SUMMATION FORMULAE, AUTOMORPHIC REALIZATIONS AND A SPECIAL VALUE OF EISENSTEIN SERIES

Yuval Z. Flicker and J. G. M. Mars

Let F be a global field of characteristic other than 2, F_v its completion at a place v, $\mathbb{A}$ its ring of adeles and $\psi : \mathbb{A} \to \mathbb{C}^\times$ a non-trivial additive character which is trivial on the discrete subgroup F of $\mathbb{A}$. Let $C(F_v)$ be the Schwartz space of F_v if v is archimedean, and the space $C_c^\infty(F_v)$ of locally constant compactly supported $\mathbb{C}$-valued functions on F_v if v is non-archimedean. Let $f_v^0 (\in C(F_v))$ be the characteristic function of the ring R_v of integers of F_v in the latter case. Denote by $C(\mathbb{A})$ the $\mathbb{C}$-span of $\otimes_v f_v$, $f_v \in C(F_v)$ for all $v, f_v = f_v^0$ for almost all v. Denote by ψ_v the component of ψ at v, and let $d_v y$ be the Haar measure of F_v normalized to have the property that the Fourier transform

$$f_v \to \mathcal{F} f_v, \qquad \mathcal{F} f_v(x) = \int_{F_v} f_v(y)\psi_v(xy) d_v y,$$

is an endomorphism of the vector space $C(F_v)$ which satisfies the Fourier inversion formula $(\mathcal{F}(\mathcal{F} f_v))(x) = f_v(-x)$. Write $\mathcal{F}(\otimes_v f_v)$ for $\otimes_v \mathcal{F} f_v$. One has the well-known

POISSON SUMMATION FORMULA. *The distribution* $D(f) = \sum_{x\in F} f(x)$ *on* $C(\mathbb{A})$ *satisfies* $D(f) = D(\mathcal{F} f)$

This formula follows easily from the Fourier inversion formula (see, e.g., [L], XIV, §6, p. 291), and has many applications. One of these applications concerns the θ-(or Weil, oscillator, smallest) representation of the unique central topological two-fold covering (metaplectic) group

$$1 \to \{\pm 1\} \to S_v \underset{s}{\overset{p}{\rightleftharpoons}} \overline{S}_v \to 1, \qquad 1 \to \{\pm 1\} \to S_{\mathbb{A}} \underset{s}{\overset{p}{\rightleftharpoons}} \overline{S}_{\mathbb{A}} \to 1$$

of $\overline{S}_v = SL(2, F_v)$, $\overline{S}_{\mathbb{A}} = SL(2, \mathbb{A})$. As usual (see [K], or [F], [FKS]), the elements of S_v and $S_{\mathbb{A}}$ will be described as pairs (g, ζ), or $\zeta s(g)$, with ζ in $ker\, p = \{\pm 1\}$ and g in $\overline{S}_v$ or $\overline{S}_{\mathbb{A}}$, and with product rule

$$\zeta s(g)\zeta' s(g') = \zeta\zeta' \beta(g, g') s(gg').$$

Department of Mathematics, The Ohio State University, 231 W. 18th Ave., Columbus, OH 43210-1174. Email: flicker@math.ohio-state.edu

Mathematisch instituut, Rijksuniversiteit Utrecht, Budapestlaan 6, Postbus 80.010, 3508 TA Utrecht, The Netherlands. Email: mars@math.ruu.nl

Partially supported by Nato grant CRG-900080

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $GL(2)$, put $t(g) = (c, d/det\, g)$ if $cd \neq 0$ and $ord\ c$ is odd, and $t(g) = 1$ otherwise; here $(.,.)$ is the Hilbert symbol. Put

$$\alpha(g,g') = \left(\frac{x(gg')}{x(g)}, \frac{x(gg')}{x(g')det\, g}\right), \qquad x\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} c, & c \neq 0, \\ d, & c = 0. \end{cases}$$

Then (the restriction to $SL(2)$ of) $\beta(g,g') = \alpha(g,g')t(g)t(g')t(gg')^{-1}$ is a two-cocycle of $\overline{S}_v$ in $\{\pm 1\}$, uniquely determined by the choice of the section s to the projection p. Define a two-cocycle $\beta_{\mathbb{A}}$ on $\overline{S}_{\mathbb{A}}$ by $\beta_{\mathbb{A}} = \Pi_v \beta_v$.

Let $\gamma_v : F_v^\times \to \mathbb{C}^\times$ be the twisted character defined by

$$\gamma_v(x)^{-1} = |x|_v^{1/2} \int \psi_v(\frac{1}{2}xy^2)d_v y \Big/ \int \psi_v(\frac{1}{2}y^2)d_v y$$

(or $\gamma_v(x) = |x|_v^{1/2} \int \psi_v(-\frac{1}{2}xy^2)d_v y / \int \psi_v(-\frac{1}{2}y^2)d_v y$) introduced by Weil [We; 1964] (see also [F], [FKS]). It satisfies $\gamma_v(a)\gamma_v(b) = \gamma_v(ab)(a,b)_v$. Then $\gamma_v : F_v^\times / F_v^{\times^2} \to \mathbb{C}^\times$ has order 4, and $\gamma_{\mathbb{A}} = \Pi_v \gamma_v$ is trivial on the subgroup $F^\times \mathbb{A}^{\times^2}$ of the group $\mathbb{A}^\times$ of ideles. The representation θ_v of S_v is defined on the space $C(F_v)$ by means of the operators

$$\left(\theta_v\left(\zeta s \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) f_v\right)(x) = \zeta \psi_v(\frac{1}{2}bx^2) f_v(x),$$

$$\left(\theta_v\left(\zeta s \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) f_v\right)(x) = \zeta c_v (\mathcal{F} f_v)(-x),$$

$$\left(\theta_v\left(\zeta s \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) f_v\right)(x) = \zeta \gamma_v(a) |a|_v^{1/2} f_v(ax)$$

$(a \in F_v^\times,\ \ b \in F_v,\ \ \zeta \in \{\pm 1\} = ker\ p)$, where $c_v = \gamma_v(-1)^{-1/2}$ is an eighth root of unity in $\mathbb{C}$ ($c_v = 1$ for almost all v and $\Pi_v c_v = 1$). Note that $SL(2, F_v)$ is generated by the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, and that the discrete subgroup $\overline{S}(F) = SL(2,F)$ of $\overline{S}_{\mathbb{A}}$ injects as a subgroup of $S_{\mathbb{A}}$ by $g \mapsto t(g)s(g)$. The representation $\theta_{\mathbb{A}}$ of $S_{\mathbb{A}}$ is defined as the restricted tensor product $\theta_{\mathbb{A}} = \otimes_v \theta_v$. A function $h : S_{\mathbb{A}} \to \mathbb{C}$ is called genuine if $h(\zeta g) = \zeta h(g)$ ($\zeta \in ker\ p$), and automorphic if $h(\gamma g) = h(g) (\gamma \in \overline{S}(F))$. An $S_{\mathbb{A}}$-module is called automorphic if it is equivalent to a subquotient of the representation of $S_{\mathbb{A}}$ on the space $L^2(\overline{S}(F)\backslash S_{\mathbb{A}})_{gen}$ of genuine square-integrable complex-valued functions on $\overline{S}(F)\backslash S_{\mathbb{A}}$, by right translation. The summation formula implies

AUTOMORPHIC REALIZATION. *For each* $f \in C(\mathbb{A})$, *the function* $D_f(g) = D(\theta_{\mathbb{A}}(g)f)$ *is automorphic.*

Namely $D(\theta_{\mathbb{A}}(\gamma g)f) = D(\theta_{\mathbb{A}}(g)f)$ for all $\gamma \in \overline{S}(F),\ g \in S_{\mathbb{A}}$. It is easy to see that D_f lies in $L^2(\overline{S}(F)\backslash S_{\mathbb{A}})_{gen}$, and that the distribution $f \mapsto D_f$ intertwines the

θ-representation $(\theta_{\mathbb{A}}, C(\mathbb{A}))$ with the regular representation of $S_{\mathbb{A}}$ on $L^2(\overline{S}(F)\backslash S_{\mathbb{A}})_{gen}$. In particular the distribution D realizes $\theta_{\mathbb{A}}$ as an automorphic representation by virtue of the Poisson summation formula.

We shall now develop a new summation formula, and relate it to the automorphic realization of a $GL(2)$-analogue of θ.

To state the new summation formula, for a finite place v let $C(F_v^\times)$ denote the space of locally constant $\mathbb{C}$-valued functions f_v on $F_v^\times$ whose support is bounded in F_v, for which there is a constant $A(f_v) > 0$ with the property that $f_{v0}(x) = |t|_v^{1/2} f_v(t^2 x)$ is independent of $t \in F_v^\times$ provided that $|t|_v \leq A(f_v)$ and $|x|_v \leq 1$. Then $|.|^{1/4} f_{v0}$ extends to a function on $F_v^\times / F_v^{\times 2}$. When v is archimedean, $C(F_v^\times)$ consists of smooth functions on $F_v^\times$ with rapid decay at ∞ and $t \mapsto |t|_v^{1/2} f_v(t^2 x)$ smooth at $t = 0$. Put $f_{v0}(x) = \lim_{t \to 0} |t|_v^{1/2} f_v(t^2 x)$. Denote by $val_v : F_v^\times \twoheadrightarrow \mathbb{Z}$ the normalized additive valuation on $F_v^\times$ when v is non-archimedean. Then $|x|_v = q_v^{-val_v(x)} (x \in F_v^\times)$, where q_v is the cardinality of the residue field of R_v. Let f_v^0 be the element of $C(F_v^\times)$ whose value at x is zero unless $val_v(x)$ is even and positive, where $f_v^0(x) = |x|_v^{-1/4}$. Put $C(\mathbb{A}^\times)$ for the $\mathbb{C}$-span of the functions $f = \otimes_v f_v$, where $f_v = f_v^0$ for almost all v. Put

$$f_0((x_v)) = \Pi_v f_{v0}(x_v) \quad \text{and} \quad \mathcal{F} f = \otimes_v \mathcal{F} f_v,$$

where

$$(\mathcal{F} f_v)(x) = c_v \gamma_v(x) |x|_v^{1/2} \int_{F_v} |y|_v^{1/2} f_v(xy^2) \psi_v(xy) d_v y.$$

NEW SUMMATION FORMULA. *The distribution* $D(f) = 2 \sum_{x \in F^\times} f(x) + \sum_{x \in F^\times / F^{\times 2}} f_0(x)$ *on* $C(\mathbb{A}^\times)$ *satisfies* $D(\mathcal{F} f) = D(f)$.

Note that given f, there are only finitely many $x \in F^\times / F^{\times 2}$ with $f_0(x) \neq 0$, since $\mathbb{A}^\times / F^\times \Pi_{v|\infty} F_v^\times \Pi_{v<\infty} R_v^\times$ is finite (its cardinality is the class number of F), and so is $R_v^\times / R_v^{\times 2}$ for each v. The rapid decay of f_v at ∞ guarantees the convergence of $\sum f(x),\ x \in F^\times$.

The distribution D can be used to construct an operator intertwining a representation θ with a space of automorphic forms. This θ will be a representation of a two-fold topological central covering group

$$1 \to \{\pm 1\} \to H_v \underset{s}{\overset{p}{\rightleftharpoons}} \overline{H}_v \to 1, \qquad 1 \to \{\pm 1\} \to H_{\mathbb{A}} \underset{s}{\overset{p}{\rightleftharpoons}} \overline{H}_{\mathbb{A}} \to 1$$

of the group $\overline{H}_v = GL(2, F_v)$ and $\overline{H}_{\mathbb{A}} = GL(2, \mathbb{A})$. Up to isomorphism, there are two such covering groups which are defined by an algebraic morphism of $GL(2)$ into $SL(n)$, and the unique covering of $SL(n)$ (see [KP], §0). They are determined by the cohomology class of the two-cocycle β_v and $\beta_{\mathbb{A}} = \Pi_v \beta_v$ which defines the product on

H_v and $H_{\mathbb{A}}$. As in [K], [F], [FKS], we choose that β (defined above) which satisfies $\beta\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right) = (a,d)$. A two-cocycle $\beta' : \overline{H} \times \overline{H} \to \{\pm 1\}$ which represents the other cohomology class is given by $\beta'(g,g') = \beta(g,g')(det\, g,\, det\, g')$. Note that the representation θ_v of S_v reduces as the direct sum of two irreducible representations θ_v^+ and θ_v^-, on the spaces $C(F_v)^+$ and $C(F_v)^-$ of even $(f_v(-x) = f_v(x))$ and odd $(f_v(-x) = -f_v(x))$ functions in $C(F_v)$. Denote by $\overline{Z}_v$ and $\overline{Z}_{\mathbb{A}}$ the groups of scalar matrices in $\overline{H}_v$ and $\overline{H}_{\mathbb{A}}$. Since $Z_v = p^{-1}(\overline{Z}_v)$ is the center of $Z_v S_v = p^{-1}(\overline{S}_v \overline{Z}_v)$, θ_v^+ extends to a $Z_v S_v$-module by $\theta_v^+(s(z)) f_v = \gamma_v(z) f_v$ $(z \in \overline{Z}_v \simeq F_v^\times)$; note that the the extension is well-defined since f_v is even. The center of H_v is $Z_v^2 = p^{-1}(\overline{Z}_v^2)$, $\overline{Z}_v^2 = \{z^2; z \in \overline{Z}_v\}$, and that of $H_{\mathbb{A}}$ is $Z_{\mathbb{A}}^2 = p^{-1}(\overline{Z}_{\mathbb{A}}^2)$.

The H_v-module in question, denoted (again) by θ_v, is the induced representation $ind(\theta_v^+; H_v, Z_v S_v)$. Choosing the section $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ to the isomorphism $S_v \backslash H_v \to F_v^\times$, $g \mapsto det\, p(g)$, the space of θ_v can be viewed (e.g. on putting $f(x,t) = |x|^{-1/2} f(s\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, t)$) as consisting of $f_v : F_v^\times \times F_v \to \mathbb{C}$ with $f_v(x,t) = |t|_v^{1/2} f_v(xt^2, 1)$ (note that f_v is even in t). Writing $f_v(x)$ for $f_v(x,1)$, the group H_v acts via

$$(\theta_v(\zeta s\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) f_v)(x) = \zeta |a|_v^{1/2} f_v(ax), \qquad (\theta_v(\zeta s\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}) f_v)(x) = \zeta (x,z)_v \gamma_v(z) f_v(x),$$

$$(\theta_v(\zeta s\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}) f_v)(x) = \zeta \psi_v(\frac{1}{2} bx) f_v(x), \qquad (\theta_v(\zeta s\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) f_v)(x) = \zeta (\mathcal{F} f_v)(x).$$

When v is non-archimedean, since $C(F_v)$ consists of functions which are constant at some neighborhood of 0 in $F_v^\times$, for each $x \in F_v^\times$ the function $f_v(x,t)$ is constant near $t = 0$; hence there is $A(f_v) > 0$ such that $f_{v0}(x) = |t|_v^{1/2} f_v(xt^2)$ is independent of t if $|x|_v \le 1$ and $|t|_v \le A(f_v)$. Similar comments apply in the archimedean case. Consequently the H_v-module θ_v can be realized on the space $C(F_v^\times)$ introduced above.

The representation θ of $H_{\mathbb{A}}$ is defined as the restricted tensor product $\theta_{\mathbb{A}} = \otimes_v \theta_v$. The discrete subgroup $\overline{H}(F) = GL(2,F)$ of $\overline{H}_{\mathbb{A}}$ embeds as a subgroup of $H_{\mathbb{A}}$. The new summation formula implies

AUTOMORPHIC REALIZATION. *For each* $f \in C(\mathbb{A}^\times)$, *the function* $D_f(g) = D(\theta_{\mathbb{A}}(g) f)$ *is automorphic.*

Namely $D(\theta_{\mathbb{A}}(\gamma g) f) = D(\theta_{\mathbb{A}}(g) f)$ for all $\gamma \in \overline{H}(F)$, $g \in H_{\mathbb{A}}$. It is easy to see that $D_f \in L = L^2(\overline{H}(F) Z_{\mathbb{A}}^2 \backslash H_{\mathbb{A}})$ (= space of genuine $\mathbb{C}$-valued functions ϕ on $\overline{H}(F) \backslash H_{\mathbb{A}}$ which transform under $s(\overline{Z}_{\mathbb{A}}^2)$ according to a unitary character, such that $|\phi|^2$ is integrable on $\overline{H}(F) Z_{\mathbb{A}}^2 \backslash H_{\mathbb{A}}$), and that $f \mapsto D_f$ intertwines $(\theta, C(\mathbb{A}^\times))$ with the representation r of $H_{\mathbb{A}}$ on L by right translation. The space L splits as a direct sum (and integral) of $H_{\mathbb{A}}$-modules, and using the trace formula it is shown in [F] that $\theta_{\mathbb{A}}$ occurs discretely in

(r,L) with multiplicity one. Thus $\theta_{\mathbb{A}}$ is an automorphic representation, and D yields the unique-up-to-scalar realization of $\theta_{\mathbb{A}}$ as an automorphic representation, intertwining $C(\mathbb{A}^\times)$ with L. The analogous multiplicity one result for the $S_{\mathbb{A}}$-module $\theta_{\mathbb{A}}$ in $L^2(\overline{S}(F)\backslash S_{\mathbb{A}})_{gen}$ is proven in Waldspurger [Wa] (see also [GP] where this result of [Wa] is deduced from the theorem of multiplicity one for $H_{\mathbb{A}}$ of [F]). In particular D is the unique-up-to-scalar operator intertwining $(\theta_{\mathbb{A}}, C(\mathbb{A}))$ with $(r, L^2(\overline{S}(F)\backslash S_{\mathbb{A}})_{gen})$.

Proof of new summation formula. Given $f=\otimes f_v$ in $C(\mathbb{A}^\times)$, define $\tilde{f}_v(t,x)=|x|_v^{1/2}f_v(tx^2)(t\in F_v^\times, x\in F_v^\times)$, and $\tilde{f}_v(t,0)=\lim_{x\to 0}\tilde{f}_v(t,x)$. Put $\tilde{f}(t,x)=\prod_v \tilde{f}_v(t,x)$ on $\mathbb{A}^\times\times\mathbb{A}$. Then $\tilde{f}(t,0)=f_0(t)$, and $\tilde{f}$ satisfies $\tilde{f}(t,ax)=|a|^{1/2}\tilde{f}(ta^2,x)$. Put $f_v^*(t,x)=\int \tilde{f}_v(t,y)\psi_v(xy)dy$. Then $(\widetilde{\mathcal{F}f_v})(t,x)=|x|_v^{1/2}(\mathcal{F}f_v)(tx^2)$ is equal to $c_v\gamma_v(t)|t|_v^{1/2}f_v^*(t,tx)$. For $\alpha\in F^\times$ and $\beta\in F$ we have $\tilde{f}(\alpha,\beta)=f(\alpha\beta^2)$ and $(\mathcal{F}f)(\alpha\beta^2)=(\widetilde{\mathcal{F}f})(\alpha,\beta)=f^*(\alpha,\alpha\beta)$. Hence for any α in $F^\times$ we have that

$$f_0(\alpha)+\sum_{\beta\in F^\times}f(\alpha\beta^2)=\sum_{\beta\in F}\tilde{f}(\alpha,\beta)$$

is equal, by virtue of the Poisson summation formula applied to the function $x\mapsto\tilde{f}(\alpha,x)$ on $\mathbb{A}$, to

$$\sum_{\beta\in F}f^*(\alpha,\beta)=\sum_{\beta\in F}f^*(\alpha,\alpha\beta)=\sum_{\beta\in F}(\widetilde{\mathcal{F}f})(\alpha,\beta)=\sum_{\beta\in F^\times}(\mathcal{F}f)(\alpha\beta^2)+(\mathcal{F}f)_0(\alpha).$$

Summing over α in $F^\times/F^{\times 2}$ we obtain that the expression

$$\sum_{\alpha\in F^\times/F^{\times 2}}f_0(\alpha)+2\sum_{\alpha\in F^\times}f(\alpha)=\sum_{\alpha\in F^\times/F^{\times 2}}[\sum_{\beta\in F^\times}f(\alpha\beta^2)+f_0(\alpha)]$$

is invariant under the replacement of f by $\mathcal{F}f$, as required.

Our final aim is to show that $D(f)$ is obtained as a special value of a standard Eisenstein series (defined below), both in the case of S and H.

EVALUATION. *The value of* $E(s,g,f)$ *at* $s=0$ *and* $g=id$ *is* $D(f)$.

The Evaluation is a Siegel-Weil formula for a quadratic form in one variable. Such formulae have been obtained by Siegel [S], Weil [We; 1965], Mars [M], Igusa [I], Rallis [R], and Kudla-Rallis [KR]. In the case of $S=SL(2)$ this Evaluation is due also to Helminck [H], p. 67, who studied the analytic properties of the Fourier coefficients of the Eisenstein series, and deduced a functional equation, holomorphy on $Re(s)>1, s\neq 3/2$, and the existence of at most a simple pole at $s=3/2$ (Theorem 16.7, p. 63, and Theorem 18.2, p. 65). Moreover, [H] computes the residue at $s=3/2$ (Theorem 17.6, p. 65). To evaluate the Eisenstein series at $s=0$, [H] uses (on p. 67) the functional equation. Our proof, which is based on computing directly the values of the Fourier series at $s=0$, is simpler.

Our main interest is in the analogous result for $H = GL(2)$. The result for H, and the technique, may turn out to be useful in constructing an automorphic embedding of the model found in [FKS] for the smallest representation of a two- fold covering of $GL(3)$. The H_v-module θ_v defined above occurs in fact as a module of coinvariants of the representation studied in [FKS], and the model of θ_v described here is used there. For this reason we decided to reprove here the Evaluation for S, in a format which seems to us to be more convenient for generalization; it is different from [H] in that we evaluate the Eisenstein series directly at $s = 0$, and we do not use the functional equation. In any case we deal not only with the non-archimedean places, but also with the archimedean places. Then we discuss the case of H, in several different ways.

As in [H], in the case of S we work with $f = \otimes f_v$, even f_v for all v. The Eisenstein series is defined (below) as a series which converges absolutely, uniformly in compact subsets of $Re(s) > 3/2$. It is well-known that it has analytic continuation to the entire complex plane, with a functional equation, and the continuation is holomorphic on $Re(s) > 1/2$, except for (at most) a simple pole at $s = 1$. We study the value at $s = 0$, in the domain of continuation. As in [H], the proof is based on computing the Fourier expansion of the Eisenstein series along the standard non-trivial parabolic subgroup. We were motivated to consider the Evaluation by the observation that our computations can be adapted to show that $E(0, g, f) = E(0, id, \theta(g)f)$, and that one has the Evaluation $E(0, g, f) = D(\theta_{\mathbb{A}}(g)f) = D_f(g)$. Then the summation formulae follow from the Evaluation. Indeed, it is clear from the definition of $E(s, g, f)$ that E is automorphic, namely when the group is S we have $E(s, g, f) = E(s, \delta g, f)$ for every δ in $\overline{S}(F) \subset S_{\mathbb{A}}$. Hence at $s = 0$ and $g = id$ we obtain $\sum_{\beta \in F} f(\beta) = \sum_{\beta \in F} (\theta(\delta)f)(\beta)$ for all $\delta \in \overline{S}(F)$. The Poisson summation formula $D(\mathcal{F}f) = D(f)$ follows on taking $\delta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, since then $\theta(\delta)f = \mathcal{F}f$ is the Fourier transform of f. The New Summation Formula similarly follows in the case of H. As noted above, this method of proof may apply to construct an automorphic embedding of the model found in [FKS] for the smallest representation of a two-fold covering of $GL(3)$. But this may require some effort, and we do not foresee ourselves studying this problem in the very near future.

I. EVALUATION FOR S.

We begin with the case of the $S_{\mathbb{A}}$-module $(\theta_{\mathbb{A}}, C(\mathbb{A}))$. To introduce the Eisenstein series on $S_{\mathbb{A}}$, recall the Iwasawa decomposition

$$\overline{S}_v = \overline{N}_v \overline{A}_v \overline{K}_v, \quad \overline{N}_v = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}, \quad \overline{A}_v = \{\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\}, \quad K_v = SL(2, R_v).$$

If $g_v = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} k_v$ then $a(g_v) = |a|_v > 0$ is uniquely determined by g_v, and so is $a(g) = \prod_v a(g_v)$ for any $g = (g_v)$ in $S_{\mathbb{A}}$. The functions $g \mapsto (\theta(g)f)(0)$ and

$g \mapsto a(g)$ are left invariant under the upper-triangular subgroup $\overline{P}(F)$ of $\overline{S}(F)$, viewed as a subgroup of $S_{\mathbb{A}}$. For every $f \in C(\mathbb{A})$ put

$$E(s,g,f) = \sum_{\gamma \in \overline{P}(F)\backslash \overline{S}(F)} (\theta(\gamma g)f)(0)a(\gamma g)^{-s}.$$

Then $E(s,g,f)$ is an automorphic function, equal to $E(s,\gamma g,f)$ for all $\gamma \in \overline{S}(F)$. Note that $\varphi(g) = (\theta(g)f)(0)a(g)^{-s}$ is left invariant under $\overline{N}_{\mathbb{A}}$, and $\varphi\left(s\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}g\right) = \gamma_{\mathbb{A}}(t)|t|_{\mathbb{A}}^{s+1/2}\varphi(g)$ $(t \in \mathbb{A}^\times)$. Consequently the series defining $E(s,g,f)$ converges absolutely, uniformly in compact subsets of $Re(s) > 3/2$ and $g \in S_{\mathbb{A}}$. It is well-known that it has analytic continuation as a meromorphic function to the entire complex plane. The proof below shows that $E(s,g,f)$, $g = id$, is holomorphic at $s = 0$. The complex parameter s, $Re(s) > 0$, is used to guarantee the convergence of the infinite products below.

To compute the Fourier expansion of $E(s,g,f)$ at $s = 0$, where $g = id$, it suffices to find the Fourier coefficients

$$E_\alpha(s,f) = \int_{\mathbb{A}\mathrm{mod} F} E(s, \begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}, f)\overline{\psi}(\alpha u)du$$

for all α in F. Here the measure du is taken to assign the compact set $\mathbb{A}\mathrm{mod}F$ the volume one. Then

$$\int_{\mathbb{A}\mathrm{mod} F} \overline{\psi}(\alpha u)du = \begin{cases} 1, & \alpha = 0, \\ 0, & \alpha \neq 0. \end{cases}$$

A set of representatives for the coset space $\overline{P}(F)\backslash\overline{S}(F)$ is given by id and $\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}$, $u \in F$. Thus for $\alpha \in F^\times$ we have

$$\begin{aligned} E_\alpha(s,f) &= \int_{\mathbb{A}} [\theta(s(\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}))f](0)\|(1,u)\|^{-s}\overline{\psi}(\alpha u)du \\ &= \int_{\mathbb{A}}\int_{\mathbb{A}} f(y)\psi(\frac{1}{2}uy^2)dy\|(1,u)\|^{-s}\overline{\psi}(\alpha u)du. \end{aligned}$$

Here $\|(1,(u_v))\| = \prod_v \|(1,u_v)\|_v$, where

$$\|(1,u_v)\|_v = \begin{cases} \max(1,|u_v|_v) & \text{if } v \neq \infty; \\ (1+u_v^2)^{1/2} & \text{if } F_v = \mathbb{R}; \\ 1 + u_v\overline{u}_v & \text{if } F_v = \mathbb{C}. \end{cases}$$

The double integral over $\mathbb{A}$ converges absolutely on $Re(s) > 2$, and is equal to the Eulerian product of the local integrals

$$C_v(\alpha, s) = \int_{F_v} \int_{F_v} f_v(y)\psi_v(u(\frac{1}{2}y^2 - \alpha))\|(1,u)\|_v^{-s} du\, dy. \qquad (\underline{1})$$

Choose $\underline{q}_v \in F_v$ with $\mathrm{val}(\underline{q}_v) = -1$ ($\underline{q}_v^{-1}$ generates the maximal ideal of the local ring R_v), when v is finite. Denote by ψ_v^0 a character on F_v which is trivial on R_v but not on $\underline{q}_v R_v$. Given ψ_v there is an integer $c(\psi_v)$ with $\psi_v(x) = \psi_v^0(x\underline{q}_v^{c(\psi_v)})$. Note that $vol(R_v, dx) = \int_{R_v} dx$ is equal to $q_v^{c(\psi_v)/2}$, and $c(\psi_v) = 0$ for almost all v.

We begin with the following local result.

PROPOSITION 1. (i) *For almost all* v, *the integral* $(\underline{1})$ *is equal to* $1 + (2\alpha, \underline{q}_v)_v q_v^{-s}$.

(ii) *For every place* v, *the integral* $(\underline{1})$ *has analytic continutation to* $\mathbb{C}$, *and its value at* $s = 0$ *is zero if* $2\alpha \notin F_v^2$, *and* $|\beta|_v^{-1}(f_v(\beta) + f_v(-\beta))$ *if* $2\alpha = \beta^2$, $\beta \in F_v^\times$.

First we note the following

LEMMA 1. *At any finite place* v, *the integral* $\int_{F_v} \psi_v^0(u\underline{q}_v^{-r})\|(1,u)\|_v^{-s} du$ *is zero unless* $r \geq 0$, *in which case it is equal to*

$$q_v^{c(\psi_v)/2} \frac{1 - q_v^{-s}}{1 - q_v^{1-s}}(1 - q_v^{(r+1)(1-s)}).$$

Proof. The first claim follows from the fact that $\int_{|u|_v \leq 1} \psi_v^0(u\underline{q}_v^r) du = 0$ if $r > 0$. If $r \geq 0$ then the integral of the lemma is equal to

$$\int\limits_{|u| \leq q^r} \|(1,u)\|^{-s} du + \int\limits_{|u| = q^{r+1}} \psi^0(u\underline{q}^{-r}) q^{-s(r+1)} du$$
$$= q^{c(\psi)/2}[1 + (1 - q^{-1})q^{1-s}\frac{q^{r(1-s)} - 1}{q^{1-s} - 1} - q^{r - s(r+1)}]$$
$$= q^{c(\psi)/2}(1 - q^{-s})(1 - q^{(r+1)(1-s)})(1 - q^{1-s})^{-1},$$

as asserted; here the index v is omitted to simplify the notations.

Consequently the integral $C_v(\alpha, s)$ of $(\underline{1})$ is equal to

$$q_v^{c(\psi_v)/2} \frac{1 - q_v^{-s}}{1 - q_v^{1-s}} \sum_{r=0}^{\infty} (1 - q_v^{(r+1)(1-s)}) \int_{|y^2 - 2\alpha| = q_v^{-r - c(\psi_v)}|2|} f_v(y) dy. \qquad (\underline{1}')$$

It follows that there are $A_v = A(f_v, \psi_v) > 0$ such that $(\underline{1})$ is zero unless $|\alpha|_v \leq A_v$ for all v; here $A_v = 1$ for all v where $f_v = f_v^0$, $\psi_v = \psi_v^0$. Hence in the function field case, for given f, ψ, there are at most finitely many non-zero $E_\alpha(s,f)$. Given α in $F^\times$, we have $f_v = f_v^0$, $\psi_v = \psi_v^0$, $\alpha \in R_v^\times$ and $2 \in R_v^\times$ for almost all v, and then $(\underline{1})$ is equal to

$$q_v^{c(\psi_v)/2}\frac{1-q_v^{-s}}{1-q_v^{1-s}}\left[(1-q_v^{1-s})\int_{|y^2-2\alpha|_v=1} f_v(y)dy + \sum_{r>0}(1-q_v^{(r+1)(1-s)})\int_{|y^2-2\alpha|_v=q_v^{-r}} f_v(y)dy\right].$$

We conclude at once the following

LEMMA 2. *If $f_v = f_v^0$, $\psi_v = \psi_v^0$, $|\alpha|_v = 1$ and $|2|_v = 1$, then $(\underline{1})$ is equal to*

$$1+q_v^{-s} = \frac{1-q_v^{-2s}}{1-q_v^{-s}} \quad \text{if} \quad 2\alpha \in F_v^{\times 2},$$

or

$$1-q_v^{-s} = 1+\chi_{2\alpha}(\underline{q}_v)q_v^{-s} = \frac{1-q_v^{-2s}}{1-\chi_{2\alpha}(\underline{q}_v)q_v^{-s}} \quad \text{if} \quad 2\alpha \notin F_v^{\times 2}.$$

Here $\chi_{2\alpha}$ denotes the quadratic character $x \mapsto (2\alpha, x)_v$ of $F_v^\times$.

Proof. In the first case note that if $2\alpha = \beta^2$, $|\beta|_v = 1$, then $|y^2 - 2\alpha|_v < 1$ implies $|y-\beta|_v < 1$ or $|y+\beta|_v < 1$. Also $\int_{|y|_v=1} dy = q_v^{c(\psi_v)/2}(1-q_v^{-1})$. In the second case note that $(2\alpha, \underline{q}_v)_v = -1$ if q_v is odd and 2α is a non-square unit in $F_v^\times$.

Lemma 2 completes the proof of Proposition 1(i). At any finite v, if $2\alpha \notin F_v^2$ then only finitely many summands of $(\underline{1}')$ are non-zero, hence $(\underline{1}')$ is $o(s)$; we write $o(s)$ for a function whose limit at $s=0$ is zero. If $2\alpha = \beta^2, \beta \in F_v^\times$, to compute the limit at $s=0$ of $(\underline{1}')$ it suffices to take the sum only over $r \geq R$ for any fixed R. We take $R = R(\alpha)$ to be sufficiently large. Then each integral in $(\underline{1}')$ ranges over the y with $|y-\beta|_v$ or $|y+\beta|_v$ equal to $q_v^{-r-c(\psi_v)}/|\beta|_v$. Up to $o(s)$ we obtain

$$\frac{1-q_v^{-s}}{1-q_v^{1-s}}(1-q_v^{-1})|\beta|_v^{-1}(f_v(\beta)+f_v(-\beta))\sum_{r=0}^{\infty}(q_v^{-r}-q_v^{1-s(r+1)}).$$

Then $(\underline{1}')$, and so also $(\underline{1})$, is equal to $2f_v(\beta)|\beta|_v^{-1}$, up to $o(s)$. This completes the proof of Proposition 1(ii) when v is finite.

LEMMA 3. *Proposition 1(ii) holds when $F_v = \mathbb{R}$.*

Proof. The integral $(\underline{1})$ is equal to

$$\begin{aligned}&\int\int_{\mathbb{R}^2} f_v(x)\, e^{-2\pi i u(\frac{1}{2}x^2-\alpha)}(1+u^2)^{-s/2}du\,dx \\ &= \frac{2\pi^{1/2}}{\Gamma\left(\frac{s}{2}\right)}\int_{\mathbb{R}} |\pi(\frac{1}{2}x^2-\alpha)|^{(s-1)/2}K_{(s-1)/2}(2\pi|\frac{1}{2}x^2-\alpha|)f_v(x)\,dx. \qquad (*)\end{aligned}$$

Here the equality follows from the well-known identity (see [B], p. 83, (27))

$$\int_{\mathbb{R}} (1+x^2)^{-t} e^{2\pi i a x} dx = 2\pi^t |a|^{t-\frac{1}{2}}, \, (t)^{-1} K_{t-\frac{1}{2}}(2\pi|a|) \qquad (a \in \mathbb{R}^\times).$$

If $\alpha < 0$, then the integral of $(*)$ over $\mathbb{R}$ is an entire function of s, and (ii) follows.

If $\alpha > 0$, define $\beta > 0$ by $\beta^2 = 2\alpha$. Then $\int_0^{\beta-\delta} + \int_{\beta+\delta}^{\infty}$ is holomorphic on $\mathbb{C}$, and, using the power series expansion of $K_t(z)$ near $z = 0$, we have

$$\int_{\beta-\delta}^{\beta+\delta} \left(\frac{1}{2}\pi|x^2-\beta^2|\right)^{(s-1)/2} K_{(s-1)/2}(\pi|x^2-\beta^2|) f_v(x)\, dx$$
$$= \int_{\beta-\delta}^{\beta+\delta} \pi[2\cos(\pi s/2), \, ((1+s)/2)]^{-1} (\pi|x^2-\beta^2|/2)^{s-1} f_v(x) dx + h(s)$$

with $h(s)$ holomorphic at $s = 0$. Consequently, up to a function which is holomorphic at $s = 0$, the integral over $\mathbb{R}$ in $(*)$ is equal twice the integral

$$\pi[2\cos(\pi s/2), \, ((1+s)/2)]^{-1} (\pi\beta)^{s-1} f_v(\beta) \int_{\beta-\delta}^{\beta+\delta} |x-\beta|^{s-1} dx,$$

whose residue at $s = 0$ is $\pi^{-1/2} f_v(\beta)/\beta$; the lemma follows.

LEMMA 4. *Proposition* 1(ii) *holds when* $F_v = \mathbb{C}$.

Proof. The integral (1) is equal to

$$\int \int_{\mathbb{C}^2} f_v(x) e^{-2\pi i tr(u(\frac{1}{2}x^2 - \alpha))} (1+u\overline{u})^{-s} du\, dx$$
$$= \frac{4\pi}{, \, (s)} \int_{\mathbb{C}} (2\pi|\frac{1}{2}x^2-\alpha|)^{s-1} K_{s-1}(4\pi|\frac{1}{2}x^2-\alpha|) f_v(x)\, dx. \qquad (*)$$

Here the equality follows from the well-known identities (see [B], p. 81, (2), and p. 95, (51))

$$\int_0^{2\pi} e^{iz\cos\theta} d\theta = 2\pi J_0(z)$$

and

$$\int_0^{\infty} J_0(ar)(1+r^2)^{-s} r\, dr = (a/2)^{s-1} K_{s-1}(a)/, \, (s) \qquad (a > 0).$$

Choose $\beta \in \mathbb{C}$ which satisfies $2\alpha = \beta^2$. Up to a function holomorphic at $s = 0$, the integral of $(*)$ is equal to

$$\int_{|x-\beta|<\delta} (\pi|x^2-\beta^2|)^{s-1} K_{s-1}(2\pi|x^2-\beta^2|) f_v(x)\, dx$$
$$\simeq \int_{|x-\beta|<\delta} \pi[2\sin(\pi s), \, (s)]^{-1} (\pi|x^2-\beta^2|)^{2s-2} f_v(x)\, dx$$
$$\simeq \pi[2\sin(\pi s), \, (s)]^{-1} (2\pi|\beta|)^{2s-2} f_v(\beta) \int_{|x-\beta|<\delta} |x-\beta|^{2s-2} dx.$$

Here again we used the power-series expansion of $K_t(z)$ at $z = 0$; $\simeq$ mean equality up to a function holomorphic at $s = 0$; $|.|$ is the usual absolute value, and dx is the measure defined by the differential form $2\,dx \wedge d\overline{x}$. Since

$$\int_{|x-\beta|<\delta} |x-\beta|^{2s-2} dx = 2\pi\delta^{2s}/s \qquad \text{if} \quad Re(s) > 0,$$

the residue at $s = 0$ of the integral in $(*)$ is $(4\pi)^{-1} f_v(\beta)/|\beta|^2$. Hence the value at $s = 0$ of $(*)$ is the sum of $f_v(\beta)/|\beta|^2$ and $f_v(-\beta)/|\beta|^2$, as required.

We can now conclude

PROPOSITION 2. *The value of the Fourier coefficient* $E_\alpha(s,f)$ *at* $s = 0$ *is* $2f(\beta) = f(\beta) + f(-\beta)$ *if* $2\alpha = \beta^2$, $\beta \in F^\times$, *and it is zero if* $2\alpha \in F - F^2$.

Proof. Note that the Γ-function $\Gamma(s)$ satisfies $\Gamma(s+1) = s\Gamma(s)$ and $\Gamma(1) = 1$, and it is analytic on $\mathrm{Re}(s) > 0$. Denote by r_1 (resp. r_2) the number of real (resp. pairs of complex) embeddings of F. The product

$$\zeta(s) = \prod_{v \neq \infty} (1 - q_v^{-s})^{-1}$$

converges absolutely, uniformly in compacts of $\mathrm{Re}(s) > 1$, has analytic continuation as a meromorphic function of s on $\mathbb{C}$, and there is a complex number $A \neq 0$ such that $\zeta(s)$ satisfies the functional equation

$$\zeta(s)\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2} A^s = A^{1-s}\Gamma\left(\frac{1-s}{2}\right)^{r_1}\Gamma(1-s)^{r_2}\zeta(1-s).$$

Since ζ has a simple pole at $s = 1$, one has

$$\lim_{s\to 0} \zeta(s)/\zeta(2s) = \lim_{s\to 0} \frac{\zeta(1-s)}{\zeta(1-2s)} \left(\frac{\Gamma(2s)}{\Gamma(s)}\right)^{r_2} \left(\frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)}\right)^{r_1} = 2^{1-r_1-r_2}.$$

Lemmas 2, 3 and 4 imply that when $\alpha = \beta^2/2$, $\beta \in F^\times$, the Fourier coefficient $E_\alpha(s,f)$ is

$$\frac{\zeta(s)}{\zeta(2s)} \prod_{v\in V, v\neq\infty} (1+q_v^{-s})^{-1} \prod_{v\in V} C_v(\alpha, s),$$

where V is a finite set of places such that each $v \notin V$ is finite and has $f_v = f_v^0$, $\psi_v = \psi_v^0$, $|\alpha|_v = 1, |2|_v = 1$. At $s = 0$ this is equal to

$$2^{1-r_1-r_2}\left(\prod_{v\in V, v<\infty} 2^{-1}\right)\left(\prod_{v\in V} 2f_v(\beta)/|\beta|_v\right) = 2f(\beta) = f(\beta) + f(-\beta).$$

Note that $\Pi_{v\in V}|\beta|_v = 1$, and $f_v(\beta) = 1$ for $v \notin V$.

When $2\alpha \in F - F^2$, define a character χ_α on $\mathbb{A}^\times$ by $\chi_\alpha(t) = \prod_v (2\alpha, t_v)_v$. The Euler product

$$\zeta(s, \chi_\alpha) = \prod (1 - \chi_\alpha(\underline{q}_v) q_v^{-s})^{-1}$$

(product over the set of finite places where χ_α is unramified) is absolutely convergent, uniformly in compact subsets of $\mathrm{Re}(s) > 1$, and has analytic continuation to the entire complex plane. Its value at $s = 1$ is a finite non-zero number. Denote by $r_1^- = r_1^-(\alpha)$ the number of real places of F where $\alpha < 0$, namely where χ_α is quadratic, and by r_1^+ the number of real places where $\alpha > 0$. From the functional equation satisfied by $\zeta(s, \chi_\alpha)$ it follows that $\zeta(s, \chi_\alpha)$ has a zero of order $r_1^+ + r_2$ at $s = 0$, and that $\zeta(2s)$ has a zero of order $r_1 + r_2 - 1$ there. Lemma 2 implies that when $\alpha \in F - F^2$, we have that

$$E_\alpha(s, f) = \prod_{v\in V} C_v(\alpha, s) \prod_{v\notin V} (1 + (2\alpha, \underline{q}_v)_v q^{-s})$$

$$= \frac{\zeta(s, \chi_\alpha)}{\zeta(2s)} \prod_{v\in V} C_v(\alpha, s) \prod_{v\in V'} (1 + q_v^{-s}(2\alpha, \underline{q}_v))^{-1} \prod_{v\in V''} (1 - q_v^{-2s})^{-1}.$$

Here V is a sufficiently large finite set of places of F, V' is the set of finite v in V where χ_α is unramified, and V'' is the set of finite v in V where χ_α is ramified. It follows that the order of zero of $E_\alpha(s, f)$ at $s = 0$ is at least

$$r_1^+ + r_2 - (r_1 + r_2 - 1) + [\{v \in V; 2\alpha \notin F_v^{\times 2}\}] - [\{v \in V'; 2\alpha \notin F_v^{\times 2}\}] - [V''] = 1.$$

Here [V] denotes the cardinality of a set V. It follows that the limit of $E_\alpha(s, f)$ at $s = 0$ is zero. The proof of proposition 2 is now complete.

PROPOSITION 3. *The value at $s = 0$ of the Fourier coefficient $E_\alpha(s, f)$ at $\alpha = 0$ is $f(0)$.*

Proof. The coset of the identity in $\overline{P}(R)\backslash\overline{S}(F)$ yields the contribution $f(0)$ to $E_0(s, f)$. Any other coset is represented by
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, and contributes the Eulerian integral

$$\int_{\mathbb{A}} \int_{\mathbb{A}} f(y) \psi(\frac{1}{2} u y^2) \|(1, u)\|^{-s} du\, dy. \qquad (\underline{2})$$

To compute the local integral which occurs in this product we use local notations (drop the index v), put $r = c(\psi)$ and write ψ for ψ^0. Since

$$\int \psi(u q^{-r-2t}) \|(1, u)\|^{-s} du$$

is zero unless $r+2t\ge 0$ where, by Lemma 1, $q^{r/2}(1-q^{-s})(1-q^{(1+r+2t)(1-s)})/(1-q^{1-s})$ is obtained, the local integral

$$\int f(y)\int \psi(uq^{-r}y^2)\|(1,u)\|^{-s}du\,dy$$

equals

$$q^{r/2}\sum_{t\ge -\frac{r}{2}}\frac{1-q^{-s}}{1-q^{1-s}}(1-q^{(1-s)(1+r+2t)})\int\limits_{|y|=q^{-t}} f(y)\,dy. \qquad (\underline{2}')$$

When $r=0$ and $f=f^0$ is the characteristic function of $|y|\le 1$, one obtains

$$q^r\frac{1-q^{-s}}{1-q^{1-s}}(1-q^{-1})\sum_{t=0}^{\infty}(q^{-t}-q^{1-s+t(1-2s)})=q^r\frac{1-q^{-2s}}{1-q^{1-2s}}.$$

It is clear that each of the summands in $(\underline{2}')$ is $o(s)$. Hence up to $o(s)$ it suffices to take $t\ge R$ in $(\underline{2}')$; for a sufficiently large R one has $f(y)=f(0)$ on $|y|\le q^{-R}$. Taking the sum over $t\ge R$ it is clear that $(\underline{2}')$ is $o(s)$. It follows that $(\underline{2})$ is equal to

$$\prod_{v\in V}C_v(0,s)\prod_{v\notin V}(1-q_v^{-2s})(1-q_v^{1-2s})^{-1}$$
$$=\frac{\zeta(2s-1)}{\zeta(2s)}\prod_{v\in V}C_v(0,s)\prod_{v\in V,v<\infty}(1-q_v^{-2s})(1-q_v^{1-2s})^{-1}.$$

Here V is a sufficiently large finite set of places. Note that $\zeta(2s-1)$ has a zero of order r_2 at $s=0$. This follows from the functional equation of $\zeta(s)$, since , $\left(\frac{1}{2}\right)$ and $\zeta(2)$ are finite and non-zero, while , $(-1+s)$ has a simple pole at $s=0$. Consequently the order of zero of $(\underline{2})$ at $s=0$ is at least $r_2-(r_1+r_2-1)+[V]-[\{v\in V;v<\infty\}]=r_2+1$. Hence $(\underline{2})$ vanishes at $s=0$, and the proposition follows.

In conclusion, the value of the Fourier expansion $\sum_{\alpha\in F}E_\alpha(s,f)$ of $E(s,g,f)$, $g=id$, at $s=0$, is

$$E(0,id,f)=\sum_{\alpha\in F}E_\alpha(0,f)=f(0)+2\sum_{\alpha\in F^{\times 2}}f(\beta_\alpha)=\sum_{\beta\in F}f(\beta),$$

where β_α is an element in $F^\times$ with $\beta_\alpha^2=\alpha$. This completes the proof of the Evaluation in the case of the group S.

As noted above, our computations can be extended to apply with any g in $S_{\mathbb{A}}$, and yield the Evaluation $E(0,g,f)=\sum_{\beta\in F}(\theta(g)f)(\beta)$. Since $E(s,g,f)=E(s,\delta g,f)$ for every δ in $\overline{S}(F) \subset S_{\mathbb{A}}$, it follows that $\sum_{\beta \in F} f(\beta) = \sum_{\beta \in F} (\theta(\delta) f)(\beta)$ for any $\delta \in \overline{S}(F)$. The Poisson summation formula is obtained on taking $\delta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, since then $\theta(\delta) f = \mathcal{F} f$ is the Fourier transform of f. Moreover, the functional $f \mapsto \sum_{\beta \in F} f(\beta)$ intertwines $\theta_{\mathbb{A}}$ with its model as a discrete series automorphic representation.

II. EVALUATION FOR H.

Next we turn to the study of the $H_{\mathbb{A}}$-module $(\theta_{\mathbb{A}}, C(\mathbb{A}^{\times}))$. For $f = \otimes f_v$, $f_v \in C(F_v^{\times})$, consider the function $f_0 = \otimes f_{v0}$, $f_{v0}(x) = \lim_{t \to 0} |t|_v^{1/2} f_v(t^2 x)$, on $\mathbb{A}^{\times}$; it satisfies $|t|_{\mathbb{A}}^{1/2} f_0(t^2 x) = f_0(x)$. The series

$$E(s, g, f) = \sum_{\gamma \in \overline{P}(F) \backslash \overline{H}(F)} \sum_{x \in F^{\times}/F^{\times 2}} (\theta(\gamma g) f)_0(x)\ a(\gamma g)^{-s}$$

is absolutely convergent, uniformly in compact subsets of $\mathrm{Re}(s) > 3/2$. Here $\overline{P}$ is the upper triangular parabolic subgroup of $\overline{H}$. The proof below implies that the analytic continuation of $E(s, g, f)$ is holomorphic at $s = 0$. We give two proofs for the Evaluation in the case of H. The first is based on reduction to the case of S. At $g = id$, one has

$$\begin{aligned}
E_H(s, id, f) &= \sum_{\gamma} \sum_{x} (\theta(\gamma) f)_0(x) a(\gamma)^{-s} \\
&= \sum_{\alpha \in F^{\times}/F^{\times 2}} f_0(\alpha) + \sum_{\beta \in F} \sum_{\alpha \in F^{\times}/F^{\times 2}} \left(\theta \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \right) f \right)_0 (\alpha) \|(1, \beta)\|^{-s} \\
&= \sum_{\alpha \in F^{\times}/F^{\times 2}} [f(\alpha, 0) + \sum_{\beta \in F} \int_{\mathbb{A}} f(\alpha, x) \psi(\frac{1}{2} \alpha \beta x^2) dx \cdot \|(1, \beta)\|^{-s}].
\end{aligned}$$

The summand in the last sum over α is no other than $E_S(s, id, f_\alpha)$, where $f_\alpha(x) = f(\alpha, x)$. By the Evaluation for S we have $E_S(0, id, f_\alpha) = \sum_{\beta \in F} f(\alpha, \beta)$. Taking the sum over α in $F^{\times}/F^{\times 2}$ we obtain

$$E_H(0, id, f) = \sum_{\alpha \in F^{\times}/F^{\times 2}} f_0(\alpha) + \sum_{\alpha \in F^{\times}/F^{\times 2}} \sum_{\beta \in F^{\times}} f(\alpha, \beta) = \sum_{\alpha \in F^{\times}/F^{\times 2}} f_0(\alpha) + 2 \sum_{\alpha \in F^{\times}} f(\alpha),$$

as required.

The second proof is analogous to that given above for S. It will now be briefly described. The Fourier expansion of $E(s, g, f)$ at $g = id$ is $\sum_{\alpha \in F} E_\alpha(s, f)$, where

$$E_\alpha(s, f) = \int_{\mathbb{A} \bmod F} E(s, \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, f) \overline{\psi}(u\alpha)\, du.$$

The coset of the identity in $\overline{P}(F)\backslash\overline{H}(F)$ contributes

$$\sum_{\alpha\in F}\int_{\mathbb{A}\bmod F}[\sum_{x\in F^\times/F^{\times 2}} f_0(x)]\overline{\psi}(u\alpha)du = \sum_{x\in F^\times/F^{\times 2}} f_0(x)$$

to the Fourier expansion. It remains to consider the contribution of the cosets of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ to $E_\alpha(s,f)$. It is the sum over $x\in F^\times/F^{\times 2}$ of the Eulerian integral

$$\int_{\mathbb{A}}\theta(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} f)_0(x)\|(1,u)\|^{-s}\ \overline{\psi}(u\alpha)\,du. \tag{$\underline{3}$}$$

To compute the local factors of $(\underline{3})$, we pass to local notations, i.e. drop the index v. Since

$$(\theta(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix})f)(x) = c\gamma(x)|x|^{1/2}\int|y|^{1/2}f(xy^2)\psi(x(\frac{1}{2}uy^2+y))dy,$$

we have

$$(\theta(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix})f)_{{}_0}(x) = c\gamma(x)|x|^{1/2}\int|y|^{1/2}f(xy^2)\psi(\frac{1}{2}uxy^2)dy.$$

Hence the local factor in $(\underline{3})$ is

$$c\gamma(x)|x|^{1/2}\int_u\int_y|y|^{1/2}f(xy^2)\psi(u(\frac{1}{2}xy^2-\alpha))\|(1,u)\|^{-s}du\,dy. \tag{$\underline{3}'$}$$

There is $A(f,\psi)>0$, with $A(f^0,\psi^0)=1$, such that $(\underline{3}')$ is zero unless $|\alpha|\le A(f,\psi)$. Hence when F is a function field the global integral $(\underline{3})$ vanishes for almost all $\alpha\in F^\times$. It is easy to see that for each of the remaining finitely many α's, for which $(\underline{3})$ may be non-zero, $(\underline{3})$ would vanish for all but finitely many x in $F^\times/F^{\times 2}$.

PROPOSITION 4. *If* $f_v=f_v^0,\quad \psi_v=\psi_v^0,\quad |\alpha|_v=1,\quad |x|_v=1$, *then* $(\underline{3}')$ *is equal to*

$$1+q_v^{-s}=\frac{1-q_v^{-2s}}{1-q_v^{-s}}\quad\text{if}\quad 2\alpha/x\in F_v^{\times 2},$$

or

$$\int\psi_v(u)\|(1,u)\|_v^{-s}du = 1-q_v^{-s} = 1+\chi_{2\alpha/x}(\underline{q}_v)q_v^{-s} = \frac{1-q_v^{-2s}}{1-\chi_{2\alpha/x}(\underline{q}_v)q_v^{-s}}$$

if $2\alpha/x\notin F_v^{\times 2}$, *where* $\chi_{2\alpha/x}(y)=(2\alpha/x,y)_v$ *is the quadratic character associated with* $2\alpha/x\in F_v^\times/F_v^{\times 2}$.

Proof. This follows at once from Lemma 2.

By Lemma 1, each of the local integrals $(\underline{3}')$ at a finite place is equal to

$$q^{c(\psi)/2}\frac{1-q^{-s}}{1-q^{1-s}}\sum_{n\geq 0}(1-q^{(1+n)(1-s)})c\gamma(x)|x|^{1/2}\int_{|y^2-2\alpha/x|=q^{-n-c(\psi)}/|2x|}|y|^{1/2}f(xy^2)\,dy.$$

Up to $o(s)$ it suffices to sum only over $n\geq R=R(\alpha,x,f)$. For a sufficiently large R we get that each integral is zero unless there is $\beta\in F^\times$ with $\beta^2=2\alpha/x$, and then we obtain

$$2c\gamma(x)|x|^{1/2}|\alpha/x|^{1/4}f(\alpha)|\beta x|^{-1}(1-q^{-1})(1-q^{-s})(1-q^{1-s})^{-1}\sum_{n\geq R}(q^{-n}-q^{1-s-ns}).$$

Up to $o(s)$ this is the same as the analogous sum over $n\geq 0$, and at $s=0$ we obtain

$$2f(\alpha)c\gamma(x)|\alpha|^{-1/4}|x|^{-3/4}.$$

The analogous result holds in the archimedean cases too.

Returning to the global notations of $(\underline{3})$, we conclude

PROPOSITION 5. *The Fourier coefficient $E_\alpha(s,f)$ is an analytic function of s near $s=0$ (which is zero, when F is a function field, for all $\alpha\in F^\times$ with only finitely many exceptions depending on f and ψ), and its value at $s=0$ is $E_\alpha(0,f)=2f(\alpha)$.*

Proof. Since $\zeta(s)/\zeta(2s)$ takes the value $2^{1-r_1-r_2}$ at $s=0$, and $\zeta(s,\chi_{2\alpha/x})/\zeta(2s)$ has a zero of order $1-r_1^-(\alpha/x)$ at $s=0$, as in the case of $SL(2)$ we conclude that given $\alpha\in F^\times$ the integral $(\underline{3})$ is zero at $s=0$ unless the class of 2α in $F^\times/F^{\times 2}$ is represented by x. Then $E_\alpha(s,f)$ is equal to the value of $(\underline{3})$ at $x=\alpha$, and this is $2f(\alpha)+o(s)$, as required.

PROPOSITION 6. *The contribution to $E_\alpha(s,f)$, $\alpha=0$, from the cosets represented by*

$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}$, *is $o(s)$.*

Proof. We have to compute the product over v of the local integrals

$$\gamma(x)|x|^{1/2}\int_y|y|^{1/2}f(xy^2)\int_u\psi(uxy^2)\|(1,u)\|^{-s}du\,dy.$$

As noted in the case of $SL(2)$, for almost all v we have $|2|=1$, $|x|=1$, $f=f^0$, $\psi=\psi^0$, $c(\psi)=0$, and the result is

$$(1-q^{-2s})/(1-q^{1-2s}).$$

In general the local integral is

$$q^{c(\psi)/2}\gamma(x)|x|^{1/2}(1-q^{-s})(1-q^{1-s})^{-1}\sum_{n\geq 0}(1-q^{(1+n)(1-s)})\int_{|y|^2=q^{-n-c(\psi)}/|2x|}|y|^{1/2}f(xy^2)dy.$$

Up to $o(s)$ we may take $n\geq R$, and when R is sufficiently large, up to $o(s)$ we obtain

$$\gamma(x)f_0(x)(1-q^{-s})(1-q^{1-s})^{-1}(1-q^{-1})\sum_{n\geq 0}(q^{-n}-q^{1-s+n(1-2s)})$$

if $val(2x)-c(\psi)$ is even, and 0 otherwise. But this expression is $o(s)$. Hence the contribution to $E_0(s,f)$ under discussion is the product of a function which vanishes at $s=0$ to the order r_1+r_2, and $\zeta(2s-1)/\zeta(2s)$, which vanishes to the order $r_2-(r_1+r_2-1)$ (see proof of Proposition 3).

It follows from Proposition 6 that $E_0(0,f)=\sum_{x\in F^\times/F^{\times 2}}f_0(x)$. Using Proposition 5 we conclude that the value of $E(s,id,f)$ at $s=0$ is

$$D(f)=\sum_{x\in F^\times/F^{\times 2}}f_0(x)+2\sum_{x\in F^\times}f(x),$$

and the proof of the Evaluation for H is complete. As noted above, one can generalize our computations to apply to $E(s,g,f), s=0$, with any g in $H_{\mathbb{A}}$. Since $E(s,\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},f)=E(s,id,f)$, this would yield another proof of the new summation formula $D(f)=D(\mathcal{F}f)$, as well as the automorphic realization of $(\theta_{\mathbb{A}},C(\mathbb{A}^\times))$.

References

[B] H. Bateman, Higher Transcendental Functions II, McGraw-Hill, New York 1953.

[F] Y. Flicker, Automorphic forms on covering groups of $GL(2)$, *Invent. Math.* 57(1980), 119–182.

[FKS] Y. Flicker, D. Kazhdan, G. Savin, Explicit realization of a metaplectic representation, *J. Analyse Math.* 55(1990), 17-39.

[GP] S. Gelbart, I. Piatetski-Shapiro, Some remarks on metaplectic cusp forms and the correspondences of Shimura and Waldspurger, *Israel J. Math.* 44(1983), 97-126

[H] G. F. Helminck, *Eisenstein series on the metaplectic group: an algebraic approach*, Mathematisch centrum, Amsterdam, 1983.

[I] J.–I. Igusa, *Lectures on forms of Higher degree*, Tata Inst., Bombay, Springer-Verlag 1978.

[KP] D. Kazhdan, S. J. Patterson, Metaplectic forms, *Publ. Math. IHES* 59(1984), 35-142.

[K] T. Kubota, Topological covering of $SL(2)$ over a local field, *J. Math. Soc. Japan* 19 (1967), 114–121; and: *Automorphic Forms and the Reciprocity law in a Number Field*, mimeographed notes, Kyoto Univ. 1969.

[KR] S. Kudla, S. Rallis, On the Weil-Siegel formula, *J.f.d.r.u.a. Math.* 387 (1988), 1-68.

[L] S. Lang, *Algebraic Number Theory,* Addison-Wesley,

[M] J.G.M. Mars, The Siegel formula for orthogonal groups, in *Algebraic Groups and Continuous Subgroups*, Proc. Symp. Pure Math. 9 (1966), 133-142.

[R] S. Rallis, *L-functions and the Oscillator Representation,* SLN 1245(1987).

[S] C.L. Siegel, Die Funktionalgleichungen einiger Dirichletscher Reihen, *Math. Z.* 63 (1956), 363–373.

[Wa] J.–L. Waldspurger, *Correspondances de Shimura et Quaternions*, 1981, unpublished.

[We] A. Weil, Sur certains groupes d'opérateurs unitaires, *Acta Math.* 111(1964), 143–211; Sur la formule de Siegel dans la theorie des groupes classiques, *Acta Math.* 113 (1965), 1-87.