z|, |t|)$. Let V^0 be the quotient of the set V^1 of $\mathbf{v}$ in V with $\|\mathbf{v}\| = 1$, by the equivalence relation $\mathbf{v} \sim \alpha\mathbf{v}$ if α is a unit in R. Denote by $\mathbb{P}V$ the projective space of lines in $V - \{0\}$. If Φ is a function on $V - \{0\}$ with $\Phi(\lambda\mathbf{v}) = |\lambda|^{-4}\Phi(\mathbf{v})$ and $d\mathbf{v} = dx\,dy\,dz\,dt$, then $\Phi(\mathbf{v})d\mathbf{v}$ is homogeneous of degree zero. Define

$$\int_{\mathbb{P}V} \Phi(\mathbf{v})d\mathbf{v} \quad \text{to be} \quad \int_{V^0} \Phi(\mathbf{v})d\mathbf{v}.$$

Clearly we have

$$\int_{\mathbb{P}V} \Phi(\mathbf{v})d\mathbf{v} = \int_{\mathbb{P}V} \Phi(g\mathbf{v})d(g\mathbf{v}) = |\det g| \int_{\mathbb{P}V} \Phi(g\mathbf{v})d\mathbf{v}.$$

Put $\nu(x) = |x|$ and $m = 2(s-1)$. Note that $\nu/\mu_s = \mu_{-s}$. Put $\langle \mathbf{w}, \mathbf{v}\rangle = {}^t\mathbf{w}J\mathbf{v}$. Then $\langle g\mathbf{w}, \theta(g)\mathbf{v}\rangle = \langle \mathbf{w}, \mathbf{v}\rangle$.

1. Proposition. *The operator*

$$T_s^Y : V_s \to V_{-s}, \qquad (T_s^Y\phi)(\mathbf{v}) = \int_{\mathbb{P}V} \phi(\mathbf{w})|\langle \mathbf{w},\mathbf{v}\rangle|^m \chi_Y(\langle \mathbf{w},\mathbf{v}\rangle)d\mathbf{w},$$

converges on $\operatorname{Re} s > 1/2$*, and satisfies there* $T_s^Y \tau_s(g) = \tau_{-s}(\theta(g))T_s^Y$ *for all* g *in* G*.*

Proof. We have

$$\begin{aligned}(T_s^Y(\tau_s(g)\phi))(\mathbf{v}) &= \textstyle\int (\tau_s(g)\phi)(\mathbf{w})|{}^t\mathbf{w}J\mathbf{v}|^m \chi_Y(\langle \mathbf{w},\mathbf{v}\rangle)d\mathbf{w}\\ &= \mu(\det g) \textstyle\int \phi({}^tg\mathbf{w})|{}^t\mathbf{w}J\mathbf{v}|^m \chi_Y(\langle \mathbf{w},\mathbf{v}\rangle)d\mathbf{w}\\ &= |\det g|^{-1}\mu(\det g) \textstyle\int \phi(\mathbf{w})|{}^t({}^tg^{-1}\mathbf{w})J\mathbf{v}|^m \chi_Y(\langle {}^tg^{-1}\mathbf{w},\mathbf{v}\rangle)d\mathbf{w}\\ &= (\mu/\nu)(\det g) \textstyle\int \phi(\mathbf{w})|{}^t\mathbf{w}J\cdot J^{-1}g^{-1}J\mathbf{v}|^m \chi_Y(\langle \mathbf{w},\theta({}^tg)\mathbf{v}\rangle)d\mathbf{w}\\ &= (\mu/\nu)(\det g) \textstyle\int \phi(\mathbf{w})|\langle \mathbf{w},\theta({}^tg)\mathbf{v}\rangle|^m \chi_Y(\langle \mathbf{w},\theta({}^tg)\mathbf{v}\rangle)d\mathbf{w}\\ &= (\nu/\mu)(\det\theta(g))\cdot (T_s^Y\phi)({}^t\theta(g)\mathbf{v}) = [(\tau_{-s}(\theta(g)))(T_s^Y\phi)](\mathbf{v})\end{aligned}$$

for the functional equation.

For the convergence, we may assume that $\phi = 1$ and ${}^t\mathbf{v} = (0,0,0,1)$, so that the integral is $\int_R |x|^m dx$, which converges for $\operatorname{Re} m > -1$. Our m is $2s-2$, as required. □

The spaces V_s are isomorphic to the space W of locally-constant complex-valued functions ϕ on V^1 with $\phi(\lambda\mathbf{v}) = \chi_Y(\lambda)\phi(\mathbf{v})$ for all $\lambda \in R^\times$, and T_s^Y is equivalent to an operator $T_s^{Y,0}$ on W. The proof of Proposition 1 implies also

1. Corollary. *The operator* $T_s^{Y,0} \circ \tau_s(g^{-1})$ *is an integral operator with kernel*

$$(\mu/\nu)(\det\theta(g))|\langle \mathbf{w},\theta({}^tg^{-1})\mathbf{v}\rangle|^m \chi_Y(\langle \mathbf{w},\theta({}^tg^{-1})\mathbf{v}\rangle) \qquad (\mathbf{v},\mathbf{w} \textit{ in } V^1)$$

and trace

$$\operatorname{tr}[T_s^{Y,0}\circ\tau_s(g^{-1})] = (\nu/\mu)(\det g)\int_{V^0} |{}^t\mathbf{v}gJ\mathbf{v}|^m \chi_Y({}^t\mathbf{v}gJ\mathbf{v})d\mathbf{v}.$$

Next we normalize the operator $T^Y = T_s^{Y,0}$. Recalling that χ_Y is unramified ($=1$ on $R^\times$, $\chi_Y(\boldsymbol{\pi}) = -1$), we normalize T^Y so that it acts trivially on the one-dimensional space of K-fixed vectors in V_s. This space is spanned by the function ϕ_0 in V_s with $\phi_0(\mathbf{v}) = 1$ for all $\mathbf{v}$ in V^0. This is the only case studied in full in this paper.

Denote again by $\boldsymbol{\pi}$ a generator of the maximal ideal of the ring R of integers in our local nonarchimedean field F of odd residual characteristic. Denote by q the number of elements of the residue field $R/\boldsymbol{\pi}R$ of R. Normalize the absolute value by $|\boldsymbol{\pi}| = q^{-1}$, and the measures by $\operatorname{vol}\{|x| \le 1\} = 1$. Then $\operatorname{vol}\{|x| = 1\} = 1 - q^{-1}$, and the volume of V^0 is $(1-q^{-4})/(1-q^{-1}) = 1 + q^{-1} + q^{-2} + q^{-3}$.

2. Proposition. *If* $\mathbf{v}_0 = {}^t(0,0,0,1)$*, we have*

$$(T^Y\phi_0)(\mathbf{v}_0) = \frac{1+q^{-2(s+1)}}{1+q^{1-2s}}\phi_0(\mathbf{v}_0).$$

When $s=0$, the constant is $(1+q^{-2})(1+q)^{-1}$.

Proof. Since χ_Y is unramified, we have

$$(T^Y\phi_0)(\mathbf{v}_0)=\int_{V^0}\phi_0(\mathbf{v})|{}^t\mathbf{v}J\mathbf{v}_0|^m\chi_Y({}^t\mathbf{v}J\mathbf{v}_0)d\mathbf{v}=\int_{V^0}|x|^m\chi_Y(x)dxdydzdt$$
$$=\left[\int_{||\mathbf{v}||\leq 1}-\int_{||\mathbf{v}||<1}\right]|x|^m\chi_Y(x)dxdydzdt/\int_{|x|=1}dx$$
$$=(1+q^{-m-4})\int_{|x|\leq 1}|x|^m\chi_Y(x)dx/\int_{|x|=1}dx=(1+q^{-2(s+1)})/(1+q^{1-2s}),$$

since $m=2(s-1)$ and

$$\int_{|x|\leq 1}|x|^m\chi_Y(x)dx=(1+q^{-m-1})^{-1}\int_{|x|=1}dx.$$

The proposition follows. □

Character computation for type I

For the θ-conjugacy class of type I, represented by $g=t\cdot\mathrm{diag}(\mathbf{r},\mathbf{s},\mathbf{s},\mathbf{r})$, the product

$${}^t\mathbf{v}gJ\mathbf{v}=(t,z,x,y)\begin{pmatrix}a_1\mathbf{r}&0&0&a_2D\mathbf{r}\\0&b_1\mathbf{s}&b_2D\mathbf{s}&0\\0&b_2\mathbf{s}&b_1\mathbf{s}&0\\a_2\mathbf{r}&0&0&a_1\mathbf{r}\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\begin{pmatrix}t\\z\\x\\y\end{pmatrix}$$

is equal to

$$-t^2a_2D\mathbf{r}-z^2b_2D\mathbf{s}+x^2b_2\mathbf{s}+y^2a_2\mathbf{r}.$$

Note that $\mathbf{r}$ and $\mathbf{s}$ range over a set of representatives for $F^\times/N_{E/F}E^\times$.

By Corollary 1, we need to compute

$$(\frac{\nu}{\mu})(\det g)\frac{\Delta(g\theta)}{\Delta_C(Ng)}\int_{V^0}|{}^t\mathbf{v}gJ\mathbf{v}|^m\chi_Y({}^t\mathbf{v}gJ\mathbf{v})d\mathbf{v}$$
$$=\frac{|\mathbf{rs}|^{1-s}|4a_2b_2D|}{|(a_1^2-a_2^2D)(b_1^2-b_2^2D)|^{s/2}}\int_{V^0}|\alpha|^{2(s-1)}\chi_Y(\alpha)dxdydzdt.$$

Here α is $x^2b_2\mathbf{s}+y^2a_2\mathbf{r}-z^2b_2D\mathbf{s}-t^2a_2D\mathbf{r}$. Put $\mathbf{r}'=-\frac{a_2}{b_2}\frac{\mathbf{r}}{\mathbf{s}}$. Thus we need to compute the value at $s=0$ of the product of

$$\chi_Y(b_2\mathbf{s})|\frac{\mathbf{r}}{\mathbf{s}}|^{-s}|4D\mathbf{r}'||((\frac{a_1}{b_2})^2-(\frac{a_2}{b_2})^2D)((\frac{b_1}{b_2})^2-D)|^{-s/2}$$

with the integral $I_s^Y(\mathbf{r}',D)$, where $Q=x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2$ and

$$I_s^Y(\mathbf{r},D)=\int_{V^0}|Q|^{2(s-1)}\chi_Y(Q)dxdydzdt.$$

I. Theorem. *When Y/F is unramified, the value of $I_s^Y(\mathbf{r},D)$ at $s=0$ is 0.*

Proof. Consider the case when the quadratic form $Q=x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2$ is anisotropic (does not represent zero). Thus $D=\boldsymbol{\pi}$ and $\mathbf{r}\in R^\times-R^{\times 2}$ (hence $|\mathbf{r}|=1$, $|D|=1/q$), or $D\in R^\times-R^{\times 2}$ and $\mathbf{r}=\boldsymbol{\pi}$. The second case being equivalent to the first, it suffices to deal with the first case.

The domain $\max\{|x|,|y|,|z|,|t|\}=1$ is the disjoint union of $\{|x|=1\}$, $\{|x|<1,|y|=1\}$, $\{|x|<1,|y|<1,|z|=1\}$ and $\{|x|<1,|y|<1,|z|<1,|t|=1\}$. Note that $\chi_Y(x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2)$ is equal to 1 on the first two subdomains and equals -1 on the other two. Thus the integral $I_s^Y(\mathbf{r},D)$ is the quotient by $\int_{|x|=1}dx$ of

$$\int_{|x|=1}dx+\int\int_{|x|<1,|y|=1}dxdy-q^{-m}\int\int\int_{|x|<1,|y|<1,|z|=1}dxdydz$$
$$-q^{-m}\int\int\int\int_{|x|<1,|y|<1,|z|<1,|t|=1}dxdydzdt$$
$$=1+q^{-1}-q^{-m-2}-q^{-m-3}=1+q^{-1}-q^{-2s}-q^{-2s-1}.$$

The value at $s=0$ is 0 and thus the theorem follows when the quadratic form is anisotropic.

We then turn to the case when the quadratic form is isotropic. Recall that $\mathbf{r}$ ranges over a set of representatives for $F^\times/N_{E/F}E^\times$, $E=F(\sqrt{D})$. Thus $D\in F-F^2$, and we may assume that $|D|$ and $|\mathbf{r}|$ lie in $\{1,q^{-1}\}$.

I.1. Proposition ([FZ4]). *When the quadratic form $x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2$ is isotropic, $\mathbf{r}$ lies in $N_{E/F}E^\times$, and we may assume that the quadratic form takes one of three shapes:*

$$x^2-y^2-Dz^2+Dt^2,\ D\in R^\times-R^{\times 2};\quad x^2+\boldsymbol{\pi}y^2-\boldsymbol{\pi}z^2-\boldsymbol{\pi}^2t^2;\quad x^2-y^2-\boldsymbol{\pi}z^2+\boldsymbol{\pi}t^2.$$

The set $V^0=V/\sim$, where $V=\{\mathbf{v}=(x,y,z,t)\in R^4;\max\{|x|,|y|,|z|,|t|\}=1\}$ and $\sim$ is the equivalence relation $\mathbf{v}\sim\alpha\mathbf{v}$ for $\alpha\in R^\times$, is the disjoint union of the subsets

$$V_n^0=V_n^0(\mathbf{r},D)=V_n(\mathbf{r},D)/\sim,$$

where

$$V_n=V_n(\mathbf{r},D)=\{\mathbf{v};\max\{|x|,|y|,|z|,|t|\}=1,|x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2|=1/q^n\},$$

over $n\ge 0$, and of $\{\mathbf{v};x^2-\mathbf{r}y^2-Dz^2+\mathbf{r}Dt^2=0\}/\sim$, a set of measure zero.

Thus the integral $I_s^Y(\mathbf{r},D)$ coincides with the sum

$$\sum_{n=0}^{\infty}(-1)^n q^{-nm}\operatorname{vol}(V_n^0(\mathbf{r},D)).$$

When the quadratic form represents zero the problem is then to compute the volumes

$$\operatorname{vol}(V_n^0(\mathbf{r},D))=\operatorname{vol}(V_n(\mathbf{r},D))/(1-1/q)\qquad(n\ge 0).$$

We need some results from [FZ4]:

I.0. Lemma ([FZ4]). *When* $c^2 \in R^{\times 2}$ *and* $n \geq 1$*, we have*

$$\int_{|c^2-x^2|=q^{-n}} dx \;=\; \frac{2}{q^n}\left(1-\frac{1}{q}\right).$$

I.1. Lemma ([FZ4]). *When* $D = \boldsymbol{\pi}$ *and* $\mathbf{r} = 1$*, thus* $|\mathbf{r}D| = 1/q$*, we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1-1/q, & \text{if } n=0,\\ q^{-1}(1-1/q)(2+1/q), & \text{if } n=1,\\ 2q^{-n}(1-1/q)(1+1/q), & \text{if } n\geq 2. \end{cases}$$

I.2. Lemma ([FZ4]). *When* $D = \boldsymbol{\pi}$ *and* $\mathbf{r} = -\boldsymbol{\pi}$*, thus* $|\mathbf{r}D| = 1/q^2$*, we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1, & \text{if } n=0,\\ q^{-1}(1-1/q), & \text{if } n=1,\\ q^{-2}(2-1/q-2/q^2), & \text{if } n=2,\\ 2q^{-n}(1-1/q)(1+1/q), & \text{if } n\geq 3. \end{cases}$$

I.3. Lemma ([FZ4]). *When* E/F *is unramified, thus* $|\mathbf{r}D| = 1$*, we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1-1/q^2, & \text{if } n=0,\\ q^{-n}(1-1/q)(1+2/q+1/q^2), & \text{if } n\geq 1. \end{cases}$$

Proof of Theorem I. We are now ready to complete the proof of Theorem I in the isotropic case. Recall that we need to compute the value at $s=0$ ($m=-2$) of $I_s^Y(\mathbf{r}, D)$. Here $I_s^Y(\mathbf{r}, D)$ coincides with the sum

$$\sum_{n=0}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0(\mathbf{r}, D))$$

which converges for $m > -1$ by Proposition 1 or alternatively by Lemmas I.1-I.3. The value at $m=-2$ is obtained then by analytic continuation of this sum.
Case of Lemma I.1. The integral $I_s^Y(\mathbf{r}, D)$ is equal to

$$\mathrm{vol}(V_0^0) - q^{-m}\,\mathrm{vol}(V_1^0) + \sum_{n=2}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$= 1-\frac{1}{q}-\frac{1}{q}\left(1-\frac{1}{q}\right)\left(2+\frac{1}{q}\right)\frac{1}{q^m}+2\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)q^{-2(m+1)}\left(1+\frac{1}{q^{m+1}}\right)^{-1}.$$

When $m=-2$, this is

$$1-\frac{1}{q}-q\left(2-\frac{1}{q}-\frac{1}{q^2}\right)+2\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)\frac{q^2}{1+q}=0.$$

Case of Lemma I.2. The integral $I_s^Y(\mathbf{r},D)$ is equal to

$$\mathrm{vol}(V_0^0)-q^{-m}\,\mathrm{vol}(V_1^0)+q^{-2m}\,\mathrm{vol}(V_2^0)+\sum_{n=3}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$=1-\frac{1}{q}\left(1-\frac{1}{q}\right)q^{-m}+\frac{1}{q^2}\left(2-\frac{1}{q}-\frac{2}{q^2}\right)q^{-2m}$$

$$-2\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)q^{-3(m+1)}\left(1+\frac{1}{q^{m+1}}\right)^{-1}.$$

When $m=-2$, this is

$$1-\frac{1}{q}\left(1-\frac{1}{q}\right)q^2+\frac{1}{q^2}\left(2-\frac{1}{q}-\frac{2}{q^2}\right)q^4-2\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)\frac{q^3}{1+q}.$$

Once simplified this is equal to 0.
Case of Lemma I.3. The integral $I_s^Y(\mathbf{r},D)$ is equal to

$$\mathrm{vol}(V_0^0)+\sum_{n=1}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$=1-\frac{1}{q^2}-\left(1-\frac{1}{q}\right)\left(1+\frac{2}{q}+\frac{1}{q^2}\right)q^{-(m+1)}\left(1+\frac{1}{q^{m+1}}\right)^{-1}.$$

When $m=-2$, this is

$$=1-\frac{1}{q^2}-\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)^2\frac{q}{1+q}=0.$$

The theorem follows. □

Character computation for type II

For the θ-conjugacy class of type II, represented by $g=t\cdot\mathrm{diag}(\mathbf{r},\mathbf{s},\mathbf{s},\mathbf{r})$, the product

$${}^t\mathbf{v}gJ\mathbf{v}=(t,z,x,y)\begin{pmatrix}a_1\mathbf{r}&0&0&a_2D\mathbf{r}\\0&b_1\mathbf{s}&b_2AD\mathbf{s}&0\\0&b_2\mathbf{s}&b_1\mathbf{s}&0\\a_2\mathbf{r}&0&0&a_1\mathbf{r}\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix}\begin{pmatrix}t\\z\\x\\y\end{pmatrix}$$

is equal to

$$-t^2a_2D\mathbf{r}-z^2b_2AD\mathbf{s}+x^2b_2\mathbf{s}+y^2a_2\mathbf{r}=b_2\mathbf{s}(x^2-y^2\mathbf{r}'-z^2AD+t^2D\mathbf{r}').$$

Here $a_1+a_2\sqrt{D}\in E_1^\times$ ($E_1=F(\sqrt{D})$) and $b_1+b_2\sqrt{AD}\in E_2^\times$ ($E_2=F(\sqrt{AD})$), and $\mathbf{r}'=-a_2\mathbf{r}/b_2\mathbf{s}$. As $\mathbf{r}$ ranges over a set of representatives for $F^\times/N_{E_1/F}E_1^\times$ (and $\mathbf{s}$ for $F^\times/N_{E_2/F}E_2^\times$), we may rename $\mathbf{r}'$ by $\mathbf{r}$.

Thus, by Corollary 1, we need to compute the value at $s=0$ of the product of

$$\chi_Y(b_2\mathbf{s})|\frac{\mathbf{r}}{\mathbf{s}}|^{-s}|4\mathbf{r}'D\sqrt{A}||((\frac{a_1}{b_2})^2-(\frac{a_2}{b_2})^2D)((\frac{b_1}{b_2})^2-AD)|^{-s/2}$$

and the value when $\mathbf{r}$ is $\mathbf{r}'$ and $Q=x^2-\mathbf{r}y^2-ADz^2+\mathbf{r}Dt^2$ of the integral

$$I_s^Y(\mathbf{r},A,D)=\int_{V^0}|Q|^{2(s-1)}\chi_Y(Q)dxdydzdt.$$

The property of the numbers A, D and AD that we need is that their square roots generate the three distinct quadratic extensions of F. Thus we may assume that $\{A,D,AD\}=\{u,\boldsymbol{\pi},u\boldsymbol{\pi}\}$, where $u\in R^\times-R^{\times2}$. Of course with this normalization AD is no longer the product of A and D, but its representative in the set $\{1,u,\boldsymbol{\pi},u\boldsymbol{\pi}\}$ mod $F^{\times2}$. Since $\mathbf{r}$ ranges over a set of representatives for $F^\times/N_{E_1/F}E_1^\times$, it can be assumed to range over $\{1,\boldsymbol{\pi}\}$ if $D=u$, and over $\{1,u\}$ if $|D|=|\boldsymbol{\pi}|$.

In this section we prove

II. Theorem. *When Y/F is unramified, the value of*

$$\chi_Y(b_2\mathbf{s})|4\mathbf{r}D\sqrt{A}|I_s^Y(\mathbf{r},A,D)/(T^Y\phi_0)(\mathbf{v}_0)$$

at $s=0$ is $-2\chi_Y(b_2\mathbf{s})\delta(Y,E_3)$.

Recall that $E_3=F(\sqrt{A})$. As usual, $\delta(Y,E_3)$ is 1 if $Y=E_3$ and 0 if $Y\neq E_3$.

The meaning of this result is that the twisted character of π_Y on elements of tori of type II relates to values of the trivial character on $\mathbf{C}_Y(F)$, $Y=E_3$, on the torus which splits over E. It does not relate to such values on $\mathbf{C}_{Y'}(F)$, $Y'\neq E_3$.

Recall Lemma II.0 from [FZ4].

II.0. Lemma ([FZ4]). *The quadratic form $x^2-\mathbf{r}y^2-ADz^2+\mathbf{r}Dt^2$ takes one of six forms: $x^2-y^2+\boldsymbol{\pi}(t^2-uz^2)$, $x^2-uy^2+u\boldsymbol{\pi}(t^2-z^2)$, $x^2-y^2+ut^2-u\boldsymbol{\pi}z^2$, $x^2-y^2-uz^2+\boldsymbol{\pi}t^2$, $x^2-\boldsymbol{\pi}y^2+u\boldsymbol{\pi}(t^2-z^2)$, $x^2-uy^2-uz^2+u\boldsymbol{\pi}t^2$, where $u\in R^\times-R^{\times2}$. It is always isotropic.*

The set $V^0=V/\sim$, where $V=\{\mathbf{v}=(x,y,z,t)\in R^4;\max\{|x|,|y|,|z|,|t|\}=1\}$ and $\sim$ is the equivalence relation $\mathbf{v}\sim\alpha\mathbf{v}$ for $\alpha\in R^\times$, is the disjoint union of the subsets

$$V_n^0=V_n^0(\mathbf{r},A,D)=V_n(\mathbf{r},A,D)/\sim,$$

where

$$V_n = V_n(\mathbf{r}, A, D) = \{\mathbf{v}; \max\{|x|, |y|, |z|, |t|\} = 1, |x^2 - \mathbf{r}y^2 - ADz^2 + \mathbf{r}Dt^2| = 1/q^n\},$$

over $n \geq 0$, and of $\{\mathbf{v}; x^2 - \mathbf{r}y^2 - ADz^2 + \mathbf{r}Dt^2 = 0\}/\sim$, a set of measure zero.

Since Y/F is unramified, the integral $I_s^Y(\mathbf{r}, A, D)$ is equal to

$$\sum_{n=0}^{\infty} (-1)^n q^{-nm} \operatorname{vol}(V_n^0(\mathbf{r}, A, D)).$$

The problem is then to compute the volumes

$$\operatorname{vol}(V_n^0(\mathbf{r}, A, D)) = \operatorname{vol}(V_n(\mathbf{r}, A, D))/(1 - 1/q) \qquad (n \geq 0).$$

Choose u to be a non square unit. To prove Theorem II, by Lemma II.0 we need precisely the following Lemmas from [FZ4]. In Lemmas II.1 and II.2, $E_3 = F(\sqrt{A})$ is unramified over F.

II.1. Lemma ([FZ4])**.** *When the quadratic form is* $x^2 - y^2 + \boldsymbol{\pi}(t^2 - uz^2)$ *(thus* $\mathbf{r} = 1$, $A = u$, $D = \boldsymbol{\pi}$ *up to squares), we have*

$$\operatorname{vol}(V_n^0) = \begin{cases} 1 - 1/q, & \textit{if } n = 0, \\ 2/q - 1/q^2 + 1/q^3, & \textit{if } n = 1, \\ 2q^{-n}(1 - 1/q), & \textit{if } n \geq 2. \end{cases}$$

II.2. Lemma ([FZ4])**.** *When the quadratic form is* $x^2 - uy^2 + u\boldsymbol{\pi}(t^2 - z^2)$ *(thus* $\mathbf{r} = u$, $A = u$, $D = \boldsymbol{\pi}$ *up to squares), we have*

$$\operatorname{vol}(V_n^0) = \begin{cases} 1 + 1/q, & \textit{if } n = 0, \\ q^{-2}(1 - 1/q), & \textit{if } n = 1, \\ 2q^{-(n+1)}(1 - 1/q), & \textit{if } n \geq 2. \end{cases}$$

II.3. Lemma ([FZ4])**.** *When the quadratic form is* $x^2 - y^2 + ut^2 - u\boldsymbol{\pi}z^2$ *or* $x^2 - y^2 - uz^2 + \boldsymbol{\pi}t^2$, *we have*

$$\operatorname{vol}(V_n^0) = \begin{cases} 1, & \textit{if } n = 0, \\ 1/q, & \textit{if } n = 1, \\ q^{-n}(1 - 1/q^2), & \textit{if } n \geq 2. \end{cases}$$

II.4. Lemma ([FZ4])**.** *When the quadratic form is* $x^2 - \boldsymbol{\pi}y^2 + u\boldsymbol{\pi}(t^2 - z^2)$, *we have*

$$\operatorname{vol}(V_n^0) = \begin{cases} 1, & \textit{if } n = 0, \\ 1/q, & \textit{if } n = 1, \\ q^{-n}(1 - 1/q^2), & \textit{if } n \geq 2. \end{cases}$$

II.5. Lemma ([FZ4]). *When the quadratic form is $x^2 - uy^2 - uz^2 + u\boldsymbol{\pi}t^2$, we have*

$$\mathrm{vol}(V_n^0) = \begin{cases} 1, & \text{if } n = 0, \\ 1/q, & \text{if } n = 1, \\ q^{-n}(1 - 1/q^2), & \text{if } n \geq 2. \end{cases}$$

Proof of Theorem II. To prove Theorem II, recall that we need to compute the value at $s = 0$ ($m = -2$) of the product

$$\chi_Y(b_2\mathbf{s})|4\mathbf{r}D\sqrt{A}|I_s^Y(\mathbf{r}, A, D)/(T^Y\phi_0)(\mathbf{v}_0).$$

Here $I_s^Y(\mathbf{r}, A, D)$ is equal to the sum

$$\sum_{n=0}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0(\mathbf{r}, D))$$

which converges for $m > -1$ by Proposition 1 or alternatively by Lemmas II.1-II.3. The value at $m = -2$ is obtained then by analytic continuation of this sum.
Case of Lemma II.1. We have $|4\mathbf{r}D\sqrt{A}| = 1/q$, and the integral $I_s^Y(\mathbf{r}, A, D)$ is equal to

$$\mathrm{vol}(V_0^0) - q^{-m}\,\mathrm{vol}(V_1^0) + \sum_{n=2}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$= 1 - \frac{1}{q} - \left(\frac{2}{q} - \frac{1}{q^2} + \frac{1}{q^3}\right)\frac{1}{q^m} + 2\left(1 - \frac{1}{q}\right)q^{-2(m+1)}\left(1 + \frac{1}{q^{m+1}}\right)^{-1}.$$

When $m = -2$, this is

$$1 - \frac{1}{q} - q^2\left(\frac{2}{q} - \frac{1}{q^2} + \frac{1}{q^3}\right) + 2\left(1 - \frac{1}{q}\right)\frac{q^2}{1+q} = \frac{-2q}{1+q}\left(1 + \frac{1}{q^2}\right).$$

Multiplying by $|4\mathbf{r}D\sqrt{A}| = 1/q$ we obtain $-2(1+1/q^2)(1+q)^{-1}$. We are done by Proposition 2.
Case of Lemma II.2. We have $|4\mathbf{r}D\sqrt{A}| = 1/q$, and the integral $I_s^Y(\mathbf{r}, A, D)$ is equal to

$$\mathrm{vol}(V_0^0) - q^{-m}\,\mathrm{vol}(V_1^0) + \sum_{n=2}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$= 1 + \frac{1}{q} - \frac{1}{q^2}\left(1 - \frac{1}{q}\right)\frac{1}{q^m} + \frac{2}{q}\left(1 - \frac{1}{q}\right)q^{-2(m+1)}\left(1 + \frac{1}{q^{m+1}}\right)^{-1}.$$

When $m=-2$, this is

$$1+\frac{1}{q}-1+\frac{1}{q}+\frac{2}{q}\left(1-\frac{1}{q}\right)\frac{q^2}{1+q}=\frac{2q}{1+q}\left(1+\frac{1}{q^2}\right).$$

Multiplying by $|4\mathbf{r}D\sqrt{A}|=1/q$ we obtain $2(1+1/q^2)(1+q)^{-1}$.
Case of Lemmas II.3, II.4, II.5. The integral $I_s^Y(\mathbf{r},A,D)$ is equal to

$$\mathrm{vol}(V_0^0)-q^{-m}\,\mathrm{vol}(V_1^0)+\sum_{n=2}^{\infty}(-1)^n q^{-nm}\,\mathrm{vol}(V_n^0)$$

$$=1-\frac{1}{q}\frac{1}{q^m}+\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)q^{-2(m+1)}\left(1+\frac{1}{q^{m+1}}\right)^{-1}.$$

When $m=-2$, this is

$$1-\frac{1}{q}q^2+\left(1-\frac{1}{q}\right)\left(1+\frac{1}{q}\right)\frac{q^2}{1+q}=0.$$

□

Character computation for type III

For the θ-conjugacy class of type III we write out the representative $g=t\cdot\mathrm{diag}(\boldsymbol{r},\boldsymbol{r})$ as

$$\begin{pmatrix} a_1r_1+a_2r_2A & (a_1r_2+a_2r_1)A & (b_1r_1+b_2r_2A)D & (b_1r_2+b_2r_1)AD\\ a_1r_2+a_2r_1 & a_1r_1+a_2r_2A & (b_1r_2+b_2r_1)D & (b_1r_1+b_2r_2A)D\\ b_1r_1+b_2r_2A & (b_1r_2+b_2r_1)A & a_1r_1+a_2r_2A & (a_1r_2+a_2r_1)A\\ b_1r_2+b_2r_1 & b_1r_1+b_2r_2A & a_1r_2+a_2r_1 & a_1r_1+a_2r_2A \end{pmatrix}.$$

The product ${}^t\mathbf{v}gJ\mathbf{v}$ (where ${}^t\mathbf{v}=(x,y,z,t)$) is equal to

$$(b_1r_2+b_2r_1)(t^2+z^2A-y^2D-x^2AD)+2(b_1r_1+b_2r_2A)(zt-xyD),$$

where $a_1+a_2\sqrt{A}\in E_3^\times$ and $b_1+b_2\sqrt{A}\in E_3^\times$. The trace is a function of g, and $r=r_1+r_2\sqrt{A}$ ranges over a set of representatives in $E_3^\times$ ($E_3=F(\sqrt{A})$) for $E_3^\times/N_{E/E_3}E^\times$.

Define the quadratic form $Q=Q(x,y,z,t)$ to be

$$\frac{rb-\tau(rb)}{2\sqrt{A}}(t^2+z^2A-y^2D-x^2AD)+(rb+\tau(rb))(zt-xyD).$$

Set $I_s^Y(r,A,D)$ to be equal to

$$\int_{V^0}|Q|^{2(s-1)}\chi_Y(Q)dxdydzdt.$$

The property of the numbers A, D and AD that we need is that their square roots generate the three distinct quadratic extensions of F. Thus we may assume that $\{A,D,AD\}=\{u,\boldsymbol{\pi},u\boldsymbol{\pi}\}$, where $u\in R^\times-R^{\times 2}$. Of course with this normalization AD is no longer the product of A and D, but its representative in the set $\{1,u,\boldsymbol{\pi},u\boldsymbol{\pi}\}$ mod $F^{\times 2}$.

III.1. Proposition. *(i) If $D = u$ and $A = \boldsymbol{\pi}$ (or $\boldsymbol{\pi}u$) then $\sqrt{A} \notin N_{E/E_3}E^\times = A^{\mathbb{Z}}R_3^\times$.*
(ii) If $A = u$ and $-1 \in R^{\times 2}$, and $D = \boldsymbol{\pi}$ (or $\boldsymbol{\pi}u$) then $\sqrt{A} \notin N_{E/E_3}E^\times = (-D)^{\mathbb{Z}}R_3^{\times 2}$.
(iii) If $A = u = -1 \notin R^{\times 2}$ and $D = \boldsymbol{\pi}$ (or $\boldsymbol{\pi}u$) then there is $d \in R^\times$ with $d^2 + 1 \in -R^{\times 2} = R^\times - R^{\times 2}$, hence $d + i \in R_3^\times - R_3^{\times 2}$ ($i = \sqrt{A}$) and so $d + i \in E_3^\times - N_{E/E_3}E^\times$.

Proof. For (iii) note that $R^\times/\{1 + \boldsymbol{\pi}R\}$ is the multiplicative group of a finite field $\mathbb{F}$ of q elements. There are $1 + \frac{1}{2}(q-1)$ elements in each of the sets $\{1 + x^2; x \in \mathbb{F}\}$ and $\{-y^2; y \in \mathbb{F}\}$. As $2(1 + \frac{1}{2}(q-1)) > q$, there are x, y with $1 + x^2 = -y^2$. But $y \neq 0$ as $-1 \notin \mathbb{F}^{\times 2}$. Hence there is x with $1 + x^2 \notin \mathbb{F}^{\times 2}$, and our d exists. □

Since r ranges over a set of representatives for $E_3^\times/N_{E/E_3}E^\times$, by Proposition III.1 we can choose br to be 1 or $\sqrt{A}$ or $d + i$. Correspondingly the quadratic form takes one of the three shapes

$$t^2 + z^2A - y^2D - x^2AD, \qquad zt - xyD, \qquad \text{or} \qquad t^2 - z^2 - y^2D + x^2D + 2d(zt - xyD).$$

III. Theorem. *When Y/F is unramified, the value of $I_s^Y(\mathbf{r}, A, D)$ at $s = 0$ is 0.*

Proof. Assume that $br = \sqrt{A} \notin N_{E/E_3}E^\times$, thus $|br\tau(br)D| = |AD|$, and the quadratic form is $t^2 + z^2A - y^2D - x^2AD$. If $|A| = 1/q$ or -1 is a square, we can replace A with $-A$. The quadratic form then becomes the same as that of type I. The result of the computation is 0, see proof of Theorem I, case of anisotropic quadratic forms and we are done in this case.

If $A = -1$, $br = d+i \notin N_{E/E_3}E^\times$, the quadratic form is $t^2 - z^2 - y^2D + x^2D + 2d(zt - xyD)$. It is equal to $X^2 - uY^2 - D(Z^2 - uT^2)$ with $X = t + dz$, $Y = z$, $Z = y + dx$, $T = x$ and $u = d^2 + 1 \in R^\times - R^{\times 2}$. Since $|D| = 1/q$ the quadratic form is anisotropic and the result of the computation is 0 by the proof of Theorem I, case of anisotropic quadratic forms.

Assume that $br = 1$, thus $|br\tau(br)D| = |D|$ and the quadratic form is $zt - xyD$. Then it is $\frac{1}{4}$ times $(z+t)^2 - (z-t)^2 - D[(x+y)^2 - (x-y)^2]$. Since $\max\{|x|, |y|, |z|, |t|\} = 1$ implies $\max\{|x+y|, |x-y|, |z+t|, |z-t|\} = 1$, the result of the computation is 0 by the proof of Theorem I, cases of Lemmas I.1 and I.3. The theorem follows. □

Character computation for type IV

For the θ-conjugacy class of type IV we write the representative $g = t \cdot \operatorname{diag}(\boldsymbol{r}, \boldsymbol{r})$ (where $t = h^{-1}t^*h$, $t^* = \operatorname{diag}(\alpha, \sigma\alpha, \sigma^3\alpha, \sigma^2\alpha)$) as

$$\begin{pmatrix} a_1r_1 + a_2r_2A & (a_1r_2 + a_2r_1)A & (b_1'r_1 + b_2'r_2A)D & (b_1'r_2 + b_2'r_1)AD \\ a_1r_2 + a_2r_1 & a_1r_1 + a_2r_2A & (b_1'r_2 + b_2'r_1)D & (b_1'r_1 + b_2'r_2A)D \\ b_1r_1 + b_2r_2A & (b_1r_2 + b_2r_1)A & a_1r_1 + a_2r_2A & (a_1r_2 + a_2r_1)A \\ b_1r_2 + b_2r_1 & b_1r_1 + b_2r_2A & a_1r_2 + a_2r_1 & a_1r_1 + a_2r_2A \end{pmatrix}.$$

Here $E_3 = F(\sqrt{A})$ is a quadratic extension of F and $E = E_3(\sqrt{D})$ is a quadratic extension of E_3, thus $A \in F - F^2$ and $D = d_1 + d_2\sqrt{A} \in E_3 - E_3^2$, $d_i \in F$.

If $-1 \in F^{\times 2}$ we can and do take $D = \sqrt{A}$, where A is a nonsquare unit u if E_3/E is unramified, or a uniformizer $\boldsymbol{\pi}$ if E_3/F is ramified. If $-1 \notin F^{\times 2}$ and E_3/F is ramified, once again we may and do take $A = \boldsymbol{\pi}$ and $D = \sqrt{A}$.

If $-1 \notin F^{\times 2}$ and E_3/F is unramified, take $A = -1$ and note that a primitive 4th root $\zeta = i$ of 1 lies in E_3 (and generates it over F). Then E/E_3 is unramified, generated by $\sqrt{D}$, $D = d_1 + id_2$, and we can (and do) take $d_2 = 1$ and a unit $d_1 = d$ in $F^\times$ such that $d^2 + 1 \notin F^{\times 2}$. Then $D = d + i \notin E_3^{\times 2}$. The existence of d is shown as in the proof of Proposition III.1.

Further $\alpha = a + b\sqrt{D} \in E^\times$, where $a = a_1 + a_2\sqrt{A} \in E_3^\times$, $b = b_1 + b_2\sqrt{A} \in E_3^\times$, and $r = r_1 + r_2\sqrt{A} \in E_3^\times / N_{E/E_3}E^\times$. The relation $bD = b_1' + b_2'\sqrt{A}$ defines $b_1' = b_1d_1 + b_2d_2A$ and $b_2' = b_2d_1 + b_1d_2$.

Recall from Corollary 1 that we need to compute

$$\left(\frac{\nu}{\mu}\right)(\det g)\frac{\Delta(g\theta)}{\Delta_C(Ng)}\int_{V^0}|{}^t\mathbf{v}gJ\mathbf{v}|^m\chi_Y({}^t\mathbf{v}gJ\mathbf{v})d\mathbf{v}. \qquad (*)$$

Since $\det g = \alpha r \cdot \sigma(\alpha r) \cdot \sigma^3(\alpha r) \cdot \sigma^2(\alpha r)$, we have

$$\left(\frac{\nu}{\mu}\right)(\det g)\frac{\Delta(g\theta)}{\Delta_C(Ng)} = |\det g|^{(1-s)/2}\left|\frac{(\alpha r - \sigma^2(\alpha r))^2}{\alpha r\sigma^2(\alpha r)} \cdot \frac{\sigma(\alpha r - \sigma^2(\alpha r))^2}{\sigma(\alpha r)\sigma^3(\alpha r)}\right|^{1/2}$$

$$= \frac{|4brD\sigma(brD)|}{|r^2(a^2 - b^2D)\sigma(r^2(a^2 - b^2D))|^{s/2}}.$$

When $s = 0$, this is $|brD\sigma(brD)|$.

The product ${}^t\mathbf{v}gJ\mathbf{v}$ (where ${}^t\mathbf{v} = (x, y, z, t)$) is then equal to

$$(b_1r_2 + b_2r_1)(t^2 + z^2A) - (b_1'r_2 + b_2'r_1)(y^2 + x^2A) + 2(b_1r_1 + b_2r_2A)zt - 2(b_1'r_1 + b_2'r_2A)xy.$$

Since $bD = b_1' + b_2'\sqrt{A}$, this is

$$\frac{br - \sigma(br)}{2\sqrt{A}}(t^2 + z^2A) + (br + \sigma(br))zt$$

$$-\frac{brD - \sigma(brD)}{2\sqrt{A}}(y^2 + x^2A) - (brD + \sigma(brD))xy.$$

Note that r ranges over a set of representatives for $E_3^\times/N_{E/E_3}E^\times$, and b lies in $E_3^\times$. As b is fixed, we may take br to range over $E_3^\times/N_{E/E_3}E^\times$.

Further, note that E_3/F is unramified if and only if E/E_3 is unramified. Hence br can be taken to range over $\{1, \boldsymbol{\pi}\}$ if E_3/F is unramified, and over $\{1, u\}$ if E_3/F is ramified, where $\boldsymbol{\pi}$ is a uniformizer in F and u is a nonsquare unit in F, in these two cases. Thus in both cases we have that $\sigma(br) = br$, and the quadratic form is equal to brQ, where

$$Q = Q(x, y, z, t) = 2zt - \frac{D - \sigma(D)}{2\sqrt{A}}(y^2 + x^2A) - (D + \sigma(D))xy.$$

Thus we need to compute the value at $s = 0$ of the product of $|brD\sigma(brD)|$, $\chi_Y(br)|br|^{2(s-1)}$ and

$$I_s^Y(\mathbf{r}, A, D) = \int_{V^0}|Q|^{2(s-1)}\chi_Y(Q)dxdydzdt.$$

IV. Theorem. *When Y/F is unramified, the the value of*

$$\chi_Y(br)|br|^{2(s-1)}|brD\sigma(brD)|I_s^Y(r,A,D)/(T^Y\phi_0)(\mathbf{v}_0)$$

at $s=0$ is $-2\chi_Y(br)\delta(Y,E_3)$.

To prove this theorem we need some results from [FZ4].

IV.1. Proposition ([FZ4])**.** *Up to a change of coordinates, the quadratic form*

$$2zt - \frac{D-\sigma(D)}{2\sqrt{A}}(y^2+x^2A) - (D+\sigma(D))xy$$

is equal to either $x^2+\boldsymbol{\pi}y^2-2zt$ or x^2-uy^2-2zt with $u\in R^\times - R^{\times 2}$. It is always isotropic.

Recall that Y/F is unramified. Then the integral $I_s^Y(\mathbf{r},A,D)$ is equal to

$$\sum_{n=0}^{\infty}(-1)^n q^{-nm}\operatorname{vol}(V_n^0(\mathbf{r},A,D)).$$

The problem is then to compute the volumes

$$\operatorname{vol}(V_n^0(\mathbf{r},A,D)) = \operatorname{vol}(V_n(\mathbf{r},A,D))/(1-1/q) \qquad (n\geq 0).$$

IV.2. Lemma ([FZ4])**.** *When the quadratic form is x^2-uy^2-2zt, we have*

$$\operatorname{vol}(V_n^0) = \begin{cases} 1+1/q^2, & \text{if } n=0,\\ q^{-n}(1-1/q)(1+1/q^2), & \text{if } n\geq 1.\end{cases}$$

Proof of Theorem IV. To prove Theorem IV, recall that we need to compute the value at $s=0$ ($m=-2$) of $I_s^Y(\mathbf{r},A,D)$. Since Y/F is unramified, the integral $I_s^Y(\mathbf{r},A,D)$ coincides with the sum

$$\sum_{n=0}^{\infty}(-1)^n q^{-nm}\operatorname{vol}(V_n^0(A,D))$$

which converges for $m>-1$. The value at $m=-2$ is obtained then by analytic continuation of this sum.

Case of $x^2+\boldsymbol{\pi}y^2-2zt$. Make a change of variables $z\mapsto 2u^{-1}z'$, followed by $z'\mapsto z$. Thus the quadratic form is equal to

$$-u^{-1}((z-t)^2-(z+t)^2-ux^2-u\boldsymbol{\pi}y^2).$$

Note that up to a multiple by a unit, this is a form of Lemma II.3. Since $\max\{|z|,|t|\}=1$ implies $\max\{|z+t|,|z-t|\}=1$, the result of that lemma holds for our quadratic form as well. In this case E_3/F is ramified, and our integral is zero.

Case of $x^2 - uy^2 - 2zt$. This is the case where E_3/F is unramified. By Lemma IV.2, the integral

$$I_s^Y(\mathbf{r}, A, D) = \mathrm{vol}(V_0^0) + \sum_{n=1}^{\infty} (-1)^n q^{-nm} \,\mathrm{vol}(V_n^0)$$

is equal to

$$1 + \frac{1}{q^2} + \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q^2}\right)\frac{-1}{q^{(m+1)}}\left(1 - \frac{-1}{q^{m+1}}\right)^{-1}.$$

When $m = -2$, this is

$$1 + \frac{1}{q^2} + \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q^2}\right)\frac{-q}{q+1} = \frac{2}{1+q}\left(1 + \frac{1}{q^2}\right).$$

The theorem follows by Proposition 2. □

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