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Automorphic Forms with Anisotropic Periods on a Unitary Group

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0. STATEMENT OF RESULTS

Let $\mathbf{G}$ be an algebraic group over a global field F with ring $\mathbb{A}$ of adeles, denote by $\mathbf{Z}$ the center of $\mathbf{G}$ and by $\mathbf{C}$ an algebraic subgroup of $\mathbf{G}$ over F such that the *cycle* $\mathbf{C}(F)\backslash\mathbf{C}(\mathbb{A})$ has finite volume. Fix unitary characters ω: $\mathbf{Z}(\mathbb{A})/\mathbf{Z}(F) \to \mathbb{C}^1$ (= unit circle in $\mathbb{C}^\times$) and ξ: $\mathbf{C}(F)\backslash\mathbf{C}(\mathbb{A}) \to \mathbb{C}^1$, and denote by ϕ: $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}) \to \mathbb{C}$ a cusp form in the cuspidal representation π of $\mathbf{G}(\mathbb{A})$, whose central character is $\omega_\pi = \omega$. By a cuspidal representation we mean an irreducible one. We say that π is $\mathbf{C}(\mathbb{A})$-*cyclic* if it has a nonzero $\mathbf{C}(\mathbb{A})$-*period* $P_{\mathbf{C}(\mathbb{A})}(\phi) = \int_{\mathbf{C}(F)\backslash\mathbf{C}(\mathbb{A})} \phi(c)\overline{\xi}(c)\,dc$. The overbar indicates complex conjugation. Studies of cyclic automorphic forms have applications to special values of L-functions (Waldspurger [W1, W2], Jacquet [J1, J2]), lifting problems [F], and studies of cohomology of symmetric spaces, in particular the Tate conjecture on algebraic cycles on some Shimura surfaces [FH]. The purpose of this paper—inspired by the applications to the Tate conjecture of [FH]—is to compare the notion of cyclicity by $\mathbf{C}(\mathbb{A})$, with cyclicity by an inner form of $\mathbf{C}(\mathbb{A})$. We let $\mathbf{G}$ be the quasi-split unitary group $U(2,1) = U(2,1; E/F)$ in three variables defined by means of a quadratic separable extension E/F of global fields. The subgroup $\mathbf{C}$ is taken to be the

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quasi-split unitary group $U(1,1) = U(1,1;E/F)$ in two variables defined by E/F (from Section 1 on—where more precise definitions are introduced—$\mathbf{C}$ and ξ of this section will be denoted by $\mathbf{C}_1^1$ and ξ_2).

The notation $U(2,1)$ and $U(1,1)$ is borrowed from the theory of real groups, but we do not discuss that theory in this paper, and for us $\mathbf{G} = U(2,1)$ and $\mathbf{C} = U(1,1)$ are the uniquely defined (up to isomorphism) *quasi-split* E/F-unitary groups in three and two variables. The anisotropic inner form of $\mathbf{C}$ over F will be denoted by $U(2) = U(2;E/F)$. To be able to concentrate on the aspects of the p-adic theory, we assume either that F is a function field (of odd characteristic), or that at each place v of F where $E_v/F_v = \mathbb{C}/\mathbb{R}$ the group $U(2)(\mathbb{R})$ (= the group of real points of $U(2,E/F)$) is quasi-split (hence isomorphic to $U(1,1)(\mathbb{R})$). Then $U(2)$ is not isomorphic to $U(1,1)$ only at a finite set ∇ of finite places of F, which stay prime in E. Indeed, at a place v which splits in E the anisotropic inner form $D(2,F_v)$ of $U(1,1;F_v) = GL(2,F_v)$ is not a subgroup of $U(2,1;F_v) = GL(3,F_v)$. Moreover, neither $GL(2,F_v)$ nor $D(2,F_v)$ is a subgroup of the anisotropic inner form $D(3,F_v)$ of $U(2,1;F_v)$. Hence our question cannot be asked with any inner form of $U(2,1)$ other than that which is split at each place v of F which splits in E. So we stick to our group $\mathbf{G}$. It is well known that the cardinality $|\nabla|$ of ∇ is even.

We shall also consider the local analogous question of cyclicity. At a place v of F which splits in E we fix characters $\boldsymbol{\xi}_v$, ω_v of $F_v^\times$, and say that an admissible irreducible $GL(3,F_v)$-module π_v with central character ω_v is $GL(2,F_v)$*-cyclic* if there is a non-zero linear form l: $\pi_v \to \mathbb{C}$ with $l(\pi_v(h)w) = \xi_v(h)l(w)$ $(h \in GL(2,F_v), w \in \pi_v)$, where $\xi_v = \boldsymbol{\xi}_v \circ \det$, namely $\mathrm{Hom}_{C_v}(\pi_v, \xi_v) \neq \{0\}$ (we put C_v for $\mathbf{C}(F_v) = U(1,1;F_v)$, which is $GL(2,F_v)$ in our case). If v is a place of F which stays prime in E then we similarly say that the irreducible admissible representation π_v of $G_v = \mathbf{G}(F_v) = U(2,1;F_v)$ is C_v*-cyclic* if $\mathrm{Hom}_{C_v}(\pi_v, \xi_v) \neq \{0\}$. Here C_v can be the quasi-split unitary group $U(1,1;F_v)$, or the anisotropic form $U(2;F_v)$. In all cases a result of the Appendix to [F6] asserts that $\dim_{\mathbb{C}}(\pi_v, \xi_v) \leq 1$. Moreover it is easy to see that all local components π_v of a $\mathbf{C}(\mathbb{A})$-cyclic cuspidal representation π are C_v-cyclic (with $\mathbf{C} = U(1,1)$ or $U(2)$). However, a cuspidal π all of whose components are C_v-cyclic, need not be $\mathbf{C}(\mathbb{A})$-cyclic. It is the global obstruction to the global cyclicity of an everywhere locally cyclic π, which is of interest. Thus our question is: When $\mathrm{Hom}_{\mathbf{C}(\mathbb{A})}(\pi, \xi)$ is non-zero, is it generated by $P_{\mathbf{C}(\mathbb{A})}$?

0.1. THEOREM. (1) *Every $U(2;\mathbb{A})$-cyclic cuspidal generic representation of $U(2,1;\mathbb{A})$ is $U(1,1;\mathbb{A})$-cyclic.*

(2) *If E_v/F_v is a local field extension, every $U(2;F_v)$-cyclic (irreducible admissible) generic representation of $U(2,1;F_v)$ is $U(1,1;F_v)$-cyclic.*

The identification of those $U(1,1)$-cyclic generic $U(2,1)$-modules which are also $U(2)$-cyclic cannot be stated simply in terms of the representation theory of the group $U(2,1)$, since this last theory is described by means of liftings. Two important such liftings are introduced in [F1] and studied in [F2, F3], namely the κ-endoscopic lifting from $U(1,1)$ to $U(2,1)$, which depends on a choice of a character κ: $\mathbb{A}_E^\times/E^\times N_{E/F}\mathbb{A}_E^\times \to \mathbb{C}^1$ whose restriction to $\mathbb{A}^\times/F^\times N\mathbb{A}_E^\times$ is non-trivial, and the base-change lifting from $U(2,1)$ to $GL(3,E)$.

In [F6] the $U(1,1)$-cyclic generic $U(2,1)$-modules are identified as the image of the κ-endoscopic lifting from $U(1,1)$. In fact [F4] establishes a correspondence between the set of packets of $U(1,1;\mathbb{A})$-cyclic generic cuspidal $U(2,1;\mathbb{A})$-modules, and the set of generic $GL(2,\mathbb{A}_E)$-modules which are cuspidal and $GL(2,\mathbb{A})$-cyclic (the adjective *"distinguished"*—instead of "cyclic"—is used in [F4] and [F6] in this context, and with $\xi = 1$, and so will it be here), or are normalizedly induced $I'(\mu_1', \mu_2')$ from the character $\left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right) \mapsto \mu_2'(a)\mu_2'(b)$ of the upper triangular subgroup $\mathrm{B}'(\mathbb{A}_E)$ of $GL(2,\mathbb{A}_E)$, where μ_i': $\mathbb{A}_E^\times/\mathbb{A}^\times E^\times \to \mathbb{C}^1$ and $\mu_1' \neq \mu_2'$. In [F4] it is shown that this last set of $GL(2,\mathbb{A})$-distinguished cuspidal $GL(2,\mathbb{A}_E)$-modules, and the $I'(\mu_1', \mu_2')$, μ_i': $\mathbb{A}_E^\times/\mathbb{A}^\times E^\times \to \mathbb{C}^1$, $\mu_1' \neq \mu_2'$, is the image of the unstable base-change lifting (this lifting also depends on a choice of κ) from $U(1,1;\mathbb{A})$ to $GL(2,\mathbb{A}_E)$. The composition of the κ-base-change lifting from $U(1,1)$ to $GL(2,E)$, and the correspondence from (distinguished generic representations on) $GL(2,E)$ to (cyclic generic representations on) $U(2,1)$, is the κ-endoscopic lifting. The analogous local results are also established in [F6] (and [F4]).

We repeat that by a $GL(2,\mathbb{A})$-distinguished cuspidal representation of $GL(2,\mathbb{A}_E)$ we mean one which is cyclic, with $\xi = 1$. Since $\mathrm{Hom}_{GL(2,\mathbb{A})}(\pi, \xi) = \mathrm{Hom}_{GL(2,\mathbb{A})}(\pi \otimes \xi^{-1}, 1)$, where ξ denotes also an extension of ξ from $\mathbb{A}^\times/F^\times$ to $\mathbb{A}_E^\times/E^\times$, there is no loss of generality in taking $\xi = 1$. We denote $\{z \in E_v^\times; z\overline{z} = 1\}$ by E_v^1.

0.2. THEOREM. (1) *A $U(1,1;\mathbb{A})$-cyclic generic cuspidal representation π of $U(2,1;\mathbb{A})$ is $U(2;\mathbb{A})$-cyclic precisely when for each $v \in \nabla$ (= the set of finite F-places where $U(2;F_v)$ is anisotropic) the component π_v of π does not correspond (in the sense of* [F6]) *to the $GL(2,E_v)$-module $I'(\mu_1', \mu_2')$, μ_i': $E_v^\times/F_v^\times \to \mathbb{C}^1$, namely π_v is not the κ_v-endoscopic lift of the $U(1,1;F_v)$-packet $\pi_0(\mu_1, \mu_2)$ of* [F2, *Sections* 3.7/8, *page* 49], *μ_i: $E_v^1 \to \mathbb{C}^1$, $\mu_i'(z) = \mu_i(z/\overline{z})(z \in E_v^\times)$ (where π_0 is denoted by ρ).*

(2) *A $U(1,1;F_v)$-cyclic generic admissible irreducible $U(2,1;F_v)$-module is $U(2;F_v)$-cyclic precisely when it does not correspond (à la* [F6]) *to $I'(\mu_1', \mu_2')$, μ_i': $E_v^\times/F_v^\times \to \mathbb{C}^1$, $\mu_1' \neq \mu_2'$, namely it is not the κ_v-endoscopic lift of $\pi_0(\mu_1, \mu_2)$, μ_i: $E_v^1 \to \mathbb{C}^1$, $\mu_1 \neq \mu_2$.*

Another way of stating these results is by means of the base change theory of [F2], from $U(2,1)$ to $GL(3,E)$. Thus [F6] asserts that a generic admissible irreducible $U(2,1;F_v)$-module is $U(1,1;F_v)$-cyclic when its base change is generic but not discrete series (= square-integrable), and Theorem 0.2(2) asserts that it is also $U(2;F_v)$-cyclic when its base-change is also not induced of the form $I(\mu_1',\mu_2',\mu_3')$, $\mu_i'\colon E_v^\times/F_v^\times \to \mathbb{C}^1$, with distinct μ_i'. Theorem 0.2(1) asserts that the $U(1,1;\mathbb{A})$-cyclic π is also $U(2;\mathbb{A})$-cyclic if each $\pi_v (v \in \nabla)$ is $U(2;F_v)$-cyclic and not of the form $I(\mu_v')$, $\mu_v'\colon E_v^\times/F_v^\times \to \mathbb{C}^1$. We denote by $I(\mu_v)$ the $U(2,1;F_v)$-module with central character ω_v normalizedly induced from the character $\mathrm{diag}(a,b,1/\overline{a}) \mapsto \mu_v(a)(\omega_v/\mu_v)(b)$ of the upper triangular subgroup $\mathrm{B}(F_v)$ of $G_v = U(2,1;F_v)$.

As a notational convention, representations of $U(1,1)$ will have an index zero (e.g., π_0, $I_0(\mu)$), those of $GL(2)$ will carry a prime (e.g., π', $I'(\mu_1,\mu_2)$), while those of $U(2,1)$ are denoted simply by π or $I(\mu)$. Also by a $U(1,1)$-cyclic generic $U(2,1)$-module we mean the set of $U(1,1)$-cyclic generic elements in its packet (as defined in [F2]). It is expected that this set consists of a single element, but this has not been shown as yet.

For an archimedean analogue of our local results the reader may like to consult Oshima and Matsuki [OM] and Kobayashi [Ko].

The statement of Theorem 0.2 suggests that its proof would be related to the variation of the notion of distinguishability (= cyclicity) of a generic cuspidal $GL(2,\mathbb{A}_E)$-module, with respect to inner forms of $GL(2,\mathbb{A})$. This question is studied in [FH], whose result which is needed for the proof of Theorem 0.2 (and Theorem 0.1) is reviewed next.

Denote by $\mathbf{D}$ the inner form of $GL(2)$ over F which is ramified precisely at the places of ∇. Then $\mathbf{D}(\mathbb{A})$ is a subgroup of $\mathbf{D}(\mathbb{A}_E) = GL(2,\mathbb{A}_E)$.

0.3. THEOREM (0.2 of [FH]). *A cuspidal (irreducible) representation π' of $GL(2,\mathbb{A}_E)$ is $\mathbf{D}(\mathbb{A})$-distinguished precisely when it is $GL(2,\mathbb{A})$-distinguished and its components π_v' at $v \in \nabla$ are not of the form $I'(\mu_1',\mu_2')$ with μ_i' trivial on $F_v^\times$.*

Our proof of Theorems 0.1 and 0.2 consists of a comparison of $U(2)$-cyclic $U(2,1)$-modules with D-distinguished $GL(2)$-modules (same ∇ for $U(2)$ and D), and an application of Theorem 0.3. We then state this comparison next. For this purpose we recall that the correspondence of [F6] relates almost everywhere locally distinguished automorphic representations π' of $GL(2,\mathbb{A}_E)$ with almost everywhere locally cyclic automorphic representations π of $U(2,1;\mathbb{A})$, and the relation is that π' and π correspond if

(1) at almost all places v of F which split, the $GL(2,E_v) = GL(2,F_v) \times GL(2,F_v)$-module $\pi_v' = \pi_{1v}' \times \pi_{2v}'$ is distinguished (has a nonzero $GL(2,F_v)$-invariant form; then $\pi_{2v}' \simeq \check{\pi}_{1v}'$(= contragredient of π_{1v}')),

and then π_v is the $GL(2,F_v)$-cyclic $GL(3,F_v)$-module $I(\pi'_{1v}\times\omega_v/\omega_{\pi'_{1v}})$ normalizedly induced from the maximal parabolic of type (2,1) as indicated, where $\omega_v=\omega_{\pi_v}$ is the central character of π_v and $\omega_{\pi'_{1v}}$ is that of π'_{1v}, and

(2) at almost all places v of F which stay prime in E, the $GL(2,F_v)$-distinguished component π'_v is of the form $I'(\mu_v)=I'(\mu_v,\overline{\mu}_v^{-1})$, and the corresponding $U(1,1;F_v)$-cyclic component π_v is $I(\mu_v)$.

Note that the central character of π_v is ω_v, and that of π'_v is ω'_v/ξ'_v, where $\xi'_v(z)=\xi_v(z/\overline{z})(z\in E_v^\times)$ is well defined since ξ_v is a character of $E_v^1=\{z\in E_v^\times; z\overline{z}=1\}$.

The theorems of [F6] establish that the correspondence relates cuspidal distinguished π', or π' of the form $I(\mu'_1,\mu'_2)$, $\mu'_1\neq\mu'_2$, with cuspidal cyclic π. Since the groups $U(2;E/F)$ and $\mathbf{D}$ are not isomorphic over F_v to $U(1,1;E/F)$ and $GL(2)$ only for v in the finite set ∇, and the definition of correspondence depends only on almost all places, the definition applies with these anisotropic groups, and we can state the following.

0.4. THEOREM. *The correspondence is a bijection from the set of* $\mathbf{D}(\mathbb{A})$-*distinguished cuspidal* $GL(2,\mathbb{A}_E)$-*modules with central character* ω'/ξ', *to the set of packets of* $U(2;\mathbb{A})$-*cyclic generic cuspidal* $U(2,1;\mathbb{A})$-*modules with central character* ω.

This global result permits extending the definition of the local correspondence (the definition depends on a certain relation of Whittaker–Period distributions) in the case that E_v/F_v is a field to show that the correspondence is a bijection from the set of D_v-distinguished generic $GL(2,E_v)$-modules π'_v to the set of packets of $U(2;F_v)$-cyclic generic $U(2,1;F_v)$-modules π_v (with $\omega_{\pi'_v}=\omega'_v/\xi'_v$ if $\omega_{\pi_v}=\omega_v$), such that π'_v is square-integrable precisely when π_v is, and π'_v is supercuspidal precisely when π_v is, and $\pi_v=I(\mu_v)$ when $\pi'_v=I'(\mu_v)$, as defined above.

Theorems 0.1 and 0.2 follow on combining the correspondence of Theorem 0.4 with that of Theorem 0.3 ([FH]) and that of [F6]. This paper will then be concerned with the proof of Theorem 0.4, which is an anisotropic analogue of the work of [F6]. Most of the technical difficulties in our present project have already been overcome in [F6], and those of the passage from an inner form to the quasi-split form, in [FH]. Theorem 0.1 is easy to prove by a direct comparison of $U(1,1)$-cyclic and $U(2)$-cyclic forms, but to prove Theorem 0.2 by such a direct comparison we would need to compute directly the Whittaker–Period distributions of the local representations mentioned in Theorem 0.2. We prefer to deduce these local computations from the global comparisons; see Proposition 11 below. Analogous local computations have been carried out for the comparison of $GL(2, F_v)$- and D_v-distinguished representations of $GL(2, E_v)$, in [FH], except that [FH] considers the bi-period distribution attached to a distinguished π' (and names it "relative"), and not our Whittaker–Period one, which is attached to a generic distinguished π'.

As in [F6] our work depends on a comparison of Fourier summation formulae on $U(2, 1; \mathbb{A})$ and $GL(2, \mathbb{A}_E)$. However these formulae simplify in our case as we take periods with respect to the anisotropic groups $U(2; \mathbb{A})$ and $\mathbf{D}(\mathbb{A})$, instead of $U(1, 1; \mathbb{A})$ and $GL(2, \mathbb{A})$ as in [F6]. The comparison is based on a transfer of Fourier orbital integrals of general (and spherical) functions between $U(2, 1)$ and $GL(2, E)$, and this was carried out in both the split and non-split cases in [F6] (and [F7]). The required analysis in the remaining finite number of places v in ∇ where $U(2)$ and $\mathbf{D}$ ramify is carried out here. It is easier than the analysis of [F6], since we deal with anisotropic groups. We also explain the transfer from $U(2, F_v)$ to $U(1, 1; F_v)$, and from D_v to $GL(2, F_v)$; these local transfers can be used in the corresponding global comparisons (the one with D_v and $GL(2, F_v)$ can replace $C1$ of [FH], but only when V'' of [FH] is empty, that with $U(2)$ and $U(1, 1)$ can be used to give an alternative proof of our Theorems 0.1 and 0.2, as noted above).

Our usage of a "Fourier" summation formula, which involves Fourier coefficients of cusp forms, limits our discussion to the case of the generic representations, those with a Whittaker model. It would be interesting to study the notion of cyclicity for degenerate $U(2, 1; F_v)$ and $U(2, 1; \mathbb{A})$-modules (where a packet of representations is expected to contain only one generic element), and perhaps a bi-period summation formula for $U(2, 1)$—analogous to that of [FH] in the case of $GL(2, E)$ and period $GL(2, F)$—would be of use. But we have not done that.

As is well known, some of our results can be obtained by the theta correspondence, but our interest is in the intrinsic approach of the summation formula.

1. THE GROUPS

We shall now define the groups which are studied in this paper. Let E/F be a quadratic extension of local or global fields of characteristic other than 2. Denote by an overbar the action of the non-trivial element of $\mathrm{Gal}(E/F)$, and write $\overline{g} = (\overline{g}_{ij})$ for $g = (g_{ij})$ in $GL(n, E)$. The unitary group in three variables of interest to us here is

$$G = U(2, 1; E/F) = \left\{ g \in GL(3, E); gJ^t\overline{g} = J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix} \right\}.$$

The quasi-split unitary group $U(1,1;E/F)$ in two variables which is considered in [F6] is related to the centralizer

$$C = U(1)\times U(1,1) = Z_G(J_0) = \{g\in G; gJ_0 = J_0 g\}$$

of

$$J_0 = \operatorname{diag}(1,-1,1).$$

The anisotropic unitary group $U(2,E/F)$ considered in this paper is related to

$$C_\theta = U(1)\times U(2) = Z_G(J_\theta), \qquad J_\theta = \begin{pmatrix} 0 & & 1/2\theta \\ & 1 & \\ 2\theta & & 0 \end{pmatrix}$$

In the local case we take $\theta \in F - N_{E/F}E$; here $N = N_{E/F}$ denotes the norm map from E to F. In the global case we take $\theta\in F$ such that $\theta \notin N_{E_v/F_v}E_v$ for all $v\in\nabla$, and $\theta\in N_{E_v/F_v}E_v$ for all $v\notin\nabla$. Then C_θ is anisotropic precisely at the places in ∇, a finite set of finite places with even cardinality. Up to isomorphism over F, the group C_θ depends only on $\theta NE^\times$; indeed $J_{\theta\lambda\bar\lambda}$ is conjugate to J_θ by diag $(\lambda, 1, 1/\bar\lambda)\in G$ $(\lambda\in E^\times)$.

In [F6, Proposition 2], it was noted that $J_1 = g_0^{-1}J_0 g_0$, and

$$G = BC\cup Bg_0C = BC_1\cup Bg_0^{-1}C_1, \qquad g_0 = \begin{pmatrix} 1 & -1 & \frac12 \\ 1 & 0 & -\frac12 \\ \frac12 & \frac12 & \frac14 \end{pmatrix}.$$

Here B denotes the upper triangular subgroup of G, and the index 1 in J_1 and C_1 indicates $\theta = 1$. When $\theta\notin NE^\times$ the group C_θ is anisotropic, and we have a simpler decomposition.

1. Proposition. *If $\theta\in F - NE$, then $G = BC_\theta$.*

Proof. The 3×3 matrices which commute with J_θ have the form

$$g = \begin{pmatrix} a & d & c/4\theta^2 \\ b & e & b/2\theta \\ c & 2\theta d & a \end{pmatrix}.$$

Then

$$J^t\bar g J = \begin{pmatrix} \bar a & -\bar b/2\theta & \bar c/4\theta^2 \\ -2\theta\bar d & \bar e & -\bar d \\ \bar c & -\bar b & \bar a \end{pmatrix},$$

and $g\in C_\theta = Z_G(J_\theta)$ if $gJ^t\bar gJ = I$. Hence $g\in Z_{GL(3,E)}(J_\theta)$ lies in C_θ when there is $\eta\in E^1$ (thus $\eta\bar\eta = 1$) with $d = \eta\bar b/2\theta$, $e = \eta(\bar a + \bar c/2\theta)$, with (1) $(a-c/2\theta)(\overline{a}-\overline{c}/2\theta)=1$ and with (2) $(a+c/2\theta)(\overline{a}+\overline{c}/2\theta)=b\overline{b}/\theta+1$ (note that any two of (1), (2), and (3) $a\overline{c}+\overline{a}c=b\overline{b}$, imply the third).

Let Y be the subvariety of $x=(x_1,x_2,x_3)$ in the projective 2-space over E with $xJ^t\overline{x}=0$. Then G acts transitively on Y by g: $x\mapsto xg^{-1}$. The stabilizer of $x_0=(0,0,1)$ is $B=\mathrm{stab}_G\, x_0$. Given $x=(\overline{c},-\overline{b},\overline{a})$ in Y we have (3), hence $a\neq c/2\theta$ (since $\theta\in F-NE$), and dividing its components by $a-c/2\theta$ we may assume that x satisfies (1), whence (2). For any choice of $\eta\in E^\times$, $\eta\overline{\eta}=1$, define $d=\eta\overline{b}/2\theta$, $e=\eta(\overline{a}+\overline{c}/2\theta)$. We then define $g\in C_\theta$ with $x=x_0g^{-1}$, and the proposition follows. ∎

COROLLARY. *When $\theta\in F-NE$, the group C_θ consists of $h\boldsymbol{\eta}$, $\boldsymbol{\eta}=\mathrm{diag}(1,\eta,1)$, $\eta\in E^1$, and*

$$h=\begin{pmatrix} a & \overline{b}/2\theta & c/4\theta^2\\ b & \overline{a}+\overline{c}/2\theta & b/2\theta\\ c & \overline{b} & a\end{pmatrix},$$

with a, b, $c\in E$ satisfying (1) $(a-c/2\theta)(\overline{a}-\overline{c}/2\theta)=1$, *and* (2) $(a+c/2\theta)(\overline{a}+\overline{c}/2\theta)=b\overline{b}/\theta+1$. ∎

Remark. (1) Note that g_0 satisfies $a=c/2\theta$ with $\theta=1$, where $(\overline{c},-\overline{b},\overline{a})=x=x_0g_0=(\frac{1}{2},\frac{1}{2},\frac{1}{4})$, hence $g_0^{-1}\notin BC_1$.

(2) We have $\det h=a-c/2\theta\in E^1$.

Any character ξ: $C_\theta\to\mathbb{C}^\times$ of the group C_θ is of the form $\xi(h\boldsymbol{\eta})=\xi_1(\det h)\xi_2(\eta)$, where ξ_1, ξ_2 are characters of E^1. Recall that the principal series representation $I(\mu\nu^s)$ of G with central character ω: $E^1\to\mathbb{C}^1$, where μ: $E^\times\to\mathbb{C}^\times$ and $\nu(x)=|x|_E$ $(x\in E^\times)$, is defined on the space of smooth functions φ on G which satisfy (for any upper triangular unipotent u)

$$\varphi\big(\mathrm{diag}(\alpha,\beta,\overline{\alpha}^{-1})ug\big)=\mu(\alpha)\left(\frac{\omega}{\mu}\right)(\beta)|\alpha|_E^{s+1}\varphi(g)$$
$$\big(g\in G,\alpha\in E^\times,\beta\in E^1\big).$$

2. PROPOSITION. *We have* $\mathrm{Hom}_{C_\theta}(I(\mu\nu^s),\xi)\neq 0$ *precisely when* $\mu=\xi_1\xi_2$ *and* $\omega=\xi_1\xi_2^2$ *on* E^1.

Proof. (a) When $\theta\in F-NE$ the group C_θ is compact, and we claim that there is a non-zero linear form L on $\pi=I(\mu\nu^s)$ which transforms under C_θ by ξ, precisely when there is a non-zero vector u in π which transforms under C_θ by ξ. Indeed, given L there is w in π with $L(w)\neq 0$, and $u=\int_{C_\theta}\pi(h)w\cdot\xi(h)^{-1}\,dh$ satisfies $\pi(t)u=\xi(t)u$ $(t\in C_\theta)$ and $L(u)\neq 0$, hence $u\neq 0$. In the opposite direction, let l be a linear form on π with

$l(u) \neq 0$, and define L by $L(w) = \int_{C_\theta} l(\pi(h)w)\xi(h)^{-1}\,dh$. Then $L(u) \neq 0$, hence L has the required properties.

(b) If $\varphi \in \pi = I(\mu\nu^s)$ satisfies $\pi(h)\varphi = \xi(h)\varphi$ $(h \in C_\theta)$, since $G = BC_\theta$, φ is determined by its values on $B \cap C_\theta = \{\mathrm{diag}(a, \eta\overline{a}, a);\ \eta\overline{\eta} = 1, a\overline{a} = 1\}$. There

$$\mu(a)(\omega/\mu)(\overline{a}\eta)\varphi(e) = \varphi\big(\mathrm{diag}(a, \eta\overline{a}, a)\big) = \xi_1(a)\xi_2(\eta)\varphi(e).$$

If $\varphi \neq 0$, then $\xi_2 = \omega/\mu$ and $\xi_1 = \mu^2/\omega = \mu/\xi_2$, so $\mu = \xi_1\xi_2$ and $\omega = \xi_1\xi_2^2$ on E^1. Conversely, if $\mu = \xi_1\xi_2$ and $\omega = \xi_1\xi_2^2$ on E^1, then $I(\mu\nu^s)$ contains a one-dimensional space of φ_0 with $\pi(h)\varphi_0 = \xi(h)\varphi_0$ $(h \in C_\theta)$, and the linear form $L(\varphi) = \int_{C_\theta} \varphi(h)\xi(h)^{-1}dh$ is non-zero on φ_0 and transforms under C_θ via ξ.

(c) When $\theta = 1$, the proposition is proven [F6, Proposition 29(a)] for $\xi_1 = 1 = \xi_2$. The extension to arbitrary ξ_i is immediate. ∎

Notations. Denote by C_θ^1 the subgroup of $h\boldsymbol{\eta}$ in C_θ with $\det h = a - c/2\theta$ equals 1. Any $h\boldsymbol{\eta} \in C_\theta$ can be written in the form $zh'\boldsymbol{\eta}\overline{\mathrm{z}}/\mathrm{z}$ with $z = a - c/2\theta$, $\mathrm{z} = \mathrm{diag}(1, z, 1)$, $\det h' = 1$. Hence for π with central character $\omega = \xi_1\xi_2^2$, we have $\mathrm{Hom}_{C_\theta}(\pi, \xi) = \mathrm{Hom}_{C_\theta^1}(\pi, \xi)$. On C_θ^1 we have $\xi(h\boldsymbol{\eta}) = \xi_2(\eta)$. Now $J_1 = g_0^{-1}J_0g_0$, and $g_0C_1g_0^{-1} = C$. Matrix multiplication shows that the (2,2) entry of $g_0\boldsymbol{\eta}g_0^{-1}$ is 1, while that of $g_0hg_0^{-1}$ is $\det h$. The character ξ on $C = \{g = (g_{ij}) \in G; g_{ij} = 0$ if $i + j$ is odd$\}$ takes then the value $\xi_2(g_{11}g_{33} - g_{13}g_{31})\xi_1(g_{22})$ at g. Note that $g_0C_1^1g_0^{-1} = C^1$ is $\{g \in G; g_{22} = 1\}$, and the restriction of ξ to C^1 is $\xi(g) = \xi_2(\det g)$. If $\xi_1 = \omega/\xi_2^2$, for π with central character ω we have $\mathrm{Hom}_C(\pi, \xi) = \mathrm{Hom}_{C^1}(\pi, \xi_2)$. As in [F6] to simplify the notations we shall often work with linear forms which transform under the subgroup C_θ^1 or C^1 by the character $\xi_2 \circ \det$. This is what we did in the Introduction (e.g., Theorem 0.4), where the cycle is taken to be $U(2)$ or $U(1,1)$, denoted C (instead of C_θ^1 as here), and the character is denoted by ξ (instead of by ξ_2 as here).

In view of these definitions, we restate Proposition 2 as follows.

2′. PROPOSITION. *The space* $\mathrm{Hom}_{C_\theta^1}(I(\mu\nu^s), \xi_2)$ *is non-zero precisely when* $\mu = \omega/\xi_2$. ∎

According to Keys [Ke] (as recorded in [F1, (3.1(3)), p. 558]), when $\omega = \beta^3$ and $\mu(z) = \beta(z/\overline{z})(z \in E^\times)$ for a character β of E^1, the induced $I(\mu\nu)$ has a square-integrable "Steinberg" subrepresentation denoted $\mathrm{St}(\mu\nu)$, and a one-dimensional quotient $\pi(\mu\nu)$: $g \mapsto \beta(\det g)$. The space of $\mathrm{St}(\mu\nu)$ consists of the φ in $I(\mu\nu)$ with $\int_{B\backslash G} \varphi(g)\beta(\det g)^{-1}\,dg = 0$, since $I(\mu\nu)$ is of length 2.

COROLLARY. *We have* $\mathrm{Hom}_{C_\theta}(\mathrm{St}(\mu\nu), \xi) = \{0\}$.

Proof. Suppose that $\theta \in F - NE$. By (a) above we need to determine the φ in $I(\mu\nu)$ with (1) $\varphi(gh) = \xi_2(h)\varphi(g)$ $(h \in C^1_\theta)$ and (2) $\int_{C^1_\theta} \varphi(h)\beta(\det h)^{-1}\,dh = 0$. It was shown in (b) that up to a scalar there is a (unique) φ satisfying (1), when $\underline{\xi_2} = \omega/\mu$ on E^1. But $\mathrm{St}(\mu\nu)$ exists only when $\omega = \beta^3$ on E^1 and ($\mu = \beta/\overline{\beta}$ on $E^\times$ hence) $\mu = \beta^2$ on E^1, namely $\xi_2 = \beta$ on E^1. Consequently (2) implies that $\varphi = 0$, since $\varphi(h) = \xi_2(\det h)$ on C^1_θ.

The case of $\theta = 1$ is proven in [F6, Proposition 29(c)]. ∎

Given $\theta \in F - NE$, where E/F is a quadratic extension of local fields with $\operatorname{char} F \neq 2$, the anisotropic quaternion (division of rank 2) algebra over F can be realized as an algebra of 2×2 matrices over E, with multiplicative group

$$D_\theta = \left\{ \begin{pmatrix} a & b\theta \\ \overline{b} & \overline{a} \end{pmatrix};\ a, b \in E,\ a\overline{a} - b\overline{b}\theta \neq 0 \right\} \subset GL(2, E).$$

This D_θ is an inner form of $GL(2, F)$: these two groups become isomorphic over E. Put $B' = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix};\ a,\ d \in E^\times,\ b \in E\}$. The group $GL(2, E)$ acts transitively by $g\colon x \mapsto xg^{-1}$ on the projective line over E, the stabilizer of $x_0 = (0, 1)$ is B'. Hence

$$GL(2, E) = B'D_\theta,$$

and

$$GL(2, E) = B'GL(2, F) \cup B'\eta_1 GL(2, F) = B'D_1 \cup B'\eta D_1.$$

Here $\eta_1 = \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}$, $\eta = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$, where $E = F(i)$ and $i^2 \in F$. These decompositions hold also when E/F are global fields. Then D_θ splits precisely at the v where $\theta \in NE^\times_v$. It is anisotropic at the finite even set ∇ of F-places where $\theta \in F_v - NE_v$.

In the local case these decompositions are used in [F4, page 169], and [F6, Proposition 28], to show the following. Put $\overline{\mu}(a) = \mu(\overline{a})$ $(a \in D^\times)$ for a character on $E^\times$.

3. Proposition. *When* $\theta \in F - NE$ *we have* $\mathrm{Hom}_{D_\theta}(I(\mu_1, \mu_2), \mathbb{1}) \neq \{0\}$ *precisely when* $\mu_2 = \overline{\mu}_1^{-1}$, *and* $\mathrm{Hom}_{D_\theta}(\mathrm{sp}(\mu), \mathbb{1}) \neq \{0\}$ *when the restriction of* μ *to* $F^\times$ *is non-trivial, but* $\mu \mid NE^\times = 1$. *Further,* $\mathrm{Hom}_{GL(2,F)}(I(\mu_1, \mu_2), \mathbb{1}) \neq \{0\}$ *precisely when* $\mu_2 = \overline{\mu}_1^{-1}$ *or* $\mu_i \mid F^\times = 1$ $(i = 1, 2)$, *and* $\mathrm{Hom}_{GL(2,F)}(\mathrm{sp}(\mu), \mathbb{1}) \neq \{0\}$ *when* $\mu \mid F^\times \neq 1$, $\mu \mid NE^\times = 1$.

Here $\mathrm{sp}(\mu)$ is the unique (square-integrable) subrepresentation of the induced $I(\mu\nu^{1/2}, \mu\nu^{-1/2})$, $\nu(x) = |x|$ $(x \in E^\times)$, and $I(\mu_1, \mu_2)$ is the $GL(2, E)$-module normalizedly induced from the character $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \mu_1(a)\mu_2(d)$ of B'.

2. FOURIER SUMMATION FORMULAE

The global tool used in our comparisons is a Fourier summation formula. Such formulae were developed and used already in [J1, J2, F4, F6], with respect to the quasi-split cycles $GL(2,F)$ in $GL(2,E)$ and $U(1,1)$ in $U(2,1)$. Here we shall have a simpler variant of these formulae, with respect to the anisotropic cycles D_θ and $C_\theta^1 = U(2)$, where $\theta \in F - NE$. Fix characters ω, ξ_2 of $\mathbb{A}_E^1/E^1$, and as usual, put $\omega'(z) = \omega(z/\overline{z})$ $(z \in \mathbb{A}_E^\times)$.

We first describe the Fourier summation formula on $\mathbb{D}' = \mathbf{D}(\mathbb{A}_E)$, $\mathbf{D} = GL(2)$, for a test function $f' = \bigotimes f'_v$ on $\mathbb{D}'$ such that f'_v is smooth and compactly supported on $D'_v = \mathbf{D}(E_v)$ modulo the center $Z'_v \simeq E_v^\times$, transforming under Z'_v via ω'_v/ξ'_{2v}, with $f'_v = f_v'^0$ for almost all v. Here $f_v'^0$ is the unit element of the convolution algebra $\mathbb{H}'_v$ of $K'_v = \mathbf{D}(R_{E_v})$-biinvariant f'_v; a choice of a Haar measure is implicit. Let $L(D')$ be the space of smooth functions $\phi\colon \mathbb{D}' \to \mathbb{C}$ with $\phi(\gamma z h) = (\omega'/\xi'_2)(z)\phi(h)$ $(h \in \mathbb{D}'$, $z \in \mathbb{Z}' \simeq \mathbb{A}_E^\times$, $\gamma \in D' = \mathbf{D}(E))$ and $\int_{\mathbb{Z}'D'\backslash\mathbb{D}'} |\phi(h)|^2\,dh < \infty$. The convolution operator

$$(r(f')\phi)(g) = \int_{\mathbb{D}'/\mathbb{Z}'} f'(h)\phi(gh)\,dh = \int_{\mathbb{Z}'D'\backslash\mathbb{D}'} K_{f'}(g,h)\phi(h)\,dh$$

is an integral operator with kernel $K_{f'}(g,h) = \sum_{\gamma\in D'/Z'} f'(g^{-1}\gamma h)$.

The space $L(D')$ decomposes as the direct sum of three mutually orthogonal invariant subspaces: the space $L_0(D')$ of cusp forms with central character ω'/ξ'_2, the space $L_1(D')$ of functions $\phi(g) = \chi(\det g)$, where χ is a character of $\mathbb{A}_E^1/E^1$ with $\chi^2 = \omega'/\xi'_2$, and the continuous spectrum $L_c(D')$. Denote the corresponding kernels by K_0, K_1, K_c. The Fourier summation formula is the equality obtained on integrating $K(n,h)\overline{\boldsymbol{\psi}}(n)$ on $h \in \mathbb{Z}D_\theta\backslash\mathbb{D}_\theta$ $(D_\theta = \mathbf{D}_\theta(F)$, $\mathbb{D}_\theta = \mathbf{D}_\theta(\mathbb{A})$, $\mathbb{Z} \simeq \mathbb{A}^\times)$ and on $n = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ in $N'\backslash\mathbb{N}'$ (i.e., on x in $\mathbb{A}_E/E$). Here $\boldsymbol{\psi}$ is a fixed non-trivial character of $\mathbb{A}/F$, and $\psi(n) = \boldsymbol{\psi}(x + \overline{x})$.

Using the disjoint decomposition $D' = \bigcup N'\left(\begin{smallmatrix} b & 0 \\ 0 & 1 \end{smallmatrix}\right)D_\theta$, union over $b \in E^\times/E^1$, we note that the "geometric " expression for the double integral $\iint K_{f'}(n,h)\overline{\psi}(n)\,dn\,dh$ is

$$\sum_{b\in E^\times/E^1} \Psi(b, f', \psi),$$

$$\Psi(b, f', \psi) = \int_{\mathbb{N}'}\int_{\mathbb{Z}\backslash\mathbb{D}_\theta} f'\left(n\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}h\right)\psi(n)\,dn\,dh.$$

Since f' is compactly supported modulo the center, the Bruhat decomposition (and an application of the map $g \mapsto g\left(\left(\begin{smallmatrix} 0 & \theta \\ 1 & 0 \end{smallmatrix}\right)\overline{g}\left(\begin{smallmatrix} 0 & \theta \\ 1 & 0 \end{smallmatrix}\right)^{-1}\right)^{-1}$), shows that for a given f' the sum is finite, and the double integral ranges over a compact in $\mathbb{N}' \times \mathbb{Z}\backslash\mathbb{D}_\theta$. In the isotropic case, where $\theta = 1$, there is another term, $\Psi(0, f', \psi)$, in the geometric side; see [F4].

The integral of $K_1(n, h)\overline{\psi}(n)$ is zero since ψ is non-trivial. The cuspidal kernel takes the form

$$K_0(g, h) = \sum_{\pi'} \sum_{\phi \in \pi'} \big(\pi'(f')\phi\big)(g)\overline{\phi}(h).$$

Here π' ranges over the set of cuspidal representations of $\mathbb{D}'$ with central character ω'/ξ'_2, and ϕ over an orthonormal basis of smooth functions in π'. The integral of $K_0(n, h)\overline{\psi}(h)$ is equal to

$$\sum_{\pi'} (W_\psi \overline{P}_\theta)_{\pi'}(f'),$$

where

$$(W_\psi \overline{P}_\theta)_{\pi'}(f') = \sum_{\phi \in \pi'} W_\psi(\pi'(f')\phi)P_\theta(\overline{\phi})$$

is independent of the choice of the basis $\{\phi\}$ of π'. Here $W_\psi(\phi) = \int_{N'\backslash\mathbb{N}'} \phi(n)\overline{\psi}(n)\, dn$ and $P_\theta(\phi) = \int_{\mathbb{Z}D_\theta\backslash\mathbb{D}_\theta} \phi(h)\, dh$.

Next we record $K_c(n, h)$. Let μ_1, μ_2 be unitary characters of $\mathbb{A}_E^\times/E^\times$ with $\mu_1\mu_2 = \omega'/\xi'_2$. For any $s \in \mathbb{C}$ consider the Hilbert space $H'(\mu_1, \mu_2, s)$ of $\phi\colon \mathbb{D}' \to \mathbb{C}$ with

$$\phi\left(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} g\right) = \left|a/b\right|_E^{s+1/2} \mu_1(a)\mu_2(b)\phi(g) \qquad \big(a, b \in \mathbb{A}_E^\times; g \in \mathbb{D}'\big)$$

and $\int_{\mathbb{K}'} |\phi(k)|^2\, dk < \infty$. Here $\mathbb{K}' = \Pi K'_v$, $K'_v =$ standard maximal compact subgroup in $D'_v = GL(2, E_v)$. The restriction to $\mathbb{K}'$ map $\phi \mapsto \phi|\mathbb{K}'$ defines an isomorphism from $H'(\mu_1, \mu_2, s)$ to $H'(\mu_1, \mu_2) = H'(\mu_1, \mu_2, 0)$. Identify $H'(\mu_1, \mu_2, s)$ with $H'(\mu_1, \mu_2)$. Denote by $\phi(\mu_1, \mu_2, s)$ the element of $H'(\mu_1, \mu_2, s)$ corresponding to $\phi(\mu_1, \mu_2)$ in $H'(\mu_1, \mu_2)$. Denote by $I'(\mu_1, \mu_2, s)$ the representation of $\mathbb{D}'$ on $H'(\mu_1, \mu_2, s)$ by right translation, and

$$E(h, \phi, \mu_1, \mu_2, s) = \sum_{\gamma \in B'\backslash D'} \phi(\gamma h, \mu_1, \mu_2, s) \qquad \big(\phi = \phi(\mu_1, \mu_2) \in H'(\mu_1, \mu_2)\big)$$

the associated Eisenstein series. The kernel on the continuous spectrum is given by

$$K_c(g, h) = \frac{1}{4\pi} \sum \sum_{\phi} \int_{-\infty}^{\infty} E\big(g, I'(\mu_1, \mu_2, it; f')\phi, \mu_2, \mu_2, it\big) \times \overline{E}(h, \phi, \mu_1, \mu_2, it)\, dt.$$

Here ϕ ranges over an orthonormal basis of $\mathbb{K}'$-finite functions in $H'(\mu_1, \mu_2)$. The first sum ranges over a set of representatives of the classes of pairs (μ_1, μ_2) of unitary characters on $\mathbb{A}_E^\times/E^\times$ under the equivalence relation $(\mu_1', \mu_2') \sim (\mu_1, \mu_2)$ if $(\mu_1', \mu_2') = (\mu_1\nu^{it}, \mu_2\nu^{-it})$ ($t \in \mathbb{R}$).

Since $N'\backslash\mathbb{N}'$ and $\mathbb{Z}D_\theta\backslash\mathbb{D}_\theta$ are compact, the integrals

$$E_\psi(\phi, \mu_1, \mu_2, s) = \int_{N'\backslash\mathbb{N}'} E(n, \phi, \mu_1, \mu_2, s)\psi(n)\,dn$$

and

$$E_\theta(\phi, \mu_1, \mu_2, s) = \int_{\mathbb{Z}D_\theta\backslash\mathbb{D}_\theta} E(h, \phi, \mu_1, \mu_2, s)\,dh$$

converge, and we conclude that $\iint K_c(n, h)\overline{\psi}(n)\,dn\,dh$ is equal to

$$\frac{1}{4\pi}\sum_{\{(\mu_1,\mu_2)\}}\sum_{\phi} \times \int_{-\infty}^{\infty} E_\psi\big(I'(\mu_1, \mu_2, it; f')\phi, \mu_1, \mu_2, it\big)\overline{E}_\theta(\phi, \mu_1, \mu_2, it)\,dt. \tag{4.1}$$

For a given f' both sums are finite. In the quasi-split case, where $\theta = 1$, the computation of $\iint K_c\overline{\psi}$ is more involved; see [F4]. In summary, the Fourier summation formula is the equality of the following.

4. Proposition. *For every test function f' we have*

$$\sum_{b\in E^\times/E^1} \Psi(b, f', \psi) = \sum_{\pi\subset L_0(D')} (W_\psi\overline{P}_\theta)_{\pi'}(f') + (4.1).$$

Next we develop the analogous formula on $\mathbf{G} = U(2, 1)$ and $\mathbf{C}_\theta = U(2)$. Let $f = \bigotimes f_v$ be a smooth compactly supported function on $\mathbb{G} = \mathbf{G}(\mathbb{A})$ modulo $\mathbb{Z}$ ($=$ center of $\mathbb{G}$, $\simeq \mathbb{A}_E^1$), which transforms under $\mathbb{Z}$ via ω^{-1}, with $f_v = f_v^0$ for almost all v. Here f_v^0 is the unit element in the convolution algebra $\mathbb{H}_v$ of the $K_v = \mathbf{G}(R_v)$-biinvariant f_v; R_v is the ring of integers in F_v; a choice of a Haar measure is implicit. Let $L(G)$ be the space of smooth functions ϕ: $\mathbb{G} \to \mathbb{C}$ with $\phi(\gamma z g) = \omega(z)\phi(g)$ ($g \in \mathbb{G}$, $z \in \mathbb{Z}$, $\gamma \in G = \mathbf{G}(F)$) and $\int_{\mathbb{Z}G\backslash\mathbb{G}} |\phi(g)|^2\,dg < \infty$. The convolution operator

$$(r(f)\phi)(g) = \int_{\mathbb{G}/\mathbb{Z}} f(h)\phi(gh)\,dh = \int_{\mathbb{Z}G\backslash\mathbb{G}} K_f(g, h)\phi(h)\,dh$$

is an integral operator with kernel $K_f(g, h) = \sum_{\gamma\in G/Z} f(g^{-1}\gamma h)$.

The space $L(G)$ decomposes as the direct sum of three mutually orthogonal invariant spaces; the space $L_0(G)$ of cusp forms with central character ω, the space $L_1(G)$ of discrete-series non-cuspidal (necessarily non-generic) representations, including the functions $\phi(g) = \chi(\det g)$, where χ is a character of $\mathbb{A}_E^1/E^1$ with $\chi^3 = \omega$, and the continuous spectrum $L_c(G)$. Correspondingly, $K_f = K_0 + K_1 + K_c$. The Fourier summation formula is the equality obtained on integrating $K(n, h)\overline{\psi}(n)\xi(h)$ on $h \in \mathbb{Z}C_\theta\backslash\mathbb{C}_\theta \simeq C_\theta^1\backslash\mathbb{C}_\theta^1$ ($\mathbb{C}_\theta$ and $\mathbb{C}_\theta^1$ denote the groups of adele points on the algebraic groups $\mathbf{C}_\theta$ and $\mathbf{C}_\theta^1$ naturally defined to have C_θ and C_θ^1 as their groups of F-points) and on

$$n = \begin{pmatrix} 1 & x & y \\ 0 & 1 & \overline{x} \\ 0 & 0 & 1 \end{pmatrix}$$

in $N\backslash\mathbb{N}$ ($\mathbb{N} = \mathbf{N}(\mathbb{A}), N = \mathbf{N}(F)$, $\mathbf{N}$ is the upper unipotent subgroup in $\mathrm{B} \subset \mathbf{G}$, thus x ranges over $\mathbb{A}_E/E$, and y over $\mathbb{A}_E/E$ with $y + \overline{y} = x\overline{x}$). Also we put $\psi(n) = \boldsymbol{\psi}(x + \overline{x})$.

The "geometric" expression for the double integral $\iint K_{f'}(n, h)\overline{\psi}(n) \cdot \xi_2(h)$ over $n \in N\backslash\mathbb{N}$ and $h \in C_\theta^1\backslash\mathbb{C}_\theta^1$ is

$$\sum_{b \in E^\times/E^1} \Psi(b, f, \psi),$$

where

$$\Psi(b, f, \psi) = \int_{\mathbb{N}} \int_{\mathbb{C}_\theta^1} f\big(n \operatorname{diag}(b, 1, \overline{b}^{-1})h\big)\psi(n)\xi_2(h)\, dn\, dh,$$

by virtue of the disjoint decomposition $\bigcup_b N \operatorname{diag}(b, 1, \overline{b}^{-1})C_\theta = \bigcup_b NZ \operatorname{diag}(b, 1, \overline{b}^{-1})C_\theta^1$ of G. Since f is compactly supported modulo the center, applying the map $g \mapsto gJ_\theta g^{-1}$ and using the Bruhat decomposition, we conclude that for a given f the sum is finite, and the double integral ranges over a compact in $\mathbb{N} \times \mathbb{C}_\theta^1$. In the isotropic case, where $\theta = 1$, one more term: $\Psi(0, f, \psi)$ of [F6], turns up.

The integral of $K_1(n, h)\overline{\psi}(n)\xi_2(h)$ over n is zero since ψ is non-trivial, and the forms ϕ contributing to K_1 are non-generic: $\int \phi(n)\overline{\psi}(n)\, dn = 0$ (see [F2]). We have

$$\int_{N\backslash\mathbb{N}} \int_{C_\theta^1\backslash\mathbb{C}_\theta^1} K_0(n, h)\overline{\psi}(n)\xi_2(h)\, dn\, dh = \sum_\pi \big(W_\psi \overline{P}_{\theta, \xi}\big)(f),$$

where

$$W_\psi(\phi) = \int_{N\backslash\mathbb{N}} \phi(n)\overline{\psi}(n)\, dn, \qquad P_{\theta, \xi}(\phi) = \int_{C_\theta^1\backslash\mathbb{C}_\theta^1} \phi(h)\xi_2(h)\, dh,$$

and

$$\big(W_\psi \overline{P}_{\theta, \xi}\big)_\pi(f) = \sum_{\phi \in \pi} W_\psi(\pi(f)\phi) P_{\theta, \xi}(\overline{\phi})$$

is independent of the choice of the orthonormal basis $\{\phi\}$ of π. The sum over π ranges over all cuspidal cyclic ($P_{\theta,\xi}(\phi) \neq 0$ for some $\phi \in \pi$) generic ($W_\psi(\phi) \neq 0$ for some $\phi \in \pi$) representations of $\mathbb{G}$.

Finally we record $K_c(n,h)$. Let μ range over a set of representatives $\{\mu\}$ of the classes ($\mu' \sim \mu$ if $\mu' = \mu\nu^{it}$ $(t \in \mathbb{R})$) of unitary characters of $\mathbb{A}_E^\times/E^\times$. For any $s \in \mathbb{C}$ consider the Hilbert space $H(\mu,s)$ of ϕ: $\mathbb{G} \to \mathbb{C}$ with (for any upper triangular unipotent u)

$$\phi\big(\mathrm{diag}(a,b,\overline{a}^{-1})g\big) = |a|_E^{s+1}\mu(a)(\omega/\mu)(b)\phi(g) \qquad \big(a \in \mathbb{A}_E^\times, b \in \mathbb{A}_E^1, g \in \mathbb{G}\big)$$

and $\int_{\mathbb{K}} |\phi(k)|^2\,dk < \infty$. Here $\mathbb{K} = \Pi K_v$, K_v being the standard maximal compact subgroup in G_v. As usual we use the map $\phi \mapsto \phi|\mathbb{K}$ to identify $H(\mu,s)$ with $H(\mu) = H(\mu,0)$, and denote by $\phi(\mu,s)$ the element of $H(\mu,s)$ corresponding to $\phi(\mu)$ in $H(\mu)$. The action of $\mathbb{G}$ on $H(\mu,s)$ by right translation is denoted by $I(\mu,s)$, and the associated Eisenstein series is

$$E(g,\phi,\mu,s) = \sum_{\gamma \in B\backslash G} \phi(\gamma g,\mu,s) \qquad (\phi = \phi(\mu) \in H(\mu)).$$

Then

$$K_c(g,h) = \frac{1}{4\pi}\sum_{\{\mu\}}\sum_{\phi}\int_{-\infty}^{\infty} E\big(g, I(\mu,it;f)\phi,\mu,it\big)\overline{E}(h,\phi,\mu,it)\,dt.$$

Here ϕ ranges over an orthonormal basis of $\mathbb{K}$-finite functions in $H(\mu)$.

Since $N\backslash\mathbb{N}$ and $C_\theta^1\backslash\mathbb{C}_\theta^1$ are compact, the integrals

$$E_\psi(\phi,\mu,s) = \int_{N\backslash\mathbb{N}} E(n,\phi,\mu,s)\psi(n)\,dn,$$

$$E_{\theta,\xi}(\phi,\mu,s) = \int_{C_\theta^1\backslash\mathbb{C}_\theta^1} E(h,\phi,\mu,s)\xi_2(h)^{-1}\,dh$$

are convergent, and we conclude that $\iint K_c(n,h)\overline{\psi}(n)\xi_2(h)\,dn\,dh$ is equal to

$$\frac{1}{4\pi}\sum_{\{\mu\}}\sum_{\phi}\int_{-\infty}^{\infty} E_\psi\big(I(\mu,it;f)\phi,\mu,it\big)\overline{E}_{\theta,\xi}(\phi,\mu,it)\,dt. \tag{5.1}$$

For a given f, both sums are finite. In the quasi-split case where $\theta = 1$ and $C_\theta^1 = U(1,1)$, the computation of $\iint K_c\overline{\psi}\xi_2$ is more involved; see [F6]. The Fourier summation formula for the pair (G, C_θ) is the equality in

5. PROPOSITION. *For every test function f we have*

$$\sum_{b \in E^\times/E^1} \Psi(b,f,\psi) = \sum_{\pi \subset L_0(G)} m(\pi)\big(W_\psi\overline{P}_{\theta,\xi}\big)_\pi(f) + (5.1);$$

here $m(\pi)$ denotes the multiplicity of π in the space of cusp forms on $G\backslash\mathbb{G}$; namely $m(\pi) = \mathrm{Hom}_{\mathbb{G}}(\pi, L_0(G))$.

3. MATCHING FUNCTIONS

For $f = \bigotimes f_v$, the global integral $\Psi(b, f, \psi)$, $b = (b_v) \in \mathbb{A}_E^\times/\mathbb{A}_E^1$, is the product over v of

$$\Psi(b_v, f_v, \psi_v) = \int_{N_v} \int_{C^1_{\theta, v}} f_v\big(\operatorname{diag}(b_v, 1, \overline{b}_v^{-1})h\big)\psi_v(n)\xi_{2v}(h)\, dn\, dh;$$

ψ_v is the component of ψ at v, ξ_{2v} is that of ξ_2, and b_v ranges over $E_v^\times/E_v^1$, in the local case. Similarly, for $f' = \bigotimes f'_v$ on $\mathbb{D}' = GL(2, \mathbb{A}_E)$ and $b = (b_v) \in \mathbb{A}_E^\times/\mathbb{A}_E^1$, the integral $\Psi(b, f', \psi)$ is the product, over all places v of F, of

$$\Psi(b_v, f'_v, \psi_v) = \int_{N'_v} \int_{D_{\theta, v}} f'_v\big(\operatorname{diag}(b_v, 1)h\big)\psi_v(n)\, dn\, dh.$$

Definition. (1) We write $\Psi_\theta(b_v, f'_v, \psi_v)$ for $\Psi(b_v, f'_v, \psi_v)$, and $\Psi_\theta(b_v, f_v, \psi_v)$ for $\Psi(b_v, f_v, \psi_v)$, when the dependence on θ needs to be made explicit.

(2) Denote by $C_v = C_c^\infty(G_v, \omega_v^{-1})$ the space of complex valued smooth functions f_v on G_v which transform via ω_v^{-1} on Z_v and are compactly supported modulo Z_v.

(3) Denote by $C'_v = C_c^\infty(D'_v, \xi'_{2v}/\omega'_v)$ the space of complex valued smooth functions f'_v on D'_v which transform via ξ'_{2v}/ω'_v on Z'_v and are compactly supported modulo Z'_v.

(4) The functions $f_v \in C_v$ and $f'_v \in C'_v$ are called *matching* if for every b in $E_v^\times/E_v^1$ we have $\Psi(b, f_v, \psi_v) = |b|_v^{1/2}\Psi(b, f'_v, \psi_v)$.

To relate the Fourier summation formulae we need to show that there are sufficiently many matching functions.

6. Proposition. *For every $f_v \in C_v$ there is a matching $f'_v \in C'_v$, and for every $f'_v \in C'_v$ there is a matching $f_v \in C_v$.*

Proof. Consider first the case of v such that $D_{\theta, v}$ and $C_{\theta, v}$ are anisotropic. Fixing such v we pass to local notations (i.e., omit v). The decomposition $D' = N'A'D_\theta$ implies that $\Psi(b, f', \psi)$ is locally constant and compactly supported on $E^\times/E^1$, and that given any locally constant and compactly supported function $\Psi'(b)$ on $E^\times/E^1$ there is such f' with $\Psi(b, f', \psi) = \Psi'(b)$ for all $b \in E^\times$. Similarly the decomposition $G = NAC^1_\theta$ implies that $\Psi(b, f, \psi)$ is locally constant and compactly supported on $b \in E^\times/E^1$, and any compactly supported locally constant function $\Psi(b)$ on $E^\times/E^1$ is so obtained. Since $b \mapsto |b|^{1/2}$ is locally constant, the proposition follows for such v.

For all other v the groups $D_{\theta, v}$ and $C_{\theta, v}$ are isotropic, and the proposition coincides with Proposition 7 of [F6]. Two cases are considered there, depending on whether v splits in E, or not. In both cases it is shown that there exists a function $\vartheta_{\psi_v}(b)$, and for each f_v there is a complex number $\Psi(0, f_v, \psi_v)$ (for each f'_v there is $\Psi(0, f'_v, \psi_v)$), such that $\Psi(b, f_v, \psi_v)$ is locally constant on $E_v^\times/E_v^1$, it is 0 for sufficiently small $|b|_v$, and equal to $|b|_v\Psi(0, f_v, \psi_v)\vartheta_{\psi_v}(b)$ for all $|b|_v \geq B(f_v)$. Moreover, all locally constant functions $\Psi(b)$ on $E_v^\times/E_v^1$ which vanish if $|b|_v$ is small and are equal to $|b|_v\Psi(0)\vartheta_{\psi_v}(b)$ for all $|b|_v \geq B(> 0)$, are of the form $\Psi(b, f_v, \psi_v)$ for some such f_v (see [F6, Lemmas 8 and 10]). Also $\Psi(b, f'_v, \psi_v)$ is locally constant on $E_v^\times/E_v^1$, it is 0 if $|b|_v$ is small enough, and equal to $|b|_v^{1/2}\Psi(0, f'_v, \psi_v)\vartheta_{\psi_v}(b)$ for all $|b|_v \geq B(f'_v)$. Moreover, all locally constant functions $\Psi'(b)$ on $E_v^\times/E_v^1$ which are zero if $|b|_v$ is small and are equal to $|b|_v^{1/2}\Psi'(0)\vartheta_{\psi_v}(b)$ for all $|b|_v \geq B'$ (> 0), are of the form $\Psi(b, f'_v, \psi_v)$ for some such f'_v (see [F6, Lemmas 9 and 11]). These characterizations imply Proposition 7 of [F6], which is our proposition, when v is a place of F such that $\theta \in N_{E_v/F_v}E_v^\times$ (in particular, if v splits in E). ∎

The characterizations described in the proof of Proposition 6 permit relating Fourier orbital integrals on G_v, with respect to different $C_{\theta, v}$, and those on D'_v, with respect to different $D_{\theta, v}$.

COROLLARY. *Let E_v/F_v be a quadratic extension of local fields*, char $F_v \neq 2$. *Fix $\theta \in F_v - NE_v$.*

(1) *For every $f_v^\theta \in C_c^\infty(G_v, \omega_v^{-1})$ there is $f_v^1 \in C_c^\infty(G_v, \omega_v^{-1})$; and for every $f_v^1 \in C_c^\infty(G_v, \omega_v^{-1})$ such that $\Psi_1(b, f_v^1, \psi_v)$ is compactly supported on $E_v^\times/E_v^1$, namely it is 0 for all $|b|_v \geq B(f_v^1)$, there is $f_v^\theta \in C_c^\infty(G_v, \omega_v^{-1})$; such that $\Psi_\theta(b, f_v^\theta, \psi_v) = \Psi_1(b, f_v^1, \psi_v)$ for all $b \in E_v^\times/E_v^1$.*

(2) *For every $f_v'^\theta \in C_c^\infty(D'_v, \xi'_{2v}/\omega'_v)$ there is $f_v'^1 \in C_c^\infty(D'_v, \xi'_{2v}/\omega'_v)$, and for every $f_v'^1 \in C_c^\infty(D'_v, \xi'_{2v}/\omega'_v)$ such that $\Psi_1(b, f_v'^1, \psi_v)$ is compactly supported on $E_v^\times/E_v^1$ there is $f_v'^\theta \in C_c^\infty(D'_v, \xi'_{2v}/\omega'_v)$ such that $\Psi_\theta(b, f_v'^\theta, \psi_v) = \Psi_1(b, f_v'^1, \psi_v)$ for all $b \in E_v^\times/E_v^1$.*

In other words, for $\theta \in F_v - NE_v$, the $\Psi_\theta(b, f_v, \psi_v)$ and $\Psi_\theta(b, f'_v, \psi_v)$ are the compactly supported functions amongst the $\Psi_1(b, f_v, \psi_v)$ and $\Psi_1(b, f'_v, \psi_v)$; the latter functions—in general—will have a specific type of asymptotic behaviour as $|b|_v \to \infty$.

The global test functions f and f', for which we need to relate the geometric sides of the Fourier summation formulae, have local components which are the unit elements in the respective Hecke algebras of spherical functions, for almost all v. For almost all v the groups C_θ and D_θ are quasisplit over F_v since $\theta(\in F - N_{E/F}E)$ lies in NE_v (for almost all v). Moreover

E_v/F_v is unramified, v is finite, and $\boldsymbol{\psi}_v$ is unramified (the maximal subring of F_v on which $\boldsymbol{\psi}_v$ is 1 is the ring R_v of integers), for almost all v. Propositions 14 and 16 of [F6] assert that these unit elements ($f_v^0 \in C_v$ and $f_v'^0 \in C_v'$) are matching. More generally, the correspondence of unramified local representations stated in the introduction defines a homomorphism of the convolution Hecke algebras $\mathbb{H}_v \subset C_v$ and $\mathbb{H}_v' \subset C_v'$ of spherical ($K_v = \mathbf{G}(R_v)$- and $K_v' = GL(2, R_v')$-biinvariant) functions. The isolation argument used to derive the representation theoretic applications from the equality of the Fourier summation formulae is based on the fact (again proven in [F6, Propositions 14 and 16]) that such corresponding spherical functions are matching. Since these results are used here, we briefly recall their statement.

In the case where v stays prime in E, and E_v/F_v, ψ_v, ω_v, ξ_v are unramified, we have $\omega_v = \xi_v = 1$, and the correspondence relates the unramified D_v'-module $I'(\mu)(= I'(\mu, \overline{\mu}^{-1}))$ with the unramified G_v-module $I(\mu)$, where μ is an unramified character of $E_v^\times$. The dual map D: $\mathbb{H}_v \to \mathbb{H}_v'$ of Hecke algebras is defined by $f_v' = D(f_v)$ if $\operatorname{tr} I'(\mu, f_v') = \operatorname{tr} I(\mu, f_v)$ for all unramified characters μ of $E_v^\times$. The theory of the Satake transform implies that the function $f_v \in \mathbb{H}_v$ (resp. $f_v' \in \mathbb{H}_v'$) is uniquely determined by the values of the traces $\operatorname{tr} I(\mu, f_v)$ (resp. $\operatorname{tr} I'(\mu, f_v')$), where μ runs through the variety of unramified characters of $E_v^\times$. Proposition 14 of [F6] asserts that the corresponding $f_v \in \mathbb{H}_v$ and $f_v' = D(f_v) \in \mathbb{H}_v'$ are matching.

In the case where v is a finite place which splits in E, we have $E_v = E \otimes_F F_v = F_v \oplus F_v$, and $E_v^1 = \{(z, z^{-1}); z \in F_v^\times\}$, since $\overline{a} = (a_2, a_1)$ if $a = (a_1, a_2) \in E_v$. Put $\omega_{0v}(z) = \omega_v((z, z^{-1}))$ and $\xi_{0v}(z) = \xi_{2v}((z, z^{-1}))$. A character μ of $E_v^\times$ is a pair (μ_1, μ_2) of characters μ_i of $F_v^\times$. The correspondence relates the unramified $GL(2, F_v)$-module $\pi_v' = I'(\mu) = I'(\mu_1, \mu_2)$ (more precisely the D_v'-module $\pi_v' \times \check{\pi}_v'$) with $\mu_1\mu_2 = \omega_{0v}/\xi_{0v}$, to the unramified $GL(2, F_v) = G_v$-module $\pi_v = I(\mu_1, \mu_2, \xi_{0v})$, whose central character is ω_{0v}. Beware: not all principal series unramified G_v-modules with central character ω_{0v} are obtained by the correspondence, since one of the three inducing characters is taken to be ξ_{0v}. The dual map D: $\mathbb{H}_v \to \mathbb{H}_v'$ of Hecke algebras is defined by $f_v' = D(f_v)$ if $\operatorname{tr} I'(\mu_1, \mu_2; f_v') = \operatorname{tr} I(\mu_1, \mu_2, \xi_{0v}; f_v)$ for all unramified characters μ_1, μ_2 of $F_v^\times$ with $\mu_1\mu_2 = \omega_{0v}/\xi_{0v}$. The theory of the Satake transform implies that the function $f_v' \in \mathbb{H}_v'$ is uniquely determined by the values of the traces $\operatorname{tr} I'(\mu_1, \mu_2; f_v')$ for all unramified μ_1, μ_2 with $\mu_1\mu_2 = \omega_{0v}/\xi_{0v}$. But the map D: $\mathbb{H}_v \to \mathbb{H}_v'$ is not injective since f_v will be uniquely determined by the traces $\operatorname{tr} I(\mu_1, \mu_2, \mu_3; f_v)$ or *all* triples of characters μ_i of $F_v^\times$, but not by the subset where μ_3 is limited to ω_{0v}/ξ_{0v}. Proposition 16 of [F6] asserts the following.

7. PROPOSITION. *Corresponding $f_v \in \mathbb{H}_v$ and $f_v' = D(f_v) \in \mathbb{H}_v'$ are matching.*

In fact the proof of [F6, Proposition 16] is carried out (1) only with $\omega_{0v} = \xi_{0v} = 1$, but this restriction was made there merely to simplify the notations and is easily removable; and (2) only for $f_v \in \mathbb{H}_v$ with value zero at $\operatorname{diag}(\pi_v, 1, 1)$ and $\operatorname{diag}(\pi_v^2, \pi_v, 1)$, but this restriction does not limit the applicability of our formula, since all $f'_v \in \mathbb{H}'_v$ are nevertheless obtained via D from this set of f_v's.

4. GLOBAL CYCLICITY

We begin with the following separation argument.

8. PROPOSITION. *Let V be a finite set of F-places containing all v which are ramified in E, those with $\theta \in F_v - NE_v$, and those where ω, ξ or ψ are ramified. For each v in V let $f_v \in C_v$ and $f'_v \in C'_v$ be matching functions. Put $f = (\bigotimes_{v\in V} f_v) \otimes (\bigotimes_{v\notin V} f_v^0)$ and $f' = (\bigotimes_{v\in V} f'_v) \otimes (\bigotimes_{v\notin V} f_v'^0)$. At each $v \notin V$ fix corresponding unramified D'_v- and G_v-modules $\tilde{\pi}'_v$ and $\tilde{\pi}_v$. Then*

$$\sum_{\pi' \subset L_0(D')} \big(W_\psi \overline{P}_\theta\big)_{\pi'}(f') = \sum_{\pi \subset L_0(G)} m(\pi)\big(W_\psi \overline{P}_{\theta,\xi}\big)_\pi(f). \tag{8.1}$$

Here π' ranges over all cuspidal representations of $\mathbb{D}'$ whose component at $v \notin V$ is $\tilde{\pi}'_v$, while π ranges over a set of representatives for the equivalence classes of cuspidal representations of $\mathbb{G}$ whose component at $v \notin V$ is $\tilde{\pi}_v$.

Proof. Consider $f^V = \bigotimes_{v\notin V} f_v$, where $f_v \in \mathbb{H}_v$ for all $v \notin V$, and $f_v = f_v^0$ for almost all $v \notin V$, and $f'^V = \bigotimes_{v\notin V} f'_v$, where $f'_v = D(f_v) \in \mathbb{H}'_v$. Note that $f_v'^0 = D(f_v^0)$, and that f_v and f'_v are matching by Proposition 7. Note that $f_v * f_v^0 = f_v$ and $f'_v * f_v'^0 = f'_v$ for all $v \notin V$. Since $\Psi(b, f * f^V, \psi) = \Psi(b, f' * f'^V, \psi)$ for all $b \in E^\times/E^1$ (since $\|b\| = 1$ by the product formula on a global field), Propositions 4 and 5 imply that

$$\sum_{\pi' \subset L_0(D')} \big(W_\psi \overline{P}_\theta\big)_{\pi'}\big(f' * f'^V\big) + (4.1)$$
$$= \sum_{\pi \subset L_0(G)} m(\pi)\big(W_\psi \overline{P}_{\theta,\xi}\big)_\pi(f * f^V) + (5.1)$$

For each $v \notin V$, the operator $\pi'_v(f'_v)$ acts on $\phi \in \pi'$ ($\subset L_0(D')$) as zero unless ϕ is $K'_v = GL(2, R'_v)$-right invariant, and then it acts as the scalar $\operatorname{tr} \pi'_v(f'_v)$. Note that if π' contributes to the sum, then $P_\theta(\overline{\phi}) \neq 0$ for some ϕ, namely π' is distinguished. Hence so is each component of π', and at the split places we have that $\pi'_v = \pi''_v \times \check{\pi}''_v$, and $\operatorname{tr} \pi'_v(f'_v) = \operatorname{tr} \pi''_v(f_{1v} * f_{2v}^*)$, if $f'_v((g_1, g_2)) = f_{1v}(g_1) f_{2v}(g_2)$, and $f_{2v}^*(g) = f_{2v}(g^{-1})$. To alleviate the notations we take $f_{2v} = f_v^0$ and write $\pi'_v = \pi'_v \times \check{\pi}'_v$ and $f'_v = (f'_v, f_v^0)$, so that π'_v denotes also the underlying $GL(2, F_v)$-module, and f'_v denotes the underlying function on $GL(2, F_v)$. It then follows that

$$\big(W_\psi \overline{P}_\theta\big)_{\pi'}(f' * f'^V) = \operatorname{tr} \pi'^V(f'^V) \cdot \big(W_\psi \overline{P}_\theta\big)_{\pi'}(f'),$$

where $\operatorname{tr} \pi'^V(f'^V) = \prod_{v \notin V} \operatorname{tr} \pi'_v(f'_v)$. This is zero unless π'_v is unramified ($v \notin V$).

Similarly we have

$$\big(W_\psi \overline{P}_{\theta, \xi}\big)_{\pi}(f * f^V) = \operatorname{tr} \pi^V(f^V) \cdot \big(W_\psi \overline{P}_{\theta, \xi}\big)_{\pi}(f),$$

and this is zero unless π_v is unramified for all $v \notin V$. At a place $v \notin V$ which stays prime in E, the component π_v is the unramified constituent of $I(\mu_v)$, for some μ_v: $E_v^\times \to \mathbb{C}^\times$, and $\operatorname{tr} \pi_v(f_v) = \operatorname{tr} \pi'_v(D(f_v))$ where π'_v is the unramified constituent of $I'(\mu_v)$.

Each π which contributes to the sum is cyclic (since $P_{\theta, \xi}(\overline{\phi}) \neq 0$ for some ϕ in π), hence so is each of its components. It is also generic (since $W_\pi(\phi) \neq 0$), and each component π_v of an automorphic representation is unitarizable. Consequently – by Proposition 0 of [F5]—if v splits in E then the generic unitarizable cyclic unramified $GL(3, F_v)$-module π_v is of the form $I(\mu_1, \mu_2, \xi_{0v})$, μ_i: $F_v^\times \to \mathbb{C}^\times$ (unramified with $\mu_1 \mu_2 \xi_{0v} = \omega_{0v}$), and $\operatorname{tr} \pi_v(f_v) = \operatorname{tr} \pi'_v(D(f_v))$ where $\pi'_v = I'(\mu_1, \mu_2)$.

The difference of the sums over π and π' can be written then as

$$\sum_{\pi' \subset L_0(D')} \big(W_\psi \overline{P}_\theta\big)_{\pi'}(f') \cdot \operatorname{tr} \pi'^V(f'^V)$$
$$= \sum_{\pi \subset L_0(G)} m(\pi) \big(W_\psi \overline{P}_{\theta, \xi}\big)_{\pi}(f) \cdot \operatorname{tr} \pi'^V(f'^V);$$

the difference (5.1)–(4.1) of integrals can similarly be expressed as an integral over $s \in \mathbb{R}/(\log q_{v_1})^{-1}\mathbb{Z}$ involving $\operatorname{tr} I'(\mu_{v_1} \nu_{v_1}^{is}, f'_{v_1})$ for some $v_1 \notin V$ which stays prime in E. A standard argument of "generalized linear independence of characters" (on $GL(2)$; see, e.g., Theorem 2 of [FK]), based on the absolute convergence of the two sums and two integrals here, the unitarity of all automorphic representations present, and the Stone–Weierstrass theorem, implies (that (4.1) = (5.1) and) the proposition as stated. ∎

Remark. By the rigidity and the multiplicity one theorems for $GL(2)$, the sum over π' consists of at most one term.

We need to relate the global distributions with products of local ones. This is done next.

If π_v is irreducible then the dimension of each of the complex spaces $\operatorname{Hom}_{N_v}(\pi_v, \psi_v)$ and $\operatorname{Hom}_{C_{\theta, v}}(\pi_v, \xi_v)$ is at most one (see the Remarks at the end of the Appendix to [F6]). If the first space is non-zero we choose a generator W_{ψ_v} and say that π_v is generic. If the second space is non-zero we choose a generator P_{θ,ξ_v} and say that π_v is $C_{\theta,v}$-cyclic. If π_v is unramified then its space contains a K_v-fixed vector ϕ_v^0. When ψ_v and ξ_v are unramified the linear forms W_{ψ_v} and P_{θ,ξ_v} are non-zero at ϕ_v^0, and they can be normalized to take the value 1 there. This ϕ_v^0 is used in the presentation of a cuspidal π as a product $\bigotimes \pi_v$; its space is spanned by local products $\bigotimes \phi_v$, with $\phi_v \in \pi_v$ for all v and $\phi_v = \phi_v^0$ for almost all v. The linear forms W_ψ and $P_{\theta,\xi}$ on π are scalar multiples of $\bigotimes W_{\psi_v}$ and $\bigotimes P_{\theta,\xi_v}$.

Let $\{\phi = \phi(\pi_v)\}$ be a K_v-finite orthonormal basis of the space of π_v. Then

$$\left(W_{\psi_v}\overline{P}_{\theta,\xi_v}\right)_{\pi_v}(f_v) = \sum_{\phi} W_{\psi_v}(\pi_v(f_v)\phi)\overline{P}_{\theta,\xi_v}(\phi)$$

defines a linear form on $C_v = C_c^\infty(G_v, \omega_v^{-1})$. This functional is independent of the choice of the basis $\{\phi\}$, it is zero unless π_v is generic and cyclic, and for inequivalent π_{vi} $(1 \le i \le k)$ the $(W_{\psi_v}\overline{P}_{\theta,\xi_v})_{\pi_{vi}}$ are linearly independent. It transforms under left translations of f_v by N_v via ψ_v, and under right translations of f_v by $C_{\theta,v}$ via ξ_v.

For every cuspidal representation π, non-trivial character ψ, element $\theta \in F^\times$, and character ξ, there is a complex number $c(\pi,\psi,\xi,\theta)$ such that

$$\left(W_\psi\overline{P}_{\theta,\xi}\right)_\pi\left(\bigotimes f_v\right) = c(\pi,\psi,\xi,\theta)\prod_v\left(W_{\psi_v}\overline{P}_{\theta,\xi_v}\right)_{\pi_v}(f_v).$$

Both sides are zero unless all π_v are generic and cyclic. The cuspidal π is (automorphically) cyclic precisely when all π_v are cyclic, and $c(\pi,\psi,\xi,\theta) \neq 0$. This constant depends on the various normalizations involved, but note that when ξ_v, ψ_v, and π_v are unramified, and π_v is generic and cyclic, we have that $(W_{\psi_v}\overline{P}_{\theta,\xi_v})_{\pi_v}(f_v) = \operatorname{tr}\pi_v(f_v)$ for spherical $f_v \in \mathbb{H}_v$.

9. Proposition. *Let π' be a $\mathbb{D}_\theta$-distinguished cuspidal representation of $\mathbb{D}' = GL(2,\mathbb{A}_E)$ with central character ω'/ξ_2'. Then π' corresponds to a $\mathbb{C}_\theta$-cyclic generic cuspidal representations π of $\mathbb{G} = U(2,1;\mathbb{A})$ with central character ω.*

Proof. We choose V and $\tilde{\pi}_v' = \pi_v'$ for all $v \notin V$ so that π' parametrizes the only term on the left of (8.1). We need to show for each $v \in V$ that there is $f_v' \in C_v' = C_v(D_v', \xi_{2v}'/\omega_v')$ (it matches some $f_v \in C_v$ by Proposition 6) with $(W_{\psi_v}\overline{P}_{\theta,v})_{\pi_v'}(f_v') \neq 0$, where this last distribution is defined in close analogy to the one on G_v. Indeed, having shown this we would conclude that the right side of (8.1) is not identically zero, and any π occurring non-trivially on the right would be cuspidal generic $\mathbb{C}_\theta$-cyclic with central character ω, corresponding to π'.

So we fix $v \in V$, let ϕ_1 be a smooth vector in π'_v with $P_{\theta,v}(\phi_1) \neq 0$, and ϕ_2 a smooth vector in π'_v with $W_{\psi_v}(\phi_2) \neq 0$. We may assume that either $\phi_1 = \phi_2$ or that ϕ_1 is orthogonal to ϕ_2. Each of ϕ_1, ϕ_2 can be multiplied by a scalar to have length 1, and we extend $\{\phi_1, \phi_2\}$ to an orthonormal basis $\{\phi_i\}$ of π'_v. Since π'_v is irreducible and admissible, the set $\{\pi'_v(f'_v); f'_v \in C'_v\}$ spans the algebra of endomorphisms of π'_v, and we may choose f'_v such that $\pi'_v(f'_v)\phi_i = \delta_{i,1}\phi_2$. Then

$$\big(W_{\psi_v}\overline{P}_{\theta,v}\big)_{\pi'_v}(f'_v) = W_{\psi_v}\big(\pi'_v(f'_v)\phi_1\big)\overline{P}_{\theta,v}(\phi_1) = W_{\psi_v}(\phi_2)\overline{P}_{\theta,v}(\phi_1) \neq 0,$$

as required. ∎

In the opposite direction, we have

10. PROPOSITION. *Let π be a $\mathbb{C}_\theta$-cyclic cuspidal generic representation of $\mathbb{G} = U(2,1;\mathbb{A})$ with central character ω. Then π corresponds to a unique $\mathbb{D}_\theta$-distinguished cuspidal representation π' of $\mathbb{D}' = GL(2,\mathbb{A}_E)$ with central character ω'/ξ'_2.*

Proof. We apply (8.1) with a suitable set V and with $\tilde{\pi}_v$ equals π_v for each $v \notin V$. To distinguish π of the proposition from the other representations which index contributions to the right side of (8.1), we denote it by $\tilde{\pi}$. As noted in the proof of Proposition 8, each component $\tilde{\pi}_v$ of $\tilde{\pi}$ is generic, unitarizable, and cyclic, hence when $v \notin V$ it is unramified and it corresponds to a generic unitarizable distinguished unramified representation $\tilde{\pi}'_v$ of $D'_v = GL(2, E_v)$. We need to show that for some choice of $f_v (v \in V)$, the right side of (8.1) is non-zero. Recall that $(W_\psi \overline{P}_{\theta,\xi})_\pi(f)$ is the product of a constant $c(\pi, \psi, \xi, \theta)$, which is non-zero when π is $\tilde{\pi}$ since $\tilde{\pi}$ is cyclic and generic, and $\prod_{v\in V}(W_{\psi_v}\overline{P}_{\theta,\xi_v})_{\pi_v}(f_v)$, since $(W_{\psi_v}\overline{P}_{\theta,\xi_v})_{\pi_v}(f_v^0) = 1$ for all $v \notin V$.

Consider a place $v \in V$ such that $\tilde{\pi}_v$ is supercuspidal. Such v is finite and it stays prime in E (since a supercuspidal $GL(3, F_v)$-module cannot be cyclic, by Proposition 0 of [F5], whose proof relies heavily on Bernstein–Zelevinsky [BZ2] and Gelfand–Kazhdan [GK]). Since $\tilde{\pi}_v$ is generic and cyclic its space contains vectors ϕ_1 and ϕ_2 of length 1 with $P_{\theta,\xi_v}(\phi_1) \neq 0$ and $W_{\psi_v}(\phi_2) \neq 0$. We may assume that $\phi_1 = \phi_2$ or that ϕ_2 is orthogonal to ϕ_1. Extend $\{\phi_1, \phi_2\}$ to an orthonormal basis of $\tilde{\pi}_v$. The matrix coefficient $\tilde{f}_v(x) = (\phi_2, \check{\tilde{\pi}}(x)\phi_1)$ is a supercusp form which satisfies $\tilde{\pi}_v(\tilde{f}_v)\phi = 0$ for all ϕ orthogonal to ϕ_1, and $\tilde{\pi}_v(\tilde{f}_v)\phi_1 = \phi_2$ (up to a non-zero multiple). Consequently $(W_{\psi_v}\overline{P}_{\theta,\xi_v})_{\pi_v}(\tilde{f}_v) = 0$ for all π_v inequivalent to $\tilde{\pi}_v$, and $(W_{\psi_v}\overline{P}_{\theta,\xi_v})_{\tilde{\pi}_v}(\tilde{f}_v) \neq 0$. Using such $\tilde{f}_v$ at each place $v \in V$ where $\tilde{\pi}_v$ is supercuspidal, we conclude that the sum on the right of (8.1) extends only over the π whose components at these $v \in V$ are the supercuspidal $\tilde{\pi}_v$.

Next we consider a place $v \in V$ such that $\tilde{\pi}_v$ is not supercuspidal. Then $\tilde{\pi}_v$ is the unique generic constituent in the composition series of an induced representation $I(\mu_v)$ if v stays prime, or $I_v \times \check{I}_v$, $I_v = I(\rho_{2v} \times \omega_{0v}/\xi_{0v})$ and ρ_{2v} is a generic unitarizable irreducible representation of $GL(2, F_v)$ with central character ξ_{0v} (if v splits in E). As usual, we choose a basis $\phi_1, \phi_2, \ldots$ for $\tilde{\pi}_v$, and $\tilde{f}_v \in C_c^\infty(G_v)$ with $\tilde{\pi}_v(\tilde{f}_v)\phi_i = \delta_{i,1}\phi_2$ and $(W_{\psi_v}\overline{P}_{\theta,\xi_v})_{\tilde{\pi}_v}(\tilde{f}_v) \neq 0$. Applying Bernstein's decomposition theorem (which is based on Bernstein's analysis of the Bernstein center, see [F4, pp. 165–166]), we may replace here $\tilde{f}_v$ by its component $\tilde{f}_{v,\Theta} \in C_v$, where Θ is the connected component $\Theta(\tilde{\pi}_v)$ of the infinitesimal character $\chi(\tilde{\pi}_v)$ of $\tilde{\pi}_v$. Then $\pi_v(\tilde{f}_{v,\Theta})$ acts as 0 on any π_v with $\chi(\pi_v) \notin \Theta(\tilde{\pi}_v)$, and $\tilde{\pi}_v(\tilde{f}_{v,\Theta})$ acts as $\tilde{\pi}_v(\tilde{f}_v)$ on $\tilde{\pi}_v$. Using this $\tilde{f}_{v,\Theta}$ for f_v in (8.1) we conclude that the sum over π on the right of (8.1) ranges precisely over all π whose components at the $v \notin V$, or at the $v \in V$ where $\tilde{\pi}_v$ is supercuspidal, are the same as that of $\tilde{\pi}$; but at the remaining finite set of places where $\tilde{\pi}_v$ is the generic constituent of the full induced $I(\mu_v)$ (if v stays prime), or $I(\mu_{1v}, \mu_{2v}, \omega_{0v}/\xi_{0v})$ or $I(\rho_{2v}, \omega_{0v}/\xi_{0v})$, ρ_{2v} supercuspidal (if v splits), we only know that π_v is a constituent of $I(\mu_v\nu_v^s)$ or $I(\mu_{1v}\nu_v^s, \mu_{2v}, \nu_v^{-s}, \omega_{0v}/\xi_{0v})$ or $I(\chi_v \otimes \rho_{2v}, \omega_{0v}/\xi_{0v})$ (χ_v unramified with $\chi_v^2 = 1$) for some $s \in \mathbb{C}$ (as usual $\nu_v(x) = |x|_v$). So far it appears that the sum over π on the right of (8.1) may range over a set larger than $\tilde{\pi}$ alone, and cancellations may cause this sum on the right of (8.1) to vanish.

At this stage we use the rigidity theorem for automorphic representations of $\mathbb{G} = U(2,1)(\mathbb{A})$ from [F2] and [F3], which asserts, in particular, that: there exists at most one (equivalence class of) cuspidal representation of $\mathbb{G}$ whose components are specified at almost all places, and such that at the remaining finite set of places the components are the generic constituents of fully induced G_v-modules. Note that this is a weak form only, of the rigidity theorems of [F2, F3]. We conclude that with our choice of $f_v (v \in V)$ there is only one non-zero term in the sum of the right side of (8.1), it is indexed by our $\tilde{\pi}$, and so the left side of (8.1) is non-zero. By the rigidity theorem for $GL(2)$ the cuspidal (hence generic) distinguished π' which parametrizes the single term on the left of (8.1), is unique. It corresponds to our $\tilde{\pi}$, and the proposition follows. ∎

Propositions 9 and 10 imply that the correspondence is a bijection between the set of $\mathbb{D}_\theta$-distinguished cuspidal representations π' of $GL(2, \mathbb{A}_E)$ with central character ω'/ξ_2', and the set of packets of generic cuspidal $\mathbb{C}_\theta^1$-cyclic representations π of $\mathbb{G}$ with central character ω.

Theorem 0.2 of [FH] (which is quoted as Theorem 0.3 in the Introduction) asserts that a cuspidal representation π' of $GL(2, \mathbb{A}_E)$ is $\mathbb{D}_\theta$-distinguished precisely when it is $\mathbb{D}_1 \simeq GL(2, \mathbb{A})$-distinguished, and its component π'_v at each v in the set ∇ (of places where $\theta \in F_v - NE_v$) is not of the form $I'(\mu_1, \mu_2)$ with μ_i trivial on $F_v^\times$.

The Main Global Theorem of [F4], which is the quasi-split ($\theta = 1$) analogue of our Propositions 9 and 10, asserts that the correspondence establishes a bijection from the set of equivalence classes of automorphic representations of $GL(2, \mathbb{A}_E)$ with central character ω'/ξ'_2 which are either cuspidal and $GL(2, \mathbb{A})$-distinguished, or induced of the form $I'(\mu'_1, \mu'_2)$, μ_i: $\mathbb{A}_E^1/E^1 \to \mathbb{C}^\times$, $\mu_1 \neq \mu_2$, $\mu'_i(z) = \mu_i(z/\bar{z})$ ($z \in \mathbb{A}_E^\times$), to the set of packets of generic cuspidal $\mathbb{C}_1^1 = U(1, 1; \mathbb{A})$-cyclic representations of $\mathbb{G} = U(2, 1; \mathbb{A})$ with central character ω. Consequently we have:

COROLLARY. *The packets of the $\mathbb{C}_\theta^1$-cyclic generic cuspidal $\mathbb{G}$-modules are precisely the packets of the $\mathbb{C}_1^1$-cyclic generic cuspidal $\mathbb{G}$-modules which correspond to the cuspidal $\mathbb{D}_1 \simeq GL(2, \mathbb{A})$-distinguished $GL(2, \mathbb{A}_E)$-modules, whose components at the v where $\theta \in F_v - NE_v$ are not of the form $I'(\mu'_1, \mu'_2)$ with μ'_i trivial on $F_v^\times$.*

This can also be stated in terms of the endoscopic κ-lifting of [F2] from $U(1, 1)$. Here κ: $\mathbb{A}_E^\times/E^\times N\mathbb{A}_E^\times \to \mathbb{C}^\times$ is a character whose restriction to $\mathbb{A}^\times/F^\times N\mathbb{A}_E^\times$ is nontrivial. The Global Theorem of [F6] asserts that the packets of the generic $\mathbb{C}_1^1$-cyclic cuspidal representations of $\mathbb{G}$ with central character ω are the image under the κ-endoscopic lifting of the packets of the cuspidal representation ρ of $\mathbb{C}_1^1 = U(1, 1; \mathbb{A})$ with central character $\omega/\xi_2\kappa$. The packets of the generic $\mathbb{C}_\theta$-cyclic cuspidal representations of $\mathbb{G}$ with central character ω are the κ-endo-lifts of those cuspidal representations π_0 of $\mathbb{C}_1^1 = U(1, 1; \mathbb{A})$ with central character $\omega/\xi_2\kappa$ which are not of the form $\pi_0(\mu_1, \mu_2)$, μ_i: $\mathbb{A}_E^\times/E^\times\mathbb{A}^\times \to \mathbb{C}^\times$, $\mu_1 \neq \mu_2$ (those which base change via the κ-unstable base change lifting to $I'(\mu'_1, \mu'_2)$ on $GL(2, \mathbb{A}_E)$), and whose components at the v where $\theta \in F_v - NE_v$ is not of the form $\pi_0(\mu_{1v}, \mu_{2v})$, μ_i: $E_v^1 \to \mathbb{C}^\times (i = 1, 2)$ (those which base-change (via the κ_v-unstable base-change lifting) to $I'(\mu'_{1v}, \mu'_{2v})$ on $GL(2, E_v)$; see [F2, Sects. 3.7, 3.8, p. 49]).

5. LOCAL CYCLICITY

As in [F6], where the case of $\theta = 1$ is considered, we define also a local correspondence, by means of an identity of Whittaker–Period distributions on the two groups in question. The case of $\theta = 1$ considered in [F6], we are mainly concerned here with $\theta \in F - NE$, and v with $\theta \in F_v - NE_v$.

11. PROPOSITION. *For every component $\tilde{\pi}_v$ of a generic $\mathbb{C}_\theta^1$-cyclic cuspidal representation $\tilde{\pi}$ of $\mathbb{G}$ with central character ω, there exists a unique $D_{\theta,v}$-distinguished generic representation $\tilde{\pi}'_v$ of $GL(2, E_v)$ with central character*

ω_v/ξ_{2v}, which is a component of a cuspidal $\mathbb{D}_\theta$-distinguished representation $\tilde{\pi}'$ of $GL(2,\mathbb{A}_E)$ with central character ω/ξ_2; and for each such component $\tilde{\pi}'_v$ there exists a unique finite set $\{\pi_v\}$ of generic $C^1_{\theta,v}$-cyclic representations of G_v, and constants $c(\pi_v,\psi_v,\xi_v,\theta)$, such that the π_v lie in one packet (*see* [F2]) *uniquely determined by $\tilde{\pi}'_v$ and are components of $\mathbb{C}^1_\theta$-cyclic generic cuspidal representations $\tilde{\pi}$ of $\mathbb{G}$; such that for all matching $f'_v \in C'_v$ and $f_v \in C_v$ we have*

$$\big(W_{\psi_v}\overline{P}_{\theta,v}\big)_{\tilde{\pi}'_v}(f'_v) = \sum_{\pi_v\in\{\pi_v\}} c\big(\pi_v,\psi_v,\xi_v,\theta\big)\big(W_{\psi_v}\overline{P}_{\theta,\xi_v}\big)_{\pi_v}(f_v). \tag{11.1}$$

Proof. Given such global $\tilde{\pi}$ or the corresponding $\tilde{\pi}'$, we set up the identity (8.1) such that $\tilde{\pi}'$ parametrizes the only term on the left, and $\tilde{\pi}$ occurs on the right. At each $v_1 \neq v$ in V we choose f_{v_2} as in the proof of Proposition 10 to have that $(W_{\psi_{v_1}}\overline{P}_{\theta,\xi_{v_1}})_{\tilde{\pi}_{v_1}}(f_{v_1}) \neq 0$, and that the π which occur on the right of (8.1) will have the component $\tilde{\pi}_{v_1}$ (at each $v_1 \in V$, $v_1 \neq v$). As in the proof of Proposition 10 we use here the rigidity theorem for $U(2,1;E/F)$ of [F2]. Starting from $\tilde{\pi}'_v$ we proceed as in Proposition 9. In any case we obtain (11.1) for all matching f'_v and f_v, where the sum on the right ranges over a subset of the packet of $\tilde{\pi}_v$ by virtue of the rigidity theorem for $U(2,1;E/F)$ of [F2]. This subset consists only of generic $C^1_{\theta,v}$-cyclic representations. The $\{\pi_v\}$ and π'_v are uniquely determined by each other since the packet of $\{\pi_v\}$ is determined by π'_v via base-change and endoscopic liftings and the Whittaker–Period distributions are linearly independent. ∎

DEFINITION. A $D_{\theta,v}$-distinguished generic representation π'_v of $GL(2,E_v)$ and a $C^1_{\theta,v}$-cyclic generic representation π_v of $G_v = U(2,1;F_v)$ are said to *correspond* if they satisfy the relation (11.1) for all matching f'_v and f_v.

We shall use this definition of correspondence only for representations for which the identity of the Whittaker–Period distributions is established, namely this is done below only for square-integrable π'_v and π_v. It is perhaps best at this stage to define the local correspondence as the composition of the (inverse of the) κ_v-unstable base change from $D_{\theta,v} = U(2,F_v)$ to $D'_v = GL(2,E_v)$, and the κ_v-endoscopic lifting from $D_{\theta,v}$ to $G_v = U(2,1;F_v)$.

Next we shall list the generic $C^1_{\theta,v}$-cyclic G_v-modules, and relate them to the generic $D_{\theta,v}$-distinguished D'_v-modules. This has already been done in [F6] when $\theta = 1$, hence we assume here that $\theta \in F_v - NE_v$, in particular that $E_v = E\otimes_F F_v$ is a field. We also compare the notion of being $C^1_{\theta,v}$-cyclic with being $C^1_{1,v}$-cyclic, for a generic representation of G_v. To simplify the notations, we use local notations (drop v) in the following.

Let E/F be a quadratic extension of non-archimedean local fields with $\operatorname{char} F \neq 2$, and $\theta \in F - NE$.

12. PROPOSITION. (a) *The correspondence is a bijection relating the generic D_θ-distinguished irreducible admissible representations π' of $GL(2,E)$ with the packets of the generic C_θ^1-cyclic irreducible admissible representations π of $U(2,1;F)$.*

(b) *The packet of a generic C_θ^1-cyclic π contains a C_1^1-cyclic generic π_1. The packet of a generic C_1^1-cyclic π_1 contains a generic C_θ^1-cyclic π precisely when π_1 does not correspond to any $\pi' = I'(\mu_1', \mu_2')$, $\mu_i'(z) = \mu_i(z/\overline{z})(z \in E^\times)$, $\mu_i \colon E^1 \to \mathbb{C}^\times$, $\mu_1 \neq \mu_2$, namely π_1 is not the κ-endoscopic lift of any $\pi_0(\mu_1, \mu_2)$, $\mu_1 \neq \mu_2$, on $U(2,F)$.*

Remark. Here $\pi = \pi_1$ when π_1 is not supercuspidal, and perhaps also for supercuspidal π_1, but this is not shown here.

Proof. We shall go through the list of induced representations, their constituents, and supercuspidals, to verify our claims.

(1) If π' is induced, then $\mathrm{Hom}_{D_\theta}(\pi', 1) \neq \{0\}$ when $\pi' = I'(\mu, \overline{\mu}^{-1})$, by Proposition 3. When $\mu = \omega/\xi_2$ on E^1 (equivalently $\mu' = \omega'/\xi_2'$ on $E^\times$), π' corresponds to $\pi = I(\mu)$. All induced π with $\mathrm{Hom}_C(\pi, \xi_2) \neq 0$ are of the form $\pi = I(\mu)$, $\mu = \omega/\xi_2$ on E^1, if $C = C_\theta^1$ and if $C = C_1^1$, by Proposition 2′. Both (a) and (b) follow in this case.

(2) If π is the Steinberg $\mathrm{St}(\mu\nu)$, then $\mathrm{Hom}_C(\mathrm{St}(\mu\nu), \xi_2) = \{0\}$ by the corollary to Proposition 2, where $C = C_\theta^1$ or $C = C_1^1$.

(3) The reducible induced $\pi = I(\mu)$ are listed in [F1, (3.2), page 558]. There are three cases of reducibility. The third in that list is the Steinberg, disposed of in (2) above. If μ is unitary, reducibility occurs precisely when $\mu|F^\times = 1$ and $\mu^3 \neq \omega'$. Since $I(\mu)$ is tempered, it is the direct sum of its two irreducible constituents, denoted in [F1, (3.2(1))], by π^+ and π^-. Then (1) above asserts that the packet $\{\pi^+, \pi^-\}$, which corresponds to $I'(\mu, \overline{\mu}^{-1})$, $\mu = \omega/\xi_2$ on E^1, $\mu \neq \xi_2'$ on $E^\times$, contains a generic C_i-cyclic representation, with $i = 1$ and with $i = \theta$. Since only one of π^+, π^- is generic, namely π^+, it is both C_θ^1-cyclic and C_1^1-cyclic.

(4) The square-integrable ("special" or "Steinberg") subrepresentation $\mathrm{sp}(\mu\kappa)$ of $I'(\mu\kappa\nu^{1/2}) = I'(\mu\kappa\nu^{1/2}, \mu\kappa\nu^{-1/2})$, where $\mu \colon E^\times \to \mathbb{C}^\times$ and $\kappa \colon E^\times/NE^\times \to \mathbb{C}^\times$ has $\kappa|F^\times \neq 1$, is D_θ-distinguished (and $GL(2,F)$-distinguished) precisely when $\mu|F^\times = 1$, by Proposition 3. It is the κ-base-change of a special representation of $U(1,1;F)$, which corresponds to a one-dimensional representation of the anisotropic inner form $U(2;F)$. Using the trace formula on an anisotropic group $U(2;\mathbb{A})$ over a global field whose component at some place is our local $U(2;F)$, we can view this special representation as a component of a global cuspidal representation, then lift the global representation via the κ-base-change map as in [F4] to a cuspidal representation of $\mathbb{D}'$ which is $\mathbb{D}_\theta$-distinguished, whose component at

our local place is our sp$(\mu\kappa)$. Proposition 11, together with the usage of the rigidity theorem on $U(2,1;\mathbb{A})$ of [F2] explained in the proof of Proposition 10, implies that sp$(\mu\kappa)$ corresponds to the square-integrable subrepresentation π_μ^+ of the induced $I(\mu\kappa\nu^{1/2})$ of [F1, (3.2(2))]; this $I(\mu\kappa\nu^{1/2})$ is reducible precisely when $\mu|F^\times = 1$, and its quotient is non-tempered and non-generic, denoted by $\pi_\mu^\times$ in [F1, (3.2(2))]. Moreover, this π_μ^+ is the only term in the sum on the right of (11.1), and it is generic and C_θ^1-cyclic (and also C_1^1-cyclic). This completes the proof of (a) and (b) for the non-supercuspidal π (and π'). Note that the packet of sp$(\mu\kappa)$ contains also a supercuspidal which we expect to be neither generic nor cyclic but have not shown this as yet. This supercuspidal is classified here according to its packet, hence it does not appear in (5) below.

(5) Given a supercuspidal D_θ-distinguished representation π' of $GL(2,E)$ it is easy to construct a global cuspidal $\mathbb{D}_\theta$-distinguished representation of $\mathbb{D}'$ whose component at some place (where the global θ is not a norm) is our local one. Applying Proposition 11 we conclude that the packet which corresponds to π' contains a generic C_θ^1-cyclic π; the packet of π consists of supercuspidals by [F2] (since our correspondence is κ-base-change composed with κ-endoscopic lifting).

Conversely, given a generic supercuspidal C_θ-cyclic π (which is not in the packet of sp$(\mu\kappa)$) we may construct a global cuspidal $\mathbb{C}_\theta$-cyclic representation whose component (at a place where the global θ is not a norm) is our local π, as in [F4, Proposition 14] (as corrected in [FH, after Proposition B17]). Applying Proposition 11, we conclude that π corresponds to a generic D_θ-cyclic representations of $GL(2,E)$, which is supercuspidal by the results of [F2] on the κ-endo-lifting.

If π is a generic supercuspidal C_1^1-cyclic representation of $G = U(2,1)$, then by [F6] it corresponds either to a supercuspidal $GL(2,F)$-distinguished representation of $GL(2,E)$, or to an induced $I'(\mu_1',\mu_2')$, μ_i': $E^\times/F^\times \to \mathbb{C}^1$, $\mu_1' \neq \mu_2'$. Theorem 0.1 of [FH] establishes that a supercuspidal representation of $GL(2,E)$ is $GL(2,F)$-distinguished if and only if it is D_θ-distinguished. This result was proven independently and by purely local means by D. Prasad [P]. Hence the packets of the generic supercuspidal C_θ^1-cyclic π are the packets of the generic supercuspidal C_1^1-cyclic π which correspond to the supercuspidal distinguished π', but not to the induced $I'(\mu_1',\mu_2')$, $\mu_1' \neq \mu_2'$, μ_i': $E^\times/F^\times \to \mathbb{C}^1$, as asserted. ∎

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