PRODUCTS OF THETA SERIES AND SPECTRAL ANALYSIS

Dedicated to the memory of Professor Hans Zassenhaus

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1. Introduction. The purpose of this note is to propose a new technique in the theory of automorphic forms which will potentially characterize those cusp forms on the general linear group whose symmetric square lifting has a one dimensional constituent. In principle, a cuspidal representation π of $\mathbb{G}_n = GL(n, \mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of a global field F, is parametrized by a complex irreducible representation ρ of dimension n of a form of the Weil group, and the symmetric square lifting $\mathrm{Sym}^2\pi$ of π is cuspidal precisely when $\mathrm{Sym}^2\rho$ is irreducible. A characterization of the ρ such that $\mathrm{Sym}^2\rho$ is reducible would suggest a parametrization of the cuspidal π whose symmetric square is expected not to be cuspidal, and in particular of the π whose symmetric square L-function $L(s, \pi, \mathrm{Sym}^2)$ – or a twist of it – will not be entire, if $\mathrm{Sym}^2\rho$ has a one-dimensional constituent.

An illuminating example is that of a three dimensional ρ with determinant 1. Its symmetric square is reducible precisely when ρ preserves a quadratic form, and ρ factorizes through the subgroup $(PGL(2, \mathbb{C}) \simeq) SO(3, \mathbb{C})$ of $SL(3, \mathbb{C})$, namely ρ is the symmetric square of some two-dimensional projective representation ρ_0. This suggests that for a cuspidal representation π of $PGL(3, \mathbb{A})$, the L-function $L(s, \pi, \mathrm{Sym}^2)$ has a pole precisely when π is the symmetric square lifting ([F1], or Gelbart-Jacquet [GJ]) of an automorphic representation of $SL(2, \mathbb{A})$. Patterson and Piatetski-Shapiro [PPS] have shown that the residue of $L(s, \varphi, \mathrm{Sym}^2)$, $\varphi \in \pi$, is $R_3(\varphi) = \int_{\mathbb{Z}_3^2 G_3 \backslash \mathbb{G}_3} \varphi(g)\Theta(g)\overline{\Theta}(g)dg$ (here $G_3 = GL(3, F)$, $\mathbb{Z}_n$ =center of $\mathbb{G}_n$), where Θ are certain "theta" functions on a two-fold covering group of $\mathbb{G}_3$. It is then natural to conjecture that the linear form R is non-zero on the cuspidal representation π of $PGL(3, \mathbb{A})$ precisely when it is the symmetric square of a cuspidal representation of $SL(2, \mathbb{A})$. A local analogue of the linear form R_3 has been studied by Savin [S] in the unramified case, using the explicit model of the theta representation of [FKS].

Analogous conjectures can be made for all n, describing the cuspidal π on which R does not vanish (Such π might be lifts from Sp_m if $n = 2m + 1$, or SO_{2m} if $n = 2m$). Here we propose a technique to prove these conjectures, by working out the case of $n = 2$. This technique is based on applying the theta-kernel to the spectral decomposition of $L^2(\mathbb{Z}_2 G_2 \backslash \mathbb{G}_2)$. It is likely to generalize to the higher n, and in particular to give a new proof of the symmetric square lifting of automorphic forms from $SL(2, \mathbb{A})$ to $PGL(3, \mathbb{A})$, and a new characterization ($R_3 \neq 0$ on $\pi_3 =$

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Nato-grant CRG-921232 is greatfully acknowledged.

1991 Mathematics Subject Classification: 11F27, 11R42, 11S40.

Typeset by $\mathcal{AMS}$-TEX

$\mathrm{Sym}^2\pi_2$) of the image of the lifting, as well as an extension of the local work of [S] to the ramified case. But this generalization will require further technical work. We decided to write up the case of $n=2$ to expose our ideas, in the simplest – least technical – case. Our main technical tool, a new type of a summation formula, is described in Proposition 5. Lemmas 2 and 3 deal with the accompanying transfer of orbital integrals.

Consider then a cuspidal representation π of $\mathbb{G}=GL(2,\mathbb{A})$. Its symmetric square lifting is an automorphic representation $\mathrm{Sym}^2\pi$ of $PGL(3,\mathbb{A})$, whose existence is proven in [GJ] by means of the converse theorem, and in [F1] by means of the trace formula. The L-function $L(s,\pi,\mathrm{Sym}^2)=L(s,\mathrm{Sym}^2\pi)$ is entire, but given a character χ of order two of $\mathbb{A}^\times/F^\times$, the twisted L-function $L(s,\pi,\chi\otimes\mathrm{Sym}^2)=L(s,\chi\otimes\mathrm{Sym}^2\pi)$ will have a pole precisely when π is associated with a character μ of $\mathbb{A}_E^\times/E^\times$, where E is the quadratic separable extension of F defined by χ using class field theory. It can be shown that the residue of this twisted-by-χ L-function is proportional to $R^\chi(\varphi)=\int_{\mathbb{Z}^2G\backslash\mathbb{G}}\varphi(g)\Theta(g)\overline{\Theta}^\chi(g)dg$, $\varphi\in\pi$, for suitable Θ-functions on a two-fold covering group of $\mathbb{G}=GL(2,\mathbb{A})$. In fact a similar linear form on $\varphi\in\pi$ appears in [GJ], where $\mathbb{G}$ is replaced by $SL(2,\mathbb{A})$. We use the linear form R^χ to characterize the image of the lifting $\mu\mapsto\pi(\mu)$.

Theorem. *Let $\mathbb{A}^\times$ be the group of ideles of a global field F, and $\chi\neq 1$ a quadratic character of $\mathbb{A}^\times/F^\times$, associated with a quadratic separable field extension E of F. Given a character μ of $\mathbb{A}_E^\times/E^\times$ whose restriction to $\mathbb{A}^\times/F^\times$ coincides with χ, there exists a unique automorphic representation $\pi(=\pi(\mu))$ of $PGL(2,\mathbb{A})$, determined as follows. At a place v of F which splits in E, there is a character μ_{1v} of $F_v^\times$ such that $\mu_v((a,b))=\mu_{1v}(a/b)((a,b)\in E_v^\times=F_v^\times\times F_v^\times)$. Then the local component $\pi_v=\pi(\mu_v)$ of $\pi=\pi(\mu)$ is defined to be the $PGL(2,F_v)$-module $I(\mu_{1v},\mu_{1v}^{-1})$ normalizedly induced from the character $\begin{pmatrix} a & * \\ 0 & b\end{pmatrix}\mapsto\mu_{1v}(a/b)$. At a non-split unramified place v of F, where μ_v is unramified, there is a character μ_{1v} of $F_v^\times$ with $\mu_v(z)=\mu_{1v}(z\overline{z})$. Define $\pi(\mu_v)$ to be $I(\mu_{1v},\mu_{1v}^{-1})$. The automorphic representation $\pi(\mu)$ is cuspidal unless $\chi=\eta^2$ for some character η of $\mathbb{A}^\times/F^\times$, and $\mu=\overline{\mu}$. In this case $\pi(\mu)$ $=I(\eta,1/\eta)$ is a principal series representation. A cuspidal representation π of $PGL(2,\mathbb{A})$ is of the form $\pi(\mu)$ precisely when $R^\chi(\varphi)=\int_{\mathbb{Z}^2G\backslash\mathbb{G}}\varphi(g)\Theta(g)\overline{\Theta}^\chi(g)dg$ is non-zero on $\varphi\in\pi$. In this case, if χ is a square then $\mu\neq\overline{\mu}$.*

The existence of the lifting $\mu\mapsto\pi(\mu)$ is well-known. It was proven using the oscillator representation (see Howe [H], or [MVW]) in Shalika-Tanaka [ST], the converse theorem in Jacquet-Langlands [JL], by stabilizing the trace formula on $SL(2)$ in Labesse-Langlands [LL], by twisting the trace formula by χ in Kazhdan [K], by quadratic base-change for $GL(2)$ in Langlands [L] (see [F2] for a simpler proof). In all of these works the image of the lifting was characterized by the requirement that $\pi\otimes\chi\simeq\pi$. Our characterization of the image, by the non-vanishing of the form R^χ on π, is different, and is at the core of our proof. Note that if exists, $\pi(\mu)$ is uniquely determined by almost all of its components – as specified in the statement of the Theorem – by virtue of the rigidity theorem for $GL(2)$ (see Jacquet-Shalika [JS]).

As noted above, the virtue of the present work is not in proving a new result, or supplying a new proof for an old result. It is in exposing a new method which may extend from the case of $GL(2)$ to the higher rank groups $GL(n)$, $n > 2$. In comparison, the method of [ST] – which we proceed to sketch – has no known projected extension to $GL(n)$. For simplicity, let us describe the method of [ST] in the case of $SL(2)$. Let θ_1 and θ_2 be two theta-functions on the two-fold topological central extension $\mathbb{S}$ of $SL(2,\mathbb{A})$. It suffices to show that $(*)$ $\theta_1(g)\theta_2(g) = \Sigma_\mu \phi_\mu(g) (g \in SL(2,\mathbb{A}))$, where $\phi_\mu \in \pi(\mu)$. Let $\boldsymbol{V}$ be a vector space over F with a quadratic form q. Put $\mathbb{V} = \boldsymbol{V}(\mathbb{A})$. Then by [H] or [MVW], the Schwartz space $C_c^\infty(\mathbb{V})$ of functions on $\mathbb{V}$ admits commuting representations of $\mathbb{S}$ and the orthogonal group $O(q,\mathbb{V})$ of q on $\mathbb{V}$. If $V = \boldsymbol{V}(F)$ is F, and $q(x) = ax^2 (a \in F^\times)$, one obtains the theta representation of $\mathbb{S}$ on $C_c^\infty(\mathbb{A})$. If $V = \boldsymbol{V}(F)$ is E, and $q(x) = x\overline{x}$ is the norm form on E, then one has a direct sum decomposition $C_c^\infty(\mathbb{A}_E) = \oplus_\mu \pi(\mu)$. Since $E = F(\tau^{1/2}) \simeq F \oplus F$ with the quadratic form $q(x,y) = x^2 - \tau y^2$, one has an isomorphism of $SL(2,\mathbb{A})$-modules $C_c^\infty(\mathbb{A}) \otimes C_c^\infty(\mathbb{A}) \simeq C_c^\infty(\mathbb{A}_E)$, and $(*)$ follows (for a complete proof see [ST], or [H], [MVW]). To repeat, this method is not known to extend to $GL(n)$, $n > 2$.

Our technique might be considered to be conceptually simpler. We consider the well-known spectral and geometric expressions for the kernel of the convolution operator $r(f)$ on $L^2(G\backslash\mathbb{G})$ for a Schwartz function f on $\mathbb{G} = PGL(2,\mathbb{A})$, multiply by $\theta_1(g)\theta_2(g)$, and by a character $\psi(n) \neq 1$ of the upper unipotent subgroup $N\backslash\mathbb{N}$, and integrate over $g \in G\backslash\mathbb{G}$ and over $n \in N\backslash\mathbb{N}$. On the spectral side we get essentially a sum over the cusp forms $(\phi \in)\pi$ of $\mathbb{G}$ of the $R^\chi(\pi(f)\phi)$, multiplied by the value at the identity of the Whittaker function of ϕ. The geometric sum is easily transformed to a sum over $\gamma \in E^\times$ (rather than $PGL(2,F)$!) of the values $f_E(\gamma)$ of a function f_E in the Schwartz space on $\mathbb{A}_E$, transferred from f compatibly with the lifting $\mu \to \pi(\mu)$ in the unramified case. The Poisson summation formula on E permits writing $\Sigma_\gamma f_E(\gamma)$ as a sum $\Sigma_\mu \mu(f_E)$, and a standard separation argument of "linear independence of characters" establishes the lifting $\mu \to \pi(\mu)$. This approach extends in principle to $GL(n), n > 2$. This we considered interesting, so we thought it was worthwhile to work out carefully the technical details in the test case of $GL(2)$, as a prototype for the general case. This is what we do in this paper.

Let us dispose at once of the degenerate case where there exists a character η of $\mathbb{A}^\times/F^\times$ such that $\chi = \eta^2$, equivalently $\chi_v(-1) = 1$ for every place v of F, and the character μ of $\mathbb{A}_E^\times/E^\times$ is equal to $\overline{\mu}$, where $\overline{\mu}(x) = \mu(\overline{x})$, $x \in \mathbb{A}_E^\times$. Since $\mu = \overline{\mu}$, there is a character μ_1 of $\mathbb{A}^\times/F^\times$ such that $\mu = \mu_1 \circ N$, where $Nx = x\overline{x}$ is the norm map from E to F. The restriction of μ to $\mathbb{A}^\times$ is χ, hence $\mu_1^2 = \chi$. Namely χ is a square when $\overline{\mu} = \mu$, and we may choose η to be μ_1. At a place v which splits in E, we have $\eta_v^2 = \chi_v = 1$, hence $\pi_v = \pi(\mu_v)$ is by definition the induced $PGL(2,F_v)$-module

$I(\eta_v, \eta_v^{-1}) = I(\eta_v, \eta_v\chi_v)$ (as $\chi_v = 1$). At a non-split place v, by definition $\pi(\mu_v)$ is $I(\eta_v, \eta_v^{-1}) = I(\eta_v, \eta_v\chi_v)$. Hence when $\chi = \eta^2$ and $\mu = \overline{\mu} = \eta \circ N$, the character μ of $\mathbb{A}_E^\times/E^\times$ lifts to the principal series (normalizedly induced) representation $I(\eta, \eta\chi)$ of $PGL(2, \mathbb{A})$.

2. Theta Kernel. Our argument uses the theta-representation of the two-fold cover of the group. For $GL(2)$, an explicit model of this representation is described in [FM]. Let v be a place of F, and $\chi_v : F_v^\times \to \mathbb{C}^\times$ a unitary character ([FM] takes $\chi_v = 1$, but the general case is similar). Let $C_{\chi_v}(F_v^\times)$ denote the space of smooth functions $u_v : F_v^\times \to \mathbb{C}$, supported in a compact of F_v (if v is finite; having rapid decay at ∞ if v is archimedean), which vanish near 0 if $\chi_v(-1) = -1$, while if $\chi_v(-1) = 1$ they have the property that $u_{v0}(x) = \chi_v(t)|t|_v^{1/2}u_v(t^2x)$ is independent of t if $|x|_v \le 1$ and $|t|_v$ is sufficiently small (if v is finite; $t \mapsto \chi_v(t)|t|_v^{1/2}u_v(t^2x)$ is smooth at $t = 0$, and $u_{v0}(x)$ is defined to be its limit at $t = 0$, when v is archimedean). Note that if $\chi_v(-1) = 1$ then there is a character χ_{1v} of $F_v^\times$ with $\chi_{1v}^2 = \chi_v$, and then $\chi_{1v}\nu_v^{1/4}u_{v0}$ extends to a function on $F_v^\times/F_v^{\times 2}$.

The Weil- or θ-representation of the 2-fold cover $\widetilde{G}_v$ of $G_v = GL(2, F_v)$ considered in [FM] acts on $C_{\chi_v}(F_v^\times)$ as follows.

$$\left(\theta_v\left(s\begin{pmatrix} z & 0 \\ 0 & z\end{pmatrix}\right)u_v\right)(\alpha) = (\alpha, z)_v\gamma_v(z)\chi_v(z)u_v(\alpha) \qquad (z, \alpha \in F_v^\times)$$
$$\left(\theta_v\left(s\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\right)u_v\right)(\alpha) = \chi_v(a)|a|_v^{1/2}u_v(a\alpha)$$
$$\left(\theta_v\left(s\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\right)u_v\right)(\alpha) = \psi_v(\frac{1}{2}b\alpha)u_v(\alpha) \qquad (b \in F_v)$$
$$\left(\theta_v\left(s\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\right)u_v\right)(\alpha) = c_v\gamma_v(\alpha)|\alpha|_v^{1/2}\int_{F_v}\chi_v(t)|t|^{1/2}u_v(\alpha t^2)\psi_v(-\alpha t)dt = (\mathcal{F}u_v)(\alpha).$$

Here ψ_v is a non-trivial character of F_v, and $c_v = \gamma_v(-1)^{-1/2}$ is an eighth root of unity in $\mathbb{C}$. Denote by R_v the ring of integers in F_v, and by $\boldsymbol{\pi}_v$ a uniformizer. When v is finite and odd, ψ_v has conductor R_v, χ_v unramified, u_v^0 is supported on the set of εt^2, $\varepsilon \in R_v^\times$, $t \in R_v$, and is given there by $u_v^0(\varepsilon\boldsymbol{\pi}^{2n}) = \chi_v(\boldsymbol{\pi})^n|\boldsymbol{\pi}|_v^{-n/2}$ ($|\varepsilon|_v = 1, n \ge 0$), then $\theta_v(s(k))u_v^0 = u_v^0$ for $k \in K_v = GL(2, R_v)$.

If $u = \otimes u_v$, then [FM] shows that the function

$$\Theta_u^\chi(g) = 2\sum_{\alpha\in F^\times}(\theta(g)u)(\alpha) + \sum_{\alpha\in F^\times/F^{\times 2}}(\theta(g)u)_0(\alpha)$$

on $\widetilde{G}(\mathbb{A})$ is automorphic, namely left-invariant under the discrete subgroup $G = GL(2, F)$ of $\widetilde{G}(\mathbb{A})$ ([FM] consider only $\chi = 1$; if $\chi_v(-1) = -1$ for some v then $(\theta(g)u)_0 \equiv 0$). Write Θ_u for Θ_u^χ when $\chi = 1$. From now on we take a non-trivial character χ of $\mathbb{A}^\times/F^\times$ of order two, as in the Theorem.

The linear form $R^\chi(\varphi) = \int\limits_{\mathbb{Z}^2G\backslash\mathbb{G}} \varphi(g)\Theta_u(g)\overline{\Theta}_w^\chi(g)dg$ $(u \in C(\mathbb{A}^\times), w \in C_\chi(\mathbb{A}^\times))$ appears in the following "spectral" expression on the space $L_0^2(\mathbb{Z}G\backslash\mathbb{G})$ of cusp forms on $\mathbb{G} = GL(2,\mathbb{A})$ which transform trivially under the center:

$$(1)\qquad \sum_{\substack{\pi \\ \text{cuspidal}}} \sum_{\substack{\varphi\in\pi \\ \text{orthonormal} \\ \text{basis}}} \int_{\mathbb{Z}^2G\backslash\mathbb{G}} (\pi(f)\varphi)(g)\Theta_u(g)\overline{\Theta}_w^\chi(g)dg \cdot \int\limits_{N\backslash\mathbb{N}} \overline{\varphi}(n)\overline{\psi}(\frac{1}{2}n)dn,$$

where $N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$ and the second integral is the value at e of the Whittaker function $\overline{W}_{\varphi,\psi}$.

We want to show that π ranges here over the $\pi = \pi(\mu)$, $\mu : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$, with $\mu \neq \overline{\mu}$ if χ is a square, and each such $\pi(\mu)$ contributes. For this end note that there are two expressions for the kernel of the convolution operator $r(f)$ on $L^2(\mathbb{Z}G\backslash\mathbb{G})$. The geometric expression is $\sum\limits_{\gamma\in Z\backslash G} f(g^{-1}\gamma n)$. The spectral expression is the sum of the contribution $\sum\limits_{\pi}\sum\limits_{\varphi\in\pi} (\pi(f)\varphi)(g)\overline{\varphi}(n)$ from the cuspidal spectrum, whose integral against $\Theta_u(g)\overline{\Theta}_w^\chi(g)dg \cdot \overline{\psi}(\frac{1}{2}n)dn$ is (1), and a contribution from the continuous spectrum.

3. Eisenstein Series. The kernel of the operator $r(f)$ on the continuous – non-discrete – spectrum, takes the form

$$\frac{1}{\pi}\sum_{\eta}\sum_{\Phi}\int\limits_{i\mathbb{R}} E(g,\pi_s(f)\Phi,\eta,s)\overline{E}(h,\Phi,\eta,s)ds.$$

The first sum ranges over a set of representatives of the classes of characters η of $\mathbb{A}^\times/F^\times$ up to multiplication with ν^{is}, $s \in \mathbb{R}$, where $\nu(x) = |x|$. The second sum ranges over an orthonormal basis of the space of right smooth functions $\Phi : \mathbb{K} \to \mathbb{C}$, with $\Phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} k\right) = \eta(a/c)\Phi(k)$, where k and $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ lie in $\mathbb{K} = \prod K_v$, K_v is the standard maximal compact subgroup in $PGL(2,F_v)$. We trivialize the vector bundle

$$I(\eta\nu^s,\eta^{-1}\nu^{-s}) = \left\{\Phi_s : \mathbb{G} \to \mathbb{C}; \Phi_s\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} g\right) = \eta(a/c)|a/c|^{s+1/2}\Phi_s(g); \quad a,c \in \mathbb{A}^\times, b \in \mathbb{A}\right\}$$

via the restriction map $\Phi_s \to \Phi = \Phi_s|\mathbb{K}$. The Eisenstein series are defined by the sum

$$E(g,\Phi,\eta,s) = \sum_{\gamma\in B\backslash G} \Phi(\gamma g;\eta,s),$$

if $\mathrm{Re}\,(s)$ is large enough, and by analytic continuation for other s in $\mathbb{C}$. We write $\Phi(g;\eta,s)$ for $\Phi_s(g)$, to emphasize also the dependence on η.

To compute the integral, I, of this kernel against $\Theta_u(g)\overline{\Theta}_w^\chi(g)dg\cdot\overline{\psi}(\frac{1}{2}n)dn$, we need to recall – and use – the truncation operator Λ^T, where $T>0$ is sufficiently large. If $g=nak$, $k\in\mathbb{K}$, $n\in\mathbb{N}$, $a=\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}$, then $dg=|a/b|^{-1}(d^\times a d^\times b)dn\,dk$, and we put $H(g)=|a/b|$. Denote by $\chi(H(g)>T)$ the characteristic function of the $g\in G$ with $H(g)>T$, and similarly with $<$ replacing $>$. The truncation of ϕ on $\mathbb{G}$ is

$$\Lambda^T\phi(g)=\phi(g)-\sum_{\delta\in B\backslash G}\phi_N(\delta g)\chi(H(\delta g)>T),$$

where

$$\phi_N(g)=\int_{N\backslash\mathbb{N}}\phi(ng)dn.$$

The truncation maps slowly increasing to rapidly decreasing functions, and standard arguments imply the following. We have that

$$I=\frac{1}{\pi}\int_{\mathbb{Z}G\backslash\mathbb{G}}\int_{N\backslash\mathbb{N}}\sum_\eta\sum_\Phi\int_{i\mathbb{R}}E(g,\pi_s(f)\Phi,\eta,s)\overline{E}(n,\Phi,\eta,s)ds\Theta_u(g)\overline{\Theta}_w^\chi(g)dg\overline{\psi}(\frac{1}{2}n)dn$$

is equal to

$$I'=\frac{1}{\pi}\sum_\eta\sum_\Phi\lim_{T\to\infty}\int_{i\mathbb{R}}\int_{\mathbb{Z}^2G\backslash\mathbb{G}}\Lambda^T E(g,\pi_s(f)\Phi,\eta,s)\Theta_u(g)\overline{\Theta}_w^\chi(g)dg\cdot\overline{E}_\psi(\Phi,\eta,s)ds,$$

where

$$E_\psi(\Phi,\eta,s)=\int_{N\backslash\mathbb{N}}E(n,\Phi,\eta,s)\psi(\frac{1}{2}n)dn.$$

Our aim is to show the following.

1. Lemma. *The integral I of the contribution from the continuous spectrum is the sum over the characters η of $\mathbb{A}^\times/F^\times$ which satisfy $\eta^2=\chi$, of I_η, defined to be the sum over Φ of*

$$\frac{1}{4}\int_{\mathbb{K}}\int_{\mathbb{A}^\times/F^\times\mathbb{A}^{\times 2}}[\eta(a)(\pi_0(f)\Phi(k)+\eta(1/a)(\pi_0(f)M\Phi)(k)]\overline{E}_\psi(\Phi,\eta,0)$$
$$\cdot\sum_{\alpha\in F^\times/F^{\times 2}}(\theta(k)u)_0(a\alpha)\overline{\theta(k)w)_0(a\alpha\tau)}|a|^{1/2}d^\times a dk.$$

Proof. To compute the inner integral, over g, in I', note that

$$E_N(g,\Phi,\eta,s)=\Phi(g,\eta,s)+(M\Phi)(g,\eta^{-1},-s),$$

where M is the standard intertwining operator from $I(\eta\nu^s, \eta^{-1}\nu^{-s})$ to $I(\eta^{-1}\nu^{-s}, \eta\nu^s)$. Hence

$$\Lambda^T E(g,\Phi,\eta,s) = \sum_{\delta\in B\backslash G} [\Phi(\delta g,\eta,s)\chi(H(\delta g)<T) - (M\Phi)(\delta g,\eta^{-1},-s)\chi(H(\delta g)>T)],$$

and the inner integral, of $\Lambda^T E\cdot\Theta_u\cdot\overline{\Theta}^\chi_w$ over g, is the difference, which we denote by $J = J(f,\Phi,\eta,s)$,

$$\int_{\mathbb{ZN}B\backslash\mathbb{G}} (\pi_s(f)\Phi)(g,\eta,s)\chi(H(g)<T)A dg - \int_{\mathbb{ZN}B\backslash\mathbb{G}} (M\pi_s(f)\Phi)(g,\eta^{-1},-s)\chi(H(g)>T)A\, dg.$$

Here

$$A = \int_{Z\mathbb{Z}^2\backslash\mathbb{Z}} \int_{N\backslash\mathbb{N}} \Theta_u(nzg)\overline{\Theta}^\chi_w(nzg) dn\, dz.$$

Substituting the two sums, over $F^\times$ and over $F^\times/F^{\times 2}$, which define each of Θ_u and Θ^χ_w, into A, we get the sum of 4 expressions. Note that we may change the order of the summations and the integrations over the compact sets, since for any compact subset C of $\widetilde{G}(\mathbb{A})$, and u, there exists a function u_1 with properties analogous to those of u, such that $|(\theta(g)u)(x)| \le |u_1(x)|$ for all $x\in\mathbb{A}^\times$ and g in C. The same remark applies to the function w. In any case, the first term, integrated over the compact $N\backslash\mathbb{N}$, is equal to

$$\int_{N\backslash\mathbb{N}} 4\sum_{\alpha,\beta\in F^\times} \psi(\frac{1}{2}n(\alpha-\beta))(\theta(zg)u)(\alpha)(\overline{\theta(zg)w})(\beta)dn$$
$$= 4\sum_{\alpha\in F^\times}(\theta(zg)u)(\alpha)(\overline{\theta(zg)w})(\alpha) = 4\sum_{\alpha\in F^\times}(\theta(g)u)(\alpha)(\overline{\theta(g)w})(\alpha)\chi(z).$$

Integrating this over the compact $Z\mathbb{Z}^2\backslash\mathbb{Z}$ we obtain 0, since $\chi\ne 1$ (is of order two). The second and third terms are similarly shown to be 0. The fourth term is 0 if there is a place v with $\chi_v(-1) = -1$, as then $(\theta(g)w)_0 \equiv 0$. Suppose then that $\chi_v(-1) = 1$ for all v.

To compute the fourth term in A, recall the following action of θ on w.

$$\left(\theta\left(s\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\right)w\right)_0(\alpha) = w_0(\alpha), \quad \left(\theta\left(s\begin{pmatrix}z & 0\\ 0 & z\end{pmatrix}\right)w\right)_0(\alpha) = (\alpha,z)\gamma(z)\chi(z)w_0(\alpha),$$
$$\left(\theta\left(s\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)w\right)_0(\alpha) = \chi(a)|a|^{1/2}w_0(a\alpha),$$
$$\left(\theta\left(s\begin{pmatrix}a^2 & 0\\ 0 & 1\end{pmatrix}\right)w\right)_0(\alpha) = |a|w_0(a^2\alpha) = \chi(a)|a|^{1/2}w_0(\alpha),$$

since $w_0(\alpha t^2) = \chi(t)|t|^{-1/2}w_0(\alpha)$. The integrand is invariant under $\mathbb{N}$. Its integral over $N\backslash\mathbb{N}$ is its product with $\mathrm{vol}\,(N\backslash\mathbb{N}) = 1$. Now, if $\ker\chi = N_{E/F}E^\times$, $E = F(\sqrt{\tau})$, then $\chi(z) = (\tau, z)$. Indeed, the Hilbert symbol satisfies $(-\frac{a}{b}, a+b) = (a,b)$, hence $(\frac{a^2}{\tau b^2}, a^2 - \tau b^2) = (a^2, -\tau b^2) = 1$, and so $(\tau, a^2 - \theta b^2) = 1$. Carrying out the integration over z in $Z\mathbb{Z}^2\backslash\mathbb{Z}$ of the fourth term, we obtain that A is

$$\int_{Z\mathbb{Z}^2\backslash\mathbb{Z}} \sum_{\alpha,\beta\in F^\times/F^{\times 2}} (\alpha,z)\gamma(z)(\beta,z)\overline{\gamma}(z)\chi(z)(\theta(g)u)_0(\alpha)(\overline{\theta(g)w})_0(\beta)dz$$

$$= \sum_{\alpha\in F^\times/F^{\times 2}} (\theta(g)u)_0(\alpha)(\overline{\theta(g)w})_0(\alpha\tau).$$

We are now in a position to compute the two terms in J. Writing $g = n\begin{pmatrix} at^2 & 0 \\ 0 & 1\end{pmatrix}k$, the first term is

$$\int_{\mathbb{K}}\int_{\mathbb{A}^\times/F^\times\mathbb{A}^{\times 2}} (\pi_s(f)\Phi)\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}k,\eta,s\right)\int_{\substack{t\in F^\times\backslash\mathbb{A}^\times\\ |t^2|<T}} \eta(t^2)|t|^{2s+1}\chi(t)|t|^{1/2}\cdot|t|^{1/2}\sum_{\alpha\in F^\times/F^{\times 2}}$$

$$\left(\theta\left(s\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}k\right)u\right)_0(\alpha)\left(\overline{\theta\left(s\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}k\right)w}\right)_0(\alpha\tau)|t|^{-2}|a|^{-1}d^\times a d^\times t\,dk.$$

The inner integral $\int(\eta^2\chi)(t)|t|^{2s}d^\times t$ over $t\in F^\times\backslash\mathbb{A}^\times$, $|t^2|<T$, is zero unless $\eta^2\chi$ is ν^λ for some $\lambda\in i\mathbb{R}$. Replacing η by $\eta\nu^{-\lambda/2}$ we may assume that $\eta^2 = \chi^{-1} = \chi$, in this case, and then the value of the integral is $\frac{1}{2s}T^s$.

The second term in J is similarly computed, and we conclude that J vanishes unless $\eta^2 = \chi$, in which case we obtain that J is equal to

$$\frac{1}{2s}T^s\int_{\mathbb{K}}\int_{\mathbb{A}^\times/F^\times\mathbb{A}^{\times 2}} (\pi_s(f)\Phi)\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}k,\eta,s\right)$$

$$\cdot\sum_{\alpha\in F^\times/F^{\times 2}}\left(\theta\left(s\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}k\right)u\right)_0(\alpha)\left(\overline{\theta\left(s\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}k\right)w}\right)_0(\alpha\tau)|a|^{-s-1}d^\times a\,dk$$

$$-\frac{1}{2s}T^{-s}\int_{\mathbb{K}}\int_{\mathbb{A}^\times/F^\times\mathbb{A}^{\times 2}} (M\pi_s(f)\Phi)\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}k,\eta^{-1},-s\right)$$

$$\cdot\sum_{\alpha\in F^\times/F^{\times 2}}\left(\theta\left(s\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}k\right)u\right)_0(\alpha)\left(\overline{\theta\left(s\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}k\right)w}\right)_0(\alpha\tau)|a|^{s-1}d^\times a\,dk.$$

For a given f, the sums over η and Φ in I' (or I) are finite. The matrix coefficients $(\pi_s(f)\Phi,\Phi')$ are rapidly decreasing holomorphic functions of $s\in i\mathbb{R}$. Hence $\sum_\eta\sum_\Phi J(f,\Phi,\eta,s)$ has the form $\frac{1}{2s}T^s h_1(s) - \frac{1}{2s}T^{-s}h_2(s)$, where h_1, h_2 are holomorphic on $i\mathbb{R}$, with $h_1(0) = h_2(0)$. Our lemma now follows from the limit formula

$$\frac{1}{2\pi}\lim_{T\to\infty}\int_{i\mathbb{R}}\{\frac{1}{s}T^s h_1(s) - \frac{1}{s}T^{-s}h_2(s)\}ds = h_1(0)$$

for such functions h_1, h_2. (The left side is $\lim_{y\to\infty}(2\pi i)^{-1}\int_{-\infty}^{\infty}[e^{ixy}h_1(ix)-e^{-ixy}h_2(ix)]x^{-1}dx$. Since $\lim_{y\to\infty}\int_{|x|>1}e^{ixy}h(ix)x^{-1}dx=0$ and $\lim_{y\to\infty}\int_{|x|<1}e^{ixy}[h(ix)-h(0)]x^{-1}dx=0$, we are left with $(2\pi i)^{-1}\int_{|x|<1}[e^{ixy}h_1(0)-e^{-ixy}h_2(0)]x^{-1}dx=(h_1(0)/\pi)\int_{|x|<1}(\sin xy/x)dx$ $=(h_1(0)/\pi)\int_{|x|<y}(\sin x/x)dx$, which has the limit $h_1(0)$ as $y\to\infty$.) □

Remark. It is clear that the integral over $N\backslash\mathbb{N}$ of the kernel of $r(f)$ on the discrete non-cuspidal (one-dimensional) spectrum, multiplied by $\psi(\frac{1}{2}n)$, is 0. Hence the one-dimensional automorphic representations do not contribute to our formulae.

4. Geometric Side. We conclude that (1) $+\sum_{\eta^2=\chi}I_\eta$ is equal to the "geometric sum":

$$\int_{\mathbb{Z}^2G\backslash\mathbb{G}}\int_{N\backslash\mathbb{N}}\sum_{\gamma\in Z\backslash G}f(g^{-1}\gamma n)\Theta_u(g)\overline{\Theta}_w^\chi(g)\overline{\psi}(\frac{1}{2}n)dn\,dg$$
$$=\int_{Z\mathbb{Z}^2\mathbb{N}\backslash\mathbb{G}}f_\psi(g^{-1})\int_{N\backslash\mathbb{N}}\Theta_u(ng)\overline{\Theta}_w^\chi(ng)\overline{\psi}(\frac{1}{2}n)dn\,dg,$$

where

$$f_\psi(g^{-1})=\int_{\mathbb{N}}f(g^{-1}m)\overline{\psi}(\frac{1}{2}m)dm.$$

The inner integral gives

$$\int_{N\backslash\mathbb{N}}\Theta_u(ng)\overline{\Theta}_w^\chi(ng)\overline{\psi}(\frac{1}{2}n)dn=\int_{N\backslash\mathbb{N}}4\sum_{\alpha,\beta\in F^\times}\psi(\frac{1}{2}n(\alpha-\beta-1))(\theta(g)u)(\alpha)(\overline{\theta(g)w})(\beta)dn$$
$$+\int_{N\backslash\mathbb{N}}2\sum_{\alpha\in F^\times/F^{\times 2}}\sum_{\beta\in F^\times}\overline{\psi}(\frac{1}{2}n(\beta+1))(\theta(g)u)_0(\alpha)(\overline{\theta(g)w})(\beta)dn$$
$$+\int_{N\backslash\mathbb{N}}2\sum_{\beta\in F^\times/F^{\times 2}}\sum_{\alpha\in F^\times}\psi(\frac{1}{2}n(\alpha-1))(\theta(g)u)(\alpha)(\overline{\theta(g)w})_0(\beta)dn$$
$$=4\sum_{-1\neq\beta\in F^\times}(\theta(g)u)(\beta+1)(\overline{\theta(g)w})(\beta)+2(\overline{\theta(g)w})(-1)\sum_{\alpha\in F^\times/F^{\times 2}}(\theta(g)u)_0(\alpha)$$
$$+2(\theta(g)u)(1)\sum_{\beta\in F^\times/F^{\times 2}}(\overline{\theta(g)w})_0(\beta).$$

Here we used the fact that $\left(\theta\begin{pmatrix}1&b\\0&1\end{pmatrix}u\right)_0(\alpha)=u_0(\alpha)$. Consequently, taking

$z \in \mathbb{Z}$, we get

$$\begin{aligned}
&\int\limits_{N\backslash\mathbb{N}} \Theta_u(nzg)\overline{\Theta}^\chi_w(nzg)\overline{\psi}(\frac{1}{2}n)dn \\
&= 4 \sum_{\beta\in F^\times-\{-1\}} (\beta+1,z)\gamma(z)(\theta(g)u)(\beta+1)(\beta,z)\overline{\gamma}(z)\overline{\chi}(z)(\overline{\theta(g)w})(\beta) \\
&\quad + 2(-1,z)\overline{\gamma}(z)\overline{\chi}(z)(\overline{\theta(g)w})(-1) \sum_{\alpha\in F^\times/F^{\times 2}} (\alpha,z)\gamma(z)(\theta(g)u)_0(\alpha) \\
&\quad + 2\gamma(z)(\theta(g)u)(1) \sum_{\beta\in F^\times/F^{\times 2}} (\beta,z)\overline{\gamma}(z)\overline{\chi}(z)(\overline{\theta(g)w})_0(\beta).
\end{aligned}$$

Integrating over $\mathbb{Z}/Z\mathbb{Z}^2$ (of course $\mathbb{Z} = \boldsymbol{Z}(\mathbb{A}), Z = \boldsymbol{Z}(F)$) we then get

$$\begin{aligned}
(2) \int\limits_{\mathbb{Z}^2 Z\backslash\mathbb{Z}} \int\limits_{N\backslash\mathbb{N}} \Theta_u(nzg)\overline{\Theta}^\chi_w(nzg)\overline{\psi}(\frac{1}{2}n)dn\, dz = 4 \sum_{\substack{\beta\in F^\times-\{-1\} \\ \frac{\beta+1}{\beta}\in\tau F^{\times 2}}} (\theta(g)u)(\beta+1)(\overline{\theta(g)w})(\beta) \\
+ 2(\theta(g)u)_0(-\tau)(\overline{\theta(g)w})(-1) + 2(\theta(g)u)(1)(\overline{\theta(g)w})_0(\tau).
\end{aligned}$$

The sum here can be expressed as

$$4 \sum_{\substack{\alpha,\beta\in F^\times \\ \alpha\beta\tau\in F^{\times 2} \\ \alpha-\beta=1}} (\theta(g)u)(\alpha)(\overline{\theta(g)w})(\beta) = \sum_{\substack{\alpha\in F^\times/F^{\times 2} \\ \xi,\eta\in F^\times \\ \alpha(\xi^2-\tau\eta^2)=1}} (\theta(g)u)(\alpha\xi^2)(\overline{\theta(g)w})(\tau\alpha\eta^2),$$

on replacing α by $\alpha\xi^2$ and β by $\alpha\tau\eta^2$. Assuming that $w = \otimes w_v \in C_\chi(\mathbb{A}^\times)$, define a function on $\mathbb{A}^\times \times \mathbb{A}$ by $w(t,x) = \prod w_v(t_v, x_v)$, where

$$w_v(t_v, x_v) = \chi_v(x_v)|x_v|^{1/2} w_v(t_v x_v^2) \quad \text{if } x_v \neq 0,$$

and

$$w_v(t_v, 0) = \lim_{x_v\to 0} w_v(t_v, x_v)(= w_{v0}(t_v)).$$

Then $x \mapsto w(t,x)$ is a Schwartz function on $\mathbb{A}$ for every t in $\mathbb{A}^\times$, and

$$w(t,x) = \chi(z)|z|^{1/2} w(z^2 t, z^{-1}x) \quad (t, z\in\mathbb{A}^\times; x\in\mathbb{A}).$$

The analogous definitions – with $\chi = 1$ – apply to $u = \otimes u_v \in C(\mathbb{A}^\times)$. Then $u(t,x) = |z|^{1/2} u(z^2 t, x/z)$. If $\alpha, \xi \in F^\times$, then $w(\alpha\xi^2) = w(\alpha, \xi)$, and $u(\alpha\xi^2) = u(\alpha, \xi)$. In these notations our sum takes the form

$$\sum_{\substack{\alpha\in F^\times/F^{\times 2} \\ \xi,\eta\in F^\times \\ \alpha(\xi^2-\tau\eta^2)=1}} (\theta(g)u)(\alpha,\xi)(\overline{\theta(g)w})(\tau\alpha,\eta).$$

Note that when $\eta=0$ we can take $\alpha=1$ (and $\xi=\pm1$), hence the missing term in the last sum is $2(\theta(g)u)(1)(\overline{\theta(g)w})_0(\tau)$. When $\xi=0$ we can take $\alpha=-\tau^{-1}$ and $\eta=\pm1$, hence the corresponding missing term is $2(\theta(g)u)_0(-\tau)(\overline{\theta(g)w})(-1)$. These are terms in our integral $\iint\Theta_u\overline{\Theta}_w^\chi\overline{\psi}dn\,dz$. We conclude that (2) equals

$$\sum_{\substack{\alpha\in F^\times/F^{\times2}\\ \xi,\eta\in F\\ \alpha(\xi^2-\tau\eta^2)=1}}(\theta(g)u)(\alpha,\xi)(\overline{\theta(g)w})(\tau\alpha,\eta)=\sum_{\alpha\in F^\times/F^{\times2}}\sum_{\substack{\gamma\in E^\times\\ \alpha N\gamma=1}}(\theta(g)F)(\alpha,\gamma;u,w),$$

where we put $(\theta(g)F)(t,z;u,w)=\prod_v(\theta_v(g_v)F_v)(t_v,z_v;u_v,w_v)$ and

$$(\theta_v(g_v)F_v)(t_v,z_v;u_v,w_v)=(\theta_v(g_v)u_v)(t_v,x_v)(\overline{\theta_v(g_v)w_v})(\tau t_v,y_v)$$

$(t_v\in F_v^\times, z_v=x_v+\sqrt{\tau}y_v\in E_v)$. Note that if $\gamma=\xi+\sqrt{\tau}\eta$, then $N\gamma=\xi^2-\tau\eta^2$. Note also that for $t\in F_v$, $z\in E_v$, we have

$$\left(\theta_v\left(s\begin{pmatrix}1&b\\0&1\end{pmatrix}\right)F_v\right)(t,z)=\psi_v\left(\frac{1}{2}btNz\right)F_v(t,z),\quad b\in F_v;$$

$$\left(\theta_v\left(s\begin{pmatrix}a&0\\0&1\end{pmatrix}\right)F_v\right)(t,z)=|a|_v\chi_v(a)F_v(at,z),\quad a\in F_v^\times;$$

$$\left(\theta_v\left(s\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)F_v\right)(t,z)=\overline{\gamma}_v(\tau)\chi_v(t)|t|_v\int_E F_v(t,\zeta)\overline{\psi}_v\left(\frac{1}{2}t\mathrm{tr}(z\overline{\zeta})\right)d\zeta,$$

and

$$\theta_v\left(s\begin{pmatrix}a&0\\0&a\end{pmatrix}\right)F_v=F_v,\quad a\in F_v^\times.$$

Moreover

$$F(t,z)=|s|\chi(s)F(s^2t,z/s)\quad(s,t\in\mathbb{A}^\times;z\in\mathbb{A}_E).$$

Note that θ_v is a representation of the group $GL(2,F_v)$ itself (even of $PGL(2,F_v)$) on the space of the functions $F_v(t_v,z_v)$, which are smooth on $F_v^\times\times E_v$. Hence (2) can be written as

$$2\sum_{\gamma\in E^\times/F^\times}(\theta(g)F)(N\gamma^{-1},\gamma;u,w),$$

and the total "geometric sum" is equal to

$$2\sum_{\gamma\in E^\times/F^\times}\int_{\mathbb{ZN}\backslash\mathbb{G}}f_\psi(g^{-1})(\theta(g)F)(N\gamma^{-1},\gamma;u,w)dg.$$

For $z\in E_v^\times$, define

$$f_{E_v}(z)=|z\overline{z}|_{F_v}^{-1/2}\int_{Z_vN_v\backslash G_v}f_{v,\psi}(g^{-1})(\theta_v(g)F_v)(Nz^{-1},z;u_v,w_v)dg.$$

The function f_{E_v} on $E_v^\times$ satisfies $f_{E_v}(az)=\chi_v(a)f_{E_v}(z)$ $(a\in F_v^\times, z\in E_v^\times)$, and $f_{E_v}(\overline{z})=\chi_v(-1)f_{E_v}(z)$.

5. Transfer of Functions.

2. Lemma. *The function $f_{E_v}(\gamma)$ extends to a smooth function on $E_v^\times$.*

Proof. This is clear, since the function F_v is smooth on $F_v^\times \times E_v$, and the integration ranges over a compact set of g, depending on f_v. Alternatively stated, let us pass to local notations – drop v – to simplify the notations. Write $\gamma = x + \sqrt{\tau}y$. Then up to a factor which is smooth in γ, our expression is the integral over k in K of

$$\chi(y)|xy|^{1/2} \int_{F^\times} \chi(a) f_\psi \left(k^{-1} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) (\theta(k)u) \left(\frac{ax^2}{\gamma\overline{\gamma}} \right) (\overline{\theta(k)w}) \left(\frac{\tau a y^2}{\gamma\overline{\gamma}} \right) d^\times a,$$

at γ with $xy \neq 0$. The only possible points where this may not be smooth are at $x = 0$ or $y = 0$. But at these points we have that

$$(\theta(k)u) \left(\frac{ax^2}{\gamma\overline{\gamma}} \right) = |x|^{-1/2} (\theta(k)u)_0 (a/\gamma\overline{\gamma}) \quad (|x| \text{ small})$$

and

$$(\theta(k)w) \left(\frac{\tau a y^2}{\gamma\overline{\gamma}} \right) = \chi(y)|y|^{-1/2} (\theta(k)w)_0 (\tau a/\gamma\overline{\gamma}) \quad (|y| \text{ small}),$$

hence the lemma again follows. $\square$

Consider next the case of a spherical function, first at a place v which splits in E.

3. Lemma. *If v splits in E, ψ_v has conductor R_v, $u_v = u_v^0$, $w_v = w_v^0$, and f_v is spherical, then $f_{E_v}((a,b)) = F_{f_v}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right)$; $a, b \in F_v^\times$.*

Corollary. *A character μ_v of $E_v^\times / F_v^\times$ is of the form $\mu_v((a,b)) = \mu_{1v}(a/b)$ for some character μ_{1v} of $F_v^\times$, and we have $\mu_v(f_{E_v}) = tr\, I(\mu_{1v}, \mu_{1v}^{-1}; f_v)$ for every spherical function f_v on $PGL(2, F_v)$.*

Proof. Note that

$$\mu_v(f_{E_v}) = \int_{E_v^\times / F_v^\times} \mu_v((a,b)) f_{E_v}((a,b)) d^\times (a/b)$$

is equal to

$$\int_{F_v^\times} \mu_{1v}(a/b) F_{f_v} \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right) d^\times (a/b) = \operatorname{tr} I(\mu_{1v}, \mu_{1v}^{-1}; f_v). \qquad \square$$

Proof of Lemma. Suppose that v splits in E, thus $E_v = F_v \oplus F_v$ and $\overline{\gamma} = (d,c)$ if $\gamma = (c,d)$, and assume that ψ_v has conductor R_v, $u_v = u_v^0$, $w_v = w_v^0$, and f_v is spherical.

Note that $\chi_v = 1$. Then using the Iwasawa decomposition $dg = dnd^\times a/|a|dk$, and noting that $\theta(k)u_v^0 = u_v^0$, and $\theta(k)w_v^0 = w_v^0$, we obtain the following expression for $f_{E_v}((c,d))$:

$$|cd|_v^{-1/2} \int_{F_v^\times} f_{v,\psi}\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) u_v^0((a/cd, \frac{1}{2}(c+d)))\overline{w}_v^0((\tau a/cd, (c-d)/2\sqrt{\tau}))d^\times a.$$

Here $\gamma = (c,d)$ in $E_v^\times$ can be expressed as $x_v + \sqrt{\tau} y_v$, where $\sqrt{\tau} = (\sqrt{\tau}, -\sqrt{\tau})$ and $x_v = \frac{1}{2}(\gamma + \overline{\gamma}) = \frac{1}{2}(c+d)$, and $y_v = (\gamma - \overline{\gamma})/2\sqrt{\tau} = (c-d)/2\sqrt{\tau}$. At $\gamma \neq \pm\overline{\gamma}$ in $E_v^\times$, we obtain

$$\left|\frac{c^2-d^2}{cd}\right|_v^{1/2} \int_{F_v^\times} f_{v,\psi}\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) u_v^0\left(\frac{a(c+d)^2}{4cd}\right) \overline{w}_v^0\left(\frac{a(c-d)^2}{4cd}\right) d^\times a.$$

This expression is not changed if (c,d) is replaced by (d,c) or $(-c,d)$, and (c,d) is taken modulo $F_v^\times$. We may take then $d=1$ and $|c| \le 1$. Consider first the case that $|c| = 1$, $c \neq \pm 1$. We may assume that $|c+1| = 1$, and $|c-1| \le 1$. Then our expression is

$$\begin{aligned} &|c-1|_v^{1/2} \int_{F_v^\times} f_{v,\psi}\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) u_v^0(a)\overline{w}_v^0(a(c-1)^2)d^\times a \\ &= \int_{F_v^\times} f_{v,\psi}\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) u_v^0(a)\overline{w}_v^0(a)d^\times a. \end{aligned}$$

If $|c| = |\boldsymbol{\pi}^n| < 1$, our expression is

$$|\boldsymbol{\pi}^n|^{-1/2} \int_{F_v^\times} f_{v,\psi}\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) u_v^0(a\boldsymbol{\pi}^{-n})\overline{w}_v^0(a\boldsymbol{\pi}^{-n})d^\times a;$$

this last expression is then valid for $n = 0$ too. The integrand is non-zero only when $a \in \boldsymbol{\pi}^{n+2m}R_v^\times$, $m \ge 0$, and we get

$$\begin{aligned} &|\boldsymbol{\pi}^n|^{-1/2} \sum_{m \ge 0} f_{v,\psi}\left(\begin{pmatrix} \boldsymbol{\pi}^{-n-2m} & 0 \\ 0 & 1 \end{pmatrix}\right) |\boldsymbol{\pi}|^{-m} \\ &= \sum_{m\ge 0} q^{m+\frac{1}{2}n}\left[f_v\left(\begin{pmatrix} \boldsymbol{\pi}^{-n-2m} & 0 \\ 0 & 1 \end{pmatrix}\right) - f_v\left(\begin{pmatrix} \boldsymbol{\pi}^{-n-2m-2} & 0 \\ 0 & 1 \end{pmatrix}\right)\right] \\ &= q^{\frac{1}{2}n}\left[f_v\left(\begin{pmatrix} \boldsymbol{\pi}^{-n} & 0 \\ 0 & 1 \end{pmatrix}\right) + \sum_{m \ge 1}\left(1 - \frac{1}{q}\right) q^m f_v\left(\begin{pmatrix} \boldsymbol{\pi}^{-n-2m} & 0 \\ 0 & 1 \end{pmatrix}\right)\right]. \end{aligned}$$

Note that

$$\begin{aligned} &F_{f_v}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \frac{|a-b|_v}{|ab|_v^{1/2}} \int_{F_v} f_v\left(\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx \\ &= |a/b|_v^{1/2} \int_{F_v} f_v\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx, \end{aligned}$$

and so, for our spherical f_v, we have

$$F_{f_v}\left(\left(\begin{pmatrix}\boldsymbol{\pi}^{-n} & 0\\ 0 & 1\end{pmatrix}\right)\right) = |\boldsymbol{\pi}^{-n}|_v^{1/2}\int f_v\left(\left(\begin{pmatrix}\boldsymbol{\pi}^{-n} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\right)dx$$

$$= |\boldsymbol{\pi}^{-n}|^{1/2}\left[f_v\left(\left(\begin{pmatrix}\boldsymbol{\pi}^{-n} & 0\\ 0 & 1\end{pmatrix}\right)\right) + \int_{|x|>1} f_v\left(\left(\begin{pmatrix}\boldsymbol{\pi}^{-n}x^2 & 0\\ 0 & 1\end{pmatrix}\right)\right)dx\right]$$

$$= q^{n/2}\left[f_v\left(\left(\begin{pmatrix}\boldsymbol{\pi}^{-n} & 0\\ 0 & 1\end{pmatrix}\right)\right) + \left(1-\frac{1}{q}\right)\sum_{m\geq 1} q^m f_v\left(\left(\begin{pmatrix}\boldsymbol{\pi}^{-n-2m} & 0\\ 0 & 1\end{pmatrix}\right)\right)\right].$$

Here we used the decomposition

$$\begin{pmatrix}1 & 0\\ 0 & \boldsymbol{\pi}^{-n}\end{pmatrix}\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & \boldsymbol{\pi}^{-n}\end{pmatrix}\begin{pmatrix}1 & \frac{1}{x}\\ 0 & 1\end{pmatrix}\begin{pmatrix}x^{-1} & 0\\ 0 & x\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1/x\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & \boldsymbol{\pi}^{n}/x\\ 0 & 1\end{pmatrix}\begin{pmatrix}x^{-1} & 0\\ 0 & x\boldsymbol{\pi}^{-n}\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1/x\\ 0 & 1\end{pmatrix} \equiv \begin{pmatrix}x^{-1} & 0\\ 0 & x\boldsymbol{\pi}^{-n}\end{pmatrix} \quad \text{if } |x|\geq 1.$$

The lemma follows. □

6. Non-split Case. Suppose now that E_v/F_v is an unramified field extension, the conductor of ψ_v is R_v (thus $\psi_v = 1$ on R_v but $\psi_v(\boldsymbol{\pi}_v^{-1}R_v) \neq 1$), $|2|_v = 1$, f_v is spherical ($K_v = GL(2,R_v)$-biinvariant), and $u_v = u_v^0$, $w_v = w_v^0$, the K_v-invariant elements in $C(F_v^\times)$ and $C_{\chi_v}(F_v^\times)$. Then $dg = dn\cdot|a|^{-1}d^\times a\cdot dk$ if $g = n\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}k$ in $Z_v\backslash G_v = N_vA_vK_v$. Hence

$$f_{E_v}(\gamma) = |\gamma\overline{\gamma}|_v^{-1/2}\int_{F_v^\times} f_{v,\psi}\left(\left(\begin{pmatrix}a^{-1} & 0\\ 0 & 1\end{pmatrix}\right)\right)$$
$$\cdot\,\chi_v(a)u_v^0(a/\gamma\overline{\gamma},\frac{1}{2}(\gamma+\overline{\gamma}))\overline{w}_v^0\left(\tau a/\gamma\overline{\gamma},\frac{\gamma-\overline{\gamma}}{2\sqrt{\tau}}\right)d^\times a.$$

This is a function on $E_v^\times$ which transforms under $F_v^\times$ according to χ_v. Hence we may assume that $\gamma\overline{\gamma}$ and τ are units in $F_v^\times$. At γ with $\overline{\gamma}\neq\pm\gamma$, we put $x = (\gamma+\overline{\gamma})/2$, $y = (\gamma-\overline{\gamma})/2\sqrt{\tau}$, and then

$$f_{E_v}(\gamma) = |xy|_{F_v}^{1/2}\chi_v(y)\int_{F_v^\times} f_{v,\psi}\left(\left(\begin{pmatrix}a^{-1} & 0\\ 0 & 1\end{pmatrix}\right)\right)\chi_v(a)u_v^0(ax^2)\overline{w}_v^0(ay^2)d^\times a$$
$$= \int_{F_v^\times} f_{v,\psi}\left(\left(\begin{pmatrix}a^{-1} & 0\\ 0 & 1\end{pmatrix}\right)\right)u_v^0(a)\overline{w}_v^0(a)d^\times a.$$

The last integral ranges over $R_v^\times\boldsymbol{\pi}^{2m}$, $m\geq 0$, where $\chi_v(a) = 1$, and we used $u_v^0(ax^2) = u_v^0(a)|x|_v^{-1/2}$ and $w_v^0(ay^2) = |y|_v^{-1/2}\chi_v(y)w_v^0(a)$ for the last equality. It follows that in the unramified-spherical case, $f_{E_v}(\gamma)$ depends only on the parity of the valuation of γ. If – moreover – f_v is the unit element f_v^0 of the Hecke algebra, and $|a|_v \le 1$, then

$$f^0_{v,\psi}\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix}\right) = \int_{F_v} f_v^0\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\right)\overline{\psi}(\frac{1}{2}x)dx$$
$$= f_v^0\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix}\right) - f_v^0\left(\begin{pmatrix} a^{-1}\boldsymbol{\pi}^{-2} & 0 \\ 0 & 1\end{pmatrix}\right)$$

is equal to $f_v^0(I) = 1$ if $|a|_v = 1$. Hence $f^0_{E_v}(\gamma) \equiv (-1)^{val_{E_v}(\gamma)}$.

4. Lemma. *When E_v/F_v is an unramified field extension, f_v is spherical, ψ_v has conductor R_v, $|2|_v = 1$, $u_v = u_v^0$, $w_v = w_v^0$, and μ_{1v} is the unramified character of $F_v^\times$ whose value at the uniformizer $\boldsymbol{\pi}_v$ of R_v is $i = \sqrt{-1}$, then*

$$\mu_v(f_{E_v}) = tr I(\mu_{1v}, \mu_{1v}^{-1}; f_v),$$

where μ_v denotes the unramified "sign" character of $E_v^\times$, whose value at a uniformizer $\boldsymbol{\pi}_v$ of R_{E_v} is -1. Here $\mu_v(f_{E_v}) = \int_{E_v^\times/F_v^\times} \mu_v(\gamma) f_{E_v}(\gamma) d^\times\gamma$ is the value of f_{E_v} at a γ in $R_{E_v}^\times$.

Proof. We drop the index v to simplify the notations, and recall that $f_E(\gamma)$, $\gamma \in R_E^\times$, is independent of γ, and is given by

$$f_E(\gamma) = \int_{F^\times}\left[f\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix}\right) - f\left(\begin{pmatrix} a^{-1}\boldsymbol{\pi}^{-2} & 0 \\ 0 & 1\end{pmatrix}\right)\right] u^0(a)\overline{w}^0(a)d^\times a$$
$$= \sum_{n\ge 0}\left[f\left(\begin{pmatrix} \boldsymbol{\pi}^{-2n} & 0 \\ 0 & 1\end{pmatrix}\right) - f\left(\begin{pmatrix} \boldsymbol{\pi}^{-2n-2} & 0 \\ 0 & 1n\end{pmatrix}\right)\right] |\boldsymbol{\pi}|^{-n}\chi(\boldsymbol{\pi}^n)$$
$$= f(I) + (1+\frac{1}{q})\sum_{n\ge 1} f\left(\begin{pmatrix} \boldsymbol{\pi}^{-2n} & 0 \\ 0 & 1\end{pmatrix}\right)(-q)^n, \quad q = |\boldsymbol{\pi}|^{-1}.$$

On the other hand, since $\mu_1(\boldsymbol{\pi}) = i$, we have

$$\operatorname{tr} I(\mu_1, \mu_1^{-1}; f) = \sum_{n\in\mathbb{Z}} F_f\left(\begin{pmatrix} \boldsymbol{\pi}^{-n} & 0 \\ 0 & 1\end{pmatrix}\right) i^n = F_f(I) + \sum_{n\ge 1} F_f\left(\begin{pmatrix} \boldsymbol{\pi}^{-n} & 0 \\ 0 & 1\end{pmatrix}\right)(i^n + i^{-n})$$
$$= F_f(I) + 2\sum_{n\ge 1}(-1)^n F_f\left(\begin{pmatrix} \boldsymbol{\pi}^{-2n} & 0 \\ 0 & 1\end{pmatrix}\right).$$

But

$$F_f\left(\begin{pmatrix} \boldsymbol{\pi}^{-n} & 0 \\ 0 & 1\end{pmatrix}\right) = q^{n/2}\left[f\left(\begin{pmatrix} \boldsymbol{\pi}^{-n} & 0 \\ 0 & 1\end{pmatrix}\right) + (1-\frac{1}{q})\sum_{m\ge 1} q^m f\left(\begin{pmatrix} \boldsymbol{\pi}^{-n-2m} & 0 \\ 0 & 1\end{pmatrix}\right)\right].$$

So we get

$$= F_f(I) + 2\sum_{n\geq 1}(-1)^n F_f\left(\begin{pmatrix}\boldsymbol{\pi}^{-2n} & 0\\ 0 & 1\end{pmatrix}\right) = f(I) + (1-\frac{1}{q})\sum_{m\geq 1} q^m f\left(\begin{pmatrix}\boldsymbol{\pi}^{-2m} & 0\\ 0 & 1\end{pmatrix}\right)$$

$$+ 2\sum_{k\geq 1}(-q)^k\left[f\left(\begin{pmatrix}\boldsymbol{\pi}^{-2k} & 0\\ 0 & 1\end{pmatrix}\right) + (1-\frac{1}{q})\sum_{m\geq 1} q^m f\left(\begin{pmatrix}\boldsymbol{\pi}^{-2k-2m} & 0\\ 0 & 1\end{pmatrix}\right)\right]$$

$$= f(I) + \sum_{n\geq 1} f\left(\begin{pmatrix}\boldsymbol{\pi}^{-2n} & 0\\ 0 & 1\end{pmatrix}\right)\left[(1-\frac{1}{q})q^n + 2(-q)^n + 2(1-\frac{1}{q})q^n\sum_{1\leq k<n}(-1)^k\right]$$

$$= f(I) + \sum_{n\geq 1}(1+\frac{1}{q})(-q)^n f\left(\begin{pmatrix}\boldsymbol{\pi}^{-2n} & 0\\ 0 & 1\end{pmatrix}\right) = f_E(\gamma),$$

as asserted. □

7. Conclusion. So far we have shown the following.

5. Proposition. *Given a finite set V of places of F containing the archimedean places and those which ramify in E, and those where the conductor of ψ is not R_v, and those where $u_v \neq u_v^0$, or $w_v \neq w_v^0$, for any test function $f = \otimes f_v$, $f_v \in C_c^\infty(Z_v\backslash G_v)$, f_v is spherical ($K_v = GL(2, R_v)$-biinvariant) for all $v \notin V$, and $f_v = f_v^0$ (= characteristic function of Z_vK_v) for almost all v, we have the equality*

$$(1) + \sum_{\eta^2=\chi} I_\eta = 2\sum_{\gamma\in E^\times/F^\times} f_E(\gamma) = \sum_\mu \mu(f_E).$$

Here I_η is defined in Lemma 1. Moreover, $f_E(a) = \prod_v f_{E_v}(a_v)$ for $a = (a_v) \in \mathbb{A}_E^\times$, where f_{E_v} is a smooth function on $E_v^\times$ (by Lemma 2) with $f_{E_v}(a\gamma) = \chi_v(a) f_{E_v}(\gamma)$ ($a \in F_v^\times, \gamma \in E_v^\times$), which is spherical ($R_{E_v}^\times$-invariant) for $v \notin V$, and for almost all v it is the unit element: $f_{E_v}^0(\gamma) = \chi_v(\boldsymbol{\pi}_v)^{val_{E_v}(\gamma)}$ (if v is non-split), and $f_{E_v}^0((a,b))$ equals 1 if $|a|_v = |b|_v$, and zero otherwise (if v splits). The sum over μ ranges over all characters of $\mathbb{A}_E^\times/E^\times$ whose restriction to $\mathbb{A}^\times/F^\times$ is χ, and

$$\mu(f_E) = \int_{\mathbb{A}_E^\times/\mathbb{A}^\times E^\times} \mu(a) f_E(a)\, da.$$

The measure da is such that $\int_{\mathbb{A}_E^\times/\mathbb{A}^\times E^\times} da = 2$, the Tamagawa number of $Res_{E/F}\mathbb{G}_m/\mathbb{G}_m$.
□

Note that the sums over γ and μ are equal by the Poisson summation formula. Since $f_E(\overline{a}) = f_E(a)$, we have $\overline{\mu}(f_E) = \mu(f_E)$, where $\overline{\mu}(a) = \mu(\overline{a})$.

Lemmas 3 and 4 assert that at $v \notin V$, we have $\mu_v(f_{E_v}) = \operatorname{tr} I(\mu_{1v}, \mu_{1v}^{-1}; f_v)$, where μ_{1v} is related to μ_v as in the Theorem. On the other hand, in (1), $\pi_v(f_v)$ acts as zero on $\varphi \in \pi$ unless φ is K_v-invariant on the right, in which case $\pi_v(f_v)$ acts as multiplication by the scalar $\operatorname{tr}\pi_v(f_v)$. A standard argument of "generalized linear independence of characters" (see, e.g., [F2], p. 758), using the absolute convergence of our sums, simple unitarity estimates, and the Stone-Weierstrass theorem, implies the following. Put $\mathbb{K}(V) = \prod_{v \notin V} K_v$, and let $\pi^{\mathbb{K}(V)}$ be the space of $\mathbb{K}(V)$-invariant vectors in the space of π.

6. Proposition. *Fix an unramified G_v-module π_v^* for each $v \notin V$. For any $f_v \in C_c^\infty(G_v)$, $v \in V$, put $f = (\bigotimes_{v \in V} f_v) \otimes (\bigotimes_{v \notin V} f_v^0)$. Then $(1) + \sum_{\eta^2=\chi} I_\eta = \sum_\mu \mu(f_E)$, where in (1) the first sum ranges over the cuspidal representations π of $PGL(2,\mathbb{A})$ with $\pi_v \simeq \pi_v^*$ for all $v \notin V$, and the second sum is over a smooth orthonormal basis $\{\varphi\}$ for the spaces $\pi^{\mathbb{K}(V)}$. The sum over η, $\eta^2 = \chi$, ranges over those characters η with $I(\eta_v, 1/\eta_v) \simeq \pi_v^*$ for all $v \notin V$. The sum over μ ranges over those characters of $\mathbb{A}_E^\times/E^\times$ such that for $v \notin V$ the component μ_v is unramified, and defines the representation $I(\mu_{1v}, \mu_{1v}^{-1})$, which is required to be equivalent to π_v^*.* □

By the Chebotarev density theorem the sum over μ consists of at most one pair $\{\mu, \overline{\mu}\}$ of non-zero contributions. Since every smooth function on $E_v^\times$ which transforms under $F_v^\times$ according to χ_v and whose values at $\gamma \in E_v^\times$ and $\overline{\gamma}$ defer by a multiple of $\chi_v(-1)$, is obtained as f_{E_v} from some f_v, for some u_v and w_v, we conclude, on choosing $\pi_v^* = I(\mu_{1v}, \mu_{1v}^{-1})$ $(v \notin V)$, that for each μ as in the Theorem there exists (a unique) $\pi(\mu)$, as in the Theorem; it is the unique π which occurs in (1), unless $\chi = \eta^2$ and $\mu = \overline{\mu}$, since the sum $\sum_\mu \mu(f_E)$ of Proposition 6 is non-zero. This $\pi = \pi(\mu)$ has the property that $R^\chi(\varphi) \neq 0$ for some $\varphi \in \pi$.

On the other hand, by the rigidity theorem for $GL(2)$ (see [JS]), at most one π can contribute to the sum (1) of Proposition 6. Let π be a cuspidal representation of $PGL(2,\mathbb{A})$ such that $\int_{\mathbb{Z}^2 G\backslash\mathbb{G}} \varphi_1(g)\Theta_u(g)\overline{\Theta}_w^\chi(g)dg$ is non-zero for some u and w, and χ, and a smooth form φ_1 in the space of π. We can choose a sufficiently large finite set V, and $\pi_v^* = \pi_v$ for $v \notin V$, such that the equality $(1) = \sum_\mu \mu(f_E)$ of Proposition 6 holds. The I_η vanish again by [JS]. We may assume that the orthonormal basis of $\pi^{\mathbb{K}(V)}$ in (1) contains φ_1. Since π is cuspidal, it is generic, namely there exists a form φ_2 in its space such that $W_{\varphi_2,\psi}(e) \neq 0$. We may assume that either φ_2 is φ_1, or φ_2 is orthogonal to φ_1. In any case, the space of endomorphisms of π_v is spanned by the operators $\pi_v(f_v)$, $f_v \in C_c^\infty(Z_v\backslash G_v)$. Hence we can choose $f_v (v \in V)$ such that $\prod_{v \in V} \pi_v(f_v)$ maps φ_2 to φ_1, and any vector in $\pi^{\mathbb{K}(V)}$ which is orthogonal to φ_2, to 0. With this choice of f in Proposition 6, the two sums of (1) consist of one term each. Our π, and φ_2, index the only possibly non-zero term:

$$\int_{\mathbb{Z}^2 G\backslash\mathbb{G}} (\pi(f)\varphi_2)(g)\Theta_u(g)\overline{\Theta}_w^\chi(g)dg \cdot \overline{W}_{\varphi_2,\psi}(e),$$

which is non-zero by our choice of φ_2 and $\pi(f)\varphi_2 = \varphi_1$. Since the sum (1) is non-zero, there is μ such that $\mu(f_E) \neq 0$, by the equality of Proposition 6, and if χ is a square, μ satisfies $\mu \neq \overline{\mu}$. This proves that π with $R^\chi(\varphi) \neq 0$ for some $\varphi \in \pi$ is necessarily of the form $\pi(\mu)$, and the Theorem follows.

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