MOTIVIC TORSORS

Yuval Z. FLICKER

Abstract. The torsor $P_\sigma = \mathrm{Hom}^\otimes(H_{\mathrm{DR}}, H_\sigma)$ under the motivic Galois group $G_\sigma = \mathrm{Aut}^\otimes H_\sigma$ of the Tannakian category $\mathcal{M}_k$ generated by one-motives related by absolute Hodge cycles over a field k with an embedding $\sigma : k \hookrightarrow \mathbb{C}$ is shown to be determined by its projection $P_\sigma \to P_\sigma/G_\sigma^0$ to a $\mathrm{Gal}(\overline{k}/k)$-torsor, and by its localizations $P_\sigma \times_k k_\xi$ at a dense subset of orderings ξ of the field k, provided k has virtual cohomological dimension (vcd) one. This result is an application of a recent local-global principle for connected linear algebraic groups over a field k of vcd ≤ 1.

The singular cohomology with coefficients in the field $\mathbb{Q}$ of rational numbers of a smooth projective – even just complete – variety over $\mathbb{C}$ has a ("pure") Hodge structure. Motives with a realization (usually by means of some cohomology theory) which has a pure Hodge structure are called pure motives. Deligne defined in [D-II] a mixed Hodge structure to be a finite dimensional vector space V over $\mathbb{Q}$ with a finite increasing (weight) filtration $W_\bullet$ and a finite decreasing (Hodge) filtration $F^\bullet$ on $V \otimes_{\mathbb{Q}} \mathbb{C}$ such that $F^\bullet$ induces a Hodge structure of weight n on the graded piece $\mathrm{Gr}_n^W V = W_n V / W_{n-1} V$ for each n. Deligne showed in [D-III] that the cohomology $H^*(E(\mathbb{C}), \mathbb{Q})$ of any variety E over $\mathbb{C}$ – not necessarily complete and smooth – carries a natural mixed Hodge structure. Motives with a realization which has a mixed Hodge structure are called mixed motives for emphasize.

Deligne introduced the notion of a one-motive M – as well as its dual $M^\vee$, and Betti: $M(\mathbb{C})_B$, de Rham: $H_{\mathrm{DR}}(M)$, and ℓ-adic: $H_\ell(M)$, realizations – in [D-III], §10, as a simple example of a motive whose Betti realization $M(\mathbb{C})_B$ has a mixed Hodge structure, but does not have a Hodge structure. Let $\sigma : k \hookrightarrow \mathbb{C}$ be an embedding of a field k in the field $\mathbb{C}$ of complex numbers, and $\overline{\sigma} : \overline{k} \hookrightarrow \mathbb{C}$ an extension to an algebraic closure $\overline{k}$. Write $\mathrm{Gal}(\overline{k}/k)$ for the Galois group. For a variety E over k, write σE for the $\mathbb{C}$-variety $E \times_{k,\sigma} \mathbb{C}$.

A one-motive over k is a complex $M = [X \xrightarrow{u} E]$ of length one placed in degrees 0 and 1, comprising of a semi-abelian variety E (namely an extension $1 \to T \to E \to A \to 0$ of an abelian variety A by a torus T) over k, a finitely generated torsion free $\mathrm{Gal}(\overline{k}/k)$-module X, and a $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphism $u : X \to E(\overline{k})$. Note that E is a commutative k-group. One-motives include the Artin motives as $[X \to 0]$ and the Tate motive as $[0 \to \mathbb{G}_m]$. We also write $M = (X, A, T, E, u)$, $M \otimes \mathbb{Q}$ for the isogeny class of M, $\sigma M = [X \xrightarrow{u} \sigma E]$ and $\sigma M(\mathbb{C}) = [X \xrightarrow{u} \sigma E(\mathbb{C})]$. A one-motive M has a "weight" filtration: $W_0 M = [X \xrightarrow{u} E]$, $W_{-1} M = [0 \to E]$, $W_{-2} M = [0 \to T]$, $W_{-3} M = [0 \to 0]$, with

Department of Mathematics, The Ohio State University, 231 W. 18th Ave., Columbus, OH 43210-1174; flicker@math.ohio-state.edu; 1991 Mathematics Subject Classification: 14L30; 14F20, 14F40, 14P25, 11G10; accepted for publication in Israel J. Math. (2000)

graded factors $\mathrm{Gr}_0 M = X$, $\mathrm{Gr}_{-1} M = (E/T)[-1] = A[-1]$, and $\mathrm{Gr}_{-2} M = T[-1]$. Put $\mathrm{Gr}^W M = [X \xrightarrow{0} A \times_k T]$.

The Betti realization $H_\sigma(M) = \sigma M(\mathbb{C})_B$ of a one-motive $M = [X \xrightarrow{u} E]$ over k is the vector space $T_\sigma(M) \otimes \mathbb{Q}$, where the lattice $T_\sigma(M)$ is the fiber product of $\mathrm{Lie}\, \sigma E(\mathbb{C})$ and X over $\sigma E(\mathbb{C})$, namely the pullback of $0 \to H_1(\sigma E(\mathbb{C})) \to \mathrm{Lie}\, \sigma E(\mathbb{C}) \xrightarrow{\exp} \sigma E(\mathbb{C}) \to 1$ by $X \xrightarrow{u} \sigma E(\mathbb{C})$. It depends on the embedding $\sigma : k \hookrightarrow \mathbb{C}$. Then $\sigma M(\mathbb{C})_B$ is a mixed Hodge structure $(V, W_\bullet, F^\bullet)$ of type $\{(0,0), (0,-1), (-1,0), (-1,-1)\}$ whose graded parts are $\mathrm{Gr}_0^W V = X \otimes \mathbb{Q}$, polarizable $\mathrm{Gr}_{-1}^W V = H_1(\sigma A(\mathbb{C}), \mathbb{Q})$, and $\mathrm{Gr}_{-2}^W V = H_1(\sigma T(\mathbb{C}), \mathbb{Q}) = X_*(\sigma T) \otimes \mathbb{Q}$; see [D-III], 10.1.3.

Denote by $\mathcal{M}_k$ the Tannakian category (Deligne-Milne [DM], Definition 2.19) generated by the isogeny classes of one-motives over k, in the category $\mathcal{MR}_k$ of mixed realizations (Jannsen [J], 2.1), related by absolute Hodge cycles (Deligne [D2], 2.10, Brylinski [Br], 2.2.5). The objects of $\mathcal{MR}_k$ are tuples $H = (H_{\mathrm{DR}}, H_\ell, H_\sigma; I_{\infty,\sigma}, I_{\ell,\overline{\sigma}})$, where ℓ ranges over the rational primes, σ over the embeddings $k \hookrightarrow \mathbb{C}$, and $\overline{\sigma}$ over the $\overline{k} \hookrightarrow \mathbb{C}$, described in [J], p. 10. In particular H_{DR} is a finite dimensional k-vector space with a decreasing (Hodge) filtration $(F^n; n \in \mathbb{Z})$ and an increasing (weight) filtration $(W_m; m \in \mathbb{Z})$; H_ℓ is a finite dimensional $\mathrm{Gal}(\overline{k}/k)$-module over $\mathbb{Q}_l$ with $\mathrm{Gal}(\overline{k}/k)$-equivariant increasing (weight) filtration $W_\bullet$; H_σ is a mixed Hodge structure (over $\mathbb{Q}$), and $I_{\infty,\sigma} : H_\sigma \otimes \mathbb{C} \tilde{\to} H_{\mathrm{DR}} \otimes_{k,\sigma} \mathbb{C}$, $I_{\ell,\overline{\sigma}} : H_\sigma \otimes \mathbb{Q}_\ell \tilde{\to} H_\ell$ $(\sigma = \overline{\sigma}|k)$ are the comparison isomorphisms.

The morphisms in $\mathcal{MR}_k$ are tuples $(f_{\mathrm{DR}}, f_\ell, f_\sigma)_{\ell,\sigma}$ described in [J], p. 11. In particular $f_\sigma : H_\sigma \to H'_\sigma$ is a morphism of mixed Hodge structures, $f_{\mathrm{DR}} : H_{\mathrm{DR}} \to H'_{\mathrm{DR}}$ is k-linear and $f_\ell : H_\ell \to H'_\ell$ is a $\mathbb{Q}_\ell$-linear $\mathrm{Gal}(\overline{k}/k)$-morphism, which correspond under the comparison isomorphisms. The category $\mathcal{MR}_k$ is abelian ([J], 2.3), tensor ([J], 2.7) with identity object $\mathbf{1} = (k, \mathbb{Q}_\ell, \mathbb{Q}; \mathrm{id}_{\infty,\sigma}, \mathrm{id}_{\ell,\overline{\sigma}})$, and it has internal $Hom(H, H') \in \mathcal{MR}_k$ for all H, H' in $\mathcal{MR}_k$ (thus $\mathrm{Hom}(H'', Hom(H, H')) = \mathrm{Hom}(H'' \otimes H, H')$ for all H, H', $H'' \in \mathcal{MR}_k$). For example, $H_{\mathrm{DR}}(Hom(H, H')) = \mathrm{Hom}_k(H_{\mathrm{DR}}, H'_{\mathrm{DR}})$, $H_\ell(Hom(H, H')) = \mathrm{Hom}_{\mathbb{Q}_\ell}(H_\ell, H'_\ell)$, $H_\sigma(Hom(H, H')) = \mathrm{Hom}_{\mathbb{Q}}(H_\sigma, H'_\sigma)$. Hence $\mathcal{MR}_k$ is rigid (each object H has a dual $H^\vee = Hom(H, 1)$).

Defining the space $\mathrm{AHC}(H)$ of absolute Hodge cycles of $H \in \mathcal{MR}_k$ to be the set of $(x_{\mathrm{DR}}, x_\ell, x_\sigma) \in H_{\mathrm{DR}} \times \prod_\ell H_\ell \times \prod_\sigma H_\sigma$ such that $I_{\infty,\sigma}(x_\sigma) = x_{\mathrm{DR}}$, $I_{\ell,\overline{\sigma}}(x_\sigma) = x_\ell$ for all σ, $\overline{\sigma}$ with $\overline{\sigma}|k = \sigma$ and $x_{\mathrm{DR}} \in F^0 H_{\mathrm{DR}} \cap W_0 H_{\mathrm{DR}}$ (it is a finite dimensional vector space over $\mathbb{Q}$), one has $\mathrm{Hom}(H, H') = \mathrm{AHC}(Hom(H, H'))$. A Hodge cycle with respect to σ is a tuple $(x_{\mathrm{DR}}, x_\ell) \in H_{\mathrm{DR}} \times \prod_\ell H_\ell$ such that there is $x_\sigma \in H_\sigma$ with $I_{\infty,\sigma}(x_\sigma) = x_{\mathrm{DR}}$, $I_{\ell,\overline{\sigma}}(x_\sigma) = x_\ell$, $x_{\mathrm{DR}} \in F^0 H_{\mathrm{DR}} \cap W_0 H_{\mathrm{DR}}$. Then $\mathcal{MR}_k$ is a Tannakian category neutral over $\mathbb{Q}$, namely a rigid abelian tensor $\mathbb{Q}$-linear category with a $\mathbb{Q}$-valued fiber ([DM], Definition 2.19: exact faithful $\mathbb{Q}$-linear tensor) functors $H_\sigma^\# : \mathcal{MR}_k \to \mathrm{Vec}_{\mathbb{Q}}$, $H \mapsto H_\sigma^\#$. The # emphasizes here that the symbol indicates the underlying vector space. In the literature, and in the abstract of this paper, # is omitted to simplify the notations for the reader who knows when H_σ is regarded as a mixed Hodge structure, and when it is regarded only as a vector space.

The mixed realization $H(M)$ of a one-motive M is $(H_{\mathrm{DR}}(M), H_\ell(M), H_\sigma(M); I_{\infty,\sigma}, I_{\ell,\overline{\sigma}})$; see [D-III], 10.1.3: the H are H_1. Note that the dual one motive $M^\vee$ (introduced in [D-III], 10.2.11) satisfies $H(M^\vee) = Hom(H(M), \mathbb{Q}(1))$. Hence $H(M)^\vee = H(M^\vee)(-1)$. From now on by a motive we mean an object in the Tannakian category $\mathcal{M}_k$ generated in $\mathcal{MR}_k$ by one-motives. The functor $H^{\#}_{\sigma}$ – which associates to a motive M the vector space underlying the mixed Hodge Betti realization $\sigma M(\mathbb{C})_B$ – is a fiber functor on $\mathcal{M}_k$, making $\mathcal{M}_k$ Tannakian and neutral over $\mathbb{Q}$. Note that an isomorphic – but not canonically – fiber functor is $H^{\#}_{\sigma}\,\mathrm{Gr}^W$. This fiber functor corresponds to a choice of a Levi decomposition of the motivic Galois group, see the end of the 5th paragraph below.

The category $\mathcal{M}_k$ is not semi-simple, but it has a semi-simple Tannakian full subcategory $\mathcal{M}_k^{\mathrm{red}}$ of motives generated by abelian varieties ($M = [0 \to A]$) and Artin motives ($M = [X \to 0]$) over k, related by absolute Hodge cycles ([DM], Propositions 6.5 and 6.21). Thus it is the subcategory of $\mathcal{MR}_k$ generated by $H(A)(= (H_{1,\mathrm{DR}}(A), H_{1,\text{ét}}(A \times_k \overline{k}, \mathbb{Q}_\ell), H_1(\sigma A(\mathbb{C}), \mathbb{Q}))$ of the abelian varieties A over k, and the Artin motives $H(X) = X \otimes \mathbf{1} = (X \otimes k, X \otimes \mathbb{Q}_\ell, X \otimes \mathbb{Q})$. Note that the realization $H(T)$ of the torus $[0 \to T]$ is the Tate twisted Artin motive $X_*(T) \otimes \mathbf{1}(1)(= (X_*(T) \otimes k(1), X_*(T) \otimes \mathbb{Q}_\ell(1), X_*(T) \otimes \mathbb{Q}(1)))$, where $X_*(T) = Hom(\mathbb{G}_m, T)$ (internal Hom in the category of one-motives). The subcategory $\mathcal{M}_k^{\mathrm{red}}$ of $\mathcal{M}_k$ is also neutral over $\mathbb{Q}$, by the fiber functor $H^{\#}_{\sigma}$.

Denote by $\mathcal{M}_k \otimes k$ the category $(\mathcal{M}_k)_{(k)}$ of [DM], Proposition 3.11, obtained on extending coefficients from $\mathbb{Q}$ to k. It is a Tannakian category neutral over k. The functors $H^{\#}_{\sigma} \otimes k$ $(: M \mapsto \sigma M(\mathbb{C})_B \otimes k)$ and $H^{\#}_{\mathrm{DR}}$ on $\mathcal{M}_k \otimes k$ are fiber functors with values in k. The groups $G_\sigma = \mathrm{Aut}^{\otimes}(H^{\#}_{\sigma} \otimes k | \mathcal{M}_k \otimes k)$ and $G_{\mathrm{DR}} = \mathrm{Aut}^{\otimes}(H^{\#}_{\mathrm{DR}} | \mathcal{M}_k \otimes k)$ of automorphisms of the fiber functors are affine group schemes over k ([DM], Theorem 2.11 and Proposition 3.11); they are inner forms of each other. Even a conjectural description of these groups is elusive. The functors $H^{\#}_{\sigma} \otimes k$ and $H^{\#}_{\mathrm{DR}}$ define equivalences $\mathcal{M}_k \otimes k \tilde{\to} \mathrm{Rep}_k\, G_\sigma$ and $\mathcal{M}_k \otimes k \tilde{\to} \mathrm{Rep}_k\, G_{\mathrm{DR}}$ of tensor categories.

Similarly we have the Tannakian category $\mathcal{M}_k^{\mathrm{red}} \otimes k$, which is semi-simple and neutral over k by the fiber functors $H^{\#}_{\sigma} \otimes k$ and $H^{\#}_{\mathrm{DR}}$, the k-groups $G_\sigma^{\mathrm{red}} = \mathrm{Aut}^{\otimes}(H^{\#}_{\sigma} \otimes k | \mathcal{M}_k^{\mathrm{red}} \otimes k)$ and $G_{\mathrm{DR}}^{\mathrm{red}} = \mathrm{Aut}^{\otimes}(H^{\#}_{\mathrm{DR}} | \mathcal{M}_k^{\mathrm{red}} \otimes k)$, and the equivalences $\mathcal{M}_k^{\mathrm{red}} \otimes k \tilde{\to} \mathrm{Rep}_k\, G_\sigma^{\mathrm{red}}$ and $\mathcal{M}_k^{\mathrm{red}} \otimes k \tilde{\to} \mathrm{Rep}_k\, G_{\mathrm{DR}}^{\mathrm{red}}$. Since the category $\mathcal{M}_k^{\mathrm{red}} \otimes k$ is semi-simple, [DM], Remark 2.28 implies that G_σ^{red} and $G_{\mathrm{DR}}^{\mathrm{red}}$ are pro-reductive (meaning that the connected component is the projective limit of connected reductive groups). The group G_σ^{red} (resp. $G_{\mathrm{DR}}^{\mathrm{red}}$) is the maximal pro-reductive quotient of the affine group scheme G_σ (resp. G_{DR}).

Note that a $\otimes$-functor $F : A \to B$ of Tannakian categories and a fiber functor β on B define a map $f : G_B = \mathrm{Aut}^{\otimes}(\beta) \to G_A = \mathrm{Aut}^{\otimes}(\beta \circ F)$ of the motivic groups (the image $g^A = (g^A_{X_A}) = f(g^B)$ is defined by $g^A_{X_A} = g^B_{F(X_A)}$), and vice versa: $f : G_B \to G_A$ defines $F : A = \mathrm{Rep}\, G_A \to B$. For relations of properties of F and f see Saavedra [Sa], II, 4.3.2.

Denote by U_σ the kernel of the projection $G_\sigma \to G_\sigma^{\mathrm{red}}$; it is pro-unipotent. By the Levi decomposition, the extension $1 \to U_\sigma \to G_\sigma \to G_\sigma^{\mathrm{red}} \to 1$ splits. More precisely, the essentially surjective functor (a functor is called *essentially surjective* if each object in the target category is isomorphic to an object in the image of the functor) $\mathrm{Gr}^W : \mathcal{M}_k \to \mathcal{M}_k^{\mathrm{red}}$, defined on one-motives by $M = (X, A, T, E, u) \mapsto H(X) \oplus H(A) \oplus H(X_*(T))(1)$, is an inverse to $\mathcal{M}_k^{\mathrm{red}} \hookrightarrow \mathcal{M}_k$. Correspondingly $G_\sigma^{\mathrm{red}} = \mathrm{Aut}^{\otimes}(H^{\#}_{\sigma} \otimes k | \mathcal{M}_k^{\mathrm{red}} \otimes k)$ is canonically a subgroup of $\mathrm{Gr}^W G_\sigma = \mathrm{Aut}^{\otimes}(H^{\#}_{\sigma}\,\mathrm{Gr}^W \otimes k | \mathcal{M}_k \otimes k)$, which is isomorphic by the Levi decomposition – but not canonically – to $G_\sigma = \mathrm{Aut}^{\otimes}(H^{\#}_{\sigma} \otimes k | \mathcal{M}_k \otimes k)$.

Our main object of study is the affine scheme $P_\sigma = \mathrm{Hom}^\otimes(H^\#_{\mathrm{DR}}, H^\#_\sigma \otimes k; \mathcal{M}_k \otimes k)$ over k of morphisms of fiber functors ([DM], Theorem 3.2). It is a G_σ-torsor (right principal homogeneous space) over k, and so it defines a class h_σ of the first Galois cohomology set $H^1(k, G_\sigma) = H^1(\mathrm{Gal}(\overline{k}/k), G_\sigma(\overline{k}))$. The group G_σ is called the (σ-)*motivic Galois group* of $\mathcal{M}_k \otimes k$, and P_σ the (σ-)*motivic torsor* of $\mathcal{M}_k \otimes k$. Analogously we have the G^{red}_σ-torsor $P^{\mathrm{red}}_\sigma = \mathrm{Hom}^\otimes(H^\#_{\mathrm{DR}}, H^\#_\sigma \otimes k; \mathcal{M}^{\mathrm{red}}_k \otimes k)$ over k, and its class h^{red}_σ in $H^1(k, G^{\mathrm{red}}_\sigma)$. The G^{red}_σ-torsor P^{red}_σ is the quotient P_σ/U_σ.

Denote by $\mathcal{M}^0_k$ the Tannakian subcategory generated by Artin motives $[X \to 0]$ in $\mathcal{M}_k$. It is equivalent to the category of [DM], Proposition 6.17, generated by the zero dimensional varieties Z over k. The motivic Galois group $\mathrm{Aut}^\otimes(H^\#_\sigma \otimes k|\ \mathcal{M}^0_k \otimes k)$ of $\mathcal{M}^0_k \otimes k$ is the constant pro-finite group scheme $\Gamma_k = \varprojlim_K \coprod_\gamma (\mathrm{Spec}\, k)_\gamma$ [$(k \subset)K$ finite Galois extensions, $\gamma \in \mathrm{Gal}(K/k)$] over k (with structure morphisms $\coprod_{\gamma\in \mathrm{Gal}(K/k)} \mathrm{id}_\gamma$). Its group of $\overline{k}$-points is $\mathrm{Gal}(\overline{k}/k)$, and the functor $H^\#_\sigma \otimes k$ (: $X \mapsto X \otimes k$, or : $Z \mapsto k^{Z(\overline{k})}$ in [DM], 6.17) induces an isomorphism $\mathcal{M}^0_k \otimes k \tilde{\to} \mathrm{Rep}_k(\Gamma_k)$ ([DM], Proposition 6.17). The group Γ_k is the group of connected components of G^{red}_σ ([DM], Proposition 6.23(a,b)). [Note that the proofs of Propositions 6.22(a), 6.23 of [DM] are incorrect for the full category of pure motives as stated there, but they do apply in our context of motives of abelian varieties and one-motives; see Remark 1 at the end of this paper.]

Thus the inclusion $\mathcal{M}^0_k \hookrightarrow \mathcal{M}^{\mathrm{red}}_k$ defines a surjection $G^{\mathrm{red}}_\sigma \overset{\pi}{\twoheadrightarrow} \Gamma_k$ (by [DM], Remark 2.29). Its kernel $G^{\mathrm{red},0}_\sigma$ is the connected component of the identity of G^{red}_σ, a connected pro-reductive affine k-group scheme which is the motivic Galois group $\mathrm{Aut}^\otimes(H^\#_\sigma \otimes k|\mathcal{M}^{\mathrm{red}}_{\overline{k}} \otimes k)$ of $H^\#_\sigma \otimes k$ on $\mathcal{M}^{\mathrm{red}}_{\overline{k}} \otimes k$. The almost surjective functor (we say that a functor is *almost surjective* if each object of the target category is isomorphic to a subquotiet of an object in the image of the functor; see [DM], Proposition 2.21(b)) $\mathcal{M}^{\mathrm{red}}_k \to \mathcal{M}^{\mathrm{red}}_{\overline{k}}$, $A \mapsto \overline{A} = A \times_k \overline{k}$, defines the injection $G^{\mathrm{red},0}_\sigma \overset{\iota}{\to} G^{\mathrm{red}}_\sigma$. In particular, denote the quotient $p : P^{\mathrm{red}}_\sigma \to P^{\mathrm{red}}_\sigma/G^{\mathrm{red},0}_\sigma$ by P^{Art}_σ. It is the Γ_k-torsor $\mathrm{Hom}^\otimes(H^\#_{\mathrm{DR}}, H^\#_\sigma \otimes k; \mathcal{M}^0_k \otimes k)$. Its class h^{Art}_σ in $H^1(k, \Gamma_k) = H^1(\mathrm{Gal}(\overline{k}/k), \Gamma_k(\overline{k}))$ is the image of $h_\sigma = \{P^{\mathrm{red}}_\sigma\}$ under the map $H^1(k, G^{\mathrm{red}}_\sigma) \to H^1(k, \Gamma_k)$.

Since G_σ is the semi-direct product of the pro-reductive G^{red}_σ and the pro-unipotent U_σ, we have that Γ_k is the group of connected components of G_σ. The inclusion $\mathcal{M}^0_k \to \mathcal{M}_k$ defines a surjection $G_\sigma \overset{\pi}{\to} \Gamma_k$ ([DM], Proposition 2.21(a)), whose kernel G^0_σ is the connected component of the identity of G_σ. This connected affine k-group scheme is the motivic Galois group $\mathrm{Aut}^\otimes(H^\#_\sigma \otimes k|\mathcal{M}_{\overline{k}} \otimes k)$ of $H^\#_\sigma \otimes k$ on $\mathcal{M}_{\overline{k}} \otimes k$. The almost surjective functor $\mathcal{M}_k \to \mathcal{M}_{\overline{k}}$, defined on one-motives by $M \mapsto \overline{M} = M \times_k \overline{k} = [X \overset{u}{\to} E \times_k \overline{k}]$, induces the injection $G^0_\sigma \overset{\iota}{\to} G_\sigma$. The quotient $p : P_\sigma \to P_\sigma/G^0_\sigma$ is P^{Art}_σ. Its class in $H^1(k, \Gamma_k)$ is the image of $h_\sigma = \{P_\sigma\}$ under the map $H^1(k, G_\sigma) \to H^1(k, \Gamma_k)$. The functor $H_\sigma \otimes k$ maps the Tannakian category $\mathcal{M}_{\overline{k}} \otimes k$ to the Tannakian category of k-mixed Hodge structures. This would help us understand what we need to know about our motivic objects, but this map is not fully faithful when $k \neq \mathbb{Q}$.

The statement of our theorem uses the set $\mathrm{Sper}\, k$ of orderings ξ of the field k. It is a compact totally disconnected topological space, where a basis of the topology is given by the sets $\{\xi;\ a > 0 \text{ in } \xi\}$, a in k (see, e.g., Scharlau [Sc], Ch. 3, §5). The space $\mathrm{Sper}\, k$ is naturally homeomorphic to the quotient of the space $\mathrm{Inv}(\mathrm{Gal}(\overline{k}/k))$ of involutions (elements of order precisely two) in $\mathrm{Gal}(\overline{k}/k)$ (endowed with the usual profinite topology) by conjugation under $\mathrm{Gal}(\overline{k}/k)$. Denote by k_ξ a real closure of k (in $\overline{k} \subset \mathbb{C}$) whose ordering induces ξ on k. Then $\mathrm{Gal}(\overline{k}/k_\xi)$ is generated by c_ξ in $\mathrm{Inv}(\mathrm{Gal}(\overline{k}/k))$. If c is an involution in $\mathrm{Gal}(\overline{k}/k)$, for any field k, then $\mathrm{char}\, k = 0$, the fixed field of c in $\overline{k}$ is a real closure k_ξ of k whose ordering induces ξ on k, $\overline{k} = k_\xi(\sqrt{-1})$, and the restriction of c to the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ is non trivial (it takes $\sqrt{-1}$ to $-\sqrt{-1}$), hence it is in the unique conjugacy class of involutions in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The ordered field k (or k_ξ) embeds in a real closed field R_ξ of codimension 2 in $\mathbb{C}$ – thus $\mathbb{C} = R_\xi(\sqrt{-1})$ – whose ordering induces ξ on k. An ordering ξ of k is called *archimedean* if the real closure k_ξ embeds in $\mathbb{R}$. When k is finitely generated, the set $\mathrm{Arch}\, k$ of archimedean orderings in k is dense in $\mathrm{Sper}\, k$; this is shown below.

The affine k_ξ-scheme $P_{\sigma,\xi} = P_\sigma \times_k k_\xi$ is a G_σ-torsor over k_ξ. Its class $h_{\sigma,\xi}$ in $H^1(k_\xi, G_\sigma) = H^1(\mathrm{Gal}(\overline{k}/k_\xi), G_\sigma(\overline{k}))$ is the image of h_σ under the natural localization map $H^1(k, G_\sigma) \to H^1(k_\xi, G_\sigma)$. Alternatively it can be described using the fact that the natural map $H^1(k_\xi, G_\sigma) \to H^1(R_\xi, G_\sigma)$ is an isomorphism (this is implied by the Artin-Lang theorem (see [BCR], Théorème 4.1.2)), as follows. The continuous map $\sigma M(\mathbb{C}) \to \sigma M(\mathbb{C})$ ($M \in \mathcal{M}_k$) defined by $c_\xi \neq 1$ in $\mathrm{Gal}(\mathbb{C}/R_\xi)$ induces an involutive endomorphism of $\sigma M(\mathbb{C})_B$. The image in $G_\sigma(\mathbb{C})$ defines a (Galois) cohomology class in $H^1(R_\xi, G_\sigma)$, which is $h_{\sigma,\xi}$.

Let k be a field with virtual cohomological dimension ≤ 1 (thus $\mathrm{vcd}(k) = \mathrm{cd}(k(\sqrt{-1}))$ is at most one). We have $\mathrm{vcd}(k) = \mathrm{cd}(k)$ precisely when k has no orderings, thus $\mathrm{Sper}\, k$ is empty. Examples of k with $\mathrm{vcd}\, k = 1 < \mathrm{cd}\, k$ are $k = \mathbb{R}(x)$ or $R(x)$, where R is a real closed field (Serre [S1], II, 3.3(b)), $\mathbb{R}((x))$ and $R((x))$ ([S1], II, 3.3, Ex. 3), and $\mathbb{Q}^{\mathrm{ab}} \cap \mathbb{R}$ ([S1], II, 3.3, Proposition 9). We assume that k embeds in $\mathbb{C}$ (to use [DM]; to embed in $\mathbb{C}$ a field k of cardinality bounded by that of $\mathbb{C}$, choose transcendence bases in both). Fix $\sigma : k \hookrightarrow \mathbb{C}$.

Theorem. *Let $p' : P' \to P_\sigma^{\mathrm{Art}}$ be a G_σ-torsor over k such that $P'_\xi = P' \times_k k_\xi$ is cohomologous to $P_{\sigma,\xi} = P_\sigma \times_k k_\xi$ for ξ in a dense subset of* $\mathrm{Sper}\, k$. *Then there exists an isomorphism of G_σ-torsors $\lambda : P_\sigma \to P'$ such that $p' \circ \lambda = p$.*

The same result holds with G_σ and P_σ replaced by G_σ^{red} and P_σ^{red}.

Our work is influenced by Blasius-Borovoi [BB] who considered the number field $\mathbb{Q}$ (whose vcd is 2) and the semi-simple Tannakian subcategory $\mathcal{M}_{\mathbb{Q}}^{\mathrm{red},H}$ generated by Artin motives and motives of abelian varieties A over $\mathbb{Q}$ for which the group $((G_\sigma^A)^0)_{\mathbb{R}}^{\mathrm{ad}}$ has no factor of type D_n^H (in the notations of Deligne [D1], (1.3.9)), and by Wintenberger [W] who had considered the field $\mathbb{Q}$ and the semi-simple Tannakian subcategory $\mathcal{M}_{\mathbb{Q}}^{\mathrm{red,CM}}$ generated by Artin motives and motives of abelian varieties with complex multiplication over $\mathbb{Q}$.

Our theorem is an application of the local-global principle for a field k with $\mathrm{vcd}(k) \leq 1$. We can work in the generality of the entire category $\mathcal{M}_k$ and the group G_σ by virtue of the local-global principle: $H^1(k, G) \hookrightarrow \prod_\xi H^1(k_\xi, G)$, proven by Scheiderer [Sch] for a perfect field k with $\mathrm{vcd}\, k \leq 1$ and a connected k-linear algebraic group G. In the number field case the analogous well known local-global principle holds only for semi-simple simply connected G.

When $\mathrm{cd}(k) \leq 1$, thus when k has no orderings, the class of P_σ is determined by P_σ^{Art}

alone. To deal with this case, we use only Steinberg's theorem ([S1], III, §2.3) on the vanishing of $H^1(k,G)$ for a perfect field k with $\mathrm{cd}(k) \leq 1$ and a connected k-linear algebraic group G.

It will be interesting to study our motivic objects over fields k with vcd ≤ 2. In this context, note that a local-global principle for k with $\mathrm{vcd}(k) \leq 2$ and semi-simple simply connected classical linear algebraic groups has recently been established by Bayer-Fluckiger and Parimala [BP].

It is my pleasure to express my deep gratitude to P. Deligne for watching over my first steps in the motivic fairyland, to M. Borovoi, U. Jannsen, R. Pink, C. Scheiderer, J.-P. Serre, R. Sujatha, and the Referee, for useful comments, to M. Jarden for invitation to talk on this work at the Gentner Symposium on Field Arithmetic, Tel-Aviv University, October 1997, and to the National University of Singapore for its hospitality in late 1999 while this paper was refereed. NATO grant CRG 970133 is gratefully acknowledged.

Proof of theorem. It is easy to adapt the proof to the context of the pro-reductive quotient group G_σ^{red}, so we discuss only the general case of the entire group G_σ.

Let $z \in Z^1(k, G_\sigma)$ be a 1-cocycle representing $h_\sigma = \{P_\sigma\} \in H^1(k, G_\sigma)$. As in [S1], I.5.3, denote by ${}_zG_\sigma$ the form of G_σ twisted by z. It is the affine group scheme over k on which $\mathrm{Gal}(\overline{k}/k)$ acts by $s : g \mapsto (\mathrm{Int}(z_s))(s(g))$ $(g \in G_\sigma(\overline{k}),\ s \in \mathrm{Gal}(\overline{k}/k))$. The natural bijection $H^1(k, {}_zG_\sigma)\tilde{\to}H^1(k, G_\sigma)$, defined by $(x_s) \mapsto (x_s z_s)$ ([S1], I.5.3, Proposition 35), takes the trivial element of $H^1(k, {}_zG_\sigma)$ to h_σ. Denote by η the class in $H^1(k, {}_zG_\sigma)$ which maps to h', the class in $H^1(k, G_\sigma)$ of the G_σ-torsor P'. By the very definition of P_σ, as relating G_σ and G_{DR}, we have that G_{DR} is ${}_{P_\sigma}G_\sigma = P_\sigma \times_{G_\sigma} G_\sigma$ (this is ${}_FP = P \times^A F$ in the notations of the first paragraph of [S1], I, §5.3; here A of [S1] is $G_\sigma(\overline{k})$, which acts on $P_\sigma(\overline{k})$ (=P in [S1]) by right multiplication and on $G_\sigma(\overline{k})$ (=F in [S1]) by conjugation). By the third paragraph of [S1], I, §5.3, we have that G_{DR} is the twist ${}_zG_\sigma$ of G_σ by z. Since $P'_\xi \simeq P_{\sigma,\xi}$, the localization $\eta_\xi = \mathrm{loc}_\xi(\eta)$ of η in $H^1(k_\xi, G_{\mathrm{DR}})$ is 1, for a dense set of ξ in $\mathrm{Sper}\, k$. Since P_σ and P' project to the same Γ_k-torsor P_σ^{Art} in $H^1(k, \Gamma_k)$, the image of η in $H^1(k, {}_{z'}\Gamma_k)$ is 1, where z' in $Z^1(k, \Gamma_k)$ is the image of $z \in Z^1(k, G_\sigma)$ under the projection $G_\sigma \to \Gamma_k$. Our aim is to show that $\eta = 1$ in $H^1(k, G_{\mathrm{DR}})$.

Consider the exact sequence of affine group schemes

$$1 \to G_{\mathrm{DR}}^0 = {}_zG_\sigma^0 \to G_{\mathrm{DR}} = {}_zG_\sigma \to \Gamma_{k,\mathrm{DR}} = {}_{z'}\Gamma_k \to 1.$$

Since the image of $\eta \in H^1(k, G_{\mathrm{DR}})$ in $H^1(k, \Gamma_{k,\mathrm{DR}})$ is trivial, there is $\eta^0 \in H^1(k, G_{\mathrm{DR}}^0)$ which maps to η. The group G_{DR}^0 is a connected pro-algebraic affine group scheme over k ([DM], Proposition 6.22(a)). Thus $G_{\mathrm{DR}}^0 = \varprojlim_N (G_{\mathrm{DR}}^N)^0$, where G_{DR}^N is the motivic Galois group $\mathrm{Aut}^\otimes(H_{\mathrm{DR}}^\#|\ \mathcal{M}_{k_N}^N \otimes k_N)$ of the Tannakian subcategory $\mathcal{M}_{k_N}^N$ of $\mathcal{M}_k$ generated by a finite set N of one-motives and their duals, the Artin motives and the Tate motive T and its dual $T^\vee$. The finite set N is defined over a finitely generated over $\mathbb{Q}$ subfield k_N of k.

As explained in the proof of [DM], Proposition 6.22(a), $(G_{\mathrm{DR}}^N)^0$ is a linear algebraic group. Correspondingly $\eta^0 = \varprojlim_N \eta_N^0$, where $\eta_N^0 \in H^1(k, (G_{\mathrm{DR}}^N)^0)$. Further, $\eta = \varprojlim_N \eta_N$, where η_N is the image of η_N^0 under the map $H^1(k, (G_{\mathrm{DR}}^N)^0) \to H^1(k, G_{\mathrm{DR}}^N)$. Since η_ξ is trivial in $H^1(k_\xi, G_{\mathrm{DR}})$, the localization $\eta_{N,\xi} = \mathrm{loc}_\xi(\eta_N)$ is trivial for all N, for the dense set of ξ in $\operatorname{Sper} k$ of the theorem.

Write $\operatorname{Arch} k$ for the set of archimedean orderings in $\operatorname{Sper} k$. The proposition below asserts that the homomorphism $G_{\mathrm{DR}}(k_\xi) \to \Gamma_{k,\mathrm{DR}}(k_\xi)$ is surjective for every $\xi \in \operatorname{Arch} k$. In particular $G^N_{\mathrm{DR}}(k_\xi) \twoheadrightarrow \Gamma_{k_N,\mathrm{DR}}(k_\xi) = \mathbb{Z}/2$ for each finite N and $\xi \in \operatorname{Arch} k$. We claim that this map is onto for all $\xi \in \operatorname{Sper} k$. The k_N-group G^N_{DR} has two connected components; denote by $C = G^{N,+}_{\mathrm{DR}}$ the component not containing the identity. The surjectivity means that $C(k_\xi)$ is non empty (for all $\xi \in \operatorname{Arch} k$). It follows from the Artin-Lang theorem that $C(k_{N,\xi})$ is non empty for all $\xi \in \operatorname{Arch} k_N$. But the set of $\xi \in \operatorname{Sper} k_N$ such that $C(k_{N,\xi})$ is non empty is open and closed in $\operatorname{Sper} k_N$ (see, e.g., [Sch], Corollary 2.2). Our claim follows once we show that for a finitely generated field k_N, the set $\operatorname{Arch} k_N$ is dense in $\operatorname{Sper} k_N$.

Lemma 0. *For a finitely generated field k_N the set* $\operatorname{Arch} k_N$ *is dense in* $\operatorname{Sper} k_N$.

Proof of Lemma 0. Choose a purely transcendental extension $F = \mathbb{Q}(t_1, \dots, t_n)$ of $\mathbb{Q}$ of finite codimension in k_N. Since the restriction of orderings is an open map $\operatorname{Sper} k_N \to \operatorname{Sper} F$, and an ordering of k_N is archimedean if its restriction to F is, it suffices to show that $\operatorname{Arch} F$ is dense in $\operatorname{Sper} F$. For this, we proceed to show that the non empty basic open set defined by $p_1, \dots, p_r \in F$ contains an archimedean ordering. The open set being non empty means that there is an ordering of F which makes the p_j positive. In other words, there are a real closed field R and $x \in R^n$ such that $p_j(x) > 0$, all j. Then the same is true for $R = \mathbb{R}$, by the Tarski principle (see, e.g., [BCR], I.1.4). That is, there is $x \in \mathbb{R}^n$ such that $p_j(x) > 0$, all j. The inequalities remain true in a neighborhood of x, hence the components $x_1, \dots, x_n$ of x can be chosen to be algebraically independent. The embedding $F \hookrightarrow \mathbb{R}$ defined by $t_i \mapsto x_i$ defines an archimedean ordering of F where the p_j are positive, namely an archimedean point in the given open set. ☐

We then have that $G^N_{\mathrm{DR}}(k_\xi) \twoheadrightarrow \Gamma_{k_N,\mathrm{DR}}(k_\xi) = \mathbb{Z}/2$ for each finite set N of one-motives, and for all $\xi \in \operatorname{Sper} k$. Consequently the kernel of the map $H^1(k_\xi, (G^N_{\mathrm{DR}})^0) \to H^1(k_\xi, G^N_{\mathrm{DR}})$ is trivial for all ξ. For the dense set of $\xi \in \operatorname{Sper} k$ given in the theorem, $\eta_{N,\xi}$ is trivial in $H^1(k_\xi, G^N_{\mathrm{DR}})$. Then for these ξ we have that $\eta^0_{N,\xi} = \mathrm{loc}_\xi\, \eta^0_N$ is trivial in $H^1(k_\xi, (G^N_{\mathrm{DR}})^0)$.

Using the local-global principle of [Sch], Theorem 4.1, which asserts that for a connected linear algebraic group G^N over a perfect field k with $\mathrm{vcd}(k) \le 1$ the map $H^1(k, G^N) \to \prod_\xi H^1(k_\xi, G^N)$ is injective where the product ranges over any dense subset of orderings ξ in $\operatorname{Sper} k$, we conclude that η^0_N is 1 for all finite sets N of one-motives. Hence $\eta^0 = \varprojlim_N \eta^0_N$ is trivial, so is its image η, and P' and P_σ define the same class in $H^1(k, G_\sigma)$. ☐

The following lemma is used in the proof of the proposition below.

Lemma. *Let K_ξ be a real closed field containing k_ξ. Then the group of K_ξ-points of $\Gamma_{k,\mathrm{DR}} = {}_{z'}\Gamma_k$ is isomorphic to* $\mathrm{Gal}(\overline{k}/k_\xi)$.

Proof. We have $\Gamma_{k,\mathrm{DR}}(K_\xi) = \Gamma_{k,\mathrm{DR}}(K)^{\mathrm{Gal}(K/K_\xi)}$, where $K = K_\xi(\sqrt{-1})$, and $\Gamma_{k,\mathrm{DR}}(K) = \Gamma_{k,\mathrm{DR}}(\overline{k})$. Moreover, the restriction to $\overline{k}$ of the non trivial element of $\mathrm{Gal}(K/K_\xi)$ is the non trivial element of $\mathrm{Gal}(\overline{k}/k_\xi)$. The group $\Gamma_{k,\mathrm{DR}}$ is the profinite group scheme attached to the identity cocycle $z'(\tau) = \tau$ in $Z^1(k, \Gamma_k)$ (this is called the Artin cocycle, see [W]).

Thus $\tau \in \mathrm{Gal}(\overline{k}/k)$ acts on $\gamma \in \Gamma_{k,\mathrm{DR}}(\overline{k}) = \mathrm{Gal}(\overline{k}/k)$ by $\tau_{\mathrm{DR}}(\gamma) = \tau\gamma\tau^{-1}$. In particular $c_\xi \in \mathrm{Gal}(\overline{k}/k_\xi)$ acts on $\gamma \in \Gamma_{k,\mathrm{DR}}(\overline{k})$ by $c_{\xi,\mathrm{DR}}(\gamma) = c_\xi\gamma c_\xi^{-1}$. Hence $\Gamma_{k,\mathrm{DR}}(k_\xi) = \{\gamma \in \mathrm{Gal}(\overline{k}/k); c_\xi\gamma c_\xi^{-1} = \gamma\}$. It remains to determine the centralizer of $c_\xi \in \mathrm{Inv}(\mathrm{Gal}(\overline{k}/k))$ in $\mathrm{Gal}(\overline{k}/k)$. We claim it is $\{1, c_\xi\}$. The field $k_\xi = \overline{k}^{c_\xi}$ of fixed points of c_ξ in $\overline{k}$ is a real closure of k whose ordering induces ξ on k. If $\gamma \in \mathrm{Gal}(\overline{k}/k)$ commutes with c_ξ then it maps k_ξ to itself. But the only automorphism of k_ξ over k is the identity (by the Artin-Schreier theorem; see, e.g., [Sc], Ch. 3, Theorem 2.1). Hence $\gamma \in \mathrm{Gal}(\overline{k}/k_\xi) = \{1, c_\xi\}$. □

The following proposition is used in the proof of the Theorem above.

Proposition. *The map $G_{\mathrm{DR}}(k_\xi) \to \Gamma_{k,\mathrm{DR}}(k_\xi)$ is surjective for every archimedean ordering ξ in* Sper k.

Proof. The lemma implies that $\Gamma_{k,\mathrm{DR}}(k_\xi) = \mathbb{Z}/2 = \Gamma_{k_\xi,\mathrm{DR}}(k_\xi)$. Write $G_{k,\sigma}$ and $G_{k,\mathrm{DR}}$ to specify the base field. Using the functor $\mathcal{M}_k \to \mathcal{M}_{k_\xi}$ which is induced from $M \mapsto M \times_k k_\xi$ (incidentally, it is almost surjective (by which we mean that each object of $\mathcal{M}_{k_\xi}$ is a subquotient of an object in the image of $\mathcal{M}_k$), by the proof of [DM], 6.23 (a)), we have a k_ξ-homomorphism $G_{k_\xi,\mathrm{DR}} \to G_{k,\mathrm{DR}}$ (in fact an injection, by [Sa], II, 4.3.2 g) ii), or [DM], Proposition 2.21 (b)) of the motivic Galois groups for the de Rham fiber functor. Hence it suffices to prove the proposition only for a real closed k. Since ξ is archimedean, k embeds in $\mathbb{R}$, and it suffices to prove the proposition for $k = \mathbb{R}$. Thus we assume from now on that k is $\mathbb{R}$, and write G_{DR} for $G_{\mathbb{R},\mathrm{DR}}$.

Recall that the functors $\mathcal{M}^0_{\mathbb{R}} \to \mathcal{M}_{\mathbb{R}}$ and $\mathcal{M}_{\mathbb{R}} \to \mathcal{M}_{\mathbb{C}}$, and the fiber functor $H^\#_\sigma$, define the exact sequence $1 \to G^0_\sigma \to G_\sigma \to \Gamma_{\mathbb{R}} \to 1$ of affine group schemes over $\mathbb{Q}$ (for the "pure" case, which implies at once the "mixed" case, see [DM], Proposition 6.23(a,b)). Using the functors $\mathcal{M}^0_{\mathbb{R}} \otimes \mathbb{R} \to \mathcal{M}_{\mathbb{R}} \otimes \mathbb{R}$ and $\mathcal{M}_{\mathbb{R}} \otimes \mathbb{R} \to \mathcal{M}_{\mathbb{C}} \otimes \mathbb{R}$, and the fiber functor $H^\#_\sigma \otimes \mathbb{R}$, the groups become groups over $\mathbb{R}$ (note that [DM], Remark 3.12, applies with any – not necessarily finite – field extension k'/k). But we do not change the notations.

For any subfield K of $\mathbb{R}$, a *K-Hodge structure* ("over $\mathbb{C}$") is a pair $(V, (V^{p,q}))$ consisting of a finite dimensional vector space V over K, and a direct sum decomposition $V \otimes_K \mathbb{C} = \oplus V^{p,q}$ with $\tau_\infty(V^{p,q}) = V^{q,p}$; $\tau_\infty \neq 1$ in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. A *$K$-Hodge structure over* $\mathbb{R}$ is a triple $(V, (V^{p,q}), F_\infty)$ where the new ingredient is an involutive endomorphism F_∞ of V whose extension to $V \otimes_K \mathbb{C}$ satisfies $F_\infty(V^{p,q}) = V^{q,p}$. With the natural definition of tensor products and morphisms, these make neutral Tannakian categories Hod_K (K-Hodge structures) and Hod^+_K (K-Hodge structures over $\mathbb{R}$) over K (for the forgetful fiber functor $\omega_K : (V, \dots) \to V$).

A *K-mixed Hodge structure* ("over $\mathbb{C}$") is a triple $(V, W_\bullet, F^\bullet)$, where V is a finite dimensional K-vector space with a finite increasing (weight) filtration $W_\bullet$ and a finite decreasing (Hodge) filtration $F^\bullet$ on $V \otimes_K \mathbb{C}$, such that $F^\bullet$ induces a K-Hodge structure of weight n on the graded piece $\mathrm{Gr}^W_n V = W_n V / W_{n-1} V$ for each n. A *K-mixed Hodge structure over* $\mathbb{R}$ is a K-mixed Hodge structure $(V, W_\bullet, F^\bullet)$ with a $W_\bullet$ preserving involutive automorphism F_∞ of V such that $F_\infty((\mathrm{Gr}^W_n V \otimes_K \mathbb{C})^{p,q}) = (\mathrm{Gr}^W_n V \otimes_K \mathbb{C})^{q,p}$. With the natural definition of $\otimes$ and morphisms, these make the Tannakian categories MHS_K and MHS^+_K.

The main Theorem 2.11 of [D2] asserts that for an algebraically closed subfield $\mathfrak{K}$ of $\mathbb{C}$, the functor $H_\sigma : \mathcal{M}_{\mathfrak{K}}^{\mathrm{red}} \to \mathrm{Hod}_{\mathbb{Q}}$ is fully faithful. It is extended in [D-III], 10.1.3, to assert that the functor $H_\sigma : M \mapsto H_\sigma(M) = \sigma M(\mathbb{C})_B$ defines an equivalence between the category of isogeny classes of one-motives over $\mathfrak{K}$ and the category of ($\mathbb{Q}$-)mixed Hodge structures of type $\{(0,0), (0,-1), (-1,0), (-1,-1)\}$ whose graded quotient Gr_{-1} is polarizable. A morphism of one-motives is a morphism (α, β) of complexes $[X \to E] \to [X' \to E']$. It is an *isogeny* if both α and β are isogenies, i. e. have finite kernels and cokernels. The functor H_σ extends to a faithful functor from the Tannakian category $\mathcal{M}_{\mathbb{C}}$ to the Tannakian category $\mathrm{MHS}_{\mathbb{Q}}$ (in this context we note Theorem 2.2.5 of [Br], which asserts that a Hodge cycle on a one-motive – and in particular a power thereof – is absolute), and from $\mathcal{M}_{\mathbb{R}}$ to $\mathrm{MHS}_{\mathbb{Q}}^+$: $\tau_\infty \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ induces an involution of $\sigma M(\mathbb{C})$, hence an involution $F_\infty = H_\sigma(\tau_\infty)$ on $H_\sigma(M)$. The restriction of H_σ to $\mathcal{M}_{\mathbb{R}}^0$ is an equivalence with the category $\mathrm{Rep}_{\mathbb{Q}} \Gamma_{\mathbb{R}}$ of representations of $\Gamma_{\mathbb{R}}$ over $\mathbb{Q}$.

The fiber functor $H_\sigma^\# \otimes \mathbb{R}$ on $\mathcal{M}_{\mathbb{R}} \otimes \mathbb{R}$ factorizes through the forgetful functor $\omega_{\mathbb{R}}$: $\mathrm{MHS}_{\mathbb{R}}^+ \to \mathrm{Rep}_{\mathbb{R}} \Gamma_{\mathbb{R}}$. The restriction of $\omega_{\mathbb{R}}$ to $\mathrm{MHS}_{\mathbb{R}}$ is the forgetful functor into the category $\mathrm{Vec}_{\mathbb{R}}$ of vector spaces over $\mathbb{R}$. The restriction of $H_\sigma^\# \otimes \mathbb{R}$ to $\mathcal{M}_{\mathbb{R}}^0 \otimes \mathbb{R}$ is an equivalence of categories with $\mathrm{Rep}_{\mathbb{R}} \Gamma_{\mathbb{R}}$.

But we are concerned with the fiber functor $H_{\mathrm{DR}}^\# \otimes \mathbb{R}$ on $\mathcal{M}_{\mathbb{R}} \otimes \mathbb{R}$ and the exact sequence $1 \to G_{\mathrm{DR}}^0 \to G_{\mathrm{DR}} \to \Gamma_{\mathbb{R},\mathrm{DR}} \to 1$ of real groups associated with the almost surjective functor $\mathcal{M}_{\mathbb{R}} \otimes \mathbb{R} \to \mathcal{M}_{\mathbb{C}} \otimes \mathbb{R}$ and the fully faithful functor $\mathcal{M}_{\mathbb{R}}^0 \otimes \mathbb{R} \to \mathcal{M}_{\mathbb{R}} \otimes \mathbb{R}$. To show the surjectivity of the map $G_{\mathrm{DR}}(\mathbb{R}) \to \Gamma_{\mathbb{R},\mathrm{DR}}(\mathbb{R})$ of groups of real points, it suffices to show that the reductive part $G_{\mathrm{DR}}^{\mathrm{red}}(\mathbb{R})$ surjects on $\Gamma_{\mathbb{R},\mathrm{DR}}(\mathbb{R})$. For this, note that the functor $H_{\mathrm{DR}}^\# \otimes \mathbb{R}$ on $\mathcal{M}_{\mathbb{R}}^{\mathrm{red}} \otimes \mathbb{R}$ factorizes via $\mathcal{M}_{\mathbb{R}}^{\mathrm{red}} \otimes \mathbb{R} \to \mathrm{Hod}_{\mathbb{R}}^+$ and a functor $\omega_{\mathrm{DR},\mathbb{R}} : \mathrm{Hod}_{\mathbb{R}}^+ \to \mathrm{Vec}_{\mathbb{R}}$ described below. This follows from the fact that for the realizations of a motive one has $c_{\mathrm{DR}} = F_\infty \circ \mathrm{bar}$, where c_{DR} and bar are respectively the deRham and the Betti complex conjugations. Defining $\mathbb{S}_{\mathrm{DR}}^+ = \mathrm{Aut}^\otimes(\omega_{\mathrm{DR},\mathbb{R}} | \mathrm{Hod}_{\mathbb{R}}^+)$ (and $\mathbb{S}_{\mathrm{DR}} = \mathrm{Aut}^\otimes(\omega_{\mathrm{DR},\mathbb{R}} | \mathrm{Hod}_{\mathbb{R}}))$, we get the vertical arrow in the commutative square

$$\begin{array}{ccc} \mathbb{S}_{\mathrm{DR}}^+ & \to & \Gamma_{\mathbb{R},\mathrm{DR}} \\ \downarrow & & \| \\ G_{\mathrm{DR}}^{\mathrm{red}} & \to & \Gamma_{\mathbb{R},\mathrm{DR}} . \end{array}$$

The horizontal arrows result from the fully faithful functors $\mathcal{M}_{\mathbb{R}}^0 \otimes \mathbb{R} \to \mathcal{M}_{\mathbb{R}}^{\mathrm{red}} \otimes \mathbb{R}$ and $\mathcal{M}_{\mathbb{R}}^0 \otimes \mathbb{R} \to \mathrm{Hod}_{\mathbb{R}}^+$. Consequently it suffices to show that $\mathbb{S}_{\mathrm{DR}}^+(\mathbb{R}) \twoheadrightarrow \Gamma_{\mathbb{R},\mathrm{DR}}(\mathbb{R})$.

Analogously we have the functor $\omega_{\mathbb{R}}$ on $\mathrm{Hod}_{\mathbb{R}}^+$, the real groups $\mathbb{S}^+ = \mathrm{Aut}^\otimes(\omega_{\mathbb{R}} | \mathrm{Hod}_{\mathbb{R}}^+)$ and $\mathbb{S} = \mathrm{Aut}^\otimes(\omega_{\mathbb{R}} | \mathrm{Hod}_{\mathbb{R}})$, and the exact sequence $1 \to \mathbb{S} \to \mathbb{S}^+ \to \Gamma_{\mathbb{R}} \to 1$. The motivic Galois group $\mathbb{S}$ of $\mathrm{Hod}_{\mathbb{R}}$ and the functor $\omega_{\mathbb{R}}$ is well known ([DM], Example 2.31). The group $\mathbb{S}$ is the connected $\mathbb{R}$-group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ obtained from the multiplicative group $\mathbb{G}_m$ on restricting scalars from $\mathbb{C}$ to $\mathbb{R}$. Thus $\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$, and the non-trivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathbb{S}(\mathbb{C})$ by $(a, b) \mapsto (\overline{b}, \overline{a})$, so $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$. Indeed, a representation $\rho : \mathbb{S} \to \mathrm{Aut}(V)$ defines $V^{p,q}$ to be the $v \in V \otimes_{\mathbb{R}} \mathbb{C}$ with $\rho(z)(v) = z^{-p} \overline{z}^{-q} v$ for all $z \in \mathbb{C}^\times$. The motivic Galois group of the subcategory $\mathrm{Hod}_{\mathbb{R}}^0$ of the V in $\mathrm{Hod}_{\mathbb{R}}^+$ with $V^{p,q} = \{0\}$ unless $p = q = 0$ is the constant group scheme $\Gamma_{\mathbb{R}}$ over $\mathbb{R}$ associated to the group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The motivic Galois group of $\mathrm{Hod}_{\mathbb{R}}^+$ (and $\omega_{\mathbb{R}}$) is an extension $\mathbb{S}^+$ of $\Gamma_{\mathbb{R}}$ by $\mathbb{S}$. Indeed, a triple $(V, (V^{p,q}), F_\infty)$ is associated with the extension of ρ from $\mathbb{S}$ to $\mathbb{S}^+$ by $\rho(1 \times \mathrm{bar}) = F_\infty$ ("bar" signifies complex conjugation). The exact sequence $1 \to \mathbb{S} \to \mathbb{S}^+ \to \Gamma_{\mathbb{R}} \to 1$ is defined by the fully faithful functor $\mathrm{Hod}^0_{\mathbb{R}} \to \mathrm{Hod}^+_{\mathbb{R}}$ and the essentially surjective "forget F_∞" functor $\mathrm{Hod}^+_{\mathbb{R}} \to \mathrm{Hod}_{\mathbb{R}}$. Note that the sequence is split, and $\mathbb{S}^+ = \mathbb{S} \ltimes \Gamma_{\mathbb{R}}$. A splitting is given by the essentially surjective functor $\mathrm{Hod}^+_{\mathbb{R}} \to \mathrm{Hod}^0_{\mathbb{R}}$, "forget the Hodge structure", and $\Gamma_{\mathbb{R}}$ acts on $\mathbb{S}$ via the Galois action.

Since $H^1(\mathbb{R}, \mathbb{S}) = 1$, the sequence $1 \to \mathbb{S}(\mathbb{R}) \to \mathbb{S}^+(\mathbb{R}) \to \Gamma_{\mathbb{R}}(\mathbb{R}) \to 1$ is exact. Since the group $\mathbb{S}_{\mathrm{DR}}$ is $\mathbb{G}_m^2$ (see the following paragraph), by Hilbert Theorem 90 we have $H^1(\mathbb{R}, \mathbb{S}_{\mathrm{DR}}) = 1$. Hence $\mathbb{S}^+_{\mathrm{DR}}(\mathbb{R}) \to \Gamma_{\mathbb{R},\mathrm{DR}}(\mathbb{R})$, which is just $H^0(\mathbb{R}, S^+_{\mathrm{DR}}) \to H^0(\mathbb{R}, \Gamma_{\mathbb{R},\mathrm{DR}})$, is onto. This completes the proof of the proposition.

Note that the structure of the entire group $\mathrm{Aut}^{\otimes}(\omega_{\mathrm{DR},\mathbb{R}}|\,\mathrm{MHS}_{\mathbb{R}})$ is computed in [D3], Construction 1.6 and Proposition 2.1, since $\omega_{\mathrm{DR},\mathbb{R}}$ is the functor Gr^W of [D3]. But by the Levi decomposition it suffices for us to work only with its reductive part. Thus we note that $\mathbb{S}^+_{\mathrm{DR}}$ is known to be $(\mathbb{G}_m \times \mathbb{G}_m) \rtimes \mathbb{Z}/2$. Indeed, the category $\mathrm{Hod}^+_{\mathbb{R}}$ is equivalent to the category $\mathrm{Hod}^*_{\mathbb{R}}$ of triples $(W, (W^{p,q}), F)$, where W is a finite dimensional real vector space with decomposition $W = \oplus W^{p,q}$ into real subspaces, and F is an involutive endomorphism of W over $\mathbb{R}$ with $F(W^{p,q}) = W^{q,p}$. In fact, $\mathrm{Hod}^+_{\mathbb{R}} \to \mathrm{Hod}^*_{\mathbb{R}}$ is given by $W^{p,q}$ = fixed points of $F_\infty \circ \mathrm{bar}$ in $V^{p,q}$, $F = F_\infty|W$, $W^{p,q} = W \cap V^{p,q}$, and $\mathrm{Hod}^*_{\mathbb{R}} \to \mathrm{Hod}^+_{\mathbb{R}}$ by: V = fixed points of $F \circ \mathrm{bar}$ in $W \otimes \mathbb{C}$, $V^{p,q} = V \cap (W^{p,q} \otimes \mathbb{C})$, $F_\infty = F|V$. The fiber functor $H^{\#}_{\mathrm{DR}} \otimes \mathbb{R}$ on $\mathcal{M}_{\mathbb{R}} \otimes \mathbb{R}$ factorizes through the fiber functor ω_{DR} on $\mathrm{Hod}^+_{\mathbb{R}}$, which is $V \mapsto W$, or $W \mapsto W$ on $\mathrm{Hod}^*_{\mathbb{R}}$. The group of automorphisms of ω_{DR} on $\mathrm{Hod}^*_{\mathbb{R}}$ is $\mathbb{S}^+_{\mathrm{DR}} = (\mathbb{G}_m \times \mathbb{G}_m) \rtimes \mathbb{Z}/2$, the product of the finite group scheme $\Gamma_{\mathbb{R},\mathrm{DR}} = \mathbb{Z}/2$ by $\mathbb{S}_{\mathrm{DR}} = \mathbb{G}_m \times \mathbb{G}_m$, the groups of automorphisms of the functor ω_{DR} on the categories $\mathrm{Hod}^0_{\mathbb{R}}$ and $\mathrm{Hod}_{\mathbb{R}}$. □

Remark 1. Proposition 6.22(b) of [DM] is wrong ($\underline{M}_k \to \underline{M}_{k'}$ there is fully faithful but not essentially surjective), but this is of no consequence for the theory. For a corrected statement and a counter example see [S2], §6. The connectedness assertion in Proposition 6.22(a) (and consequently 6.23) of [DM] – which is a consequence of the standard conjectures – is out of reach of current technology (in Deligne's opinion) in the context of the whole category of (even only pure) motives. In particular, (6.1) of [S2] should be (6.1?), and similarly for [J], Theorem 4.7, p. 50. The proof of [DM], 6.22(a), implicitly assumes that Hodge cycles are absolute. It works in our setting (of motives of abelian varieties, and one-motives) since Hodge cycles on abelian varieties are absolute, by [D2], Theorem 2.11. Thus we use [DM], 6.22(a) and 6.23, replacing $\underline{M}_k$, $\underline{M}_{\overline{k}}$ by $\mathcal{M}^{\mathrm{red}}_k$, $\mathcal{M}^{\mathrm{red}}_{\overline{k}}$ in [DM], p. 213, l. -7 to p. 216, l. -9; in particular the group $G(\sigma)$ of [DM], p. 213, l. -6 (denoted G_σ here) should be $\mathrm{Aut}^{\otimes}(H^{\#}_\sigma|\mathcal{M}^{\mathrm{red}}_k)$, and in the proof of [DM], 6.22(a), X should be in $\mathcal{M}^{\mathrm{red}}_k$ (to use (I 3.4)).

Yet the full Galois group $G(\sigma)$ of [DM], 6.22(a) ($= \mathrm{Aut}^{\otimes}(H^{\#}_\sigma|\underline{M}_{\overline{k}})$) is pro-reductive (as asserted in [DM], 6.22(a)) – meaning that its connected component G^0 is the projective limit of connected reductive groups – by [DM], Remark 2.28 ("G^0 is pro-reductive iff $\mathrm{Rep}_{\mathbb{Q}}\, G(\sigma)$ is semi-simple") and [DM], Proposition 6.5 ("$\underline{M}_{\overline{k}} = \mathrm{Rep}_{\mathbb{Q}}\, G(\sigma)$ is semi-simple").

In an attempt to clarify the proof of [DM], 6.22(a), note that it uses the following well-known assertion. Only the special case of pure Hodge structures is used in [DM], and this suffices for our purposes too, since an algebraic group is connected if its (Levi) reductive component is. As in [DM], Proof of Proposition 2.8, let C_H be the full (Tannakian) subcategory of the category Hod of $\mathbb{Q}$-Hodge structures generated by $\mathbb{Q}(1)$ and an object H. The objects of C_H are by definition the subquotients of sums of $T = H^{\otimes m_1} \otimes (H^\vee)^{\otimes m_2} \otimes \mathbb{Q}(1)^{\otimes m_3}$, and $a \in \mathbb{G}_m$ acts on $\mathbb{Q}(1)^{\otimes m}$ by multiplication by a^{-m}. Let ω be the fiber (forgetful) functor to the category of vector spaces over $\mathbb{Q}$. Suppose that H is a polarizable Hodge structure. Then C_H is semi-simple. Write G' for the subgroup $GL(H) \times \mathbb{G}_m$ over $\mathbb{Q}$ which fixes all $(0,0)$-vectors t in every object T of C_H.

Assertion. *The group $G = \mathrm{Aut}^{\otimes}(\omega|C_H)$ is isomorphic to the group G'.*

Proof. A morphism $g = (g_X : \Phi(X) \to \Phi'(X))$ of functors Φ, Φ' on a category satisfies $\Phi'(f)g_X = g_Y\Phi(f)$ for every morphism $f : X \to Y$. In C_H, an endomorphism of the fiber functor ω is an element g of $GL(H) \times \mathbb{G}_m$ which – extended to $H_{\mathbb{C}} = H \otimes \mathbb{C}$ – commutes with $\omega(f)$, thus $g\omega(f) = \omega(f)g$, for every morphism $f : V \to U$ in Hod, namely with all linear maps $f : V \to U$ with $f(V^{p,q}) \subset U^{p,q}$. Thus for each V, g commutes with $\mathrm{Hom}_{\mathrm{Hod}}(\mathbb{Q}(0), V) = V^{0,0}$, namely it fixes $V^{0,0}$, so $g \in G'$.

Conversely, if $g \in G'$ then for any $V, U \in C_H$, g fixes $(V^\vee \otimes U)^{0,0} = \mathrm{Hom}(V,U)^{0,0}$, thus $g : H \to H$ commutes with every morphism $f : V \to U$ in Hod, so $g \in G$. □

Now the problem in the proof of 6.22(a) in [DM] is that for X in the Tannakian category $\underline{M}_{\mathfrak{K}}$ of motives of absolute Hodge cycles, the full subcategory C_X of $\underline{M}_{\mathfrak{K}}$ embeds via H_σ in the Tannakian category Hod of $\mathbb{Q}$-Hodge structures, but it is not a full subcategory unless each σ-Hodge cycle on X is absolute. If C_X is a full subcategory of Hod (via H_σ, namely each σ-Hodge cycle is absolute), then $G_X = \mathrm{Aut}^{\otimes}(H_\sigma|C_X)$ of [DM], 6.22(a), becomes the group G of the Assertion above, and it can be compared with G', the connected group which features in the second half of [DM], proof of 6.22(a) (and (I 3.4) there). In general, the group G_X consists of those automorphisms of the vector space $H_\sigma(X)$ which commute with each automorphism of the absolute Hodge structure $H(X)$. Not every automorphism f_σ of the Hodge structure $H_\sigma(X)$ extends to an automorphism $(f_{\mathrm{DR}}, f_\ell, f_\tau)$ of absolute Hodge structures, so the group G_X – being the commutator of absolute Hodge morphisms – may be larger than the commutator G of the larger family of σ-Hodge morphisms. The two groups are equal (and the *a-priori* possibly bigger G_X is connected) for abelian varieties X, for which Hodge cycles are absolute.

Remark 2. An extension E of an abelian variety by a torus T is commutative: (a) T is central: the action by inner automorphism of $A = E/T$ on T is trivial, because it amounts to an action on the character group, which is discrete; (b) the commutator $E \times E \to E$ has image in $T = \ker[E \to A]$, and it factors via $A \times A = E/T \times E/T \to T$ by (a); it is trivial since the image is proper and reduced in the affine T.

References

[BP] E. Bayer-Fluckiger, R. Parimala, Classical groups and the Hasse principle, *Ann. of Math.* 147 (1998), 651-693.

[BB] D. Blasius, M. Borovoi, On period torsors, *Automorphic forms, automorphic representations, and arithmetic*, Proc Symp. Pure Math. 66, Part 1, AMS, Providence, RI, 1999.

[BCR] J. Bochnak, M. Coste, M.-F. Roy, *Géométrie algébrique réelle*, Ergebnisse der Math. 12, Springer-Verlag (1987).

[Br] J.-L. Brylinski, "1-Motifs" et formes automorphes, in *Journées Automorphes*, Publ. Math. Univ. Paris VII 15 (1983), 43-106.

[D-II] P. Deligne, Théorie de Hodge, II, *Publ. Math. IHES* 40 (1971), 5-58.

[D-III] P. Deligne, Théorie de Hodge, III, *Publ. Math. IHES* 44 (1975), 5-77.

[D1] P. Deligne, Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques, in *Automorphic Forms, Representations and L-functions*, Proc. Sympos. Pure Math. 33 II (1979), 247-290.

[D2] P. Deligne, (Notes by J. Milne), Hodge cycles on abelian varieties, in *Hodge Cycles, Motives, and Shimura Varieties*, Lecture Notes in Mathematics 900, Springer-Verlag (1982), 9-100.

[D3] P. Deligne, Structures de Hodge mixtes réelles, in *Motives*, Proc. Sympos. Pure Math. 55 (1994), 509-514.

[DM] P. Deligne, J. Milne, Tannakian categories, in *Hodge Cycles, Motives, and Shimura Varieties*, Lecture Notes in Mathematics 900, Springer-Verlag (1982), 101-228.

[J] U. Jannsen, Mixed motives and algebraic K-theory, Lecture Notes in Mathematics 1400, Springer-Verlag (1990).

[M] J. Milne, Canonical models of (mixed) Shimura varieties and automorphic vector bundles, in *Automorphic Forms, Shimura Varieties, and L-functions* I (1990), 283-414.

[Sa] N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics 265, Springer-Verlag (1972).

[Sc] W. Scharlau, *Quadratic and Hermitian Forms*, Grundlehren 270, Springer-Verlag (1985).

[Sch] C. Scheiderer, Hasse principles and approximation theorems for homogeneous spaces over fields of virtual cohomological dimension one, *Invent. Math.* 125 (1996), 307-365.

[S1] J.-P. Serre, *Cohomologie Galoisienne*, Cinquième édition, Lecture Notes in Mathematics 5, Springer-Verlag (1994).

[S2] J.-P. Serre, Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ-adiques, in *Motives*, Proc. Sympos. Pure Math. 55 I (1994), 377-400.

[W] J.-P. Wintenberger, Torseurs pour les motifs et pour les représentations p-adiques potentiellement de type CM, *Math. Ann.* 288 (1990), 1-8.