DISTINGUISHED REPRESENTATIONS AND A FOURIER SUMMATION FORMULA

Yuval Z. Flicker

Titre Courant. **A FOURIER SUMMATION FORMULA**

Résumé en français. Une nouvelle formule sommatoire de type Fourier est développée dans le cadre du groupe linéaire, $GL(n, E)$, et du groupe unitaire quasi-déployé $U(n, E/F)$, où E est une extension quadratique d'un corps global F. Cette formule est utilisée pour réduire la forme précise de la conjecture de [F1] réexprimée ci-dessous à une hypothèse technique locale concernant les intégrales orbitales de type Fourier. La conjecture est que le changement de base stable (si n est impair) et instable (si n est pair) est une surjection de l'ensemble (a) des représentations irréductibles automorphes, séries-discrètes non dégénérées, π, du groupe de points adéliques de $U(n, E/F)$, sur l'ensemble (b) des représentations automorphes irréductibles, non dégénérées, π', du groupe des points adéliques de $GL(n, E)$, induites normalisées d'une représentation, $\rho_1 \times \cdots \times \rho_a$, d'un sous-groupe parabolique de type $(n_1, \ldots, n_a)$, où les ρ_i sont des représentations mutuellement inéquivalentes, distinguées cuspidales et non dégénérées du groupe des points adéliques de $GL(n_i, E)$.

Résumé en anglais. A new "Fourier" summation formula is developed in the context of both $GL(n, E)$ and the quasi-split unitary group $U(n, E/F)$ associated with a quadratic extension E/F of global fields. It is used to reduce to a local technical conjecture concerning matching "Fourier" orbital integrals, the following precise form of the conjecture of [F1]. The stable (if n is odd) and the unstable (if n is even) base-change lifting is a surjection from (a) the set of irreducible automorphic discrete-series non-degenerate representations π of the group of adele points on $U(n, E/F)$, to (b) the set of automorphic irreducible non-degenerate representations π' of $GL(n, \mathbb{A}_E)$ normalizedly induced from a representation $\rho_1 \times \cdots \times \rho_a$ of a parabolic subgroup of type $(n_1, \ldots, n_a)$, where the ρ_i are pairwise inequivalent distinguished cuspidal non-degenerate representations of $GL(n_i, \mathbb{A}_E)$.

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Let E/F be a quadratic extension of global fields with char $F \neq 2$, $\mathbb{A}_E$ and $\mathbb{A} = \mathbb{A}_F$ their rings of adeles, and put $\underline{G} = GL(n)$, viewed as an F-group. Put $G = \underline{G}(F)$, $G' = \underline{G}(E)$, $\mathbb{G} = \underline{G}(\mathbb{A})$, $\mathbb{G}' = \underline{G}(\mathbb{A}_E)$, and denote their centers by $Z \simeq F^\times$, $Z' \simeq E^\times$, $\mathbb{Z} \simeq \mathbb{A}^\times$, $\mathbb{Z}' \simeq \mathbb{A}_E^\times$. Fix a unitary character ω' of $\mathbb{Z}'/Z'$, and denote by $L_{\omega'}(G'\backslash\mathbb{G}')$ the space of (smooth, absolutely square-integrable on $G'\mathbb{Z}'\backslash\mathbb{G}'$) automorphic forms on $\mathbb{G}'$ which transform on $\mathbb{Z}'$ according to ω'. An irreducible constituent π' of the right representation r' of $\mathbb{G}'$ on $L_{\omega'}$ is called *automorphic* (it is unitary since ω' is), in the *discrete series* if π' is a subrepresentation of r', and *cuspidal* if π' is a constituent, necessarily a direct summand, of the restriction of r' to the space $L_{0,\omega'}$ of cusp forms in $L_{\omega'}$. A discrete-series $\mathbb{G}'$-module π' is called ($\mathbb{G}$-)*distinguished* if there exists a form $\phi \in \pi' \subset L_{\omega'}$ such that $A(\phi) = \int_{\mathbb{Z}G\backslash\mathbb{G}} \phi(x)dx$ is non-zero; naturally we require ω to be trivial on $\mathbb{Z} \simeq \mathbb{A}^\times$. Distinguished representations have been studied in various contexts by Waldspurger [W] and others.

Every irreducible admissible G'-module π' factorizes as a restricted tensor product $\pi' = \otimes\pi'_v$ of admissible irreducible representations π'_v of $G'_v = \underline{G}(E_v)$, $E_v = E \otimes_F F_v$, where v runs through the places of F. Put $G_v = \underline{G}(F_v)$. The G'_v-module π'_v is called G_v-*distinguished* if there exists a non-zero G_v-invariant complex valued linear form D_v on the space of π'_v. Such modules have been studied in the archimedean case by Flensted-Jensen [FJ], Oshima-Matsuki [OM], Bien [B], and others. If exists, the form D_v is unique up to a scalar multiple ([F1], Prop. 11). Given a $\mathbb{G}'$-module π', for almost all v the component π'_v contains a unique-up-to-scalar K'_v-fixed non-zero vector ξ^0_v, used in the definition of the tensor product $\otimes\pi'_v$. Here $K'_v = \underline{G}(R'_v)$, $K_v = \underline{G}(R_v)$, R_v is the ring of integers in field F_v when v is non-archimedean, and $R'_v = R_v \otimes_F E$. If each component π'_v of π' is G_v-distinguished, and $D_v(\xi^0_v) = 1$ for almost all v, then $D = \otimes D_v$ is a $\mathbb{G}$-invariant complex-valued non-zero linear form on π', and we say that π' is *abstractly distinguished*.

If π' is a distinguished (in the automorphic sense) discrete-series $\mathbb{G}'$-module, it is clear that each of its components is G_v-distinguished, and by the uniqueness property mentioned above there exists $c \neq 0$ such that $A = cD$, where $D = \otimes D_v$ as above, and $\phi \mapsto A(\phi)$ is the automorphic functional. However, there are abstractly (locally everywhere) distinguished cusp forms π' on $\mathbb{G}'$ which are not (automorphically) distinguished. In this case $A(\phi) = 0$ for all ϕ in π', but $D \neq 0$. Examples are constructed in [F1] when $n = 2$.

The distinguished cuspidal π' are characterized in [F2] by a property of their twisted tensor L-function $L(s, \pi', r)$, that it has a pole at $s = 1$. The introduction of [F1] states and motivates a precise conjectural characterization of the distinguished cuspidal π' and irreducible admissible π'_v as unstable (when n is even) or stable (when n is odd) base-change lifts of representations of the quasi-split unitary group. This conjecture is proven in [F1] when $n = 2$ (and $n = 1$).

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The purpose of this paper is to reduce this conjecture, on developing the techniques introduced in [F1], *to a local conjecture concerning matching of Fourier-orbital integrals.* In particular we obtain a precise form of this technical conjecture, and our representation theoretic results (which are reduced to this local conjecture) establish that each generic cuspidal or discrete series representation of the unitary group base-change lifts to an automorphic or irreducible $\mathbb{G}'$- or G'_v-module; moreover the image is determined (in the global case) to be the automorphic $\mathbb{G}'$-modules parabolically induced from the generic discrete-series distinguished modules of the Levi factors.

The structure of the paper is as follows. We shall first state our global representation theoretic results, then state the conjectural statement of matching Fourier-orbital integrals, then reduce the global results to this conjecture in harmonic analysis, and finally briefly state and prove our local representation theoretic results.

The *unitary group* $U(n, E/F)$ consists of all g in $\underline{G}(E) = GL(n, E)$ with $\sigma(g) = g$, where $\sigma(g) = J^t\overline{g}^{-1}J^{-1}$, J is the $n\times n$ matrix whose (i, j) entry is $(-1)^{n-i}\delta_{i,n-j+1}$, ${}^tg$ indicates the transpose of g, and $\overline{g} = (\overline{g}_{ij})$ if $g = (g_{ij})$; the non-trivial automorphism of E over F is denoted by $a \mapsto \overline{a}$. Similarly we have $U_v = U(n, E_v/F_v)$ ($\simeq GL(n, F_v) = G_v$ if v splits in E) and $\mathbb{U} = U(n, \mathbb{A}_E/\mathbb{A}_F)$. The base-change lifting is defined in terms of a homomorphism of dual groups (see Langlands [L1]), where the *dual group* $\hat{U}$ of $\underline{U}$ is $\underline{G}(\mathbb{C}) \rtimes W_{E/F}$. The Weil group $W_{E/F}$ is an extension of the galois group $\mathrm{Gal}(E/F)$ by $W_{E/E}(= E^\times$ if E is local, $= \mathbb{A}_E^\times/E^\times$ if E is global), explicitly $W_{E/F} = \langle z \in W_{E/E},\ \sigma;\ \sigma z\sigma^{-1} = \overline{z},\ \sigma^2 \in W_{F/F} - N_{E/F}W_{E/E}\rangle$ where $N_{E/F}$ is the norm from E to F. The Weil group $W_{E/F}$ acts on the connected component $\hat{U}^0 = \underline{G}(\mathbb{C})$ of $\hat{U}$ via its quotient $\mathrm{Gal}(E/F)$, the non-trivial element acting as $g \mapsto \sigma(g) = J^tg^{-1}J^{-1}$. Let $\underline{G}' = \mathrm{Res}_{E/F}\underline{G}$ be the F-group obtained from $\underline{G}$ on restricting scalars from E to F (then $\underline{G}'(F) = G' = \underline{G}(E)$). On the connected component $\hat{G}'^0 = \underline{G}(\mathbb{C}) \times \underline{G}(\mathbb{C})$ of the dual group $\hat{G}'$ of $\underline{G}'$, the Weil group $W_{E/F}$ again acts via the quotient $\mathrm{Gal}(E/F)$, and $\sigma \neq 1$ acts by $\sigma(g_1, g_2) = (g_2, g_1)$. Fix a character $\kappa : \mathbb{A}_E^\times \to \mathbb{C}^\times$ whose restriction to $E^\times N_{E/F}\mathbb{A}_E^\times$ is trivial, but whose restriction to $\mathbb{A}^\times$ is non-trivial. The *stable base-change* homomorphism is

$$b : \hat{U} = \underline{G}(\mathbb{C}) \rtimes W_{E/F} \to \hat{G}' = [\underline{G}(\mathbb{C}) \times \underline{G}(\mathbb{C})] \rtimes W_{E/F}, \quad b(g, w) = (g, \sigma g, w).$$

The *unstable base-change* homomorphism $b_\kappa : \hat{U} \to \hat{G}'$ depends on κ. It maps $g \in \underline{G}(\mathbb{C})$ again to $(g, \sigma g) \in \hat{G}'^0$, and $\sigma(\neq 1) \in \mathrm{Gal}(E/F)$ to $(I_n, -I_n)\sigma \in \hat{G}'$. Here I_n is the identity $n \times n$ matrix. Further, b_κ maps $z \in W_{E/E} \subset W_{E/F} \subset \hat{U}$ to $(\kappa(z), \kappa(\overline{z}))z$ in $\hat{G}'$.

To define the lifting, recall that given an irreducible admissible $\mathbb{G}'$-module $\pi' = \otimes\pi'_v$, there is a finite set V of places of F, containing the archimedean places and those which ramify in E, such that: for each place v' of E above a place $v \notin V$ of F, the component $\pi'_{v'}$ of π' at v' is unramified. For each such v' there is an unramified character $(a_{ij}) \mapsto \prod_{1\leq i\leq n} \mu_{iv'}(a_{ii})$ of the upper triangular subgroup $B_{v'}$ of $G_{v'} = GL(n, E_{v'})$, such that $\pi'_{v'}$ is the unique irreducible unramified constituent in the composition series of the unramified $G_{v'}$-module $I((\mu_{iv'}))$ normalizedly induced from $(\mu_{iv'})$. Let $\underline{\pi} = \underline{\pi}_v$ be a uniformizer of F_v. Denote by $t_{v'} = t(\pi'_{v'})$ the semi-simple conjugacy class in $\underline{G}(\mathbb{C})$ with eigenvalues $(\mu_{iv'}(\underline{\pi}))$. For each v' the map $\pi'_{v'} \mapsto t(\pi'_{v'})$ is a bijection from the set of equivalence classes of irreducible unramified $G_{v'}$-modules to the set of semi-simple conjugacy classes in $\underline{G}(\mathbb{C})$. If v splits into v', v'' in E, the component $\pi'_v = \pi'_{v'} \times \pi'_{v''}$ defines a conjugacy class $t_{v'} \times t_{v''}$ in $\underline{G}(\mathbb{C}) \times \underline{G}(\mathbb{C})$, and a conjugacy class $t_v = t(\pi'_v) = (t_{v'} \times t_{v''}) \times 1$ in $\hat{G}'$. If $v(\notin V)$ is inert in E, and v' is the place of E above v, then we put π'_v for $\pi'_{v'}$. This π'_v defines a conjugacy class $t_{v'}$ in $\underline{G}(\mathbb{C})$, and a conjugacy class $t_v = (t_{v'} \times 1) \times \sigma$ in $\hat{G}'$ ($\sigma \neq 1$ in $\mathrm{Gal}(E/F)$).

Similarly given an unramified irreducible U_v-module π_v there exists an $[n/2]$-tuple (μ_i) of unramified characters of $E_v^\times$ such that π_v is the unique unramified irreducible constituent in the composition series of the U_v-module $I((\mu_i))$ normalizedly induced from the character $(a_{ij}) \mapsto \prod_{1 \leq i \leq n/2} \mu_i(a_{ii})$ of the upper triangular subgroup. Then π_v is parametrized by the conjugacy class $t_v = \mathrm{diag}(\mu_1(\underline{\pi}_v), \dots, \mu_{[n/2]}(\underline{\pi}_v), 1, \dots, 1) \times \sigma$ in $\hat{U}$. At a place v which splits in E we have $U_v = \underline{G}(F_v)$, an unramified irreducible π_v is again associated with an induced $I((\mu_i(1 \leq i \leq n)))$ and a conjugacy class $t_v = \mathrm{diag}(\mu_i(\underline{\pi})) \times 1$ in $\hat{U}$.

When v splits into v', v'' in E, we define the *stable base-change lift* $b(\pi_v)$ of a $U_v = G_v$-module π_v to be $\pi_v \times {}^\sigma\pi_v$, where ${}^\sigma\pi_v(g) = \pi_v(\sigma g)$ and $\sigma g = J{}^t g^{-1} J^{-1}$. The lift is a $G'_v = G_v \times G_v$-module, and the lifting is compatible with the stable base-change homomorphism $b : \hat{U} \to \hat{G}'$ and the parametrization of unramified U_v- and G'_v-modules. Moreover, we have $\sigma(x, y) = (y, x)$ on $E_v = F_v \times F_v$, hence $N_{E/F} E_v = F_v$, and the component κ_v of κ is of the form $\kappa_{v'} \times \kappa_{v''}$ with $\kappa_{v''} = \kappa_{v'}^{-1}$. We define the unstable base-change lift $b_{\kappa_v}(\pi_v)$ of π_v to be $\pi_v \kappa_{v'} \times {}^\sigma\pi_v \kappa_{v'}^{-1}$. This definition is again compatible with the parametrization of unramified modules and the unstable base-change homomorphism.

When E_v is a non-archimedean field we define the *stable base-change lift* $b(\pi_v)$ of the unramified irreducible U_v-module π_v to be the irreducible unramified G'_v-module π'_v which is parametrized by the conjugacy class $b(t(\pi_v))$ in $\hat{G}'$. The *unstable base-change lift* $b_{\kappa_v}(\pi_v)$ of such π_v is the unramified irreducible G'_v-module π'_v parametrized by the conjugacy class $b_{\kappa_v}(t(\pi_v))$ in $\hat{G}'_v$. Globally if $\pi = \otimes \pi_v$ is an irreducible admissible $\mathbb{U}$-module we define its *stable base-change lift* $b(\pi)$ to be an automorphic $\mathbb{G}'$-module $\pi' = \otimes \pi'_v$ with $\pi'_v = b(\pi_v)$ for almost all v; the *unstable base-change lift* $b_\kappa(\pi)$ is similarly defined to be an automorphic $\mathbb{G}'$-module $\pi' = \otimes \pi'_v$ with $\pi'_v = b_{\kappa_v}(\pi_v)$ for almost all v.

Let $\underline{\psi}'$ be a non-trivial complex-valued (additive) character of $\mathbb{A}_E$ modulo E (later we will take $\underline{\psi}'$ to be $\mathbb{A}$-invariant), and ψ' the character of the unipotent upper triangular subgroup $\mathbb{N}'$ of $\mathbb{G}'$ defined by $\psi'(m) = \underline{\psi}'(\sum_{1 \leq i < n} m_{i+1})(m = (m_{ij}) \in \mathbb{N}')$. An irreducible $\mathbb{G}'$-module π' is called ψ'-*generic* if $Hom_{\mathbb{N}'}(\pi', \psi') = Hom_{\mathbb{G}'}(\pi', \mathrm{ind}(\psi'; \mathbb{G}', \mathbb{N}'))$ is non-zero (see Bernstein-Zelevinski [BZ] for a definition of induction and for Frobenius reciprocity). By the rigidity theorem of Jacquet-Shalika [JS], if π'_1 is a generic automorphic $\mathbb{G}'$-module, and π'_2 is an automorphic $\mathbb{G}'$-module with $\pi'_{2v} \simeq \pi'_{1v}$ for almost all v, then $\pi'_2 \simeq \pi'_1$. Hence if the lift exists and it is generic, then it is unique.

The notion of a local generic G'_v-module is analogously defined, using a character $\psi'_v \neq 1$ of N'_v. The notion of a generic $\mathbb{U}$-module is similarly defined. Let $\underline{\psi} \neq 1$

be a character of $\mathbb{A}/F$, and define a character ψ of the upper triangular unipotent subgroup $\mathbb{N}(=\mathbb{U}\cap\mathbb{N}')$ of $\mathbb{U}$ by $\psi(m)=\underline{\psi}(\sum_{1\le i<n}m_{i,i+1})$; note that $m_{i,i+1}=\overline{m}_{n-i,n-i+1}(1\le i<n)$, hence the sum is in $\mathbb{A}$. An irreducible $\mathbb{U}$-module π is called *ψ-generic* if $Hom_{\mathbb{N}}(\pi,\psi)=Hom_{\mathbb{U}}(\pi,\mathrm{ind}(\psi;\mathbb{U},\mathbb{N}))$ is non zero. Given x_0 in $E-F$, and $\underline{\psi}$, the character $\underline{\psi}'$ can be defined by $\underline{\psi}'(x)=\underline{\psi}((x-\overline{x})/(x_0-\overline{x}_0))$. We fix these $\underline{\psi}$ and $\underline{\psi}'$.

While every cuspidal $\mathbb{G}'$-module is generic, the conjectural analogue for $\mathbb{U}$ asserts only that in every packet of cuspidal $\mathbb{U}$-modules which lifts to a cuspidal (or more generally, generic) $\mathbb{G}'$-module, there is a unique generic element. This last statement assumes knowledge of base-change lifting, and the definition of packets; these are studied in [F3] and [F4] when $n=2$ and $n=3$, but have not yet been analyzed for $n>3$. In [F4] a U_v-packet is defined as the set of U_v-modules which lift to a G'_v-module. The lifting is defined via a character relation. The coefficients in the germ expansion of the characters encode the dimension of the space of Whittaker vectors, and the character relation is used to imply that if the G'_v-module is generic, so will be precisely one element in the U_v-packet. Such an argument was first used in the general rank case of the metaplectic group in the joint paper [FK1] with Kazhdan, § 22, Theorem, p. 90. The "conjectural analogue" mentioned above is suggested by the work of [FK1], § 22. It can be made in the context of any reductive group.

Let ω be a fixed unitary character of $\mathbb{A}_E^\times/E^\times$; then $\omega'(z)=\omega(z/\overline{z})$ is a unitary character of $\mathbb{A}_E^\times/E^\times\mathbb{A}^\times$. Put $\kappa'(z)=\kappa(z/\overline{z})$. Given $GL(n_i,\mathbb{A}_E)$-modules ρ_i $(1\le i\le a)$ with $\sum\limits_{1\le i\le a}n_i=n$, denote by $I(\rho_1,\dots,\rho_a)$ the $\mathbb{G}'$-module normalizedly induced from the corresponding representation of the parabolic subgroup of type $(n_1,\dots,n_a)$. If the ρ_i are irreducible, unitarizable and generic, it follows from well-known results of Bernstein-Zelevinsky [BZ], [Z], Tadic [T] and Vogan [V], that $I(\rho_1,\dots,\rho_a)$ is irreducible, unitarizable and generic.

The proofs of Propositions 16 and 19 (see also the Remark following Lemma 20) rely on results of [JS], [JS1], [F2], [F5], which concern the twisted-tensor L-function. These results are proven only for an F-place which splits in E or is non-archimedean. Hence we now restrict attention to E/F such that each archimedean place of F splits in E. Our techniques extend to deal with all E/F once the archimedean analogue of [F5], namely the twisted-tensor analogue of [JS1], is carried out.

Our main global result is

1. Theorem*. *When n is even, the unstable base-change lifting (via b_κ) is a surjection from the set of discrete-series generic $\mathbb{U}$-modules π with central character ω, to the set of the automorphic irreducible generic $\mathbb{G}'$-modules π' of the form $I(\rho_1,\dots,\rho_a)$, where ρ_i are all distinguished discrete-series $GL(n_i,\mathbb{A}_E)$-modules ($\sum\limits_{1\le i\le a}n_i=n$) which are pairwise inequivalent, whose central character is $\omega'\kappa^n$. When n is odd the same assertion remains true when "unstable" is replaced by "stable" and κ is erased.*

It is clear that the lifting be an isomorphism between the two sets once a rigidity theorem for generic automorphic (or discrete-series) $\mathbb{U}$-modules is proven. Such a theorem would assert that two such generic modules π_1 and π_2 are equivalent if their components π_{1v} and π_{2v} are equivalent for almost all v. This theorem follows from [F3] and [F4] when $n=2$ and $n=3$, but it is possible that such a rigidity theorem for generic discrete series modules be proven using elementary "Whittaker model" techniques.

The superscript $*$ on Theorem 1^* indicates that this assertion is not proven here, but merely is reduced to a conjecture (5 and 6 below) in harmonic analysis, concerning matching of some Fourier-orbital integrals, which we proceed to state. We will then reduce Theorem 1^* to the local conjecture on generalizing the techniques introduced in [F1] in the case of $n=2$. Theorem 1 is proven in [F1] when n is (1 or) 2. The other $*$-superscripted results here are Proposition 28^*, on which the proof of Theorem 1^* relies, and Proposition 29^*, which states the local results. The rest of the paper, which discusses the Fourier summation formulae for U and G' is independent of Conjectures 5 and 6.

The local analogue of the global Theorem 1^* in the archimedean case where $E/F=\mathbb{C}/\mathbb{R}$ is briefly discussed in a remark at the end of this paper.

To state our local conjecture let E/F be now a quadratic extension of local fields, $G=GL(n,F)$, $G'=GL(n,E)$, $U=U(n,E/F)=\{g\in G';\sigma g=g\}$, N the unipotent upper triangular subgroup of U and N' that of G', A the diagonal subgroup of U and A' that of G', $B=AN$ and $B'=A'N'$. Denote by W the Weyl group of G' (or G); it can be represented by $n\times n$ matrices w each of whose rows and columns consists of a single non-zero entry equal to 1; W is isomorphic to the symmetric group S_n on n letters. Let $\underline{\psi}\neq 1$ be a character of F in $\mathbb{C}^\times$, fix x_0 in $E-F$ and define a character of E/F in $\mathbb{C}^\times$ by $\underline{\psi}'(x)=\underline{\psi}((x-\overline{x})/(x_0-\overline{x}_0))$. These define characters ψ and ψ' of N and N' as above.

Denote by E^1 the kernel of the norm map $N_{E/F}:E^\times\to F^\times$. Fix a character ω of the center $Z\simeq E^1$ of U; then $\omega'(z)=\omega(z/\overline{z})$ is a character of the center $Z'\simeq E^\times$ of G'. Let κ be a character of $E^\times$ which is trivial on $N_{E/F}E^\times$ but not on $F^\times$, and put $\kappa'(z)=\kappa(z/\overline{z})$. We shall state the conjecture below when n is even. When n is odd κ (and κ') should be erased (or replaced by 1) from all formulae.

Let $C(G')=C_c^\infty(G',(\omega'\kappa^n)^{-1})$ be the convolution algebra of smooth (locally constant when F is non-archimedean) complex valued functions on G' which transform under Z' by $(\omega'\kappa^n)^{-1}$ and are compactly supported modulo Z'. Implicit is a choice of a Haar measure on G'. We also fix Haar measures on $G,N,N',A,A',Z,Z',K'=GL(n,R')$ and $K=U\cap K'$ (R' is the ring of integers in E if E is non-archimedean) and specify a normalization below if needed. Denote by $H(G')$ the convolution subalgebra of spherical (K'-biinvariant) measures in $C(G')$, and by f'^0 its unit element. $H(G')$ is zero unless $\omega'\kappa'$ is unramified, namely invariant under $R'^\times$. Similarly we have $C(U)=C_c^\infty(U,\omega^{-1})$ and its subalgebra $H(U)$ of K-biinvariant functions; its unit element f^0 is supported on K.

The local conjecture relates the Fourier orbital integral

$$\Psi(u,f;\psi)=\int\int_{(N\times N)/Z(u)} f(n_1un_2)\psi(n_1n_2)dn_1dn_2$$

of $f\in C(U)$ at $u\in U$, where $Z(u)=\{(n_1,n_2)\in N\times N;n_1un_2=u\}$, with the Fourier-orbital integral

$$\Psi'(g',f';\psi')=\int\int_{(N'\times G)/Z(g')} f'(n'g'g)\psi'(n')dn'dg$$

of $f' \in C(G')$ at $g' \in G'$, where $Z(g') = \{(n', g) \in N' \times G; n'g'g = g'\}$. To relate these, we first find sets of representatives for the double coset spaces $N\backslash U/N$ and $N'\backslash G'/G$ where Ψ and Ψ' are not identically zero. We shall show that these two sets are isomorphic, and then make our conjecture.

Note that by the Bruhat decomposition, $N\backslash U/N = N(A) = AW = WA$, where A is the diagonal in U, $N(A)$ its normalizer, and $W = N(A)/A$ the Weyl group. Write
antidiag (a, b) for $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$.

2. Proposition. *Let $w \in W$ and $a \in A$ be such that $\Psi(wa, f; \psi) \neq 0$ for some f, ψ. Then ($w^2 = 1$ and) $w = antidiag(I_1, I_2, \dots, I_k)$ with $I_i = I_{k+1-i} (1 \leq i \leq k)$, where I_i is the identity $n_i \times n_i$ matrix (and $\sum_{1 \leq i \leq k} n_i = n$), and $\alpha = diag(\alpha_1 I_1, \dots, \alpha_k I_k)$, where $\alpha_i = \overline{\alpha}_{k+1-i}{}^{-1}$ is a scalar in $E^\times$.*

Proof. Given $t \in N \cap w^{-1}Nw$, we have

$$\psi(t^{-1}) \int\int f(mwan)\psi(mn)dmdn = \int\int f(mwt(a)an)\psi(mn)dmdn \qquad (t(a) = ata^{-1}),$$

$$= \int\int f(mt(w, a)wan)\psi(mn)dmdn = \psi(t(w, a)^{-1}) \int\int f(mwan)\psi(mn)dmdn$$

$(t(w, a) = wt(a)w^{-1})$. If this is non-zero then $\psi(t) = \psi(wata^{-1}w^{-1})$ for all t. As $W \simeq S_n$, the permutation w of $\{1, \dots, n\}$ has the property that if $w(i + 1) > w(i)$ then $w(i + 1) = 1 + w(i)$ (since ψ is a non-degenerate character of N). Hence $w = \mathrm{antidiag}(I_1, \dots, I_k)$. Then as permutations $J^t wJ = w$, where J is the longest element $(1, n)(2, n-1)\dots$ of S_n, and the transpose acts as inverse on w. Since w lies in U, we have $w = \sigma w = J^t w^{-1} J$. Hence $w^2 = 1$ and so $I_{k+1-i} = I_i$ $(1 \leq i \leq k)$. This establishes that w has the asserted form. Now t in $N \cap w^{-1}Nw$ has the form $t = \mathrm{diag}(t_1, \dots, t_k)$, where t_{k+1-i} is $\sigma_i(t_i) = J_i{}^t\overline{t}_i^{-1} J_i^{-1}$, t_i is an upper triangular unipotent matrix of size $n_i \times n_i$, and J_i is the matrix "J" of size $n_i \times n_i$. The matrix $wata^{-1}w^{-1}$ is then $\mathrm{diag}(a_k t_k a_k^{-1}, \dots, a_1 t_1 a_1^{-1})$, and $a_{k+1-i} t_{k+1-i} a_{k+1-i}^{-1} = \sigma_i(a_i t_i a_i^{-1})$. Then the identity $\psi(t) = \psi(wata^{-1}w^{-1})$ for all t in $N \cap w^{-1}Nw$ implies that $\psi(t_i) = \psi(a_i t_i a_i^{-1})$ for all t_i. It follows that the diagonal $n_i \times n_i$ matrix a_i is a scalar $\alpha_i I_i$, $\alpha_i \in E^\times$, as required.

To determine where Ψ' is not necessarily zero, we first describe the double coset space $N'\backslash G'/G$. In fact, we begin with $B'\backslash G'/G$. The following is the same as Lemma 6 of [F5].

3. Proposition. *The group G' is the disjoint union of the double cosets $B'\eta_w G$ over all $w \in W$, $w^2 = 1$, where $\eta_w \in G'$ satisfies $\eta_w \overline{\eta}_w^{-1} = w$ (here w is the representative in G' whose entries are 0 or 1). The double coset is independent of the choice of the representative η_w.*

Proof. As noted in [F1], Proposition 10(1), the map $G'/G \to S = \{g \in G'; g\overline{g} = 1\}$, by $g \mapsto g\overline{g}^{-1}$, is a bijection. Indeed, it is clearly well defined and injective, and the surjectivity follows at once from the triviality of $H^1(\mathrm{Gal}(E/F), GL(n, E))$ (if $g\overline{g} = 1$, $a_\sigma = g$ defines a cocycle, which is then a coboundary, namely there is $x \in G'$ with $g = a_\sigma = x\overline{x}^{-1}$).

If $g \in G'$ maps to $s \in S$, then $bg \mapsto bs\overline{b}^{-1}$. By the Bruhat decomposition $G' = B'WB'$ applied to S, varying g in its double coset $B'gG$ we may assume that $g \mapsto wb \in S$, where $w \in W$ and $b \in B'$. Since wb lies in S, $1 = wbw\overline{b}$. Hence $w^{-1} = bw\overline{b}$, and the uniqueness of the Bruhat decomposition implies that $w^{-1} = w$. Write now $b = an$ with $a \in A'$, $n \in N'$. Since $1 = wbw\overline{b}$, we have $1 = waw\overline{a}$. Define an action σ of $\mathrm{Gal}(E/F)$ on A' by $\sigma(a') = w\overline{a}'w^{-1}$. Since $a\sigma(a) = 1$, $\{\sigma \mapsto a\}$ defines an element of $H^1(\mathrm{Gal}(E/F), A')$. This last group is trivial, hence there exists some $c \in A'$ with $a = w\overline{c}^{-1}wc$. Since $\overline{c}wanc^{-1} = wcnc^{-1}$, replacing g by $\overline{c}g$ we may assume that $g \mapsto wn$. Again $wn \in S$ implies $1 = wnw\overline{n}$, so if we define a galois action σ on $N' \cap wN'w$ by $\sigma(n') = w\overline{n}'w$, the map $\{\sigma \mapsto n\}$ defines an element of $H^1(\mathrm{Gal}(E/F), N' \cap wN'w)$. Since this last group is trivial, there exists an $m \in N'(\cap wN'w)$ with $n = w\overline{m}^{-1}wm$. Hence $\overline{m}wnm^{-1} = w$, and replacing g by $\overline{m}g$ we may assume that $g \mapsto g\overline{g}^{-1} = w$. Since $G'/G \simeq S$ the existence of g, and the independence of $B'\eta_w G$ of the choice of η_w, are clear.

Remark. To fix ideas, note that w with $w^2 = 1$ is a product of disjoint transpositions, and when $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\eta = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ satisfies $\eta\overline{\eta}^{-1} = w$. Here i is a non zero element of E with $\overline{i} = -i$. Also, $\eta^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1/i & -1/i \end{pmatrix}$.

4. Proposition. *Let $w = w^{-1} \in W$ and $a \in A'$ be such that $\Psi'(a\eta_w, f'; \psi') \neq 0$ for some f', ψ'. Then $w = antidiag(I_1, \dots, I_k)$ (necessarily $I_{k+1-i} = I_k$) and $a = b\, diag(\alpha_1 I_1, \dots, \alpha_k I_k)$, $\alpha_i \in E^\times$ and $b \in A'$ with $\overline{b} = wbw$.*

Proof. Given $m \in N'$ put $m_a = a^{-1}ma$, and note that there exists $g \in G$ with $m_a\eta_w g = \eta_w$ if and only if $m_a w\overline{m}_a^{-1} = w$. Suppose that $\overline{m}_a = wm_a w$ for $m \in N'$. If

$$\int_{N'}\int_G f'(na\eta_w g)\psi'(n)dndg = \int_{N'}\int_G f'(nma\eta_w g)\psi'(n)dndg$$
$$= \psi'(m^{-1})\int\int f'(na\eta_w g)\psi'(n)dndg$$

is non zero, then $\psi'(m) = 1$. Since ψ' is non-degenerate it follows that w must map any entry above the diagonal indexed by $(i, i+1)$ either below the diagonal or to another entry indexed by $(j, j+1)$. In other words, if $w(i+1) > w(i)$ then $w(i+1) = 1 + w(i)$. Consequently w has the asserted form.

To continue, note that if $m = w\overline{m}w$ then $m = \mathrm{diag}(m_1, \dots, m_k)$, m_i of the same size as I_i, and $m_{i+1-i} = \overline{m}_i$. Write $a = \mathrm{diag}(b_1, \dots, b_k)$ with b_i also of the same size as I_i. If $\psi'(ama^{-1}) = 1$ for all $m = w\overline{m}w$, then

$$\psi'(b_i m_i b_i^{-1})\psi'(b_j m_j b_j^{-1}) = 1 \qquad (j = k+1-i)$$

for all (i and) m_i. It follows that for each i we have that $b_j = \beta_i\overline{b}_i$ for some $\beta_i \in E^\times$. Putting $b = \mathrm{diag}(b_1, \dots, b_k)$ and $\alpha_i = 1$ $(i \leq k/2)$, $\alpha_i = \beta_{k+1-i}$ $(i > k/2)$, we deduce that a has the asserted form.

In view of Propositions 2 and 4 we can redefine the Fourier-orbital integrals as functions on the set Ω of matrices aw in U, $w = \mathrm{antidiag}(I_1, \dots, I_k)$, $I_{k+1-i} = I_i$,

$a = \operatorname{diag}(a_1 I_1, \dots, a_k I_k)$, $a_{k+1-i} = \overline{a}_i^{-1} \in E^\times$, by

$$\Psi(aw, f; \psi) = \int_{N/N \cap wNw^{-1}} \int_N f(n_1 awn_2)\psi(n_1 n_2) dn_1 dn_2$$

and

$$\Psi'(aw, f'; w') = \int_{N'/N_w} \int_G f'(n'\alpha\eta_w g)\psi'(n') dn' dg.$$

Here $\eta_w \overline{\eta}_w^{-1} = w$ and $\alpha = \operatorname{diag}(\alpha_1 I_1, \dots, \alpha_k I_k)$ with $a_i = \alpha_i/\overline{\alpha}_{k+1-i}$, and $N_w = \{n' \in N'; \overline{n}' = wn'w\}$. Since $G'/G \simeq S$, the integral Ψ' is independent of the choice of $\alpha\eta_w$.

5. Conjecture. *There exists a complex-valued function $\gamma = \gamma_\psi$ on Ω such that on writing $\Psi_1(aw, f; \psi)$ for $\gamma(aw)\Psi(aw, f; \psi)$, for every f in $C(U)$ there exists f' in $C(G')$, and for every f' there exists an f, with*

$$\Psi'(aw, f'; \psi') = \Psi_1(aw, f; \psi).$$

If F is non-archimedean and E/F and ψ are unramified, then $\gamma(aw) = 1$ for $a \in A \cap K$. If E/F is an extension of global fields and γ_v is the function on Ω_v associated with the local extension E_v/F_v, then $\gamma = \otimes\gamma_v$ is a function on $\Omega(\mathbb{A})$ which is invariant under $A\mathbb{Z}$.

Definition. Functions f and f' which satisfy the identity $\Psi'(f'; \psi') = \Psi_1(f; \psi)$ are called *matching*.

At a place v which splits we have $E = F \oplus F$, where F is the local completion, $G' = G \times G$ and G embeds diagonally in G'. A function f' on G' is a pair (f_1, f_2) of functions on G, and $U = G$ embeds via $g \mapsto (g, \sigma g)$ in G'. Define f_2^* by $f_2^*(x) = f_2(x^{-1})$, and $f = f_1 * f_2^*$. Write $n' = (n_1, n_2), n_i \in N$; $\eta_w = (\eta_1, \eta_2)$ with $\eta_1\eta_2^{-1} = w$ (the galois action is $(x, y) \mapsto (y, x)$, hence $\eta_w \overline{\eta}_w^{-1} = (\eta_1\eta_2^{-1}, \eta_2\eta_1^{-1}) = (w, \sigma w)$, $\sigma w = J^t w^{-1} J$); and $\alpha = (\alpha_1, \alpha_2)$. Put $a = \alpha_1 w \alpha_2^{-1} w$. Then

$$\Psi'(aw, f'; \psi') = \int\int\int (f_1, f_2)((n_1, n_2)(\alpha_1, \alpha_2)(\eta_1, \eta_2)(g, g))\psi(n_1 n_2^{-1}) dn_1 dn_2 dg$$
$$= \int\int\int f_1(n_1 \alpha_1 \eta_1 \eta_2^{-1} \alpha_2^{-1} n_2^{-1} g) f_2(g) \psi(n_1 n_2^{-1}) dn_1 dn_2 dg$$
$$= \int\int (f_1 * f_2^*)(n_1 awn_2)\psi(n_1 n_2) dn_1 dn_2,$$

and the Conjecture 5 is trivial in this case (note that given f there are f_1, f_2 with $f = f_1 * f_2^*$). We shall then concentrate on the non-split case, where E is a field.

A stronger form of Conjecture 5 is needed in the spherical case, as follows. Let $f^\vee$ be the Satake transform of $f \in H(U)$. It is the function on the manifold of unramified irreducible $Z\backslash U$-modules π defined by $f^\vee(\pi) = tr\,\pi(f)$, the trace of the convolution operator $\pi(f) = \int_{Z\backslash U} f(g)\pi(g) dg$. We assume that ω is trivial (and so is the central character of π). The function f is uniquely determined by $f^\vee$. Similarly, the Satake transform $f'^\vee$ of $f' \in H(G')$ is the function on the manifold of irreducible unramified G'-modules π' with a trivial central character defined by $f'^{\vee}(\pi') = tr\,\pi'(f')$; f' is uniquely determined by $f'^{\vee}$. The dual group homomorphism $b_\kappa : \hat{U} \to \hat{G}'$ can be viewed as a morphism of the manifolds of unramified U and G'-modules, and we write $\pi' = b_\kappa(\pi)$ for the image of π. Define a dual map b_κ^* from $H(G')$ to $H(U)$ by $b_\kappa^*(f') = f$ if $f^\vee(\pi) = f'^\vee(b_\kappa(\pi))$. We say that f and f' are *corresponding* (spherical functions) if $f = b_\kappa^*(f')$. In particular the unit elements correspond: $f^0 = b_\kappa^*(f'^0)$. The required stronger form asserts:

6. Conjecture. *Corresponding spherical functions are matching.*

Further analysis of the definition of f, f' being corresponding is carried out in [F1], end of proof of Proposition 3, when $n = 2$. This analysis is easily generalizable to all n, and yields a relationship between the orbital integrals of f and f'. This may be useful in any attempt to prove the Conjecture 6. In this work we assume Conjectures 5 and 6, and mark any result which depends on them by a superscript $*$, as we have done in Theorem 1^*. The other starred results here are Propositions 28^* and 29^*.

To reduce Theorem 1^* to Conjectures 5 and 6 we use the technique employed in [F1] in the case where $n = 2$, namely compare the Fourier summation formulae for G' and U. We proceed to describe these formulae. The statements of the formulae are independent of Conjectures 5 and 6; these Conjectures are used only in their comparison.

Let E/F be a global quadratic extension, and r the representation of $\mathbb{U} = U(n, \mathbb{A}_E/\mathbb{A}_F)$ by right translation on the space $L_\omega(U\backslash\mathbb{U})$ of absolutely square integrable, smooth automorphic forms on $\mathbb{U}$ which transform under the center $\mathbb{Z} \simeq \mathbb{A}_E^1$ via the unitary character ω. Denote by $C(\mathbb{U})$ the linear span of the functions $f = \otimes f_v$ on $\mathbb{U}$ where $f_v \in C(U_v)$ for every place v of F (when v splits in E we have $U_v = G_v (= GL(n, F_v))$ and $C(G_v) = C_c^\infty(G_v, \omega_v^{-1})$), and f_v is the unit element f_v^0 in $H(U_v)$ for almost all v. The convolution operator

$$r(f) = \int_{\mathbb{Z}\backslash\mathbb{U}} f(g) r(g) dg \quad \text{on} \quad L_\omega(U\backslash\mathbb{U})$$

is an integral operator: $(r(f)\phi)(x) = \int_{\mathbb{Z}\backslash\mathbb{U}} K_f(x,y)\phi(y)dy$, with kernel

$$K_f(x,y) = \sum_{\gamma \in Z\backslash U} f(x^{-1}\gamma y).$$

There is another expression for this kernel, see Arthur [A1], p. 935, whose definition we shall now recall, from [A1]. More precisely, via restriction of scalars the case of a number field F can be reduced to that of the field Q of rational numbers, which is discussed in [A]. The work of [A] is based on [L2], which uses the language of real groups, instead of that of adele groups. But the passage between these languages is well-known. Further comments on [L2] by its author can be found in [L3]. For a discussion of Eisenstein series in the analogous function field case one has the reference Morris [M]. A clear and comprehensive study of the theory of Eisenstein series for any global field is given in the recent manuscript [MW2].

Let $\underline{P}$ denote a standard parabolic subgroup of $\underline{U}$, one which contains the upper triangular subgroup, $\underline{N}$ its unipotent radical and $\underline{M}$ its Levi subgroup which contains the diagonal subgroup $\underline{A}$. We then have $M, \mathbb{M}, N, \mathbb{N}$, etc. Let $\prod(\mathbb{M})$ be the set of equivalence classes of irreducible unitary discrete series representations of $\mathbb{M}$ which transform under $\mathbb{Z}$ via ω. Put $X(\underline{M}) = \mathrm{Hom}_{\mathbb{Q}}(\underline{M}, GL(1))$, $\mathfrak{A}_P = \mathrm{Hom}(X(\underline{M}), \mathbb{R})$ the Lie algebra of $\underline{M}$, and $\mathfrak{A}_P^* = X(\underline{M}) \otimes_{\mathbb{Q}} \mathbb{R}$ its dual space. For $m = (m_v)$ in $\mathbb{M}$ define a vector $H_M(m)$ in $\mathfrak{A}_P$ by

$$e^{\langle H_M(m), \chi \rangle} = |\chi(m)| = \prod_v |\chi(m_v)|_v, \qquad \chi \in X(\underline{M}).$$

Extend H_M to a function on $\mathbb{U} = \mathbb{N}\mathbb{M}\mathbb{K}$ by $H_M(nmk) = H_M(m)$. If $\mathbb{M}^1$ is the kernel of H_M on $\mathbb{M}$ and $\underline{A_M}$ is the center of $\underline{M}$, then $\mathbb{M}$ is the direct product of $\mathbb{M}^1$ and $\underline{A_M}(\mathbb{R})$, and $H_M : \underline{A_M}(\mathbb{R}) \xrightarrow{\sim} \mathfrak{A}_P$. For any $\lambda \in \mathfrak{A}_{\mathbb{C}}^* = \mathfrak{A}_P^* \otimes_{\mathbb{R}} \mathbb{C}$ consider the character $x \mapsto e^{\langle \lambda, H_M(x) \rangle}$ on $\mathbb{G}$, and denote its tensor product with $\rho \in \prod(\mathbb{M})$ by ρ_λ. If $\lambda \in i\mathfrak{A}_P^*$ then ρ_λ is unitary, and we obtain a free action of the group $i\mathfrak{A}_P^*$ on $\prod(\mathbb{M})$, making $\prod(\mathbb{M})$ a differential manifold whose connected components are the orbits of $i\mathfrak{A}_P^*$.

For $\rho \in \prod(\mathbb{M})$ denote by $H_P(\rho)$ the Hilbert space completion of the space $H_P^0(\rho)$ of smooth functions $\Phi : \mathbb{N}M\backslash\mathbb{U} \to \mathbb{C}$ which are $\mathbb{K}$-finite, transform under $\mathbb{Z}$ via ω, have the property that

$$\int_{\mathbb{K}} \int_{M\mathbb{Z}\backslash\mathbb{M}} |\Phi(mk)|^2 dm dk$$

is finite, and that for every $x \in \mathbb{U}$ the function $m \to \Phi(mx)$ on $\mathbb{M}$ is a matrix coefficient of ρ. Let ρ_P be the vector in $\mathfrak{A}_P^*$ such that the modular function $\delta_P(p) = |\det(Ad(p)|\widetilde{N})|$ on $\mathbb{P}$ is equal to $e^{2\langle \rho_P, H_P(p) \rangle}$; here $\widetilde{N}$ is the Lie algebra of $\underline{N}$. For $\Phi \in H_P(\rho)$ and $\lambda \in \mathfrak{A}_{\mathbb{C}}^*$ put $\Phi(x, \lambda) = \Phi(x) e^{\langle \rho_P + \lambda, H_P(x) \rangle} (x \in \mathbb{U})$ and denote by $I(\lambda, \rho)$ the right representation, $(I(y, \lambda, \rho)\Phi)(x, \lambda) = \Phi(xy, \lambda)$, of $(y \in)\mathbb{U}$. The $\mathbb{U}$-module $I(\lambda, \rho)$ is unitary for $\lambda \in i\mathfrak{A}_P^*$. Denote by Δ_P the set of simple roots of $\underline{A_M}$ in $\underline{P}$. These are elements of $X(\underline{M}) \subset \mathfrak{A}_P^*$. For each root $\alpha \in \Delta_P$ denote by $\alpha^\vee$ the corresponding coroot in $\mathfrak{A}_P$. Define $\mathfrak{A}_P^+ = \{H \in \mathfrak{A}_P; \langle \alpha, H \rangle > 0, \alpha \in \Delta_P\}$, and $(\mathfrak{A}_P^*)^+ = \{\lambda \in \mathfrak{A}_P^*; \langle \lambda, \alpha^\vee \rangle > 0, \alpha \in \Delta_P\}$. Then $\rho_P \in (\mathfrak{A}_P^*)^+$.

If Q is also a standard parabolic subgroup, denote by $W(\mathfrak{A}_P, \mathfrak{A}_Q)$ the set of elements s in the Weyl group W with $s\mathfrak{A}_P = \mathfrak{A}_Q$. Denote by w_s a representative in U for the element s of G. For $\rho \in \prod(\mathbb{M})$ and $\Phi \in H_P^0(\rho)$, and $\lambda \in \mathfrak{A}_{P,\mathbb{C}}^*$ with real part $\mathrm{Re}\,\lambda \in \rho_P + (\mathfrak{A}_P^*)^+$, define the Eisenstein series

$$E(x, \Phi, \rho, \lambda) = \sum_{\delta \in P\backslash U} \Phi(\delta x, \lambda)$$

and intertwining operator

$$(M(s, \rho, \lambda)\Phi)(x, s\lambda) = \int_{\mathbb{N}_Q \cap w_s \mathbb{N}_P w_s^{-1} \backslash \mathbb{N}_Q} \Phi(w_s^{-1} n x, \lambda) dn.$$

The functions $E(x, \Phi, \rho, \lambda)$ and $M(s, \rho, \lambda)\Phi$ can be continued as meromorphic functions in λ to $\mathfrak{A}_{\mathbb{C}}^*$. If $\lambda \in i\mathfrak{A}^*$, $E(x, \Phi, \rho, \lambda)$ is smooth in x, and $M(s, \rho, \lambda)$ is a unitary operator from $H_P(\rho_\lambda)$ to $H_Q(s\rho_{s\lambda})$. Also denote by $n(P)$ the number of *chambers* of $\mathfrak{A}$, namely the connected component of the complement to the union of the hyperplanes orthogonal to the roots of Δ_P.

The representation theoretic expression for the kernel $K_f(x,y)$ is

$$\sum_P n(P)^{-1}\sum_\rho \int_{i\mathfrak{A}_P^*} \sum_{\alpha,\beta}(I(f,\lambda,\rho)\Phi_\alpha,\Phi_\beta)E(x,\Phi_\beta,\rho,\lambda)\overline{E}(y,\Phi_\alpha,\rho,\lambda)d\lambda.$$

Here ρ ranges over a set of representatives for the connected components ($i\mathfrak{A}_P^*$-orbits) of $\prod(\mathbb{M})$, and Φ_α, Φ_β over an orthonormal basis (chosen to have the finiteness properties of [A1], p. 926, $\ell.-12$) for the space $H_P(\rho)$; $I(f,\lambda,\rho)$ is the convolution operator, and $(\cdot,\cdot)$ indicates the inner product on $H_P(\rho)$. By [A1], Lemma 4.4, p. 929, the sum over P and ρ and the integral over $i\mathfrak{A}_P^*$ is absolutely convergent. In fact $(I(f,\lambda,\rho)\Phi_\alpha,\Phi_\beta)$ is a rapidly decreasing function in $|\lambda|\to\infty$, and $E(x,\Phi,\rho,\lambda)$ is slowly increasing, on $i\mathfrak{A}_P^*$.

Now that we have the two expressions, geometric and representation theoretic, for the kernel $K_f(x,y)$ of $r(f)$ on $L_\omega(U\backslash\mathbb{U})$, we shall obtain the Fourier summation formula on integrating both sides over x and y in $N\backslash\mathbb{N}$, after multiplying by $\psi(x^{-1}y)$.

7. Proposition. *For every $f\in C(\mathbb{U})$ we have*

$$\sum_w\sum_a \Psi(aw,f;\psi)=\sum_P n(P)^{-1}\sum_\rho\int_{i\mathfrak{A}_P^*}\sum_\Phi E_\psi(I(f,\lambda,\rho)\Phi,\rho,\lambda)\overline{E}_\psi(\Phi,\rho,\lambda)d\lambda,$$

where the sum over w ranges over $w=diag(I_1,\dots,I_k)$, $w=w^{-1}\in W$, a over $diag(\alpha_1 I_1,\dots,\alpha_k I_k)$ in A (thus $\alpha_{k+1-i}=\overline{\alpha}_i^{-1}\in E^\times$), and

$$\Psi(aw,f;\psi)=\prod_v \Psi(aw,f_v;\psi_v)\ \textit{if}\ f=\otimes f_v.$$

Also,

$$E_\psi(\Phi,\rho,\lambda)=\int_{N\backslash\mathbb{N}}E(x,\Phi,\rho,\lambda)\overline{\psi}(x)dx.$$

This follows without difficulty from the above descriptions of $K_f(x,y)$, and Proposition 2. Note that the sum over P contains also the standard parabolic subgroup U, and this is the only group for which $\mathfrak{A}^*$ is zero dimensional. Here $n(U)=1$, the ρ corresponding to $P=U$ range over the discrete-series representations π of $\mathbb{U}$, and $E_\psi(\Phi,\rho,\lambda)$ is the value at the identity I of the Whittaker function

$$W_{\Phi,\psi}(g)=\int_{N\backslash\mathbb{N}}\Phi(xg)\psi(x)dx$$

attached to Φ, and ψ. The distribution

$$W_\pi(f)=\sum_\Phi W_{\pi(f)\Phi,\psi}(I)\overline{W}_{\Phi,\psi}(I)$$

on $C(\mathbb{U})$ satisfies $W_\pi({}^x f^y) = \overline{\psi}(xy)W_\pi(f)$, where ${}^x f^y(g) = f(xgy)$. Note that conjecturally the discrete series generic (having Whittaker model) π are cuspidal, but for the unitary group this is known only for $n = 2$ ([F3]) and $n = 3$ ([F4]). In conclusion the term corresponding to the non-proper parabolic $P = U$ in the representation theoretic side of the Fourier summation formula of Proposition 7 is

$$\sum_\pi W_\pi(f),$$

where π ranges over all discrete series generic $\mathbb{U}$-modules, counted according to their multiplicities in $L_\omega(U\backslash\mathbb{U})$; these multiplicities are finite (e.g. since the representation theoretic side of the Fourier summation formula is absolutely convergent).

Let v be a place of F where ω and E/F are unramified, and let $f_v \in H(U_v)$ be a spherical function. For any irreducible admissible U_v-module π_v the convolution operator $\pi_v(f_v)$ is zero unless π_v is unramified, namely has a non-zero K_v-fixed vector, and then $\pi_v(f_v)$ acts as the scalar $f_v^\vee(t(\pi_v))$ on the K_v-fixed vector, and as zero on any vector in π_v orthogonal to the K_v-fixed one.

Let V be a finite set of places of F which contains the archimedean places and those where ω or E/F ramify. The Fourier summation formula will be used with $f = \otimes f_v$ in $C(\mathbb{U})$ such that f_v is spherical for all $v \notin V$. Denote by $t(\lambda, \rho_v)$ the class in $\hat{U}_v$ parametrizing the irreducible unramified subquotient of the induced U_v-module $I(\lambda, \rho_v)$. Put $f_V = \underset{v\in V}{\otimes} f_v$, $f^V = \underset{v\notin V}{\otimes} f_v$, $\pi_V = \underset{v\in V}{\otimes} \pi_v$, $\pi^V = \underset{v\notin V}{\otimes} \pi_v$, etc. Also write $f^\vee(t(\pi^V))$ for $\prod_{v\notin V} f_v^\vee(t(\pi_v))$, and $f^\vee(t(\lambda,\rho^V))$ for $\prod_{v\notin V} f_v^\vee(t(\lambda,\rho_v))$.

8. Corollary. *For $f = \otimes f_v$ spherical outside V we have*

$$\sum_w \sum_a \Psi(aw, f; \psi) = \sum_\pi f^\vee(t(\pi^V)) W_{\pi_V}(f_V)$$
$$+ \sum_{P\neq U} n(P)^{-1} \sum_\rho \int_{i\mathfrak{a}_P^*} f^\vee(t(\lambda,\rho^V)) \cdot \sum_\Phi E_\psi(I(f_V,\lambda,\rho_V)\Phi,\rho,\lambda)\overline{E}_\psi(\Phi,\rho,\lambda)d\lambda.$$

The sums over w and a are as in Proposition 7. The π and ρ range over the discrete series representations of $\mathbb{U}$ and $\mathbb{M}(\neq \mathbb{U})$ which are unramified outside V, and Φ over the vectors fixed by K_v for all $v \notin V$.

Proof. This follows from the discussion above once we note that Φ can be chosen to be a product $\Phi^V \otimes \Phi_V$, where $\Phi^V = \underset{v\notin V}{\otimes} \Phi_v^0$ and Φ_v^0 is a K_v-fixed vector in the space of π_v. Then $W_{\Phi,\psi} = W_{\Phi_V,\psi_V} \otimes (\underset{v\notin V}{\otimes} W_{\Phi_v^0,\psi_v})$, and for a suitable choice of measures $W_{\Phi_v^0,\psi_v}(I) = 1$, and so $W_{\pi_v}(f_v^0) = 1$. Hence $W_\pi(f)$ is the product of $f^\vee(t(\pi^V))$ and $W_{\pi^V}(f^V)$.

The Fourier summation formula for U and f will be compared with a Fourier summation formula for f' and G' which we proceed to describe. Since all terms are defined analogously to the case of U, we simply note that once again there are two expressions for the kernel $K_{f'}(x,y)$ of the convolution operator $r'(f')$ on $L_{\omega'\kappa'}(G'\backslash\mathbb{G}')$, the geometric one being

$$K_{f'}(x,y) = \sum_{\gamma\in Z'\backslash G'} f'(x^{-1}\gamma y).$$

The representation theoretic expression is

$$\sum_{P'\subset G'} n(P')^{-1}\sum_{\rho'}\int_{i\mathfrak{A}^*_{P'}}\sum_{\alpha,\beta}(I(f',\lambda,\rho')\Phi_\alpha,\Phi_\beta)E(x,\Phi_\beta,\rho',\lambda)\overline{E}(y,\Phi_\alpha,\rho',\lambda)d\lambda.$$

We will multiply these two expressions by $\psi'(x^{-1}) = \overline{\psi}'(x)$, and let x range over $N'\backslash\mathbb{N}'$, and y over $\mathbb{Z}G\backslash\mathbb{G}$. We would like to integrate both sides over x and y. But to do this we will need to discuss the convergence of the integral over y of $E(y,\Phi_\alpha,\rho',\lambda)$, and change the order of summation and integration (see the Remark following Proposition 9). To avoid that, we use the truncation operator Λ^T introduced in [A2], p. 89. Here T is a fixed, suitably regular point in $\mathfrak{A}_0^+$, where $\mathfrak{A}_0 = \mathfrak{A}_{P'_0}$, and $\underline{P}'_0$ is the upper triangular subgroup of $\underline{G}'$. The function $\Lambda_2^T K_{f'}(x,y)$, where the index 2 refers to the second variable, y, is rapidly decreasing in y by [A2], Lemma 1.4, p. 95. Hence the integral $\int_{N'\backslash\mathbb{N}'}\int_{\mathbb{Z}G\backslash\mathbb{G}}\Lambda_2^T K_{f'}(x,y)\overline{\psi}'(x)dx\,dy$ is absolutely convergent, and

$$\int\int \Lambda_2^T K_{f'}(x,y)\overline{\psi}'(x)dx\,dy \to \int\int K_{f'}(x,y)\overline{\psi}'(x)dx\,dy$$

as $T\to\infty$ (in the positive Weyl chamber). Applying then Λ_2^T to both expressions for the kernel, integrating over x and y, and taking the limit as $T\to\infty$, elementary considerations based on Proposition 4 imply

9. Proposition. *For every $f' = \otimes f'_v \in C(\mathbb{G}')$ we have that the sum of*

$$\Psi'(aw,f';\psi') = \prod_v \Psi'(aw,f'_v;\psi'_v)$$

over the w and a as in Proposition 7, is equal to

$$\sum_{P'} n(P')^{-1}\sum_{\rho'}\lim_{T\to\infty}\int_{i\mathfrak{A}^*_{P'}} d\lambda\cdot\sum_{\Phi} E_{\psi'}(I(f',\lambda,\rho')\Phi,\rho',\lambda)\cdot\int_{\mathbb{Z}G\backslash\mathbb{G}}\Lambda^T\overline{E}(y,\Phi,\rho',\lambda)dy.$$

Here

$$E_{\psi'}(\Phi,\rho',\lambda) = \int_{N'\backslash\mathbb{N}'} E(x,\Phi,\rho',\lambda)\overline{\psi}'(x)dx.$$

Remark. The integral $\int_{\mathbb{Z}G\backslash\mathbb{G}} E(x,\Phi,\rho',\lambda)dx$ converges for λ in $i\mathfrak{A}^*_{P'}$. Indeed, let $\underline{P}'_1$ be any standard parabolic subgroup associated to $\underline{P}'$, and $E_1(x,\Phi,\rho',\lambda)$ the constant term of $E(x,\Phi,\rho',\lambda)$ along $\mathbb{N}'_1$. Thus E_1 is the image of E under the (projection) operator π_1 which maps h on $N'\backslash\mathbb{G}'$ to $(\pi_1 h)(g) = \int_{N_1'\backslash\mathbb{N}_1'} h(ug)du$ on $N'\mathbb{N}_1'\backslash\mathbb{G}'$. Note that the operators π_1 – which are associated to different parabolic subgroups – commute. By the "principle of the constant term" of [L2], the difference between E and some linear combination of its constant terms E_1, namely $\prod_{\underline{P}_1'}(1-\pi_1)E$, is rapidly decreasing in x in a Siegel domain for $\mathbb{G}'$. Indeed, all constant terms of $\prod_{\underline{P}_1'}(1-\pi_1)E$ are zero, and a cusp form is rapidly decreasing. Hence it suffices to study the convergence of $\int_{\mathbb{Z}G\backslash\mathbb{G}} E_1(\dots)dx$.

The standard expression for the constant term (cf. [A2], p. 113, ℓ. -8) is recalled in (16.2) below. Each of the summands on the right of (16.2) is $\mathbb{N}_1'$-invariant. Each of the functions $(M(s,\rho',\lambda)\Phi)(x)$, $s \in W(\mathfrak{A},\mathfrak{A}_1)$, is integrable over the closed subset $\mathbb{A}_1 M_1\backslash\mathbb{M}_1$ when $\lambda \in i\,\mathfrak{A}^*$, where the operator $M(s,\rho',\lambda)$ is unitary. In fact the integral may be non-zero for some Φ only when ρ' is $\mathbb{M}$-distinguished. It then suffices to integrate (16.2) over a Siegel domain for $\mathbb{G}'$ modulo $\mathbb{P}_1'$, equivalently over the set of a in $\mathbb{Z}A_1\backslash\mathbb{A}_1$ with $\langle\mu_\alpha, H(a)\rangle \geq 0$ for all $\alpha \in \Delta_1 = \Delta_{P_1'}$. Here $\{\mu_\alpha;\alpha\in\Delta_1\}$ is the basis of $\mathfrak{A}^*_{P_1'}$ dual to the basis $\{\alpha^\vee;\alpha\in\Delta_1\}$ of $\mathfrak{A}_{P_1'}$. Namely we need to integrate

$$e^{\langle s\lambda+\rho_{P_1'},H(a)\rangle}\delta_{P_1'}^{-1}(a) = e^{\langle s\lambda-\rho_{P_1'},H(a)\rangle}$$

over the set of $H = H(a) = \sum_{\alpha\in\Delta_1} h_\alpha\alpha^\vee$, $h_\alpha \geq 0$, for each $s \in W(\mathfrak{A},\mathfrak{A}_1)$.

The integral is finite if $\langle sRe\lambda,\alpha^\vee\rangle < \langle\rho_{P_1'},\alpha^\vee\rangle$ for all $\alpha\in\Delta_1$. If $Re\,\lambda = 0$ then the last inequality is satisfied with any $\underline{P}_1'$ and $s \in W(\mathfrak{A},\mathfrak{A}_1)$, and the desired convergence follows. Consequently we also have $\lim_{T\to\infty}\int_{\mathbb{Z}G\backslash\mathbb{G}}\Lambda^T E(y)dy = \int_{\mathbb{Z}G\backslash\mathbb{G}} E(y)dy$. However we prefer to work with the integral of the truncated Eisenstein series in order to change the order of integration and the summation which defines the Eisenstein series. This last sum converges absolutely only for λ with $Re\,\lambda \in \rho_{P'} + (\mathfrak{A}^*_{P'})^+$, but there our integral $\int_{\mathbb{Z}G\backslash\mathbb{G}} E(x)dx$ does *not* converge.

We first make the following observation.

10. Lemma. *The discrete series $\mathbb{M}'$-module ρ' contributes a zero term to the identity of Proposition 9 unless ρ' is cuspidal.*

Proof. The Fourier coefficient $E_{\psi'}(\Phi,\rho',\lambda)$ is the value at I of the Whittaker function

$$W_{\Phi,\rho',\lambda}(g) = \int_{N'\backslash\mathbb{N}'} E(xg,\Phi,\rho',\lambda)\overline{\psi}'(x)dx,$$

attached to the automorphic form $E(g,\Phi,\rho',\lambda)$, which generates the space of the induced $\mathbb{G}'$-module $I(\lambda,\rho')$. The Whittaker function is identically zero unless $I(\lambda,\rho')$ is generic, and $I(\lambda,\rho')$ is generic precisely when ρ' is generic. The claim follows from the fact (see Moeglin-Waldspurger [MW1]) that a discrete series $\mathbb{G}'$-module is generic only when it is cuspidal.

Our approach will be to compute $\int\Lambda^T E(y)dy$, at least partially. We begin with recalling the explicit expression for $\Lambda^T E$ given in [A2], Lemma 4.1, p. 114.

Let P_2' be a standard parabolic subgroup of G', $\Delta_2 = \Delta_{P_2'}$, and $\lambda \in \mathfrak{A}_0^*$. Let $\epsilon_2(\lambda)$ be $(-1)^a$, where a is the cardinality of the set of $\alpha\in\Delta_2$ with $\langle\lambda,\alpha^\vee\rangle \leq 0$.

Let $\phi_2(\lambda, H)$ be the characteristic function of the $H \in \mathfrak{A}_0$ such that for any $\alpha \in \Delta_2$, if $\langle \lambda, \alpha^\vee \rangle \leq 0$ then $\langle \mu_\alpha, H \rangle > 0$, and if $\langle \lambda, \alpha^\vee \rangle > 0$ then $\langle \mu_\alpha, H \rangle \leq 0$; here $\{\mu_\alpha; \alpha \in \Delta_2\}$ is the basis of $\mathfrak{A}^*_{P'_2}$ dual to the basis $\{\alpha^\vee; \alpha \in \Delta_2\}$ of $\mathfrak{A}_{P'_2}$.

11. Lemma. *([A2], Lemma 4.1, p. 114). For P', $\rho' \in \prod(\mathbb{M}')$, $\Phi \in H^0_{P'}(\rho')$ and $\lambda \in \mathfrak{A}^*_{\mathbb{C}}$ with real part $Re\lambda$ in $\rho_{P'} + (\mathfrak{A}^*_{P'})^+$, we have that $\Lambda^T E(x, \Phi, \rho', \lambda) = \sum_{P'_2} \sum_{\delta \in P'_2 \backslash G'} \psi_2(\delta x)$, where*

$$\psi_2(x) = \sum_{s \in W(\mathfrak{A}, \mathfrak{A}_2)} \epsilon_2(s\, Re\, \lambda) \phi_2(s\, Re\, \lambda, H_0(x) - T) e^{\langle s\lambda + \rho_2, H_0(x) \rangle} (M(s, \rho', \lambda)\Phi)(x)$$

and the sum over δ converges absolutely.

We shall integrate $\Lambda^T E$ over $\mathbb{Z}G \backslash \mathbb{G}$. Note that $\Lambda^T E(x, \Phi, \rho', \lambda)$ is a rapidly decreasing function of x ([A2], p. 108, ℓ. 5-8), hence the integral converges. To analyze this integral, note that by Proposition 3 the quotient $P'_2 \backslash G'$ is the disjoint union of $\eta_w^{-1} P'_2 \eta_w \cap G \backslash G$ over the $w \in W_{M'_2} \backslash W$ of order 2. Each of the coset spaces is isomorphic to $P_{2w} \backslash G_w$, where $G_w = \eta_w G \eta_w^{-1}$ and $P_{2w} = G_w \cap P'_2$. Note that $G_w = \{g \in G'; \overline{g} = wgw\}$, and $P_{2w} = \{p \in P'_2; \overline{p} = wpw\}$. Since the sum over P'_2 and $\delta \in P'_2 \backslash G'$ in Lemma 11 is absolutely convergent, we conclude that $\int_{\mathbb{Z}G \backslash \mathbb{G}} \Lambda^T E(x, \Phi, \rho', \lambda) dx$ is the sum over $P'_2 \subset G'$ and w of order ≤ 2 in $W_{M'_2} \backslash W$ of

$$(11.1) \int \sum \epsilon_2(s\, Re\lambda) \phi_2(s\, Re\, \lambda, H_0(x) - T) e^{\langle s\lambda + \rho_2, H_0(x) \rangle} (M(s\rho', \lambda)\Phi)(x\eta_w) dx;$$

(integral over $\mathbb{Z}P_{2w} \backslash \mathbb{G}_w$, sum over $s \in W(\mathfrak{A}, \mathfrak{A}_2)$).

12. Lemma. *The integral (11.1) vanishes unless $wM'_2 w = M'_2$.*

Proof. Consider $P'_{2w} = P'_2 \cap wP'_2 w$, and its subgroup $P_{2w} = \{p \in P'_{2w}; \overline{p} = wpw\}$. Note that (i, j) are the (row, column) coordinates of a non-zero entry of $\tilde{M}'_2 \cap \tilde{P}'_{2w}$ (we denote here by a tilde the Lie algebra of a group) $= \tilde{M}'_2 \cap w\tilde{P}'_2 w$, unless (wi, wj) are the coordinates of a non-zero entry in ${}^t\tilde{N}'_2$, namely (wj, wi) are the coordinates of a non-zero entry of $\tilde{N}'_2$. Denote by a_{ij} the matrix in $\tilde{G}'$ whose only non-zero entry is a at (i, j). Namely: a_{ij} in $\tilde{M}'_2 \cap w\tilde{P}'_2 w$ must have $a = 0$ if and only if $a_{wj,wi}$ lies in $\tilde{N}'_2$ for all a; in other words: w takes $a_{ji} \neq 0$ in $\tilde{M}'_2 \cap w\tilde{P}'_2 w$ to $\tilde{N}'_2$ if a_{ji} lies in the unipotent radical of the parabolic subalgebra $\tilde{M}'_2 \cap w\tilde{P}'_2 w$ of $\tilde{M}'_2$ (in which case a_{ij} is identically zero). Thus up to conjugation by M'_2, if the Levi M'_2 consists of blocks of size k_i along the diagonal, then the parabolic subgroup $M'_2 \cap P'_{2w}$ is of type $(\dots, \ell_1(i), \dots, \ell_{t_i}(i), \dots)$ with $\sum_{1 \leq j \leq t_i} \ell_j(i) = k_i$, and its unipotent entries (those above the blocks of size $\ell_j(i)$ along the diagonal) are mapped by w to unipotent entries in the unipotent radical N'_2 of P'_2.

Consequently $N'_2 \backslash P_{2w} N'_2$ is a product LU' of a subgroup L of the Levi of type $(\ell_j(i))$ and the unipotent radical U' of the parabolic subgroup $M'_2 \cap P'_{2w}$ of M'_2 (i.e. of type $(\ell_j(i))$ in (k_i)). We conclude that unless $M'_2 \cap P'_{2w} = M'_2 \cap wP'_2 w$ is M'_2, namely $wM'_2 w = M'_2$, the integral (11.1) factorizes through the integral

$$\int_{U' \backslash \mathbb{U}'} (M(s, \rho', \lambda)\Phi)(umk) du,$$

which is zero by the cuspidality of $M(s,\rho',\lambda)\Phi$. The lemma follows.

Thus we need to consider only w with $wM_2'w = M_2'$. For such w we have $P_{2w} = M_{2w}N_{2w}$, where $M_{2w} = G_w \cap M_2'$, $N_{2w} = G_w \cap N_2'$, and we put $A_{2w} = G_w \cap A_2'$. The integral (11.1) can be written as

$$\int_{\mathbb{P}_{2w}\backslash\mathbb{G}_w} \sum_{s\in W(\mathfrak{A},\mathfrak{A}_2)} \epsilon_2(s\,Re\,\lambda)\cdot(12.1)\cdot(12.2)dk,$$

where

$$(12.1) \qquad \int\limits_{\mathbb{Z}\,A_{2w}\backslash\mathbb{A}_{2w}} \phi_2(s\,Re\,\lambda, H_0(a)-T)e^{\langle s\lambda+\rho_2,H_0(a)\rangle}\delta_{P_{2w}}(a)^{-1}da,$$

and

$$(12.2) \qquad \int_{\mathbb{A}_{2w}\,M_{2w}\backslash\mathbb{M}_{2w}} (M(s,\rho',\lambda)\Phi)(mk\eta_w)dm;$$

(12.1) depends on w, s, λ, and (12.2) also on Φ and k. To simplify (12.1) we prove

13. Lemma. *For w with $wM_2'w = M_2'$ and a in $\mathbb{A}_{2w}$ we have $\delta_{P_{2w}}(a) = e^{\langle\rho_2,H_0(a)\rangle}$.*

Proof. The element w of W is of order 2 and is taken to be of shortest length modulo $W_{M_2'}$, thus it acts as I on the square blocks. We need to compare

$$e^{\langle\rho_2,H_0(a)\rangle} = |\det(Ad(a)|\tilde{N}_2')|^{1/2}$$

and

$$\delta_{P_{2w}}(a) = |\det(Ad(a)|\tilde{N}_{2w})|, \qquad N_{2w} = \{u\in N_2'\cap wN_2'w; wuw = \overline{u}\},$$

for a in $\mathbb{A}_{2w} = \{a\in\mathbb{A}_2'; a = w\overline{a}w\}$. It suffices to deal with a of the following special form. If (ij) is a non-trivial transposition occurring in w, thus $wi = j \neq i$, consider a whose diagonal entries are 1 except at the ith place, where it is $\alpha\in\mathbb{A}_E^\times$, and the jth, where it is $\overline{\alpha}$. If $i(1\le i\le n)$ is fixed by w, consider a whose diagonal entries are 1 except at the ith place where the entry is $\alpha\in\mathbb{A}^\times$.

In the first case where $wi = j\neq i$, the entry (i,i) lies in a block of size $k\times k$ of the Levi M_2', and this block is mapped by w to a $k\times k$ block containing the entry (j,j). Put $U' = M_2'\cap N_0'$, where N_0' is the unipotent upper triangular subgroup, and $U = \{u\in U'; u = w\overline{u}w\}$. Since $|\det(Ad(a)|\tilde{U}')| = |\det(Ad(a)|\tilde{U})|^2$, we may assume that $k=1$. Thus we may assume that $N_2' = N_0'$, both denoted now by N', and in computing $\det(Ad(a)|\tilde{N}')$ we need consider only the ith and jth rows and columns. We obtain, in absolute value,

$$|\alpha^{n-i}\overline{\alpha}^{n-j}\alpha^{-(i-1)}\overline{\alpha}^{-(j-1)}|_E = |\alpha|_E^{2n+2-2i-2j}.$$

We also have to consider the action of $Ad(a)$ on $\tilde{N}$, where $N = \{u = w\overline{u}w\in N'\}$. The action of $Ad(a)$ multiplies each entry above the diagonal on the ith row by α, on the jth row by $\overline{\alpha}$, on the ith column by α^{-1}, on the jth column by $\overline{\alpha}^{-1}$. Note that:

(1) The (i,k), $i<k$, entry on the ith row of N is necessarily zero precisely when $w(i,k)=(j,\ell)$ satisfies $\ell<j$. But then the (ℓ,j), $\ell<j$, entry on the jth column of N is necessarily zero, since $w(\ell,j)=(k,i)$ and $k>i$.

(2) The (j,k) entry on the jth row of N, with $j<k$, is identically zero if $w(j,k)=(i,\ell)$ and $\ell<i$. But then the (ℓ,i) entry on the ith column of N has $\ell<i$, and it is identically zero as $w(\ell,i)=(k,j)$ and $k>j$.

In other words, the number of spots on the ith and jth rows where α (or $\overline{\alpha}$) is *not* contributed to $\det(Ad(a)|\tilde{N})$ is equal to the number of places on the ith and jth columns where α^{-1} (or $\overline{\alpha}^{-1}$) is not counted.

(3) For every non-zero entry x at (i,k), $i<k$, on the ith row of N, there is an entry $\overline{x}$ at (j,ℓ) on the jth row with $\ell=w(k)>j$.

(4) For every non-zero entry x at (k,i), $k<i$, on the ith column of N, there is an entry $\overline{x}$ at (ℓ,j) on the jth column of N with $\ell=w(k)<j$.

In other words, the number of times α (or $\overline{\alpha}$, or α^{-1} or $\overline{\alpha}^{-1}$) is counted into $\det(Ad(a)|\tilde{N})$ is precisely half the times it is counted into $\det(Ad(a)|\tilde{N}')$. Hence

$$|\det(Ad(a)|\tilde{N})|=|det(Ad(a)|\tilde{N}')|^{1/2},$$

as required.

In the second case $wi=i$, and the only diagonal entry of a which may be not 1 is the ith, and it is $\alpha\in\mathbb{A}^{\times}$. We need consider the action of $Ad(a)$ only on the ith row and column of N_2' and N_2. Note that the diagonal block B' in M_2' containing the entry (i,i) is not moved by w, hence the action of $Ad(a)$ on $B'\cap N_0'$ has Jacobian $|x|_E$ for some $x\in\mathbb{A}^{\times}$, while its action on $B'\cap N=\{b'\in B'\cap N';b'=\overline{b}'\}$ has the Jacobian $|x|_F$, with the same x. Consequently it suffices to consider the action of $Ad(a)$ on $\tilde{N}'=\tilde{N}_0'$ and $\tilde{N}=\tilde{N}_0$.

On $\tilde{N}'$ the Jacobian is $|\alpha^{n-i}\alpha^{-(i-1)}|_E=|\alpha|_F^{2n+2-4i}$. We have to consider the action of $Ad(a)$ on $\tilde{N}$, where $N=\{u=w\overline{u}w\in N'\}$. This $Ad(a)$ leaves its mark only on the ith row and the ith column, by multiplying the ith row by α, and the ith column by α^{-1}.

We claim that there is a bijection between the entries of $\tilde{N}$ on the ith row indexed by $(i,k), i<k$, which are zero, and the entries of $\tilde{N}$ on the ith column indexed by (ℓ,i), $\ell<i$, which are zero. Indeed, the entry (i,k), $i<k$, is identically zero precisely when it is mapped by w to (i,ℓ) (i.e. $wk=\ell$) with $\ell<i$. But then the entry of $\tilde{N}$ at (ℓ,i), $\ell<i$, is necessarily zero, since it is mapped by w to (k,i), and $k>i$. The claim follows.

Further, if the entry of $\tilde{N}$ at (i,k), $i<k$, is x, and $wk=\ell>i$, then the entry of $\tilde{N}$ at (i,ℓ) is $\overline{x}$. The contribution to the Jacobian corresponding to (i,k) and (i,ℓ) is then $|\alpha|_E$ on $\tilde{N}$ and $|\alpha|_E^2$ on $\tilde{N}'$. Alternatively, if $wk=k$, the entry of $\tilde{N}$ at (i,k), $i<k$, is $x=\overline{x}$ in F, hence the contribution of (i,k) to the Jacobian on $\tilde{N}$ is $|\alpha|_F$, while on $\tilde{N}'$ it is $|\alpha|_E=|\alpha|_F^2$. With this the proof of the lemma is complete.

The integral (12.1) can now be written in the form

$$\int_{\mathbb{Z}A_{2w}\backslash\mathbb{A}_{2w}}\phi_2(s\,Re\,\lambda,H_0(a)-T)e^{\langle s\lambda,H_0(a)\rangle}da. \tag{13.1}$$

Note that $a \mapsto H_2(ax)$ (any $x \in \mathbb{G}'$) is a measure preserving isomorphism $H_2 : A_2'\mathbb{Z}\backslash\mathbb{A}_2' \to \mathfrak{A}_2$.

Definition. Denote by $\mathfrak{A}_{2w}$ the image of $A_{2w}\mathbb{Z}\backslash\mathbb{A}_{2w}$ under H_2; it is a subspace of $\mathfrak{A}_2$, whose dimension we denote by $d(w)(d/2 \leq d(w) \leq d = \dim_{\mathbb{R}}\mathfrak{A}_2)$. Note that $d(w) = d$ when $w = I$.

We then rewrite the product of $\epsilon_2(s\,Re\,\lambda)$ and (13.1) as

$$\epsilon_2(s\,Re\,\lambda)\int_{\mathfrak{A}_{2w}} \phi_2(s\,Re\,\lambda, H-T)e^{\langle\lambda,H\rangle}dH; \tag{13.2}$$

up to a Jacobian factor this is

$$\epsilon_2(s\,Re\,\lambda)\int_{\mathfrak{A}_2} \phi_2(s\,Re\,\lambda, H+wH-T)e^{\langle s\lambda,H+wH-T\rangle}dH.$$

It converges for $\lambda \in \mathfrak{A}_{\mathbb{C}}^*$ with $Re\,\lambda \in (\mathfrak{A}_P^*)^+$ since so does

$$\epsilon_2(s\,Re\,\lambda)\int_{\mathfrak{A}_2} \phi_2(s\,Re\,\lambda, H-T)e^{\langle s\lambda,H\rangle}dH.$$

Indeed, write $H = \sum_{\alpha\in\Delta_2} x_\alpha\alpha^\vee(x_\alpha \in \mathbb{R})$. Under this change of variables we need to multiply by the Jacobian, which is the volume $|\mathfrak{A}_2/L_2|$ of $\mathfrak{A}_2$ modulo the lattice L_2 spanned by $\{\alpha^\vee; \alpha \in \Delta_2\}$. The integral becomes a product of integrals of decreasing functions over half lines. It is easily evaluated to be the product of $|\mathfrak{A}_2/L_2|$ and

$$\prod_{\alpha\in\Delta_2}\frac{e^{\lambda_\alpha t_\alpha}}{\lambda_\alpha} = \frac{e^{\langle s\lambda,T\rangle}}{\prod_{\alpha\in\Delta_2}\langle s\lambda,\alpha^\vee\rangle}\quad,$$

where $s\lambda = \sum_{\alpha\in\Delta_2}\lambda_\alpha\mu_\alpha \in \mathfrak{A}_{2,\mathbb{C}}^*$, and $t_\alpha = \langle\mu_\alpha, T\rangle$, $\alpha \in \Delta_2$.

To clarify the analytic nature of the integral (13.2), we prove

14. Lemma. *The integral (13.2) is a finite linear combination of terms of the form*

$$\prod_{1\leq i\leq d(w)} h_{iw}(s\underline{\lambda})^{-1}e^{h_{iw}(s\underline{\lambda})\ell_{iw}(\underline{t})},$$

where the $h_{iw}(\underline{\lambda})$ are $d(w)$ homogeneous linearly independent linear forms in $\underline{\lambda} = (\lambda_\alpha; \alpha \in \Delta_2)$, and the $\ell_{iw}(\underline{t})$ are $d(w)$ linearly independent homogeneous forms in $\underline{t} = (t_\alpha; \alpha \in \Delta_2)$.

Proof. Since $H = \sum_{\alpha\in\Delta_2} x_\alpha\alpha^\vee$ ranges over a $d(w)$-dimensional space $\mathfrak{A}_{2w}$, renaming the x_α's as x_i's we may assume that $x_1, \ldots, x_{d(w)}$ are linearly independent, and $x_{d(w)+1}, \ldots, x_d$ are linearly dependent on them: $x_i = a_i(\underline{x})$ $(d(w) < i \leq d)$, where $a_i(\underline{x}) = a_{iw}(\underline{x})$ are linear homogeneous forms in $\underline{x} = (x_1, \ldots, x_{d(w)})$. Put $t_i = t_{\alpha_i}$ where α_i is the α with $x_\alpha = x_i$. There are homogeneous linear forms $b_i(s\underline{\lambda}) =$

$b_{iw}(s\underline{\lambda})(1 \leq i \leq d(w))$ in $s\underline{\lambda} = (\lambda_\alpha; \alpha \in \Delta_2)$ such that (13.2) is equal, up to a sign, to an integral of

$$\prod_{1 \leq i \leq d(w)} e^{b_i(s\underline{\lambda})x_i} \prod_{1 \leq i \leq d(w)} dx_i.$$

The integration is taken over the subset of the $d(w)$-dimensional Euclidean space in $(x_1, \ldots, x_{d(w)})$ which is bounded from above or from below by the hyperplanes $x_i = t_i (1 \leq i \leq d(w))$, and also by the $d - d(w)$ hyperplanes $a_i(\underline{x}) = t_i$ $(d(w) < i \leq d)$. We may cut this subset into finitely many subsets defined by

$$c_i(x_1, \ldots, x_{i-1}, t_1, \ldots, t_d) < x_i < d_i(x_1, \ldots, x_{i-1}, t_1, \ldots, t_d) \qquad (1 \leq i \leq d(w)),$$

where c_i and d_i are homogeneous linear forms, $d_i \leq t_i$ or $t_i \leq c_i$ for all i, and $-c_i$ or d_i may be identically ∞, as suitable.

It is clear that any chain $(e_1, \ldots, e_{d(w)})$ of linear forms $e_i = c_i(0, \ldots, 0, \underline{t})$ or $d_i(0, \ldots, 0, \underline{t})$ in $\underline{t}$, with e_i discarded if e_i is ∞ or $-\infty$, makes a set of linearly independent forms. For real numbers $-\infty \leq b \leq a < \infty$ and complex μ with $Re\, \mu > 0$ we have

$$\int_{b<x<a} e^{\mu x} dx = \frac{e^{\mu a} - e^{\mu b}}{\mu}.$$

Hence integrating the x_i's we obtain a finite linear combination of terms as described by the lemma, whose proof is now complete.

Let $\rho' = \rho'_1 \times \cdots \times \rho'_a$ be a cuspidal $\mathbb{M}'_2$-module; here $\underline{M}_2$ is a product of $GL(n_i)$ with $\sum_{i=1}^{a} n_i = n$, and ρ'_i is a cuspidal $GL(n_i, \mathbb{A}_E)$-module $(1 \leq i \leq a)$. Let w be an element of the Weyl group W with $w^2 = 1$ and $wM'_2w = M'_2$. This w is chosen modulo $W_{M'_2}$, and we fix a representative of shortest length. For the following discussion it will be convenient to introduce

Definition. The cuspidal $\mathbb{M}'_2$-module ρ' is $\mathbb{M}_{2w}$-*distinguished* if the functional

$$D_{M_{2w}}(\Phi) = \int_{\mathbb{A}_{2w}\, M_{2w} \backslash \mathbb{M}_{2w}} \Phi(m) dm$$

is not identically zero on the space of ρ'.

Such ρ' is abstractly $\mathbb{M}_{2w}$-distinguished, namely there is a non-zero $\mathbb{M}_{2w}$-invariant form on its space. Each of its local components ρ'_v is M_{2w}-distinguished.

15. Lemma. *If $\int_{\mathbb{Z}G \backslash \mathbb{G}} \Lambda^T E(x, \Phi, \rho', \lambda) dx$ is not identically zero, then the cuspidal $\mathbb{M}'$-module $\rho' = \rho'_1 \times \cdots \times \rho'_a$ has the property that for each i $(1 \leq i \leq a)$, ρ'_i is $GL(n_i, \mathbb{A})$-distinguished or there is $j \neq i$ with $\rho'_j \simeq \check{\overline{\rho}}'_i$ (these two possibilities are not mutually exclusive, since a distinguished ρ'_i may occur more than once).*

Proof. The assumption implies that the integral (12.2) is not identically zero. Hence $s\rho'$ is $\mathbb{M}_{2w}$-distinguished, since the function $m \mapsto (M(s, \rho', \lambda)\Phi)(mk\eta_w)$ is a cusp form in the space of $s\rho'$. The element $w' = s^{-1}ws$ of W is of order ≤ 2 with $w'M'w' = M'$, and we conclude that ρ' is $\mathbb{M}_{w'}$-distinguished.

For such ρ', if the Weyl group element w' interchanges the ith block with the jth, and $j \neq i$, then $\rho'_j \simeq \check{\bar{\rho}}'_i$, where as usual check indicates contragredient and bar the action of $\mathrm{Gal}(E/F)$ on the entries of the matrix. If w' fixes the ith block then ρ'_i is $GL(n_i, \mathbb{A})$-distinguished, hence $\rho'_i \simeq \check{\bar{\rho}}'_i$ by [F1], Proposition 12. The lemma follows.

Let us summarize what we have at this stage.

Corollary. *The integral $\int_{\mathbb{Z}G\backslash\mathbb{G}} \Lambda^T E(x, \Phi, \rho', \lambda)dx$ is zero unless ρ' is as in Lemma 15, and then it is the sum over $P'_2 \subset G'$ and w in W with $w^2 = 1$ and $wM'_2w = M'_2$, taken modulo $W_{M'_2}$, of the integrals*

$$I(w, \Phi, \rho', \lambda, T) = \int_{\mathbb{P}_{2w}\backslash\mathbb{G}_w} \sum_{s\in W(\mathfrak{A},\mathfrak{A}_2)} (13.2) \cdot (12.2)dk.$$

When $w = 1$ we have $\eta_w = 1$, and $\mathbb{P}_{2w}\backslash\mathbb{G}_w = \mathbb{K} \cap \mathbb{P}_2\backslash\mathbb{K}$ is compact. The sum over $P'_2 \subset G'$ of the terms corresponding to $w = 1$ can be expressed as

$$(15.1)\sum_{P'_2\subset G'} \sum_{s\in W(\mathfrak{A},\mathfrak{A}_2)} \frac{e^{\langle s\lambda,T\rangle}}{\prod_{\alpha\in\Delta_2} \langle s\lambda, \alpha^\vee\rangle} \int_{\mathbb{K}\cap\mathbb{P}_2\backslash\mathbb{K}} \int_{\mathbb{A}_2 M_2\backslash\mathbb{M}_2} (M(s, \rho', \lambda)\Phi)(mk)dm\, dk.$$

The coset space $\mathbb{P}_{2w}\backslash\mathbb{G}_w$ is not compact when $w \neq 1$. It is a subset of the compact coset space $\mathbb{P}'_2\backslash\mathbb{G}'$ (see Proposition 3). Since (13.2) is independent of $k \in \mathbb{P}_{2w}\backslash\mathbb{G}_w$, one would expect the integral of (12.2) over $\mathbb{P}_{2w}\backslash\mathbb{G}_w$ to converge for λ with a sufficiently large $Re(\lambda)$. In the case of $\underline{G} = GL(2)$ and $\underline{P} \neq \underline{G}$, and $w \neq 1$, $\mathfrak{A}_{P'}$ is one dimensional and the convergence is shown for λ with $Re\,\lambda \geq \frac{1}{2}$ in [F1], proof of Proposition 9 (the local argument there is easy to globalize). When $n > 2$ we do not have a proof of the convergence of $\int_{\mathbb{P}_{2w}\backslash\mathbb{G}_w}$ (12.2) for a large $Re(\lambda)$. Instead, we argue as follows.

The terms (13.2) are linear combinations of exponential functions in λ and T, as described by Lemma 14. They are independent of the integration variable $k \in \mathbb{P}_{2w}\backslash\mathbb{G}_w$. The linear independence of the exponentials implies that each $I(w, \Phi, \rho', \lambda, T)$, for $w \neq 1$, is a sum of products of an exponential function in λ and T as displayed in Lemma 14, and a function in λ, which depends also on w, Φ, ρ', which is meromorphic in $\lambda \in \mathfrak{A}^*_{P',\mathbb{C}}$. Namely, for $j = 1, \ldots, J(w)$, there are linearly independent homogeneous linear forms $\ell_{ijw}(\underline{t})$ $(1 \leq i \leq d(w))$ in $\underline{t} = (t_\alpha)$, and linearly independent homogeneous linear forms $h_{ijw}(\underline{\lambda})$ in $\underline{\lambda} = (\langle\lambda, \alpha^\vee\rangle)$, and meromorphic functions $F_j(\lambda, w, \Phi, \rho')$ in $\lambda \in \mathfrak{A}^*_{P',\mathbb{C}}$, holomorphic in λ with λ sufficiently large (in the positive Weyl chamber), such that

$$I(w, \Phi, \rho', \lambda, T) = \sum_{1\leq j\leq J(w)} F_j(\lambda, w, \Phi, \rho') \cdot \prod_{1\leq i\leq d(w)} h_{ijw}(\underline{\lambda})^{-1} e^{h_{ijw}(\underline{\lambda})\ell_{ijw}(\underline{t})}.$$

The sum over $w \neq 1$ of the $I(w, \Phi, \rho', \lambda, T)$ can be written in the form

$$I(\lambda, \Phi, \rho', T) = \sum_{1\leq j\leq J} F_j(\lambda, \Phi, \rho') \cdot \prod_{1\leq i\leq d(j)} h_{ij}(\underline{\lambda})^{-1} e^{h_{ij}(\underline{\lambda})\ell_{ij}(\underline{t})},$$

where $d(j) < d$, and the F, h and ℓ have the properties described above.

The function $E(x, \Phi, \rho', \lambda)$ is holomorphic in λ on $i\mathfrak{A}^*_{P'}$, and so is $\Lambda^T E(x, \Phi, \rho', \lambda)$. This truncated Eisenstein series is rapidly decreasing as a function of x ([A2], p. 108, ℓ. 5-8). Hence the integral $\int_{\mathbb{Z}G\backslash\mathbb{G}} \Lambda^T E(x, \Phi, \rho', \lambda)dx$ is holomorphic in λ on $i\mathfrak{A}^*_{P'}$. In other words, the sum of (15.1) $= I(1, \Phi, \rho', \lambda, T)$ and $I(\lambda, \Phi, \rho', T)$ is holomorphic in λ on $i\mathfrak{A}^*_{P'}$. The argument to be employed below requires us to identify the possible singularities on $i\mathfrak{A}^*_{P'}$ of the meromorphic functions $F_j(\lambda, \Phi, \rho')$. These are described by

16. Proposition. *For each $\rho', \Phi, \psi', f', j$, the product of $F_j(\lambda, \Phi, \rho')$ and $E_{\psi'}(I(f', \lambda, \rho')\Phi, \rho', \lambda)$ is holomorphic in λ on $i\mathfrak{A}^*_{P'}$.*

Proof. Since the integrals in (15.1) converge and represent holomorphic functions in λ (always in $i\,\mathfrak{A}^*_{P'}$), the only possible singularities of (15.1) are on the hyperplanes $\{\lambda; \langle s\lambda, \alpha^\vee\rangle = 0\}$ (for some $s \in W(\mathfrak{A}, \mathfrak{A}_2)$ and α in Δ_2). To identify some of these hyperplanes which do *not* contribute singularities, we argue analogously to [A2], p. 117.

Given α in Δ_2, there is a parabolic subgroup $P'_3 \subset G'$, and a simple reflection $s_\alpha \in W(\mathfrak{A}_2, \mathfrak{A}_3)$ "belonging" to α (see [A2], p. 117, ℓ. 9, and the definition in [L2], p. 35, ℓ. 6, and the preceding pages 33–34). As in [L2], denote the elements of Δ_2 by α_i, and those of Δ_3 by β_i. Given $\alpha_\ell \in \Delta_2$, the reflection s_ℓ belonging to α_ℓ has the property that $\beta_\ell = -s_\ell\alpha_\ell$ lies in Δ_3, and $s_\ell\alpha_i = \beta_i + b_{i\ell}\beta_\ell$ for some $b_{i\ell} \geq 0$ $(i \neq \ell)$; see [L2], p. 34, ℓ. 6. In particular,

$$\langle s_\ell s\lambda, \beta_\ell^\vee\rangle = \langle s\lambda, s_\ell\beta_\ell^\vee\rangle = -\langle s\lambda, \alpha_\ell^\vee\rangle.$$

Moreover, on $\{\lambda; \langle s\lambda, \alpha_\ell^\vee\rangle = 0\} = \{\lambda; \langle s_\ell s\lambda, \beta_\ell^\vee\rangle = 0\}$, we have

$$\prod_{i\neq\ell}\langle s_\ell s\lambda, \beta_i^\vee\rangle = \prod_{i\neq\ell}\langle s\lambda, \alpha_i^\vee + b_{i\ell}\alpha_\ell^\vee\rangle = \prod_{i\neq\ell}\langle s\lambda, \alpha_i^\vee\rangle$$

and $\langle s_\alpha s\lambda, T\rangle = \langle s\lambda, T\rangle$ for any T in $\mathfrak{A}_{P'}$. Consequently the summands of (15.1) which are singular along a given hyperplane occur naturally in pairs, indexed by (P'_2, s) and $(P'_3, s_\alpha s)$, where from now on we write α for α_ℓ, s_α for s_ℓ, β for α_i $(i \neq \ell)$, and α_3 for β_ℓ.

We will show that the residues are equal, for some ρ', using in particular the functional equation for the intertwining operators (e.g., [A1], p. 927, (iii), ℓ. 10), which asserts

$$M(s_\alpha s, \rho', \lambda) = M(s_\alpha, s\rho', s\lambda)M(s, \rho', \lambda).$$

Put $\Phi' = M(s, \rho', \lambda)\Phi$, and denote the integral which appears in (15.1) by

$$D_2(\Phi') = \int\limits_{M_2\mathbb{A}_2\mathbb{N}_2\backslash\mathbb{G}} \Phi'(g)dy.$$

Write $s\rho'$ as a product $\rho'_1 \times \cdots \times \rho'_a$ of $GL(n_i, \mathbb{A}_E)$-modules $(1 \leq i \leq a)$. The element $s_\alpha \in W(\mathfrak{A}_2, \mathfrak{A}_3)$ interchanges ρ'_i and ρ'_j for some pair $i \neq j$ of indices.

16.1 Lemma. *If* $\langle s\lambda, \alpha^\vee\rangle = 0$*, then*

$$D_{3,\alpha}(\Phi') = \int\limits_{M_3\mathbb{A}_3\mathbb{N}_3\backslash\mathbb{G}} (M(s_\alpha, s\rho', s\lambda)\Phi')(g)dg$$

is equal to $\epsilon(s_\alpha, s\rho')D_2(\Phi')$*, where* $\epsilon(s_\alpha, s\rho')$ *is* 1 *if* $\rho'_i \not\simeq \rho'_j$*, and* -1 *if* $\rho'_i \simeq \rho'_j$*.*

We delay the proof of this Lemma to the proof of Proposition 19, where the same result is needed in the study of the discrete terms in the Fourier summation formula. Note that the intertwining operator $M(s_\alpha)$ depends only on the two components ρ'_i and ρ'_j which are interchanged, and the corresponding components $\lambda_i\mu_i$ and $\lambda_j\mu_j$ in $s\lambda = \sum_{\beta\in\Delta_2} \lambda_\beta\mu_\beta$, which are affected by the action of the Weyl group element s_α. On the hyperplane $\langle s\lambda, \alpha^\vee\rangle = 0$ we have $\lambda_i = \lambda_j$. Then Lemma 22 below asserts that $D_2(\Phi') = c\Psi(W_{\Phi'})$, and $D_{3,\alpha}(\Phi') = c\Psi(W_{M(s_\alpha,s\rho',s\lambda)\Phi'})$ for some complex number $c \neq 0$ and functional Ψ on the space of Whittaker functions of the $\mathbb{G}'$-modules $I(s\rho')$ or $I(s_\alpha s\rho')$. Corollary 25.1 below asserts that the function $W_{M(s_\alpha,s\rho',s\lambda)\Phi'}$ is the product of $W_{\Phi'}$ with $\epsilon(s_\alpha, s\rho')$, which establishes the Lemma here.

It follows that the two terms associated with the singular hyperplane $\langle s\lambda, \alpha^\vee\rangle = 0$ have equal residues, and so the residues cancel each other and do not contribute a singularity to (15.1), when the factors ρ'_i and ρ'_j which are interchanged by the reflection s_α are inequivalent. When $\rho'_i \simeq \rho'_j$ the residues have different signs, canceling the different sign of $\langle s\lambda, \alpha^\vee\rangle$ and $\langle s_\alpha s\lambda, \alpha_3^\vee\rangle = -\langle s\lambda, \alpha^\vee\rangle$, and then (15.1) may indeed have a singularity of the form $\langle s\lambda, \alpha^\vee\rangle^{-1}$ on the hyperplane $\langle s\lambda, \alpha^\vee\rangle = 0$.

We shall show that $E_{\psi'}(\Phi, \rho', \lambda)$ has a zero on the hyperplane $\langle s\lambda, \alpha^\vee\rangle = 0$ when the factors ρ'_i and ρ'_j of $s\rho'$ which are interchanged by s_α are equivalent. It suffices to show this for the Eisenstein series $E(x, \Phi, \rho', \lambda)$, whose Fourier coefficient is our $E_{\psi'}$, at x in $N'_0\backslash\mathbb{N}'_0$. By the "principle of the constant term" of [L2], it suffices to show the vanishing under the same assumptions on λ, x and ρ', of the constant term

$$E_{N_1}(x, \Phi, \rho', \lambda) = \sum_{t\in W(\mathfrak{A},\mathfrak{A}_1)} (M(t, \rho', \lambda)\Phi)(x)e^{\langle t\lambda+\rho_1, H(x)\rangle} \tag{16.2}$$

of E with respect to any parabolic subgroup $P'_1 \subset G'$ associate to P' (for the equality, see, e.g., [A2], p. 113, ℓ. -8). Note that $H(x) = 0$ on $N'_0\backslash\mathbb{N}'_0$, and recall that "associate" means that $W(\mathfrak{A}, \mathfrak{A}_1)$ is non-empty.

16.3. Lemma. *Suppose that* ρ' *and* λ *have the property that for some* $s \in W(\mathfrak{A}, \mathfrak{A}_2)$ *(and* P'_2*) and* $\alpha \in \Delta_2$*, we have* $\langle s\lambda, \alpha^\vee\rangle = 0$ *and* $\rho'_{s,i} \simeq \rho'_{s,j}$*, where* $\rho'_{s,i}$ *and* $\rho'_{s,j}$ *are the factors of* $s\rho'$ *interchanged by the simple reflection* s_α *belonging to* α*. Then* $E_{N_1}(I, \Phi, \rho', \lambda) = 0$ *for any parabolic subgroup* P'_1 *associated to* P'*.*

Proof of Lemma. For any $t \in W(\mathfrak{A}, \mathfrak{A}_1)$, the reflection $ts^{-1}s_\alpha st^{-1}$ lies in $W(\mathfrak{A}_1, \mathfrak{A}_1)$, and it interchanges the factors $\rho'_{t,i}$ and $\rho'_{t,j}$ of the $\mathbb{M}'_1$-module $t\rho'$. Our assumption implies that $\rho'_{t,i} \simeq \rho'_{t,j}$, and that $\langle s\lambda, ts^{-1}\alpha^\vee\rangle (= \langle t\lambda, \alpha^\vee\rangle)$ is zero. The functional equation

$$M(ts^{-1}s_\alpha st^{-1} \cdot t, \rho', \lambda) = M(ts^{-1}s_\alpha st^{-1}, t\rho', t\lambda)M(t, \rho', \lambda)$$

implies that the terms in $E_{N_1}(I, \Phi, \rho', \lambda)$ (I denotes the identity element) come in pairs, indexed by t and $ts^{-1}s_\alpha s$, whose value differ by a sign since $M(ts^{-1}s_\alpha st^{-1}, t\rho', t\lambda)$ is -1 for our $t\rho'$ and $t\lambda$. This last evaluation is again delayed to Corollary 25.1 below, since it is needed also in the discussion of the discrete contribution to the Fourier summation formula. A sum of zeroes is zero, hence the lemma follows.

At this stage we conclude that the product of (15.1) and $E_{\psi'}(\Phi, \rho', \lambda)$ is holomorphic on $i\mathfrak{A}^*_{P'}$. Hence the product of $I(\lambda, \Phi, \rho', T)$ and $E_{\psi'}(\Phi, \rho', \lambda)$ is holomorphic there (since (15.1) + $I(\lambda, \Phi, \rho', T)$ is holomorphic). By the linear independence of the J exponentials in T in $I(\lambda, \Phi, \rho', T)$ we conclude that each product $F_j(\lambda, \Phi, \rho')E_{\psi'}(\Phi, \rho', \lambda)$ is holomorphic on $i\mathfrak{A}^*_{P'}$ $(1 \leq j \leq J)$, and the proposition follows.

We shall use the following

16.4. Lemma. *Let f be a Schwartz (smooth, rapidly decreasing as $|\mu| \to \infty$) function on $i\mathbb{R}$. Then $\lim\limits_{t\to\infty} \int_{\mathbb{R}} f(i\mu)\mu^{-1} exp(\pm i\mu t)d\mu = \pm f(0)$.*

Proof. Elementary.

On mapping T to ∞ in the positive Weyl chamber, this Lemma 16.4, together with Proposition 16, permits deducing from Proposition 9, Corollary to Lemma 15, and the expression (15.2) for the sum of terms indexed by $w \neq 1$, the following

17. Proposition. *For every $f' = \otimes_v f'_v \in C(\mathbb{G}')$, the sum $\sum_w \sum_a \Psi'(aw, f'; \psi')$ of Proposition 9 is equal to the sum of (a)*

$$|\mathfrak{A}_2/L_2| \sum_{P' \subset G'} n(P')^{-1} \sum_{\rho' \in L_0(\mathbb{M}')} \sum_{\Phi \in H_{P'}(\rho')} E_{\psi'}(I(f', 0, \rho')\Phi, \rho', 0) \sum_{P'_2 \subset G'} \sum_{s \in W(\mathfrak{A}, \mathfrak{A}_2)} \qquad (17.1),$$

where

$$(17.1) \qquad \int_{\mathbb{A}_2\mathbb{N}_2 M_2\backslash\mathbb{G}} (M(s, \rho', 0)\Phi)(g)dg,$$

and (b) the sum over $P' \subset G'$, certain cuspidal $\mathbb{M}'$-modules ρ', and over $j (1 \leq j \leq J)$, of the integral over the $\lambda \in i\mathfrak{A}^*_{P'}$ such that $h_{ij}(\underline{\lambda}) = 0$ for all $i (1 \leq i \leq d(j))$, of the sum over Φ in an orthonormal basis of $H_{P'}(\rho')$, of the product of $F_j(\lambda, \Phi, \rho')$ and $E_{\psi'}(I(f', \lambda, \rho')\Phi, \rho', \lambda)$.

Lemma 16.4 is used to reduce the domain of integration $i\mathfrak{A}^*_{P'}$ to the hyperplanes $h_{ij}(\underline{\lambda}) = 0$ $(1 \leq i \leq d(j))$, since the forms $\ell_{ij}(\underline{t})$ are linearly independent. The fact that $d(j) < d$ implies that the terms index by j contain an integral over a non-zero dimensional space. The linear forms $\langle s\lambda, \alpha^\vee\rangle$ in λ which occur in the term (15.1) are linearly independent, and there are $d = \dim \mathfrak{A}_{P'} = [\Delta_2]$ such forms. Consequently the integral over $i\,\mathfrak{A}^*_{P'}$ reduces to the value of the integrand at $\lambda = 0$, by Lemma 16.4. The first summand of the Proposition is thus obtained.

We now continue by analyzing the discrete contribution to the representation theoretic part of the Fourier summation formula, namely the sum corresponding to $w = 1$ in Proposition 17.

18. Lemma. *The integral (17.1) is non-zero (for some $\Phi \in H_{P'}(\rho')$) if and only if $s\rho'$ is a distinguished $s\mathbb{M}'$-module (, and ρ' is a distinguished $\mathbb{M}'$-module).*

Proof. The operator $M(s, \rho', 0)$ is an isomorphism from the space $H_{P'}(\rho')$ of the induced $\mathbb{G}'$-module $I(\rho'; \mathbb{G}', \mathbb{P}')$ to the space $H_{P_2'}(s\rho')$ of $I(s\rho')$. Since $s\rho'$ is distinguished if and only if ρ' is, it suffices to show the lemma when $s = 1$. Then $P_2' = P'$. If $D(\Phi) = \int_{\mathbb{A}\mathbb{N}M\backslash\mathbb{G}} \Phi(g)dg$ is non-zero, then the functional $\Phi \mapsto D(\Phi)$ is non-zero and $\mathbb{G}$-invariant on $I(\rho')$, hence $I(\rho')$ is distinguished. Moreover, $D(\Phi) = D_M(\Phi^K)$, where $\Phi^K(g) = \int_{\mathbb{K}} \Phi(gk)dk$, and $D_M(\phi) = \int_{\mathbb{A}M\backslash\mathbb{M}} \phi(m)dm$. Then Φ^K is an element in the space of $\rho' \subset L_0(M'\backslash\mathbb{M}')$, and $D_M(\Phi^K) \neq 0$ for some Φ^K means that ρ' is distinguished.

On the other hand, if ρ' is distinguished then there is a cusp form ϕ in the space $\rho' \subset L_0(M'\backslash\mathbb{M}')$, ϕ necessarily transforms trivially under $\mathbb{A}$ (a subgroup of the center of $\mathbb{M}'$), such that $D_M(\phi) \neq 0$. Let $\mathbb{K}_\epsilon'$ be a sufficiently small congruence subgroup of the maximal compact subgroup $\mathbb{K}'$ of $\mathbb{G}'$. Define Φ on $\mathbb{G}'$ by $\Phi(g) = \phi(m)$ if $g = mnk$ with $m \in \mathbb{M}'$, $n \in \mathbb{N}'$, $k \in \mathbb{K}_\epsilon'$, and $\Phi(g) = 0$ otherwise. Then $D(\Phi)$ is $D_M(\phi)$ up to a non-zero (integration) scalar, hence (17.1) is non-zero, as asserted.

Our next aim is to show

19. Proposition. *The sum of (17.1) over s in $W(\mathfrak{A}, \mathfrak{A}_2)$ is zero if there is $s \neq 1$ with $s\rho' = \rho'$, and it is the product of the cardinality $[W(\mathfrak{A}, \mathfrak{A}_2)]$ and*

$$D(\Phi) = \int_{M\mathbb{A}\mathbb{N}\backslash\mathbb{G}} \Phi(g)dg \tag{19.1}$$

if $s\rho' = \rho'$ only when $s = 1$.

In other words, if $\underline{M}' = GL(n_1) \times \cdots \times GL(n_a)$, and correspondingly $\rho' = \rho_1' \times \cdots \times \rho_a'$, the Proposition distinguishes between the case where $\rho_i' \simeq \rho_j'$ for some $i \neq j$, and the case where the ρ_i's are pairwise inequivalent.

The proof of this requires several lemmas. The main step is to rewrite the integral (19.1) in terms of the Whittaker function

$$W_\Phi(g) = \int_{J^{-1}M'J\cap N_0'\backslash\mathbb{N}_0'} \Phi(Jng)\psi'(n)dn$$

attached to Φ. Here $J = ((-1)^{n-i}\delta_{i,n-j+1})$, and ψ' the non-trivial character of $\mathbb{N}_0 N_0'\backslash\mathbb{N}_0'$ fixed above ($\underline{N}_0$ is the upper triangular unipotent subgroup in $\underline{G}$). In fact, we would have liked to argue, using the uniqueness of the distinguished functional ([F1], Proposition 11), that $D(\Phi)$ is a multiple of the functional $E(\Phi) = \int_{\mathbb{Z}\mathbb{N}_0\backslash\mathbb{G}} W_\Phi(g)dg$, which is also $\mathbb{G}$-invariant. The only problem is that this last integral does not converge. To overcome the problem we regularize this integral. A few notations have to be recalled, mainly from [F2], for this purpose.

Denote by $\underline{Q}$ the upper triangular parabolic subgroup of $\underline{G}$ of type $(n-1, 1)$ (in [F2] this Q is denoted by ZP). Denote by δ_Q the modular function on G attached to Q, thus if $g = qk$, $q \in \mathbb{Q}$, $k \in \mathbb{K}$, and $q = \begin{pmatrix} a & * \\ 0 & b \end{pmatrix}$, $a \in GL(n-1, \mathbb{A})$ and $b \in \mathbb{A}^\times$, then

$$\delta_Q(g) = \delta_Q(q) = |\det(Ad(q)|\tilde{N}_Q)| = |\det(b^{-1}a)| = |b^{-(n-1)}\det a|.$$

Let (π'_v, V) be an irreducible admissible generic G'_v-module, thus there exists a non-zero linear form λ on V with $\lambda(\pi'_v(n)v) = \psi'_v(n)\lambda(v)$ for all v in V and n in N'_{0v}. There exists (in general at most) only one such λ up to a scalar (see Gelfand-Kazhdan [GK]). Denote by $W(\pi'_v; \psi'_v)$ the space of all functions W_w on G'_v of the form $W_w(g) = \lambda(\pi'_v(g)w)$ (w in V). The space $W(\pi'_v; \psi'_v)$, called the *Whittaker model* of π'_v (with respect to ψ'_v), is invariant under right translations by G'_v, and it is equivalent to (π'_v, V) as a G'_v-module. For W_v in $W(\pi'_v; \psi'_v)$ we have $W_v(ng) = \psi'_v(n)W_v(g)$ ($n \in N'_{0v}, g \in G'_v$). Suppose that the central character of π'_v is trivial on Z_v. This is the case if $\pi'_v = I(\rho'_v)$, and ρ'_v is a distinguished M'_v-module.

20. Lemma. *The integral* $\Psi(t, W_v) = \int_{Z_v N_{0v}\backslash G_v} W_v(g)\delta_Q(g)^t dg$ *converges (absolutely) for* $Re(t) \geq 1$.

Proof. Note that ψ'_v is trivial on N_{0v}. Hence W_v is $Z_v N_{0v}$-invariant, and the integral is defined. We shall use below only the fact that Ψ converges for a sufficiently large $Re(t)$. This is a fact since W_v is majorized by a function ξ on G'_v which is left-$N'_{0v}Z'_v$ and right-K'_v invariant, and given on A'_v by

$$\xi(a) = T\left(\frac{a_1}{a_2}, \frac{a_2}{a_3}, \ldots, \frac{a_{n-1}}{a_n}\right)\left|\frac{a_1}{a_n}\right|^n \qquad (a = \operatorname{diag}(a_1, \ldots, a_n)),$$

for some $u \leq 0$ and a smooth compactly supported function T on $E_v \times \cdots \times E_v$ ($n-1$ copies); see Jacquet, Piatetski-Shapiro, Shalika [JPS, (2.3.6)].

To show the convergence for $Re(t) \geq 1$, we use the decomposition $G_v = Q_v K_v$. Then

$$\int\limits_{Z_v N_{0v}\backslash G_v} |W_v(g)|\delta_Q(g)^t dg = \int\limits_{K_v} dk \int\limits_{N_{0v}\backslash Q_{0v}} |W_v(qk)||\det a|_v^{t-1} dq,$$

where $Q_0 = \{q = \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \in Q; b = 1\}$. The integral over the subset of q with $|\det a|_v \leq 1$ is bounded by $\int_{K_v}\int_{N_{0v}\backslash Q_{0v}} |W_v(qk)| dq\, dk$, which is finite by the Lemma of [F2], p. 306. Since $|W_v(qk)|$ is bounded by a function ξ, the integral over the set of $q \in N_{0v}\backslash Q_{0v}$ with $|\det a|_v \geq 1$ is taken over a compact set; hence it converges. The lemma follows.

Remark. We shall discuss here only the case of a non-archimedean F-prime which stays prime in E. In the case of v which splits into v', v'', thus $E_v = F_{v'} \oplus F_{v'}$, we have $W_v(g_v, g_{v''}) = W_{v'}(g_{v'})W_{v''}(g_{v''})$, and Lemma 20, as well as the entire discussion below, analogously follows from [JS] in the non-archimedean case, and from [JS1] when F_v is archimedean. This forces us to restrict attention to extensions E/F such that each archimedean place of F splits in E. The final Remark in [F5] suggests how to remove this restriction, but the work suggested there has not been carried out as yet.

We are interested in the value of $\Psi(t, W_v)$ at $t = 0$. This value is not in the domain of convergence, but it is defined by analytic continuation. Our concern is with the possible poles, or zeroes, beyond the line of absolute convergence. To examine this, let us compute $\Psi(t, W_v)$ when v is unramified in E and π'_v is unramified, namely has a non-zero K'_v-fixed vector, and W_v *is* the K'_v-fixed vector W_v^0 in $W(\pi'_v; \psi'_v)$, normalized by $W_v^0(I) = 1$.

To state the result, denote by r the twisted tensor representation of the dual group $\hat{G}'$ of G', introduced in [F2]. At a place v which splits it is simply the tensor product representation of $\hat{G}'_v = G(\mathbb{C}) \times G(\mathbb{C})$ on $\mathbb{C}^n \otimes \mathbb{C}^n : r(x, y)(u \otimes v) = xu \otimes yv$. At a non-split place v, $\hat{G}'_v = (G(\mathbb{C}) \times G(\mathbb{C})) \rtimes \mathrm{Gal}(E_v/F_v)$, and r acts on $u \otimes v \in \mathbb{C}^n \otimes \mathbb{C}^n$ by $r(x, y)(u \otimes v) = xu \otimes yv$ and $r(\sigma)(u \otimes v) = v \otimes u$.

The L-function $L(t, \pi'_v, r)$ attached to r and the unramified G'_v-module π'_v is defined (see [F2]) to be

$$\det[1 - q_v^{-t} r(t(\pi'_v))]^{-1},$$

where q_v is the cardinality of the residue field of F_v. If v splits and the Hecke eigenvalues of $\pi'_v = \pi_{v'} \times \pi_{v''}$ are x_i and $y_i (1 \le i \le n)$, then

$$L(t, \pi'_v, r) = L(t, \pi_{v'} \times \pi_{v''}, \otimes) = \prod_{i,j} (1 - q_v^{-t} x_i y_j)^{-1}.$$

If v stays prime and the Hecke eigenvalues of π'_v are $z_1, \dots, z_n$, then

$$L(t, \pi'_v, r) = \prod_i (1 - q_v^{-t} z_i)^{-1} \cdot \prod_{j<k} (1 - q_v^{-2t} z_j z_k)^{-1}.$$

In fact, another L-function is needed. Denote by ω'_v the central character of π'_v and put

$$L(t, \omega'_v) = (1 - q_v^{-t} \omega'_v(\underline{\pi}_v))^{-1}$$

if v stays prime ($\underline{\pi}_v$ is a uniformizer (generator of the maximal ideal in the ring of integers) of F_v, and also of E_v, since E_v/F_v is unramified), and

$$L(t, \omega'_v) = (1 - q_v^{-t} \omega_{v'}(\underline{\pi}_{v'}) \omega_{v''}(\underline{\pi}_{v''}))^{-1}$$

if v splits and $\omega_{v'}$, $\omega_{v''}$ are the central characters of $\pi_{v'}$ and $\pi_{v''}$. Note that $L(nt, \omega'_v)$ "divides" $L(t, \pi'_v, r)$, namely $L(nt, \omega'_v)/L(t, \pi'_v, r)$ is a polynomial in q_v^{-t} whose value at 0 is 1.

With these notations we prove

21. Lemma. *When v and π'_v are unramified, $\Psi(t, W_v^0)$ is equal to $L(t, \pi'_v, r)/L(nt, \omega'_v)$.*

Proof. This is the Proposition of [F2], p. 305, with a minor modification: the integral $\Psi(t, \Phi, W_v^0)$ of [F2] is taken over $N_{0v} \backslash G_v$ (instead of our $Z_v N_{0v} \backslash G_v$), and its integrand contains the characteristic function Φ. As explained in [F2], pp. 305/6, the integral of [F2] is a sum which ranges over $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$ and adds up to $L(t, \pi'_v, r)$ (when $Re(t) \ge 1$). The presence of Φ in [F2] causes the inequality $\lambda_n \ge 0$. In our case the integration is taken modulo the center Z_v, and so we may choose $\lambda_n = 0$ and sum over $\overline{\lambda}_1 \ge \overline{\lambda}_2 \ge \cdots \ge \overline{\lambda}_{n-1} \ge \overline{\lambda}_n = 0$. The Schur function $s_\lambda(x)$, $x = (x_1, \ldots, x_n)$, used in [F2], satisfies $s_\lambda(x) = (x_1 \ldots x_n)^{\lambda_n} s_{\overline{\lambda}}(x)$, if $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_n)$, $\overline{\lambda}_i = \lambda_i - \lambda_n$. Since $\omega'_v(\underline{\pi}_v) = z_1 \ldots z_n$, $\omega_{v'}(\underline{\pi}_{v'}) = x_1 \ldots x_n$, $\omega_{v''}(\underline{\pi}_{v''}) = y_1 \ldots y_n$, and $\sum_{j \geq 0} z^j = (1-z)^{-1}$ for $|z| < 1$, the lemma follows.

In the general non-archimedean case, that is when v or π_v are not necessarily unramified, or W_v is not a multiple of W_v^0, Theorem 2.7(ii) of [JSP] in the split case, and the Theorem (ii) of [F5] in the non-split case, establish - after a simple modification, taking into account the denominator $L(nt, \omega'_v)$ - that there is a polynomial $P(X)$ in $\mathbb{C}[X], X = q_v^{-t}$, with $P(0) = 1$, such that the p-adic integrals $\Psi(t, W_v)$ span the fractional ideal $P(X)^{-1}\mathbb{C}[X, X^{-1}]$ of the ring $\mathbb{C}[X, X^{-1}]$. Define

$$L(t, \pi'_v, r)/L(nt, \omega'_v) = P(X)^{-1}.$$

When $E_v/F_v, \omega'_v$ and π'_v are unramified, it is clear from the proofs of [JSP] and [F5] that the L-factors so defined coincide with those of Lemma 21.

We now return to the global notations used in Proposition 19, where ρ' is a distinguished cuspidal $\mathbb{M}'$-module, and $\Phi \in H_{P'}(\rho')$. For $W(g) = \prod_v W_v(g_v)$ define

$$\Psi(t, W) = \prod_v L(t, \pi'_v, r) L(nt, \omega'_v)^{-1} \Psi(t, W_v).$$

By Lemma 21 almost all factors here are equal to 1. The remaining finite number of factors are regular in (q_v^{-t} and in particular in) t.

Note that in the split archimedean case the L-factor is defined in [JS1] to be that which is associated to the representation of the Weil group which parametrizes the $\pi_{v'} \times \pi_{v''}$. The corresponding local factor of $\Psi(t, W)$ is shown in [JS1], Theorem 5.1(i), to be regular in t. The L-factor can of course be analogously defined when $E_v/F_v = \mathbb{C}/\mathbb{R}$, but then $-$ as remarked after Lemma 20 $-$ the regularity of the corresponding factor of $\Psi(t, W)$ has not been established as yet.

Consequently, for E/F in which each archimedean place of F splits, $\Psi(t, W)$ can be evaluated at $t = 0$. If $W_\Phi = \sum_i a_i W_i$, where $W_i = \prod_v W_{iv}$, put

$$\Psi(W_\Phi) = \sum_i a_i \Psi(0, W_i).$$

Then $\Phi \mapsto \Psi(W_\Phi)$ is a non-zero complex-valued linear form on the $\mathbb{G}'$-module $\pi' = I(\rho')$ (each of the local forms $W_v \mapsto L(t, \pi'_v, r) L(nt, \omega'_v)^{-1} \Psi(t, W_v)$ is non-zero). Since $\delta_{Q_v}(gq_1) = \delta_{Q_v}(qq_1)\delta_{Q_v}(q_1^{-1}kq_1)$ for $g = qk$, and $\delta_{Q_v}(q_1^{-1}kq_1)$ is bounded over $k \in K_v$, it is easy to see that the distribution $\Psi(W_\Phi)$ is $\mathbb{G}$-invariant.

22. Lemma. *For every ρ' and ψ' there exists a non-zero constant $c = c(\rho', \psi')$, depending on ρ' up to conjugacy, such that $D(\Phi) = c\Psi(W_\Phi)$ for all Φ in $H_{P'}(\rho')$.*

Proof. By [F1], Proposition 11, there is a unique (up to a scalar) non-zero G_v-invariant complex-valued form on a distinguished irreducible admissible G'_v-module such as $\pi'_v = I(\rho'_v)$. Both $W_{\psi'}(\Phi)$ and $D(\Phi)$ are $\mathbb{G}$-invariant and non-zero, hence the lemma follows.

Consequently, for any $s \in W(\mathfrak{A}, \mathfrak{A}_2)$ as in Proposition 19, the expression (17.1) is

$$D(M(s, \rho', 0)\Phi) = c\Psi(W_{M(s,\rho',0)\Phi}).$$

To relate (17.1) and (19.1) we thus need to relate W_Φ with $W_{M(s,\rho',0)\Phi}$. By the functional equation

$$M(s_1 s_2, \rho', 0) = M(s_1, s_2\rho', 0)M(s_2, \rho', 0)$$

(due to [L2], see also [A1], p. 927, (iii), in the number field case, and [M] in the function field case), we are reduced to the case where s interchanges two blocks, namely P' is of type (r_1, r_2) $(r_1 + r_2 = n)$, P'_2 is of type (r_2, r_1), and $s = \begin{pmatrix} 0 & I_{r_1} \\ I_{r_2} & 0 \end{pmatrix}$, so that $s^{-1}M's = M'_2$.

To deal with this case, let $\rho' = \rho'_1 \times \rho'_2$ be an irreducible cuspidal $\mathbb{M}'$-module, and $\rho'_t = \rho' \otimes \delta^t_{P'}$. For any $\Phi \in H_{P'}(\rho')$ put $\Phi_t(g) = \Phi(g)\delta_{P'}^{t+1/2}(g)$, and denote W_{Φ_t} by $W_{\Phi,\rho',t,\psi'}$. Denote the central character of ρ'_i by ω'_i. Put $\tilde{\rho}' = \rho'_2 \times \rho'_1$. We shall also need the local analogues of these notations. In particular, denote by V_i the space of ρ'_{iv}, and by $\Phi_{t,v}$ a right-smooth function from G'_v to $V_1 \otimes V_2$ satisfying

$$\Phi_{t,v}(pg) = \delta_{P'_v}(p)^{t+1/2}(\rho'_{1,v}(a) \otimes \rho'_{2,v}(b))\Phi_{t,v}(g) \qquad (p = \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \in P'_v,\ g \in G'_v).$$

If $\lambda_i^{\psi'}$ is a (non-zero) Whittaker functional on $(\rho'_{i,v}, V_i)$, thus $\lambda_i^{\psi'}(\rho_{i,v}(u)w) = \psi'_v(u)\lambda_i^{\psi'}(w)$ for $u \in N'_{0,v}$ and $w \in V_i$, then

$$W_{\Phi_v,\rho'_v,t,\psi'_v}(g) = \int_{N'_{0,v} \cap s^{-1}N'_{0,v}s \backslash N'_{0,v}} \langle \lambda_1^{\psi'} \otimes \lambda_2^{\psi'}, \Phi_{t,v}(sug)\rangle \overline{\psi}'(u)du$$

and

$$(M(s, \rho'_v, t)\Phi_{t,v})(g) = \int_{N'_{0,v} \cap s^{-1}N'_{0,v}s \backslash N'_{0,v}} \Phi_{t,v}(sug)du$$

(cf. [Sh2], p. 72). Denote by $L(t, \rho'_{1,v} \otimes \rho'_{2,v})$ and $\epsilon(t, \rho'_{1,v} \otimes \rho'_{2,v}, \psi'_v)$ the local L-function and ϵ-factor attached in [JPS], Theorem 2.7, and [JS1], Theorem 5.1, to the pair $(\rho'_{1,v}, \rho'_{2,v})$ of $GL(r_1, E_v)$ and $GL(r_2, E_v)$-modules (which are generic, being as they are components of cuspidal representations), and the character ψ'_v of $N'_{0,v}$, and the tensor representation on $\mathbb{C}^{r_1} \otimes \mathbb{C}^{r_2}$.

In the non-archimedean case the ϵ-factor is a monomial in q_v^{-t}, while the L-function, which has been introduced after Lemma 21 above (when v splits), is of the form $P(q_v^{-t})^{-1}$, where P is a polynomial over $\mathbb{C}$ with $P(0) = 1$. In the archimedean case the L and ϵ-factors are those associated to the representations of the Weil group which parametrize the $\rho'_{i,v}$; see [JS1].

23. Lemma. *For an admissible irreducible generic $\rho'_v = \rho'_{1,v} \times \rho'_{2,v}$ one has*

$$W_{\Phi,\rho'_v,t,\psi'_v}(g)\omega'_{2,v}{}^{r_1}(-1)\frac{L(t, \rho'_{1,v} \otimes \check{\rho}'_{2,v})}{\epsilon(t, \rho'_{1,v} \otimes \check{\rho}'_{2,v}, \psi'_v)L(1-t, \check{\rho}'_{1,v} \otimes \rho'_{2,v})} = W_{M(s,\rho'_v,t)\Phi,\tilde{\rho}'_v,-t,\psi'_v}(g)$$

for all $g \in G'_v$, where $\check{\rho}'_{i,v}$ is the contragredient of $\rho'_{i,v}$ and $\tilde{\rho}'_v = \rho'_{2,v} \times \rho'_{1,v}$.

Proof. This is the main result of Shahidi [Sh2], see the first, third and fourth displayed formulae on p. 68 of [Sh2].

Definition. ([Sh3], p. 272): The *normalized* intertwining operator is

$$R(s, \rho'_v, t) = \frac{\epsilon(t, \rho'_{1,v} \otimes \check{\rho}'_{2,v}, \psi'_v) L(1+t, \rho_{1,v} \otimes \check{\rho}'_{2,v})}{L(t, \rho'_{1,v} \otimes \check{\rho}'_{2,v})} M(s, \rho'_v, 0).$$

Corollary. *For any g in G'_v one has*

$$W_{\Phi,\rho'_v,0,\psi'_v}(g) {\omega'_{2,v}}^{r_1}(-1) \frac{L(1, \rho'_{1,v} \otimes \check{\rho}'_{2,v})}{L(1, \check{\rho}'_{1,v} \otimes \rho'_{2,v})} = W_{R(s,\rho'_v,0)\Phi,\tilde{\rho}'_v,0,\psi'_v}(g).$$

Note that by Lemma 18 the only ρ' which contribute to our formulae are the distinguished ones.

24. Lemma. *If $\rho'_{1,v}$ and $\rho'_{2,v}$ are distinguished then $L(t, \rho'_{1,v} \otimes \check{\rho}'_{2,v}) = L(t, \check{\rho}'_{1,v} \otimes \rho'_{2,v})$.*

Proof. The assumption implies that $\rho'_{i,v} \simeq \check{\overline{\rho}}'_{i,v}$, where $\overline{\rho}'_{i,v}(g) = \rho'_{i,v}(\overline{g})$, $\overline{g} = (\overline{a}_{ij})$ if $g = (a_{ij})$, and $x \mapsto \overline{x}$ is the non-trivial element of $\mathrm{Gal}(E_v/F_v)$, by virtue of [F1], Proposition 12. This is well-known in the archimedean case. To simplify the notations, in the rest of this proof the index v is omitted. In the archimedean case the identity is clear. In the non-archimedean case, by definition (see [JPS], Theorem 2.7(ii)), the L-factor $L(t, \rho'_1 \otimes \check{\rho}'_2) = L(t, \rho'_1 \otimes \overline{\rho}'_2)$ is the g.c.d. of the integrals

$$\int_{N'_0 \backslash G'} W_1(g) W_2(\overline{g}) \phi(\epsilon_n g) |\det g|^t \, dg,$$

where $\epsilon_n = (0, \dots, 0, 1) \in E^n$, and ϕ ranges over $C_c^\infty(E^n)$ and W_i over the Whittaker space $W(\rho'_i; \psi')$ of ρ'_i with respect to ψ'. Of course, the factor $L(t, \check{\rho}'_1 \otimes \rho'_2) = L(t, \overline{\rho}'_1 \otimes \rho'_2)$ is the g.c.d. of the same integrals with $W_1(g)$ replaced by $W_1(\overline{g})$ and $W_2(\overline{g})$ by $W_2(g)$. Since $|\det \overline{g}| = |\det g|$, and $g \mapsto \phi(\overline{g})$ lies in $C_c^\infty(E^n)$ if ϕ does, the lemma follows.

Corollary. *Globally, for any Φ in $H_{P'}(\rho')$ we have*

$$W_{\Phi,\rho',0,\psi'}(g) = W_{R(s,\rho',0)\Phi,\tilde{\rho}',0,\psi'}(g).$$

Indeed, $H_{P'}(\rho')$ is spanned by $\otimes \Phi_v$, and $\omega'_2(-1) = 1$ since ρ'_2 is automorphic. The global normalized intertwining operator is defined by the product of the local operators:

$$R(s, \rho', t) = \underset{v}{\otimes} R(s, \rho'_v, t).$$

The global operators are related by

$$R(s, \rho', t) = \frac{\epsilon(t, \rho'_1 \otimes \check{\rho}'_2) L(1+t, \rho'_1 \otimes \check{\rho}'_2)}{L(t, \rho'_1 \otimes \check{\rho}'_2)} M(s, \rho', t).$$

The global ϵ and L functions are defined to be the product of the local factors. Note that the global ϵ-function is independent of the choice of the additive character $\psi' \neq 1$ on $\mathbb{N}'_0$, as long as ψ' is trivial on N'_0.

The functional equations $L(t, \rho'_1 \otimes \check{\rho}'_2) = \epsilon(t, \rho'_1 \otimes \check{\rho}'_2) L(1-t, \check{\rho}'_1 \otimes \rho'_2)$ for the tensor product L-function (proven first in the Appendix to [MW1], and then by [JS1], Theorem 5.1, which completes the work of [JPS] in the archimedean case) implies that

$$R(s, \rho', t) = \frac{L(1+t, \rho'_1 \otimes \check{\rho}'_2)}{L(1-t, \check{\rho}'_1 \otimes \rho'_2)} M(s, \rho', t).$$

By Lemma 18 the ρ'_i are distinguished (hence $\check{\rho}'_i \simeq \overline{\rho}'_i$ by [F1], Proposition 12). By Lemma 24 we then have $L(t, \rho'_1 \otimes \check{\rho}'_2) = L(t, \check{\rho}'_1 \otimes \rho'_2)$. One more Lemma is needed.

25. Lemma. *If $\rho'_1 \simeq \rho'_2$ then $L(t, \rho'_1 \otimes \check{\rho}'_2)$ has a simple pole at $t = 1$. If not then the L-function is finite at $t = 1$; moreover it is non-zero at $t = 1$.*

Proof. Both assertions follow from [$\mathrm{JS_{II}}$], Proposition 3.6, p. 802, for the partial (product over almost all local places) L-function. This result of [$\mathrm{JS_{II}}$] extends to the full L-function too by the Appendix to [MW1], or alternatively by virtue of [JPS], Theorem 2.7(ii), and [JS1], Theorem 5.1, where the remaining local factors are introduced and related to the local integrals of Whittaker functions from [$\mathrm{JS_{II}}$]. The "moreover" part of the Lemma, and more generally the claim that $L(t)$ is non zero on $Re(t) = 1$, is due to [Sh1].

25.1 Corollary. *At $t = 0$, the quotient $L(1+t, \rho'_1 \otimes \check{\rho}'_2)/L(1-t, \check{\rho}'_1 \otimes \rho'_2)$ takes the value $\epsilon(\rho')$, which is 1 if $\rho'_1 \not\simeq \rho'_2$, and -1 if $\rho'_1 \simeq \rho'_2$. Consequently we have*

$$W_{M(s,\rho',0)\Phi, \tilde{\rho}', 0, \psi'}(g) = \epsilon(\rho') W_{\Phi, \rho', 0, \psi'}(g).$$

We now return to the

Proof of Proposition 19. Here $\rho' = \rho'_1 \times \cdots \times \rho'_a$, and $s = s_1 s_2 \dots s_b$, where $s_i \in W_{G'}/W_{M'}$ permutes two blocks only. If $u = u(\rho', s)$ is the number (modulo 2) of the i such that $s_i s_{i+1} \dots s_b \rho' \simeq s_{i+1} \dots s_b \rho'$, then by Corollary 25 and the functional equation we have

$$W_{M(s,\rho',0)\Phi, s\rho', 0, \psi'}(g) = (-1)^u W_{\Phi, \rho', 0, \psi'}(g).$$

Put $M'_s = sM's^{-1}$, and denote by N'_s the unipotent radical of $P'_s = M'_s P'_0$ (P'_0 is the upper triangular subgroup). By Lemma 22 we have

$$\int_{M'_s \mathbb{A}_s \mathbb{N}_s \backslash \mathbb{G}} (M(s, \rho', 0)\Phi)(g) dg = c\Psi(W_{M(s,\rho',0)\Phi, s\rho', 0, \psi'})$$

$$= (-1)^u c\Psi(W_{\Phi, \rho', 0, \psi'}) = (-1)^u \int_{M\mathbb{A}\mathbb{N}\backslash\mathbb{G}} \Phi(g) dg.$$

Thus if $\rho'_i \not\simeq \rho'_j$ for all $i \neq j$, the integral (17.1) is equal to $\int_{M\mathbb{A}\mathbb{N}\backslash\mathbb{G}} \Phi(g)dg$ and is independent of s. But if there are $i \neq j$ with $\rho'_i \simeq \rho'_j$ then there is s_0 in $W(\mathfrak{A}, \mathfrak{A})$ which permute the two, and the element s_0 acts by $s \mapsto ss_0$ on the set $W(\mathfrak{A}, \mathfrak{A}_2)$.

The sum over s in $W(\mathfrak{A},\mathfrak{A}_2)$ of Proposition 19 contains then with each term its negative, so the sum is zero, as asserted.

If $s\rho' \neq \rho'$ for any $s \neq 1$ in $W(\mathfrak{A},\mathfrak{A}_2)$, then the cardinality of the set $\{s\rho'; P_2' \subset G', s \in W(\mathfrak{A},\mathfrak{A}_2)\}$ is $n(P')$ (see [A1], end of p. 919). It follows that the terms corresponding to $w=1$ in Proposition 17 add up to

$$\sum_{P' \subset G'} |\mathfrak{A}/L| \sum_{\rho'} \sum_{\Phi \in H_{P'}(\rho')} E_{\psi'}(I(f',\rho',0)\Phi,\rho',0) \cdot \int_{M\mathbb{A}\mathbb{N}\backslash\mathbb{G}} \overline{\Phi}(g,0)dg.$$

The second sum ranges over the cuspidal distinguished $\mathbb{M}'$-modules ρ' with $s\rho'=\rho'$ implying $s=1$ for any s in $W(\mathfrak{A},\mathfrak{A}_2)$, $P_2' \subset G'$. These are the ρ' which occur in Theorem 1^*, and the above sum describes the discrete part of the representation theoretic side of the Fourier summation formula. This discrete part is responsible to the applications concerning liftings.

It will be convenient to express this discrete part of the representation theoretic side of the Fourier summation formula as $\sum_{P'\subset G'} |\mathfrak{A}/L| \sum_{\rho'} WD_{I(\rho'),\psi'}(f')$. Here we put

$$D(\Phi) = \int_{M\mathbb{A}\mathbb{N}\backslash\mathbb{G}} \Phi(g)dg;$$

this is a distinguished functional on $H_{P'}(\rho')$. Also we put

$$W_{I(\rho'),\psi'}(\Phi) = E_{\psi'}(\Phi,\rho',0);$$

this functional satisfies $W_{I(\rho'),\psi'}(\Phi^x) = \psi'(x)W_{I(\rho'),\psi'}(\Phi)$ for $x \in \mathbb{N}'$, where $\Phi^x(g) = \Phi(gx)$. By the uniqueness of the Whittaker model and of the distinguished functional, the distribution

$$WD_{I(\rho'),\psi'}(f') = \sum_{\Phi \in H_{P'}(\rho')} W_{I(\rho'),\psi'}(I(f',\rho',0)\Phi) \cdot D(\overline{\Phi}), \qquad f' \in C(\mathbb{G}'),$$

where the sum ranges over an orthonormal basis $\{\Phi\}$ of $H_{P'}(\rho')$, is well-defined, namely is independent of the choice of the basis $\{\Phi\}$. It satisfies

$$WD_{I(\rho'),\psi'}({}^x f'^y) = \overline{\psi}'(x) WD_{I(\rho'),\psi'}(f'), \qquad {}^x f'^y(g) = f'(xgy),$$

for $x \in \mathbb{N}'$ and $y \in \mathbb{G}$, since $\pi'({}^x f'^y)\Phi = (\pi'(f')\Phi^{y^{-1}})^{x^{-1}}$, and $\Phi \mapsto \Phi^y$ maps $\{\Phi\}$ to the orthonormal basis $\{\Phi^y\}$ of the space $H_{P'}(\rho')$.

Note also that the proof of Proposition 19, together with the functional equation $E(g, M(s,\rho',\lambda)\Phi, s\lambda) = E(g,\Phi,\lambda)$ (due to [L2], but see also [A1], (ii) on p. 927, ℓ. 9, in the characteristic zero case, and [M] in the positive characteristic case), implies that the distribution $WD_{I(\rho'),\psi'}$ depends only on the equivalence class of $I(\rho')$. The discrete part can be written then as $\sum_{I(\rho')} |\mathfrak{A}/L| \cdot n(\rho') \cdot WD_{I(\rho'),\psi'}(f')$, where the sum ranges over all $\mathbb{G}'$-modules $I(\rho')$, where ρ' is a cuspidal $\mathbb{M}$-distinguished $\mathbb{M}'$-module, up to equivalence, and $n(\rho')$ is the number of distinct pairs P', ρ' yielding the same equivalence class $I(\rho')$.

Before launching into the comparison of the Fourier summation formulae we show that the functor of induction respects the notion of being distinguished.

26. Proposition. *(1) Let F be a local field. Let (ρ'_i, V_i) be $G_i = GL(n_i, F)$-distinguished G'_i-modules $(1 \leq i \leq b)$, and $\sum n_i = n$. Then $\pi' = I(\rho'_1, \ldots, \rho'_b)$ is a G-distinguished G'-module. (2) Let E/F be global fields. Suppose that $n = 2m$, and ρ' is a cuspidal $GL(m, \mathbb{A}_E)$-module with $\rho' \neq \check{\bar{\rho}}'$. Then $\pi' = I(\rho', \check{\bar{\rho}}')$ is a $\mathbb{G}$-distinguished $\mathbb{G}'$-module. In particular, each of its components is G_v-distinguished.*

Proof. (1) Let L_i be a non-zero G_i-invariant linear form on ρ'_i. A vector Φ in π is a function $\Phi : G' \to V_1 \otimes \cdots \otimes V_b$ satisfying

$$\Phi(nmg) = \delta_{P'}^{1/2}(m)\rho'_1(m_1) \otimes \cdots \otimes \rho'_b(m_b)\Phi(g) \qquad (m = \mathrm{diag}(m_1, \ldots, m_b))$$

where P' is the standard parabolic subgroup of G' of type $(n_1, \ldots, n_b)$, and $P' = M'N'$, $n \in N'$, $m \in M'$. Define

$$D(\Phi) = \int\limits_{P\backslash G} (L_1 \otimes \cdots \otimes L_b)(\Phi(g))dg.$$

This integral is well-defined since (a) $L_i(\rho'_i(a)v_i) = L_i(v_i)$ $(v_i \in V_i, a \in G_i$ viewed naturally as embedded in the standard Levi factor M of $P)$, (b) the integrand is continuous and $P\backslash G = K \cap P\backslash K$ is compact, and (c) $\delta_P^2 = \delta_{P'}$ and

$$\int_G f(g)dg = \int_N \int_M \int_K f(n\,m\,k)\delta_P^{-1}(m)dn\,dm\,dk.$$

The distribution $D(\Phi)$ is G-invariant by construction, and it is non-zero. Indeed, fix $v_i \neq 0$ in V_i and a congruence subgroup K'_ϵ of K' such that $\rho'_i(k)v_i = v_i$ for all $k \in G'_i \cap K'_\epsilon$ (G'_i is viewed as a subgroup of M'). The v_i can be chosen so that $L_i(v_i) \neq 0$. Let Φ_0 be a function on G' supported on $P'K'_\epsilon$, and given there by $\Phi_0(pk) = \delta_{P'}^{1/2}(m)\rho'_1(m_1)v_1 \otimes \cdots \otimes \rho'_b(m_b)v_b$ $(p = nm \in P' = N'M', k \in K'_\epsilon)$. Then $D(\Phi_0)$ is $\prod_i L_i(v_i)$ up to a volume factor, and (1) follows.

Remark. (a) If the ρ'_i are unramified, then so is π'. If each L_i takes a non-zero value on the K'_i-fixed vector v_i in ρ'_i, then D is non-zero on the K'-fixed vector in π', which is defined as Φ_0 is, but with K'_ϵ replaced by K'.
(b) The proof of (1) is easy to "globalize". Suppose that $\rho'_1, \ldots, \rho'_b$ are $\mathbb{G}_i$-distinguished cuspidal $\mathbb{G}'_i$-modules, and Φ is an element in the normalizedly induced $\mathbb{G}'$-module $I(\rho')$, thus

$$\Phi(nmg) = \delta_{P'}^{1/2}(m)\phi_1(m_1)\cdots\phi_b(m_b)\Phi(g) \qquad (n \in \mathbb{N}', m \in \mathbb{M}', g \in \mathbb{G}')$$

for some $\phi_i \in \rho'_i$. Then

$$D(\Phi) = \int_{\mathbb{A}P\backslash\mathbb{G}} \Phi(g)dg = \int_{\mathbb{A}M\backslash\mathbb{M}} \int_{\mathbb{K}\cap\mathbb{M}\backslash\mathbb{K}} \Phi(mk)dm\,dk$$

(the last equality follows from the decomposition $G = NMK$ and the equality $\delta_{P'} = \delta_P^2$) defines a $\mathbb{G}$-invariant linear form on $I(\rho')$, which is easily seen to be non-zero.

(2) Let $\underline{P}$ denote the parabolic subgroup of type (m,m) in $\underline{G}=GL(n)$, and consider the Eisenstein series $E(x,\Phi,\lambda)$ associated with an element Φ in the space of the induced $\pi'=\pi'_0$, where $\pi'_\lambda=I(\rho'\nu^\lambda,\check{\bar{\rho}}'\nu^{-\lambda})$. Here $\mathfrak{A}$ is one-dimensional, and the series converges for λ in $\mathbb{C}$ with $Re\,\lambda>\frac{1}{2}$. The map $\Phi\mapsto E(\Phi,\lambda)$ yields an embedding of π'_λ into the space of automorphic forms. We shall construct a $\mathbb{G}$-invariant functional on π' on studying the form

$$D(\Phi,\lambda,T)=\int_{\mathbb{Z}G\backslash\mathbb{G}}\Lambda^T E(x,\Phi,\lambda)dx$$

at $\lambda=0$ and $T=\infty$ (T is now a sufficiently large real number). Denote by χ_T the characteristic function of (T,∞) in $\mathbb{R}$. By Lemma 11, $\Lambda^T E(x,\Phi,\lambda)$ is the sum over $\delta\in P'\backslash G'$ of

$$\Phi(\delta x)e^{\langle\lambda+\rho_{P'},H(\delta x)\rangle}[1-\chi_T(H(\delta x))]-(M(s,\lambda)\Phi)(\delta x)e^{\langle s\lambda+\rho_{P'},H(\delta x)\rangle}\chi_T(H(\delta x)).$$

By Proposition 3, the space $P'\backslash G'$ is the union of $P\backslash G$ and $\eta\cdot(\eta^{-1}P'\eta\cap G)\backslash G$. Integrating over $\mathbb{Z}G\backslash\mathbb{G}$, we obtain the sum of two terms. The one corresponding to $P\backslash G$ is easily evaluated to be

$$D_M(\Phi)\cdot e^{\lambda T}/\lambda-D_M(M(s,\lambda)\Phi)\cdot e^{-\lambda T}/\lambda,$$

where

$$D_M(\Phi)=\int_{\mathbb{K}}\int_{\mathbb{A}M\backslash\mathbb{M}}\Phi(mk)dm\,dk.$$

The second term, denoted by $J(\Phi,\lambda,T)$, is the integral over $\mathbb{Z}(G\cap\eta^{-1}P'\eta)\backslash\mathbb{G}$ of

$$\Phi(\eta x)e^{\langle\lambda+\rho_{P'},H(\eta x)\rangle}[1-\chi_T(H(\eta x))]-(M(s,\lambda)\Phi)(\eta x)e^{\langle s\lambda+\rho_{P'},H(\eta x)\rangle}\chi_T(H(\eta x)).$$

Note that $\eta\overline{\eta}^{-1}=s$, $P_s=P'\cap G_s$, and $x\mapsto H(x\eta)$ is left invariant under $\mathbb{P}_s$. Suppose that $\rho'\not\simeq\check{\bar{\rho}}'$. Then the integral of $\Phi(xy)$ over x in $P_s\backslash\mathbb{P}_s$ is zero (same for $M(s,\lambda)\Phi$, at least at $\lambda=0$). Hence $J(\Phi,0,T)=0$. Moreover, as in the proof of Proposition 19, we have $D_M(M(s,0)\Phi)=D_M(\Phi)$. Since

$$(e^{\lambda T}-e^{-\lambda T})/\lambda=2T+\lambda T^2/3+\dots$$

is holomorphic in $\lambda\in\mathbb{C}$ we conclude that the required (non-zero, $\mathbb{G}$-invariant) functional is given by $\Phi\mapsto\lim_{T\to\infty}(D(\Phi,0,T)/4T)=D_M(\Phi)$.

Remark. (a) A purely local proof of (2) could be attempted as follows. In local notations, consider an admissible irreducible $GL(m,E)$-module (ρ',V), and the normalizedly induced G'-module $\pi'=I(\rho',\check{\bar{\rho}}')$. An element Φ in π' is a function $\Phi:G'\to V\otimes\check{V}$ with

$$\Phi(\begin{pmatrix}a&*\\0&b\end{pmatrix}g)=|det\,ab^{-1}|_E^{1/2}\rho'(a)\otimes\check{\rho}'(\bar{b})\Phi(g)\quad(a,b\in GL(m,E);g\in G').$$

Note that $\check{\overline{\rho}}'(b) = \rho'({}^t\overline{b}^{-1})$. Introduce a form $L : V \otimes \check{V} \to \mathbb{C}$ by $L(v \otimes \lambda) = \langle v, \lambda \rangle$. Put $J_0 = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$, $P_w = \{p \in P'; \overline{p} = wpw\}$ for $w \in W$, $w^2 = 1$, where P' is the standard parabolic subgroup of type (m, m), and $G_w = \{g \in G'; \overline{g} = wgw\}$. The integral

$$D(\Phi) = \int_{P_{J_0} \backslash G_{J_0}} L(\Phi(g)) dg$$

is well-defined. Indeed, any element of P_{J_0} is of the form $p = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}$, $a \in GL(m, E)$, and

$$L((\rho'(a) \otimes \check{\rho}'(\overline{b}))(v \otimes \lambda)) = \langle \rho'(a)v, \check{\rho}'(\overline{b})\lambda \rangle = \langle \rho'(\overline{b})^{-1}\rho'(a)v, \lambda \rangle$$

is $\langle v, \lambda \rangle = L(v \otimes \lambda)$ if $b = \overline{a}$.

By construction the functional $\Phi \mapsto D(\Phi)$ is distinguished, namely invariant under G, or more precisely the conjugate group $G_{J_0} = \eta_{J_0}^{-1} G \eta_{J_0}$. Moreover, D is non-zero. To see this, choose $v \in V$ and $\lambda \in \check{V}$ such that $\langle v, \lambda \rangle \neq 0$. Then there is some congruence subgroup K'_ϵ of K' such that $\rho'(a)v = v$ and $\check{\rho}'(b)\lambda = \lambda$ for any $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ in $K'_\epsilon \cap M'$. Define a function Φ_0 on G' which is supported on $P'K'_\epsilon$, with

$$\Phi_0\left(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} k\right) = |\det ab^{-1}|_E^{1/2} \rho'(a)v \otimes \check{\rho}'(\overline{b})\lambda \qquad \left(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \in P', k \in K'_\epsilon\right).$$

Then $D(\Phi_0)$ is equal to $\langle v, \lambda \rangle (\neq 0)$ up to a volume factor.

The claim (2) would follow once it is shown that the integral which defines $D(\Phi)$ is always convergent, but we do not have a proof for that.

In this context, note that by Proposition 3 we have the disjoint decomposition $G' = \cup B'\eta_w G$ over $w \in W$, $w^2 = 1$ (where $\eta_w \overline{\eta}_w^{-1} = w$), implying that $P'\backslash G' = \cup P'\backslash P'\eta_w G$ where w ranges over W modulo W_M and $w^2 = 1$. The quotient $P'\backslash G' = P' \cap K' \backslash K'$ is compact, and $P'\backslash P'\eta_{J_0} G \simeq P_{J_0}\backslash G_{J_0}$ is an open dense subset of $P'\backslash G'$. However, $P_{J_0}\backslash G_{J_0}$ is not compact in itself.

(b) An alternate approach to constructing a global distinguished functional will be to consider the integral over $G\mathbb{Z}\backslash\mathbb{G}$ of the untruncated Eisenstein series $E(x, \Phi, \rho', \lambda)$ for those λ where the integral converges, as discussed in the Remark following Proposition 9. The proof of the convergence of $\int E$ in that Remark is based on a comparison to the integral of a constant term E_1. To show the non-vanishing of $\int E$ one would need to show that $\int(E - \Sigma E_1)$ does not cancel $\Sigma \int E_1$. But we have not tried to pursue this line of reasoning.

We now return to the expression for the Fourier summation formula of Proposition 17. It will be presented in the format of Corollary 8. As there, let v be a place of F where ω, κ and E/F are unramified, and let $f'_v \in H(G'_v)$ be a spherical function. For any irreducible admissible G'_v-module π'_v the operator $\pi'_v(f'_v)$ is zero unless π'_v is unramified, and then $\pi'_v(f'_v)$ acts as the scalar $f'^\vee_v(t(\pi'_v))$ on the unique (up to scalar) non-zero K'_v-fixed vector, and as zero on any vector in π'_v orthogonal to the K'_v-fixed one. Let V be a set of F-places containing the archimedean places and those where ω, κ or E/F ramify. The Fourier summation formula will be used with $f' = \otimes f'_v$ in $C(\mathbb{G}')$ such that f'_v is spherical for all $v \notin V$. Denote by $t(\lambda, \rho'_v)$ the class in $\hat{G}'_v$ parametrizing the induced G'_v-module $I(\lambda, \rho'_v)$. Put $f'_V = \underset{v \in V}{\otimes} f'_v$, $f'^V = \underset{v \notin V}{\otimes} f'_v$, $\pi'_V = \underset{v \in V}{\otimes} \pi'_v$, $\pi'^V = \underset{v \notin V}{\otimes} \pi'_v$, etc. Also write $f'^\vee(t(\pi'^V))$ for $\prod_{v \notin V} f'^\vee(t(\pi'_v))$ and $f'^\vee(t(\lambda, \rho'^V))$ for $\prod_{v \notin V} f_v^\vee(t(\lambda, \rho'_v))$.

27. Corollary. *Suppose that each archimedean place of F splits in E. For $f' = \otimes f'_v$ with f'_v spherical for all v outside V we have*

$$\sum_w \sum_a \Psi'(aw, f'; \psi') = \sum_{I(\rho')} |\mathfrak{A}_P / L_P| \cdot n(\rho') \cdot f'^\vee(t(I(\rho')^V)) \cdot WD_{I(\rho'),\psi'}(f'_V)$$

$$+ \sum_{P' \subset G'} \sum_{1 \le j \le J} \sum_{\rho', \Phi} \int f'^\vee(t(\lambda, \pi'^V)) \cdot F_j(\lambda, \Phi, \rho') \cdot E_{\psi'}(I(f'_V, \lambda, \rho'_V)\Phi, \rho', \lambda) \cdot d\lambda.$$

The sums over a and w are as in Propositions 7 and 9; the ρ' are cuspidal distinguished $\mathbb{M}'$-modules, which are unramified outside V, where in the first sum (over $I(\rho')$) $s\rho' = \rho'$ implies $s = 1$ for any $s \in W(\mathfrak{A}, \mathfrak{A}_2)$, $P'_2 \subset G'$. The Φ range over an orthonormal basis for the vectors in $H_{P'}(\rho')$ which are K'_v-invariant for all v outside V. The integrals range over non zero dimensional subspaces $i\mathfrak{A}_j^*$ of $i\mathfrak{A}_{P'}^*$.

This is the final form of the Fourier summation formula for f' on $\mathbb{G}'$, to be used in the comparison, and our last result which does not depend on Conjecture 6. We now assume Conjecture 6, and in particular that the unit elements f_v^0 and f'^0_v in the Hecke algebras $H(U_v)$ and $H(G'_v)$ of spherical functions on U_v and G'_v are matching (the definition was given after Conjecture 5). For the reason explained in the Remark following Lemma 20, we assume that each archimedean place of F splits in E.

28. Proposition*. *Let V be a finite set of F-places containing the archimedean places and those where ω, κ or E/F ramify. For any $v \notin V$ fix a conjugacy class $t_v = t(\pi_v)$ in $\hat{U}_v$, parametrizing an unramified irreducible generic U_v-module π_v^*, and put $t'_v = b_{\kappa_v}(t_v) \in \widehat{G}'_v$; t'_v parametrizes an unramified irreducible generic G'_v-module π'^*_v. For any v in V let $f_v \in C(U_v)$ and $f'_v \in C(G'_v)$ be matching functions on U_v and G'_v. Then*

$$\sum_\pi n(\pi) \cdot W_{\pi_V}(f_V) = \sum_{I(\rho')} vol(\rho') \cdot n(\rho') \cdot WD_{I(\rho'),\psi'}(f'_V). \tag{28.1}$$

The sum over π ranges over the equivalence classes of cuspidal generic $\mathbb{U}$-modules π with $t(\pi_v) = t_v$ for all v outside V; $n(\pi)$ denotes the multiplicity of π in $L_{0,\omega}(U\backslash\mathbb{U})$. The sum over $I(\rho')$ ranges over all equivalence classes of $\mathbb{G}'$-modules normalizedly induced from cuspidal generic $\mathbb{M}$-distinguished $\mathbb{M}'$-modules ρ', such that $I(\rho'_v) = I(\rho'_v; G'_v, M'_v)$ is (equivalent to) π'^*_v (thus $t(\rho'_v, 0) = t'_v$) for all $v \notin V$, and $s\rho' = \rho'$ for $s \in W(\mathfrak{A}, \mathfrak{A}_2)$, $P'_2 \subset G'$, implies $s = 1$; $vol(\rho')$ is a volume factor depending only on M'.

Proof. By Conjecture 6 we may consider matching $f = \otimes f_v \in C(\mathbb{U})$ and $f' = \otimes f'_v \in C(\mathbb{G}')$ whose components at $v \in V$ are as in the proposition, while at $v \notin V$

they are matching spherical functions, almost all the unit elements in the Hecke algebras. By definition of matching, and since the sets over which w and a are taken in Corollaries 8 and 27 are equal, the geometric sides of the two Fourier summation formulae, for U and for G', are equal. For such f and f' we then have that the representation theoretic sides are equal. Namely the difference of their discrete parts is equal to the difference of their continuous parts (which involve integrals over non-zero dimensional spaces). By Theorem 2, and especially the Proposition on p. 198, of [FK2], the unitarity of the modules involved, the Stone-Weierstrass theorem, and the absolute convergence of the sums which appear in the summation formulae, imply that each of these differences is zero, and moreover the identity stated in the proposition holds.

Remark. By Zelevinsky [Z], (9.7b), an unramified irreducible generic G'_v-module π'_v is a full-induced (from a character of the upper triangular subgroup) representation. The rigidity theorem of [JS$_{\rm II}$] implies that $\{\pi'_v; v \notin V\}$ specifies uniquely the conjugacy class of $\underline{M}'$ (namely the associated class of $\underline{P}'$) and the cuspidal $\mathbb{M}'$-module ρ'. Hence the right side of (28.1) has at most one nonzero entry for a fixed choice of $\{t'_v; v \notin V\}$. Reducibility properties of U_v-modules π_v normalizedly induced from unitary characters μ_v of the upper triangular subgroup are listed in Keys [K1], Theorem, p. 127, when n is odd (when the character is unramified, the induced representation is irreducible), and in [K2], Theorem 3.6, p. 48, when n is even. In the even case, reducibility occurs: e.g., when $n = 2$, and the restriction of μ_v to $\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}$ with $a \in F_v^\times$ is the sign (unramified) character, then π_v is reducible, and only one of its constituents is generic.

Proof of Theorem 1.* Suppose that $\rho' = \rho'_1 \times \cdots \times \rho'_a$ is a cuspidal $\mathbb{M}$-distinguished $\mathbb{M}'$-module with $\rho'_i \not\simeq \rho'_j$ for any $i \neq j$. Since $\pi' = I(\rho')$ is non-degenerate, there exists some Φ in $H_{P'}(\rho')$ with $E_{\psi'}(\Phi, \rho', 0) \neq 0$. Since ρ' is distinguished, by Lemma 18 there is a Φ' with $D(\Phi') \neq 0$. Choose a sufficiently large finite set V of F-places such that both Φ and Φ' are K'_v-invariant for all $v \notin V$. Write $\pi'^{\mathbb{K}'(V)}$ for the space of vectors in π' which are K'_v-fixed for all $v \notin V$. The set $\{\pi'_V(f'_V);$ all $f'_V\}$ of operators act transitively on $\pi'^{\mathbb{K}'(V)}$. Hence we may and do choose f'_V with $\pi'_V(f'_V)\Phi' = \Phi$, and $\pi'_V(f'_V)\Phi_1 = 0$ for every Φ_1 orthogonal to Φ'. Apply Proposition 28* with our V and with $\{t'_v = t(I(\rho'_v)); v \notin V\}$. Then the right side of the identity (28.1) is equal to

$$\mathrm{vol}(\rho') \cdot n(\rho') \cdot W_{I(\rho'),\psi'}(\Phi) \cdot D(\overline{\Phi}') \neq 0.$$

Hence the sum on the left is non-empty, and there is a cuspidal generic $\mathbb{U}$-module π which base-changes via b_κ to the generic automorphic $\mathbb{G}'$-module $I(\rho')$.

In the opposite direction, let π_0 be a cuspidal generic $\mathbb{U}$-module. Then there is some vector Φ_0 in π_0 with $W_{\Phi_0,\psi}(I) \neq 0$ (this Whittaker functional was defined after Proposition 7). Choose a sufficiently large finite set V such that Φ_0 is $\mathbb{K}(V) = \prod_{v \notin V} K_v$-invariant. For each $v \in V$ choose a compact open subgroup K_{1v} in K_v such that Φ_0 is K_{1v}-fixed. Put $\mathbb{K}_1 = \mathbb{K}(V) \prod_{v \in V} K_{1v}$. Choose an orthonormal basis to the space of $\mathbb{K}_1$-fixed vectors in $L^2_{0,\omega}(U\backslash\mathbb{U})$; it can be viewed as a subset of an orthonormal basis of the space of $\mathbb{K}(V)$-fixed vectors in $L^2_{0,\omega}(U\backslash\mathbb{U})$. Let H_{1v} be the convolution algebra of K_{1v}-biinvariant complex valued measures on U_v which transform under Z_v via ω_v^{-1} and are compactly supported modulo Z_v (note that Z_v is not compact when v splits). Put $f_V = \underset{v\in V}{\otimes} f_v$, where f_v is the unit element in H_{1v}. If Φ lies in $\pi^{\mathbb{K}(V)}$ ($\subset L_{0,\omega}(U\backslash\mathbb{U})$), then $\pi_V(f_V)$ acts trivially on Φ if Φ is $\mathbb{K}_1$-invariant, and its maps Φ to 0 if Φ is in the orthogonal complement to this subspace. Apply Proposition 28* with the set V and the sequence $\{t_v = t(\pi_{0v}); v \notin V\}$. The left side of the identity (28.1) is

$$\sum_\pi n(\pi) \sum_{\Phi\in\pi^{\mathbb{K}(V)}} W_{\pi_V(f_V)\Phi,\psi}(I)\cdot \overline{W}_{\Phi,\psi}(I) = \sum_\pi n(\pi) \sum_{\Phi\in\pi^{\mathbb{K}_1}} |W_{\Phi,\psi}(I)|^2.$$

Since $\pi_0^{\mathbb{K}_1}$ contains Φ_0, this is positive. Consequently the sum on the right of (28.1) is non-empty. By the rigidity theorem of [JS$_{\rm II}$] the sequence $\{t'_v = b_{\kappa_v}(t(\pi_{0v})); v \notin V\}$ determines only one equivalence class of $\mathbb{G}'$-modules $I(\rho')$ on the right. Hence π_0 base changes via b_k to $I(\rho')$, as required.

A few local conclusions can be drawn from the global theory.

Definition. An irreducible admissible G'_v-module π'_v is said to be *in the image* of the *suitable*, namely stable (if n is odd) or unstable ($= b_{\kappa_v}$, if n is even), *base change lifting* if it is a component of an automorphic $\mathbb{G}'$-module π' which is the base change lift via b_v (n odd) or b_{κ_v} (n even) of an automorphic generic $\mathbb{U}$-module.

Note that this definition is special to the present context. Better definitions, purely local, are given in [F3] for $n = 2$ and in [F4] for $n = 3$, and can easily be stated for a general n. The main drawback of the definition here is that it concerns only generic π'_v (since π' is a lift of a generic (cuspidal) $\mathbb{U}$-module). There are also interesting non-generic $\mathbb{G}'$-modules which can be defined to be a base-change lift from cuspidal $\mathbb{U}$-modules, which are, however, non-generic. When $n = 3$ an example of this is given by some $GL(3,\mathbb{A}_E)$-modules normalizedly induced from some characters of the maximal parabolic subgroup; see [F4].

It follows at once from Theorem 1* that if π'_v is a G'_v-module in the image of the suitable base change lifting, then π'_v is G_v-distinguished. Indeed, any component of a globally distinguished $\mathbb{G}'$-module is G_v-distinguished. Conversely, suppose that π'_v is a generic admissible irreducible G_v-distinguished G'_v-module. By [Z], (9.7b), since π'_v is generic, there are square-integrable G'_{iv}-modules ρ'_{iv}, and complex numbers s_i, such that $\pi'_v = I((\rho'_{iv}\nu_v^{s_i}))$, where $\nu_v(x) = |x|_{E_v}$. Since π'_v is distinguished, it satisfies $\check{\overline{\pi}}'_v \simeq \pi'_v$ by [F1], Proposition 12. Hence for each i there is $j \neq i$ with $\check{\overline{\rho}}'_{jv} \simeq \rho'_{iv}$ and $s_j + s_i = 0$, and if not then $s_i = 0$ and $\check{\overline{\rho}}'_{iv} \simeq \rho'_{iv}$ and ρ'_{iv} is G_{iv}-distinguished (the last fact is easily seen). It is easy to see that the U_{iv}-module $I(\rho'_{iv})$, normalizedly induced from the representation $\begin{pmatrix} a & * \\ 0 & b\end{pmatrix} \mapsto \rho'_{iv}(a)$ of the standard parabolic subgroup of type (n_i, n_i) of the unitary group formed from $GL(2n_i)$ and E_v/F_v, base changes to $I(\rho'_{iv} \times \check{\overline{\rho}}'_{iv})$ via the stable map and to $I(\rho'_{iv}\otimes\kappa_v, \check{\overline{\rho}}'_{iv}\otimes\kappa_v)$ via the unstable map. Generalizing this comment to the case of other parabolic subgroups (symmetric with respect to the anti-diagonal), we may assume that the ρ'_{iv} are all G_{iv}-distinguished and pairwise inequivalent.

Now Proposition 14 of [F1] asserts that there exists a cuspidal $\mathbb{M}$-distinguished $\mathbb{M}'$-module ρ' whose component at v is our (ρ'_{iv}). Applying Theorem 1* to $\pi' = I(\rho')$, we deduce that π' is a suitable base change lift of a generic cuspidal $\mathbb{U}$-module. Hence π'_v is in the image of the suitable base change lifting. In conclusion, we proved

29. Proposition*. *An admissible irreducible generic G'_v-module π'_v is in the image of the suitable base change lifting if and only if it is G_v-distinguished.*

Remark. Note that in the local archimedean case where $E/F = \mathbb{C}/\mathbb{R}$, the following statement seems to be known (cf. Bien [B]). Each irreducible admissible principal series $H = GL(n,\mathbb{R})$-distinguished $G = GL(n,\mathbb{C})$-module is of the form

$$I(\mu_1, \overline{\mu}_1^{-1}, \dots, \mu_k, \overline{\mu}_k^{-1}, \mu_{k+1}/\overline{\mu}_{k+1}, \dots, \mu_{n-2k}/\overline{\mu}_{n-2k}),$$

namely it is normalizedly induced from a character of the upper triangular parabolic subgroup which (is trivial on the unipotent subgroup and) maps the diagonal matrix

$$(z_1, \dots, z_n) \qquad \text{to} \qquad \prod_{1 \le i \le k} \mu_i(z_{2i-1}/\overline{z}_{2i}) \prod_{1 \le i \le n-2k} \mu_i(z_{2k+i}/\overline{z}_{2k+i}) \;\; (z_i \in \mathbb{C}^\times).$$

In other words, these representations occur in $C^\infty(G/H)$. More generally, it appears – but we have not checked this – that known techniques (e.g. of [B]) would imply that all irreducible admissible H-distinguished G-modules are induced from a parabolic subgroup P where on the Levi factor the (irreducible admissible) representations which occur are distinguished or occur in pairs consisting of ρ and $\check{\overline{\rho}}$. Taking P to be minimal the local archimedean analogue of our Theorem 1* could be checked.

It is also known that when the μ_i are unitary the principal series representations listed above occur in $L^2(G/H)$, discretely precisely when $k = 0$ and $\mu_i(z/\overline{z}) \neq \mu_j(z/\overline{z})$ for all $i \neq j$ as characters of z in $\mathbb{C}^\times$; see Sano [S], and Bopp-Harinck [BH] in the analogous case of $U(p,q)$-distinguished $GL(n,\mathbb{C})$-modules.

It will be interesting to make - and prove - a conjecture characterizing the H-distinguished G-modules, where G/H is a pseudo-Riemannian symmetric space (H is an open subgroup of the group of fixed points of an involution on the real form G of a complex reductive Lie group $G_{\mathbb{C}}$) as in [FJ], [OM], [B], [S] and [BH], possibly as the image of a lifting map from another group. At least in our (and [S]'s) case of $G/H = GL(n,\mathbb{C})/GL(n,\mathbb{R})$, and in the dual case of [BH], the representations occuring in $C^\infty(G/H)$ seem to be described by the lifting of the Theorem 1*.

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