Introduction.

Langlands' principle of functoriality [B] conjectures that there is a parametrization of the set $\mathrm{Rep}_F(G)$ of admissible [BZ] or automorphic [BJ] representations of a reductive group G over a local or global field F, by admissible homomorphisms $\rho : W_F \to \hat{G} \rtimes W_F$. Here W_F is a form of the Weil group [T] of F, and $\hat{G}$ is the connected (complex) Langlands dual group [B] of G, on which W_F acts via the absolute Galois group of F. If H is another reductive group over F and there is an admissible map $\hat{H} \rtimes W_F \to \hat{G} \rtimes W_F$, then composing with $\rho_H : W_F \to \hat{H} \rtimes W_F$ we get $\rho : W_F \to \hat{G} \rtimes W_F$, and by the functoriality conjecture we would expect a "lifting" map $\mathrm{Rep}_F(H) \to \mathrm{Rep}_F(G)$.

The trace formula has been used to establish the lifting in a few cases. For a test function $f = \otimes f_v \in C_c^\infty(G(\mathbb{A}))$, the convolution operator $r(f)$ maps ϕ in $L^2(G(F)\backslash G(\mathbb{A}))$ to the function whose value at $h \in G(\mathbb{A})$ is $\int_{G(\mathbb{A})} f(g)\phi(hg)dg$. It is an integral operator with kernel $K_f(x,y)$ which has geometric expansion $\sum_{\gamma\in G(F)} f(x^{-1}\gamma y)$, and spectral expansion $\sum_\pi \sum_\phi r(f)\phi(x)\overline{\phi}(y)$. Here π ranges over the set of the irreducible direct summands of L^2 as a module under the action of $G(\mathbb{A})$ by multiplication on the right, and ϕ ranges over an orthonormal basis of smooth vectors. Integrating over $x = y \in G(F)\backslash G(\mathbb{A})$ we obtain the trace formula $\sum_\pi \operatorname{tr}\pi(f) = \sum_{G/\sim} \Phi_f(\gamma)$. Here $G/\sim$ denotes the set of conjugacy classes in $G(F)$, and $\Phi_f(\gamma) = \int_{G(\mathbb{A})/Z(\gamma)} f(x\gamma x^{-1})dx$ is an orbital integral of f. In this outline we ignore all questions of convergence, which make the development of the trace formula such a formidable task.

To develop a theory of liftings of representations from the group H to G, one develops a trace formula for a test function f_H on $H(\mathbb{A})$, of the form $\sum_{\pi_H} \operatorname{tr}\pi_H(f_H) = \sum_{H/\sim} \Phi_{f_H}(\gamma_H)$. One then tries to compare the geometric sides of the two trace formulae. For this one needs:

(1) A notion of a norm map $N : \{G/\sim\} \to \{H/\sim\}$, sending a stable conjugacy class γ in $G(F)$ to γ_H in $H(F)$, locally and globally. This has been defined by Kottwitz-Shelstad [KS] in our context.

(2) A statement of transfer of orbital integrals, asserting that given a test function $f \in C_c^\infty(G(F))$, where F is a local field, there exists a test function f_H, and given f_H there is an f, with "matching orbital integrals", namely $\Phi_f(\gamma) = \Phi_{f_H}(N\gamma)$.

The global test function f is a product of local functions which are almost all the unit element 1_K of the Hecke algebra of spherical (bi-invariant by a standard maximal compact subgroup K of the local group $G(F)$ (K is hyperspecial, [Ti, 3.9.1]) functions on $G(F)$. Hence one must have also the statement that:

(3) $\Phi_{1_K}(\gamma) = \Phi_{1_{K_H}}(N\gamma)$ for all (regular) γ. This statement is called the fundamental lemma. It is a necessary initial point for the comparison to exist.

Further, the admissible map $\hat{H} \rtimes W_F \to \hat{G} \rtimes W_F$ defines a lifting map for unramified representations from $H(F)$ to $G(F)$, and via the Satake transform a dual map from the Hecke algebra of G (locally) to the Hecke algebra of H, and one needs:

(4) An extended fundamental lemma, relating the orbital integrals of the corresponding spherical functions.

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The statements (4) and (2) follow – or should follow – from (3); perhaps (2) implies (3).

Once all this is accomplished, the spectral sides of the trace formulae are equal for sufficiently many corresponding test functions, which are used to isolate individual contributions to the formula, and thus derive the lifting of global and local representations.

The technique of comparison of trace formulae has been applied to lift representations of the multiplicative group of a central simple algebra of degree n, to $GL(n)$. Note that inner forms of G all have the same dual group $\hat{G}$. This is due to Jacquet-Langlands for $n = 2$, Deligne-Kazhdan for all n and local as well as automorphic representations with two supercuspidal components, and [FK2] with "one" rather than "two" such constraints (see [F1] for the special case of a division algebra). However, in this case the two groups under comparison are isomorphic for almost all completions of the global field F, and the fundamental lemma holds automatically.

The next case of such a comparison concerns endoscopy for $G = GL(n, F)$, where $H = GL(m, E)$, E/F is a cyclic field extension of degree n/m. Labesse-Langlands dealt with $n = 2$, Kazhdan [K] with all n and $m = 1$, and Waldspurger [W1] with the general case. The fundamental lemma in this endoscopic case implies the fundamental lemma needed to establish the metaplectic correspondence of [FK1], between $GL(n)$ and any central topological covering group of it. This lifting generalizes Shimura's in the case of $n = 2$. The extended fundamental lemma follows (as in [F2]) from the fundamental lemma of [W1] by means of the (simple) regular functions technique introduced in [FK1], or alternatively by using the spherical functions technique of Clozel.

For a cyclic extension E/F one has the base change lifting from $H(F)$ to $H(E)$. Viewing $H(E)$ as the group of F-points of the F-group $G = \mathrm{Res}_{E/F} H$ obtained by restricting scalars from E to F, the lifting is compatible with the diagonal map of $\hat{H} \rtimes W_F$ to $\hat{G} \rtimes W_F$. Here $\hat{G}$ is a product of $[E : F]$ copies of $\hat{H}$, on which W_F acts via its quotient $\mathrm{Gal}(E/F)$. H. Saito used (in the context of modular forms) the twisted (by a generator σ of the Galois group $\mathrm{Gal}(E/F)$) trace formula $\sum \mathrm{tr}\, \pi(f\sigma) = \sum \Phi_f(\gamma\sigma)$, for the convolution operator $r(f\sigma)$. Here the twisted orbital integrals are $\int f(x^{-1}\gamma\sigma(x))dx$. For $n = 2$ the base change lifting for $GL(n)$ has been carried out by Saito, Shintani, Langlands, and for general n by Arthur-Clozel [AC]. The stable fundamental lemma, matching stable orbital integrals and stable twisted ones, has been proven by Kottwitz [Ko] for any G. Regular functions are used in [F3] to give a simple proof of the (unconditional) base change lifting for $GL(2)$, and in [F4] for cusp forms on $GL(n)$ with a supercuspidal component.

Naturally one can consider actions other than that of the Galois group. Twisting by the outer automorphism $\theta(g) = {}^t g^{-1}$ (t for "transpose") of $GL(n)$ would lead to liftings from symplectic and orthogonal groups to $GL(n)$. The first example in this line concerns the symmetric square lifting [F6] from $H = SL(2)$ to $G = PGL(3)$, which is associated with the dual group homomorphism embedding $\hat{H} = PGL(2, \mathbb{C}) = SO(3, \mathbb{C}) = \hat{G}^{\hat{\theta}}$ in $\hat{G} = SL(3, \mathbb{C})$. Here $\hat{H} = Z_{\hat{G}}(\hat{\theta})$ is a twisted endoscopic group. More generally, for $n \geq 3$, $\hat{G} = GL(n, \mathbb{C})$, $\theta(g) = J^t g^{-1} J^{-1}$ for some symmetric or anti-symmetric matrix J, since $\hat{H} = Sp(n/2, \mathbb{C})$ or $SO(n, \mathbb{C})$, one expects to obtain liftings from orthogonal or symplectic groups to the general linear group. The purpose of this work is to prove the fundamental lemma in the next case, of $GL(4)$, by means of a new technique, which also provides a more elementary proof in other (known) cases, and a hope for extension.

The orbital integral $\int_G 1_K(x^{-1}\gamma x)dx$ is the number of cosets xK in G/K (G is a p-adic group and K denotes a hyperspecial maximal compact subgroup), which are fixed by the action of γ. Since G/K is the Bruhat-Tits building of G, Langlands interpreted the computation of the orbital integral as a problem of counting points on the building. This led to a satisfactory proof of the stable fundamental lemma for base change [Ko], and to a counting proof for the symmetric square lifting [F5, §4]. Langlands and Shelstad then studied the asymptotic expansion of orbital integrals of general (C_c^∞) functions for a general G on developing an "Igusa data" approach, and Hales [H1] in the context of $Sp(2)$. The recent coherence result of Waldspurger [W3] for the unit element 1_K (and standard endoscopy) is used in [H2] to deduce from [H1] the fundamental lemma for $Sp(2)$.

Our – elementary – approach is entirely different. It involves neither buildings nor germs. Our expression for the orbital integral is entirely explicit. Our results for 1_K in the context of $GSp(2)$ and $Sp(2)$ imply – using the reduction of Waldspurger [W2] – the transfer of general functions on $GSp(2)$ and $Sp(2)$ to their endoscopic groups, recovering the results of [H1] and [H2]. Further, we prove the fundamental lemma in the twisted case.

To start with, we note that a useful reduction of the computation of the orbital integral of 1_K at an element k of K is given by Kazhdan's decomposition [K] of k as a commuting product of an absolutely semi-simple element s, and a topologically unipotent element u. The integral is then reduced to that of u, where G and K are replaced by the centralizers of s in these groups. A twisted analogue of this result is developed in [F7], where – taking the group to be the semi direct product of $PGL(3,F)$ and the group generated by the twisting σ – the twisted orbital integrals of 1_K are reduced to orbital integrals on forms of $GL(2)$, which can be directly computed, and compared with the orbital integrals on the "lifted" groups ($SL(2)$ and $PGL(2)$). This reduction is carried out in the context of $GL(4)$ rather than $GL(3)$ in the present work. It permits us to compare the resulting integrals on the group $Sp(2)$ of fixed points of $\sigma(g) = J^t g^{-1} J^{-1}$ on $GL(4)$, with the integrals of 1_K on $GSp(2)$ at the norm of the element u.

The basic idea for the computation of the non twisted orbital integrals comes from the work of Weissauer [We]. Since the orbital integral is an integral over $T\backslash G/K$, where T is the centralizer of our regular element in G, it suffices to find a double coset decomposition for $H\backslash G/K$, for a subgroup H of G which contains T, and then the computation of the orbital integral is reduced to one on the subgroup H, which should be simpler than G. Weissauer [We] proved the fundamental lemma for $GSp(2)$ and its endoscopic group $SO(4)$. We prove here this lemma from $GL(4)$ to all of its twisted endoscopic groups, including $GSp(2)$, using this approach. Of course here we consider all tori T of $GSp(2)$, not only those which transfer to its endoscopic group, and compute the norm map.

Our work is entirely explicit. We exhibit a set of representatives for the twisted conjugacy classes in G, in families of types which we call (I), (II), (III), and (IV). We list those in the same stable twisted conjugacy class. The listing is done on computing the Galois hypercohomology groups used in [KS], or simply on using low brow Galois cohomology, but it is important for us to exhibit explicit representatives, not just to describe the abstract structure of the conjugacy classes within the stable class. Further we describe the norm map explicitly for each type, and find representatives for the stable conjugacy classes and the conjugacy classes in it, for $GSp(2)$. The stable orbital integral is simply the sum over the orbits in the stable orbit. Thus our computations can be used to compute the unstable orbital integrals. In the case of $GSp(2)$ we recover the results of Weissauer [We]. In the twisted case, this is done here too for all unstable twisted endoscopic groups. We compute all unstable orbital integrals of 1_K on the group $Sp(2)$, which has more endoscopic groups than $GSp(2)$, and deduce all endoscopic transfers of orbital integrals.

In [F8] we obtain a double coset decompositions in the context of $(U(2)\times U(1))\backslash U(3)/K$, where U denote unitary groups of a quadratic field extension E/F, and use these to prove the fundamental lemma for $U(2,1)$ and its endoscopic group $U(1,1)\times U(1)$, for a torus T split over E, a quadratic unramified extension of F, and for a torus T which splits over a biquadratic extension of F.

The results and techniques of this work were described in the talk [F9] at the conference "Automorphic Forms on Algebraic Groups", RIMS 1995. At the end of my talk Takayuki Oda pointed out that results of Murase and Sugano [MS] on double coset decompositions of the form $H\backslash G/K$ existed for all classical quasi-split groups, and our direct and elementary approach might extend to deal with twisted $GL(n)$ for all n, namely with all symplectic and orthogonal groups.

This work started and was completed at Mannheim, supported by DAAD and the Humboldt Stiftung. I wish to express my very deep gratitude to Rainer Weissauer for his hospitality, inspiration and help, to J.-L. Waldspurger for locating an error at my request, and to J.G.M. Mars for developing an alternative technique – based on usage of lattices – and verifying that the result of our computations coincide.

Our work concerns an example, and we worked out all related objects. It will be useful to list here *informally* the main objects. These are the twisted elliptic endoscopic groups; the elliptic twisted stable conjugacy classes, listed according to the elliptic tori T; the group structure of the conjugacy classes within the stable conjugacy classes; the characters κ on these groups, and the endoscopic groups attached to a regular element of T and to κ. The "fundamental lemma" takes the form: the κ-linear combination of θ-orbital integrals of the unit element 1_K at a θ-regular element t – multiplied by a suitable transfer factor – is equal to the stable (trivial κ) orbital integral of 1_K on the θ-endoscopic group determined by t and κ at the norm of t.

Thus our group is $\mathbf{G}=GL(4)\times GL(1)$; our automorphism is $\theta(g,x)=(J\,{}^tg^{-1}J^{-1},x\ \det(g))$. In Section I.F (i.e. Section F of Part I) we show that the stable θ-endoscopic group is $\mathbf{H}=GSp(2)$. It would have been $Sp(2)$ had we taken $\mathbf{G}=GL(4)$. But while $GSp(2)$ has only one elliptic endoscopic group: $(GL(2)\times GL(2))/GL(1)$, $Sp(2)$ has the elliptic endoscopic groups $(GL(2)\times GL(2))'/GL(1)$ (the prime indicates: equal determinants), $\mathrm{Res}_{E/F}\,GL(2)'/GL(1)$ for each quadratic extension E/F (its group of F-points is $GL(2,E)'/F^\times$, the prime indicates: determinant in $F^\times$), $SL(2)\times U(1,E/F)$ for each quadratic extension E/F (its group of F-points is $SL(2,E)\times E^1$, $E^1=\ker\mathrm{Norm}_{E/F}$). The unstable θ-endoscopic groups are "of type I.F.2": $\mathbf{C}=(GL(2)\times GL(2))'$ and $\mathbf{C}_E=\mathrm{Res}_{E/F}\,GL(2)'$ for each quadratic extension E/F, and "of type I.F.3": $C_+=GL(2,F)\times E^1$, again all $[E:F]=2$.

The θ-elliptic strongly θ-regular elements are classified in Section I.D according to tori of types (I), (II), (III), (IV) in $GSp(2)$. We list the tori of $GSp(2)$ reversing the order of (II) and (III), so that the norm map from $\mathbf{G}$ to $\mathbf{H} = GSp(2)$ preserves the type. Tori of type (I) are isomorphic to $E^\times \times E^\times$, $[E : F] = 2$, those of type (II) are $\simeq E_1^\times \times E_2^\times$, $[E_i : F] = 2$, $E_1 \not\simeq E_2$, E_2/F ramified. They lie in the group C_0 of F-rational points in $\mathbf{C}_0 \simeq (GL(2) \times GL(2))'$, where $\mathbf{C}_0$ is the group of $[(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), (\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix})] = \begin{pmatrix} a & 0 & 0 & b \\ 0 & \alpha & \beta & 0 \\ 0 & \gamma & \delta & 0 \\ c & 0 & 0 & d \end{pmatrix} \in \mathbf{H}$. Tori of type (III) are isomorphic to $E^\times$, where $E = E_1E_2$ is a biquadratic extension ($[E_i : F] = 2$) of F. The choice of the quadratic extensions E_1, E_2, E_3 of F, is implicit in our presentation of the tori. Tori of type (IV) are isomorphic to $E^\times$, where E is a cyclic or a non Galois extension of F of degree 4. Put $E_3 = F(\sqrt{A})$, $A \in F - F^2$, for the quadratic extension of F in E. These tori embed in the group $C_A \simeq GL(2, E_3)'$ of rational points over E_3 of the group $\mathbf{C}_A$ of $(\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}) \in \mathbf{H} = GSp(2)$, where $\mathbf{a} = (\begin{smallmatrix} a_1 & a_2 \\ a_2 A & a_1 \end{smallmatrix})$, $\mathbf{b} = \dots$. The double coset decompositions (see Section I.J) of $C_0 \backslash GSp(2, F)/K$, $C_A \backslash GSp(2, F)/K$, and the analogues with $Sp(2)$ instead of $GSp(2)$, play key roles in our analysis.

The θ-conjugacy classes within a stable θ-conjugacy class of a θ-elliptic strongly θ-regular element are the following groups. When the class is of type (I), the group is $F^\times/N_{E/F}E^\times \times F^\times/N_{E/F}E^\times$. Type (II): $F^\times/N_{E_1/F}E_1^\times \times F^\times/N_{E_2/F}E_2^\times$. Types (III) and (IV): $E_3^\times/N_{E/E_3}E_3^\times$. The κ combinations of θ-orbital integrals of 1_K are related to stable orbital integrals of 1_K on the θ-endoscopic groups determined as follows. If κ is trivial, we are in the stable case, and $GSp(2)$ is obtained. In type (I), $\kappa = \kappa_1 \times \kappa_2$. If both $\kappa_i \neq 1$, the group is $\mathbf{C} = (GL(2) \times GL(2))'$. If precisely one of the κ_i is non trivial, then the group is $\mathbf{C}_+ = GL(2) \times U(1, E/F)$ if E/F is unramified, but the κ-θ-integral vanishes when E/F is ramified: this is a general phenomenon, that the integral of 1_K would vanish when it should relate to a *ramified* endoscopic group. In type (II), $\kappa = \kappa_1 \times \kappa_2$. If both $\kappa_i \neq 1$, the group is $\mathbf{C}_{E_3} = \mathrm{Res}_{E_3/F}\, GL(2)'$ when E_3/F is unramified; the integral vanishes when E_3/F is ramified. If $\kappa_1 \neq 1$, $\kappa_2 = 1$, and E_1/F is unramified, the group is $\mathbf{C}_+ = GL(2) \times U(1, E_1/F)$, but the κ-integral vanishes when E_1/F is ramified. In type (III), if $\kappa \neq 1$, the group is $\mathbf{C}$. In type (IV), if $\kappa \neq 1$, the group is $\mathbf{C}_{E_3}$ when E_3/F is unramified; the κ-integral vanishes when E_3/F is ramified.

To repeat, elliptic conjugacy classes in $\mathbf{C} = (GL(2) \times GL(2))'$ lie in $E_1^\times \times E_2^\times$ come from type (I) when $E_1 = E_2$, and from type (III) if $E_1 \neq E_2$. Those in $\mathbf{C}_{E_3} = \mathrm{Res}_{E_3/F}\, GL(2)'$ lie in a quadratic extension E of the quadratic extension E_3 of F; they come from type (II) if E is biquadratic ($=E_1E_2$) over F, and from type (IV) if E is cyclic or non Galois over F. An elliptic conjugacy classes in $\mathbf{C}_+ = GL(2) \times U(1, E_1/F)$, unramified E_1/F, defines a quadratic extension E_2/F (in its $GL(2)$ part); it comes from type (I) if $E_1 = E_2$, and from type (II) if $E_1 \neq E_2$, and a $\kappa = \kappa_1 \times \kappa_2$ with only one non trivial factor.

Our analysis applies to establish the fundamental lemma for the group $Sp(2)$, except that types (II) and (III) need to change names, as they are interchanged under the norm map. The lists of endoscopic groups, elliptic elements, κ and even statement of results are essentially the same, since the θ-integrals on G are integrals on $Sp(2, F)$. The analysis in the case of $GSp(2, F)$ is simpler, there is a unique endoscopic group, essentially $GL(2) \times GL(2)$, and tori of type (I), (II), yield the tori $E^\times \times E^\times$ and $E_1^\times \times E_2^\times$ of the endoscopic group.

PART I. Preparations.

A. Statement of Theorem.

Let R denote the ring of integers in a local non archimedean field F. Let $\mathbf{G}$ be the F-group $\mathbf{G}_1 \times \mathbf{G}_m$, where $\mathbf{G}_1 = GL(4)$ and $\mathbf{G}_m = GL(1)$. Put ${}^t g_1$ for the transpose of $g_1 \in \mathbf{G}_1$. Define $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix} 0 & w \\ -w & 0 \end{smallmatrix}\right)$, $\theta(g_1) = J {}^t g_1^{-1} J^{-1}$, and $\theta(g_1, e) = \big(\theta(g_1), e\|g_1\|\big)$ for $g = (g_1, e) \in \mathbf{G}$; $\|g_1\|$ denotes the determinant of g_1. Put $\mathbf{H} = GSp(2) = GSp(J)$ for the group $\{g_1 \in \mathbf{G}_1; \theta(g_1) = eg_1$ for some $e = e(g_1) \in GL(1)\}$ of symplectic similitudes. We write $G = \mathbf{G}(F)$ and $H = \mathbf{H}(F)$ for the groups of F-points, and $K = \mathbf{G}(R)$ and $K_H = \mathbf{H}(R)$ for the standard maximal compact subgroups. Similarly we have $G_1, K_1, \dots$.

We choose Haar measures dg, $dh, \dots$ on G, $H, \dots$, and denote by $1_K = 1_{K_G}$ the quotient by the volume $|K|$ of K of the characteristic function of $K = K_G$ in G, by 1_{K_H} the analogous object for K_H, 1_{K_1} for K_1 in G_1, etc. Then 1_K lies in the space $C_c^\infty(G)$ of locally constant compactly supported functions on G. We often omit the subscript of K, when it is clear from the context. Identify $C_c^\infty(G)$ with $C_c^\infty(G\theta)$ by $f(g) = f(g\theta)$, put $\mathrm{Int}(g)(t\theta) = gt\theta g^{-1} = gt\theta(g^{-1})\theta$, and introduce the *orbital integral*

$$\Phi_f^G(t\theta) = \Phi_f^G(t\theta; d_G/d_{Z_G(t\theta)}) = \int_{G/Z_G(t\theta)} f((\mathrm{Int}(g))(t\theta)) dg/d_{Z_G(t\theta)}$$

of $f \in C_c^\infty(G)$ at $t\theta, t \in G$ (it is also called the θ-*orbital integral* of f at t). Here

$$Z_G(t\theta) = \{g \in G; \mathrm{Int}(g)(t\theta) = t\theta\}$$

is the θ-*centralizer* of t in G, or the centralizer of $t\theta$ in G.

The elements t, t' of G are called *stably θ-conjugate* if $t'\theta = \mathrm{Int}(g)(t\theta)$ for some $g \in \mathbf{G}(= \mathbf{G}(\overline{F}), \overline{F} =$ algebraic closure of $F)$. There are finitely many θ-conjugacy classes $\big(\mathrm{Int}(g)(t\theta), g \in G\big)$ in a stable θ-conjugacy class, and we define the stable orbital integral $\Phi_f^{G,st}(t\theta)$ of f at $t\theta$ to be the sum $\sum \Phi_f^G(t'\theta)$ over a set of representatives t' for the θ-conjugacy classes within the stable θ-conjugacy class of t (in G). Note that $Z_{\mathbf{G}}(t\theta)$ and $Z_{\mathbf{G}}(t'\theta)$ are isomorphic when t, t' are stably θ-conjugate, this isomorphism is used to relate the measures on these groups. Similarly we have the stable orbital integral $\Phi_f^{H,st}(h; d_H/d_{Z_H(h)})$ of $f \in C_c^\infty(H)$ at $h \in H$.

The purpose of this paper is to prove the following.

Theorem. *For any strongly θ-regular $t \in G$ we have*

$$\Phi_{1_K}^{G,st}(t\theta; d_G/d_{T^\theta}) = \Phi_{1_{K_H}}^{H,st}(Nt; d_H/d_{T^\theta} \circ (1+\theta) \circ N^{-1}).$$

An element t of G is called *θ-semi-simple* if $t\theta$ is semi-simple in the group $G \rtimes \langle\theta\rangle$ (θ is an automorphism of G of order two). Such an element is called *θ-regular* if $Z_{\mathbf{G}}(t\theta)^0$, the connected component of the identity in $Z_{\mathbf{G}}(t\theta)$, is a torus. Further it is called *strongly θ-regular* if $Z_{\mathbf{G}}(t\theta)$ is abelian. In this case $Z_{\mathbf{G}}\big(Z_{\mathbf{G}}(t\theta)^0\big)$ is a maximal torus $\mathbf{T}$ in $\mathbf{G}$ which is stable under $\mathrm{Int}(t\theta)$, and $Z_{\mathbf{G}}(t\theta) = \mathbf{T}^{\mathrm{Int}(t\theta)}$ (see Kottwitz-Shelstad [KS, 3.3]). According to [KS, Lemma 3.2.A(a)], we may assume that the strongly θ-regular t lies in a θ-stable F-torus $\mathbf{T}$. Thus $t \in T = \theta(T)$.

To define the norm map – which appears in the statement of the Theorem – following [KS] we fix a θ-stable F-pair $(\mathbf{T}^*,\mathbf{B}^*)$ consisting of a minimal θ-stable F-parabolic subgroup $\mathbf{B}^*$ of $\mathbf{G}$, and a maximal θ-stable F-torus $\mathbf{T}^*$ in $\mathbf{B}^*$. Namely we take $\mathbf{B}^*$ to be the upper triangular subgroup of $\mathbf{G}$, and $\mathbf{T}^*$ to be the diagonal subgroup (thus $\mathbf{T}^*=\mathbf{T}_1^*\times\mathbf{G}_m$). Any two θ-stable F-tori $\mathbf{T}^*$ and $\mathbf{T}$ are θ-conjugate in $\mathbf{G}$, thus given $\mathbf{T}$ ($\mathbf{T}^*$ is fixed) there is $h\in\mathbf{G}$ with $\mathbf{T}=h^{-1}\mathbf{T}^*\theta(h)$, and in particular $t^*\in\mathbf{T}^*$ such that $t=h^{-1}t^*\theta(h)$. The norm of t is defined to be the stable conjugacy class in H which is conjugate to Nt^* over $\overline{F}$, where Nt^* is defined as follows.

Put $\mathbf{V}=(1-\theta)\mathbf{T}^*$ and $\mathbf{U}=\mathbf{T}^*_\theta=\mathbf{T}^*/\mathbf{V}$. Here $\mathbf{T}^*$ consists of $(a,b,c,d;e)$ $(=(\mathrm{diag}(a,b,c,d),e))$, and $\theta(a,b,c,d;e)=(d^{-1},c^{-1},b^{-1},a^{-1};eabcd)$. Then $\mathbf{V}$ consists of $(\alpha,\beta,\beta,\alpha;1/\alpha\beta)$. Choose the isomorphism $N:\mathbf{U}\tilde{\to}\mathbf{T}_H^*$ given by

$$(x,y,z,t;w)\bmod\{(\alpha,\beta,\beta,\alpha;1/\alpha\beta)\}\mapsto(xyw,xzw,tyw,tzw;xyztw^2)=(a,b,e/b,e/a;e).$$

It is surjective since $(b,a/b,1,e/a;1)\mapsto(a,b,e/b,e/a;e)$. Of course $\mathbf{T}_H^*$ is the diagonal subgroup in $\mathbf{H}$, and any torus $\mathbf{T}_H$ in $\mathbf{H}$ is conjugate to $\mathbf{T}_H^*$ over $\overline{F}$. The stable conjugacy class of a regular element in H is the intersection with H of its conjugacy class over $\overline{F}$. The choice of the isomorphism $\mathbf{U}\tilde{\to}\mathbf{T}_H^*$ is dictated by dual groups considerations, namely that $\mathbf{H}$ is an endoscopic group in $\mathbf{G}$; this we explain in Section F below.

The orbital integrals on $G=GL(4,F)$ and $H=GSp(2,F)$ depend on a choice of Haar measures. These are chosen compatibly, as follows. A Haar measure is unique up to a scalar, determined by the volume of the maximal compact subgroup. The function 1_{K_G} is the unit element in the Hecke algebra $C_c(K_G\backslash G/K_G)$, thus it is the quotient of the characteristic function of K_G in G by the volume of K_G. The product $1_{K_G}d_G$ is the constant measure with support K_G and total volume 1; it is independent of the choice of the Haar measure d_G. Thus we may and do assume that $|K_G|=1$ and 1_{K_G} is the characteristic function of K_G. This simplifies our computations below. The same comment applies to $1_{K_H}d_H$.

It remains to relate the measures on $Z_G(t\theta)$ and on $Z_H(Nt)$, for a strongly θ-regular element t in G. We shall use the observation that if $N:\mathbf{T}_1\to\mathbf{T}_2$ is an epimorphism of F-tori with kernel $\mathbf{T}_0$, and if d_{T_i} denotes the Haar measure on $T_i=\mathbf{T}_i(F)$ which assigns the maximal compact subgroup $T_i(R)$ the volume $|T_i(R)|=d_{T_i}\big(T_i(R)\big)$ one, then $d_{T_1}=\mu N^*(d_{T_2})$ for some $\mu>0$, where $N^*(d_{T_2})=d_{T_2}\circ N$ is the measure on T_1 obtained from d_{T_2} via N. Computing the volume of $T_1(R)$ we see that $\mu=[T_2(R):N\big(T_1(R)\big)]$. We shall relate an orbital integral $\Phi(d_{G_2},d_{T_2})$ with $\Phi(d_{G_2},d_{T_2}\circ N)=\mu\Phi(d_{G_1},d_{T_1})\big(T_i\subset G_i(i=1,2)\big)$.

Applying this principle to the norm map $N:\mathbf{T}^*\to\mathbf{T}_H^*$, where $\mathbf{T}_H^*=\{(x,y,z,t);xt=yz)\}$, defined by $N(x,y,z,t)=(xy,xz,yt,zt)$, whose kernel is $\mathbf{V}$, we see that $d_{T^*}=[T_H^*(R):N\big(T^*(R)\big)]d_{T_H^*}\circ N$. Applying the principle to the map $1+\theta:\mathbf{T}^*\to\mathbf{T}^{*\theta}$, whose kernel is $\mathbf{V}$, where $\theta(x,y,z,t)=(t^{-1},z^{-1},y^{-1},x^{-1})$, thus $(1+\theta)(x,y,z,t)=(x/t,y/z,z/y,t/x)$, and $\mathbf{T}^{*\theta}=\{(x,y,y^{-1},x^{-1})\}$, we see that $d_{T^*}=[T^{*\theta}(R):(1+\theta)\big(T^*(R)\big)]d_{T^{*\theta}}\circ(1+\theta)$. In conclusion

$$d_{T^{*\theta}}\circ(1+\theta)=\frac{[T_H^*(R):N\big(T^*(R)\big)]}{[T^{*\theta}(R):(1+\theta)T^*(R)]}d_{T_H^*}\circ N,$$

and the (stable) θ-orbital integral $\Phi(1_Kd_G,d_{T^{*\theta}})$ on G is related to the (stable) orbital integral $\Big([T^{*\theta}(R):(1+\theta)T^*(R)]/[T_H^*(R):N\big(T^*(R)\big)]\Big)\Phi(1_Kd_H,d_{T_H})=\Phi\big(1_Kd_H,d_{T^{*\theta}}\circ(1+\theta)\circ N^{-1}\big)$.

This is the relation of measures which appears in the Theorem. We shall see below that $Z_G(t\theta)$ takes the form $T^{*\theta}$ (up to isomorphism; $T^{*\theta}$ = θ-fixed points in T^*), and the measure used in the integration over H is pulled back from the measure $d_{T^{*\theta}}$ on $T^{*\theta}$ via the isomorphism $\mathbf{T}_H^* \overset{N}{\tilde{\leftarrow}} \mathbf{T}^*/\mathbf{V} \overset{1+\theta}{\tilde{\longrightarrow}} \mathbf{T}^{*\theta}$. The factor $[T^{*\theta}(R) : (1+\theta)\big(T^*(R)\big)]/[T_H^*(R) : N\big(T^*(R)\big)]$ which relates d_{T_H} with $d_{T^\theta} \circ (1+\theta) \circ N^{-1}$, will be computed for each torus considered in the course of the proof below.

B. Stable conjugacy.

Let us recall the structure of the set of (F-rational) conjugacy classes within the stable ($\overline{F}$-) conjugacy class of a regular element t in H. By definition, the centralizer $Z_{\mathbf{H}}(t)$ of t in $\mathbf{H}$ is a maximal F-torus $\mathbf{T}_H$. The elements t, t' of H are *conjugate* if there is g in H with $t' = \mathrm{Int}(g^{-1})t (= g^{-1}tg)$. They are *stably conjugate* if there is such g in $\mathbf{H}\big(= \mathbf{H}(\overline{F})\big)$. Then $g_\sigma = g\sigma(g^{-1})$ lies in $\mathbf{T}_H$ for every σ in the Galois group , $= \mathrm{Gal}(\overline{F}/F)$, and $g \mapsto \{\sigma \mapsto g_\sigma\}$ defines an isomorphism from the set of conjugacy classes within the stable conjugacy class of t to the pointed set $D(T_H/F) = \ker[H^1(F, \mathbf{T}_H) \to H^1(F, \mathbf{H})]$. In our case $H^1(F, \mathbf{H})$ is trivial, hence $D(T_H/F)$ is a group.

1. Lemma. *The set of stable conjugacy classes of F-tori in $\mathbf{H}$ injects naturally in the image in $H^1(F, W)$ of $\ker[H^1(F, \mathbf{N}) \to H^1(F, \mathbf{H})]$, where $\mathbf{N} = \mathrm{Norm}(\mathbf{T}_H^*, \mathbf{H})$, and W is the Weyl group of $\mathbf{T}_H^*$ in $\mathbf{H}$. This map is an isomorphism when $\mathbf{H}$ is quasi-split. Note that the image is $H^1(F, W)$ when $H^1(F, \mathbf{H})$ is trivial, and $H^1(F, W)$ is the group of continuous homomorphisms $\rho : , \to W$, when , acts trivially on W.*

Proof. Indeed, the tori $\mathbf{T}$ and $\mathbf{T}_H^*$ are conjugate in $\mathbf{H}$, thus $\mathbf{T} = g^{-1}\mathbf{T}_H^* g$ for some g in $\mathbf{H}$. For any t in $\mathbf{T}$ there is t^* in $\mathbf{T}_H^*$ with $t = g^{-1}t^*g$. For t in T, $\sigma g^{-1} \sigma t^* \sigma g = \sigma t = t = g^{-1}t^*g$, thus $\sigma t^* = g_\sigma^{-1} t^* g_\sigma \in \mathbf{T}_H^*$, and $g_\sigma \in \mathrm{Norm}(\mathbf{T}_H^*, \mathbf{H})$. Since t (and so t^*) is regular, g_σ is uniquely determined modulo $\mathbf{T}_H^*$, namely in W. For a general t^* in $\mathbf{T}_H^*$ we then have $\sigma(g^{-1}t^*g) = g^{-1}\big(g\sigma(g^{-1})\big)\sigma(t^*)\big(\sigma(g)g^{-1}\big)g$, so that the induced action on $\mathbf{T}_H^*$ is given by $\sigma^*(t^*) = \mathrm{Int}(g_\sigma)\big(\sigma(t^*)\big)$. The cocycle $\rho = \rho(T) : , \to W$ is given by $\rho(\sigma) = g_\sigma \bmod \mathbf{T}_H^*$. It determines $\mathbf{T}$ up to stable conjugacy. Conversely, a $\{g_\sigma\}$ in $\ker[H^1(F, \mathbf{N}) \to H^1(F, \mathbf{H})]$ determines an action $\sigma^*(t^*) = \mathrm{Int}(g_\sigma)\big(\sigma(t^*)\big)$ on $\mathbf{T}_H^*$. By a well-known theorem of Steinberg, when $\mathbf{H}$ is quasi split over F, an F-conjugacy class in $\mathbf{H}$ of a regular t^* contains a rational element $h^{-1}t^*h$ (in H), whose centralizer is an F-torus which defines g_σ. $\square$

In our case of $\mathbf{H} = GSp(2)$, the Weyl group W is the dihedral group D_4, generated by the reflections $s_1 = (12)(34)$ and $s_2 = (23)$. Its other elements are $1, (12)(34)(23) = (3421)$ (which takes 1 to 2, 2 to 4, 4 to 3, 3 to 1), $(23)(12)(34) = (2431)$, $(23)(3421) = (42)(31)$, $(3421)^2 = (23)(41)$, $(23)(23)(41) = (41)$. Let us list the F-tori $\mathbf{T}$ according to the subgroups of W, the split torus corresponding to $\{1\}$, and conclude the following.

2. Lemma. *We have that $H^1(F, \mathbf{T})$ is trivial except when $\rho(,)$ is the subgroup of W of the form $\langle(14)(23)\rangle$ or $\langle(14)(23), (12)(34), (13)(24)\rangle$, where $H^1(F, \mathbf{T}) = \mathbb{Z}/2$.*

Proof. Recall that if $\mathbf{T}_H$ splits over the Galois extension E of F then $H^1(F, \mathbf{T}_H) =$ $H^1\big(\mathrm{Gal}(E/F), \mathbf{T}_H^*(E)\big)$, where $\mathbf{T}_H^*(E) = \{\mathrm{diag}(a, b, \lambda/b, \lambda/a); a, b, \lambda \in E^\times\}$, and $\mathrm{Gal}(E/F)$

acts via ρ. Thus H^1 is the quotient of the group C^1 of cocycles: $a_\tau \in \mathbf{T}_H^*(E)$ with $a_1 = 1$ and $a_{\sigma\tau} = a_\sigma \sigma^*(a_\tau)$ for all $\sigma, \tau \in \mathrm{Gal}(E/F)$, by the group of coboundaries: $c\sigma^*(c^{-1}), c \in \mathbf{T}_H^*(E)$. Here $\sigma^* = \rho(\sigma) \circ \sigma$, thus $\sigma^*(a) = g_\sigma \cdot \sigma a \cdot g_\sigma^{-1}$ if $\rho(\sigma) = \mathrm{Int}(g_\sigma)$. When $\rho(\,,\,) = \{1\}$, the group H^1 is trivial since $E = F$. The other cases are:

(1) $\rho(\,,\,) = \langle(23)\rangle, [E:F] = 2, a_\sigma = (a, b, \lambda/b, \lambda/a)$ with $a_\sigma \sigma^*(a_\sigma) = I$ satisfies $a\sigma a = 1$, $\lambda\sigma\lambda = 1$, $b\sigma\lambda = \sigma b$. Choosing $\alpha, \mu \in E^\times$ with $a = \alpha/\sigma\alpha, \mu = \sigma b^{-1}$, we have $\lambda = \mu/\sigma\mu$, and $c = (\alpha, 1, \mu, \mu/\alpha)$ satisfies $c\sigma^*(c)^{-1} = a_\sigma$. Hence H^1 is trivial. The same result holds for $\rho(\,,\,) = \langle(14)\rangle$.

(2) $\rho(\,,\,) = \langle(12)(34)\rangle, [E:F] = 2, a_\sigma$ satisfies $a\sigma b = 1$, and $\lambda\sigma\lambda = 1$. Choosing $\mu \in E^\times$ with $\lambda = \mu/\sigma\mu$, we have that $c = (a, 1, \mu, \mu/a)$ satisfies $c\sigma^*(c^{-1}) = a_\sigma$. Hence H^1 is trivial.

(3) $\rho(\,,\,) = \langle(13)(24)\rangle, [E:F] = 2, a_\sigma$ satisfies $\lambda\sigma\lambda = 1$ and $b = \lambda\sigma a$. Take $\mu \in E^\times$ with $\lambda = \mu/\sigma\mu$, and $c = (a, \mu, 1, \mu/a)$. Then $c\sigma^*(c^{-1}) = a_\sigma$ and H^1 is trivial.

These tori are not elliptic – their quotient by the center of H is not compact. The elliptic tori are:

(I) $\rho(\,,\,) = \langle(14)(23)\rangle, [E:F] = 2, a_\sigma$ satisfies $\lambda = b/\sigma b = a/\sigma a = \sigma\lambda^{-1}$. Thus $a/b \in F^\times$. If $c = (1, \beta, 1/\beta\sigma a, 1/\sigma a)$, then $c\sigma^*(c^{-1}) = (a, a\beta\sigma\beta, \lambda/a\beta\sigma\beta, \lambda/a)$. Then $H^1 = \{a_\sigma\}/\{c\sigma^*(c)^{-1}\} = F^\times/N_{E/F}E^\times$.

(II) $\rho(\,,\,) = \langle(14)(23), (12)(34), (13)(24)\rangle$, E is the composition of the different quadratic extensions E_1, E_2, E_3 of F, and so $\mathrm{Gal}(E/F) = \mathbb{Z}/2 \times \mathbb{Z}/2$ is generated by σ and τ whose fixed fields are $E_3 = E^{\langle\sigma\rangle}$, $E_2 = E^{\langle\sigma\tau\rangle}$, $E_1 = E^{\langle\tau\rangle}$. Say $\rho(\sigma) = (14)(23)$ and $\rho(\tau) = (12)(34)$. Then $a_\tau = c\tau^*(c^{-1})$, as seen in (2) above. We shall replace the cocycle $\{a_\alpha\}$ by the equivalent $\{a_\alpha c^{-1}\alpha^*(c)\}$. Then we may assume that $a_\tau = I$. The relation $a_\sigma = a_\sigma\sigma^*(a_\tau) = a_{\sigma\tau} = a_{\tau\sigma} = a_\tau\tau^*(a_\sigma) = \tau^*(a_\sigma)$ implies that $a_\sigma = (a, \tau a, \lambda/\tau a, \lambda/a)(a \in E^\times, \lambda \in E_1^\times)$. The relation $a_\sigma\sigma^*(a_\sigma) = I$ implies that $\lambda = a/\sigma a$. Hence $a/\sigma a = \lambda = \tau\lambda = \tau a/\sigma\tau a$, and $a\sigma\tau a = \sigma a\tau a$ lies in $F^\times$. Since $N_{E_1/F}E_1^\times \neq N_{E_2/F}E_2^\times$ and $F^\times/N_{E_i/F}E_i^\times$ is of order two, $a\sigma\tau a$ can take any value in $F^\times$. For $c = (\alpha, \tau\alpha, \mu/\tau a, \mu/\alpha)$, $\mu = \tau\mu \in E_1^\times$, we have $c\sigma^*(c)^{-1} = (d, \tau d, \lambda/\tau d, \lambda/d)$ with $d = \alpha\sigma\alpha/\sigma\mu$ and $\lambda = \mu/\sigma\mu = d/\sigma d$. However, $d\sigma\tau d \in N_{E_1/F}E_1^\times$, since $\alpha\sigma\alpha\tau(\alpha\sigma\alpha) \in N_{E/F}E^\times$ and $\mu\sigma\mu \in N_{E_1/F}E_1^\times$. Hence $H^1 = F^\times/N_{E_1/F}E_1^\times$.

(III) $\rho(\,,\,) = \langle(14), (23)\rangle$, again $E = E_1E_2$ and $\mathrm{Gal}(E/F) = \mathbb{Z}/2 \times \mathbb{Z}/2$ is generated by σ and τ whose fixed fields are $E_3 = E^{\langle\sigma\rangle}, E_2 = E^{\langle\sigma\tau\rangle}$ and $E_1 = E^{\langle\tau\rangle}$, and $\rho(\tau) = (23), \rho(\tau\sigma) = (14)$. Using (1) above we may replace $\{a_\alpha\}$ by an equivalent cocycle with $a_{\tau_1} = I$, where $\tau_1 = \sigma\tau$. Then $a_\tau = \tau_1^*(a_\tau)$ (since $a_\tau\tau^*(a_{\tau_1}) = a_\sigma = a_{\tau_1}\tau_1^*(a_\tau)$), hence $b = \tau_1 b \in E_2^\times$ and $\lambda = a\tau_1 a \in N_{E/E_2}E^\times$. Further $I = a_\tau\tau^*(a_\tau)$ implies that $a\tau a = 1$ and $\lambda = b/\tau b$. Take $\alpha \in E^\times$ with $a = \alpha/\tau\alpha$ and $c = (\alpha, 1, \alpha\tau_1\alpha, \tau_1\alpha)$. Since $c = \tau_1^*(c)$ we can replace $\{a_\sigma\}$ by $\{a_\sigma c^{-1}\sigma^*(c)\}$, to assume that a_τ has $a = 1$. Thus $a_\tau = (1, b, 1/b, 1)$ and $b = \tau_1 b = \tau b$. Now taking $c = (\alpha, \beta, \alpha\tau_1\alpha/\beta, \tau_1\alpha)$ with $\alpha = \tau\alpha$ and $\beta = \tau_1\beta$, we have $c = \tau_1^* c$, and $c\tau^* c^{-1} = (1, \beta\tau\beta/\alpha\tau_1\alpha, \alpha\tau_1\alpha/\beta\tau\beta, 1)$. Since $\beta\tau\beta$ ranges over $N_{E_2/F}E_2^\times$ and $\alpha\tau_1\alpha$ over $N_{E_1/F}E_1^\times, E_1 \neq E_2$, and $F^\times/N_{E_i/F}E_i^\times$ have order two, we conclude that $a_\sigma = c\sigma^* c^{-1}$ for some $\alpha \in E_1^\times, \beta \in E_2^\times$, hence H^1 is trivial.

Remark. In the situation of (II) and (III), where E is the composition of the quadratic extensions of F, we have $N_{E/F}E^\times = F^{\times 2}$, hence $N_{E_3/F}$ followed by inclusion yields the isomorphism $E_3^\times/N_{E/E_3}E^\times \tilde{\to} N_{E_3/F}E_3^\times/F^{\times 2} \hookrightarrow F^\times/N_{E_1/F}E_1^\times$. Indeed, $N_{E/F}E^\times \supset N_{E/F}E_i^\times =$

$(N_{E_i/F}E_i^\times)^2$ implies $N_{E/F}E^\times \supset F^{\times 2}$, and $N_{E/F}E^\times = N_{E_i/F}N_{E/E_i}E^\times \subset N_{E_i/F}E_i^\times$ implies $N_{E/F}E^\times \subset F^{\times 2}$, since $N_{E_1/F}E_1^\times \cap N_{E_3/F}E_3^\times = F^{\times 2}$.

(IV) $\rho(\Gamma)$ contains an element of order 4. There are two cases here. If $\rho(\Gamma) = W$, then the splitting field E is a Galois extension of F with Galois group $W = D_4$. Suppose $\rho(\sigma_1) = (23)$ and $\rho(\sigma_2) = (14)$. As in (III), we can multiply the cocycle by a coboundary so that $a_{\sigma_1} = I = a_{\sigma_2}$, and so $a_{\sigma_1\sigma_2} = I = a_{\sigma_2\sigma_1}$. If $\rho(\sigma) = (3421), \rho(\sigma^2) = \rho(\sigma_1)\rho(\sigma_2)$, and $I = a_{\sigma^2} = a_\sigma\sigma^*(a_\sigma) = (a, b, \lambda/b, \lambda/a)(\sigma\lambda/\sigma b, \sigma a, \sigma\lambda/\sigma a, \sigma b)$. Then $b\sigma a = 1 = \lambda\sigma\lambda$, and $\sigma b = a\sigma\lambda$, thus $\lambda = a/\sigma b = a\sigma^2(a)$, and $a\sigma(a)\sigma^2(a)\sigma^3(a) = 1$, so that $a = \alpha/\sigma^3\alpha$ for some $\alpha \in E^\times$. Now $a_\sigma = (a, 1/\sigma a, 1/\sigma^3 a, \sigma^2 a)$, and c is equal to $\sigma^{*2}(c)$ (thus $a_{\sigma^2} = a_{\sigma^2}c\sigma^{*2}(c^{-1})$) if $c = (\alpha, \beta, \sigma^2\beta, \sigma^2\alpha)$ and $\alpha\sigma^2\alpha = \beta\sigma^2\beta$. As $c\sigma^*(c)^{-1} = (\alpha/\sigma^3\beta, \beta/\sigma\alpha, \sigma^2\beta/\sigma^3\alpha, \sigma^2\alpha/\sigma\beta)$, we have $a_\sigma = c\sigma^*(c)^{-1}$ for $\beta = \alpha$. Then H^1 is trivial.

The other case is when $\rho(\Gamma)$ is $\mathbb{Z}/4$, say $\rho(\sigma) = (3421)$. The splitting field E is a cyclic extension of F of degree 4. Put $E_3 = E^{\langle\sigma^2\rangle}$. By case (I), we may assume that $a_{\sigma^2} = (1, f, f^{-1}, 1), f \in E_3^\times/N_{E/E_3}E^\times$ (as $\rho(\sigma^2) = (14)(23)$). If $a_\sigma = (a, b, \lambda/b, \lambda/a)$ then $a_{\sigma^2} = a_\sigma\sigma^*(a_\sigma) = (a\sigma\lambda/\sigma b, b\sigma a, \lambda\sigma\lambda/b\sigma a, \lambda\sigma b/a)$. Hence $a = \sigma b/\sigma\lambda, \lambda\sigma\lambda = 1$, and $b\sigma a = f$. Hence $\sigma\lambda = \sigma b/a, \lambda = b/\sigma^3 a = f/\sigma(a)\sigma^3(a)$, and $a_\sigma = \big(a, f/\sigma a, 1/\sigma^3 a, f/a\sigma(a)\sigma^3(a)\big)$. The relation $a_{\sigma^2} = a_\sigma\sigma^*(a_\sigma)$ amounts to $f\sigma(f) = a\sigma(a)\sigma^2(a)\sigma^3(a)$, hence $f \in N_{E/E_3}E^\times$, we may assume $f = 1$, and we are done as in the case where $\rho(\Gamma)$ contains an element of order four. □

There is an easier way of computing the Galois cohomology groups above, using the Tate-Nakayama duality, which identifies $H^1(F, \mathbf{T}_H)$ with the Tate cohomology group $\hat{H}^{-1}\big(F, X_*(\mathbf{T}_H)\big)$. The group $X_*(\mathbf{T}_H)$ of cocharacters is $\{(x, y, z-y, z-x);\, x, y, z \in \mathbb{Z}\}$, and $\hat{H}^{-1}$ is the quotient of $\{X \in X_*(\mathbf{T}_H); NX = 0\}$, where N is the norm from a splitting field of F to F, by the span of $X - \sigma X, X \in X_*(\mathbf{T}_H), \sigma \in \Gamma$. Thus for example in case (IV), $NX = 0$ means $z = 0$, and $X - (3421)X = (x + y - z, y - x, x - y, z - x - y)$, hence $\hat{H}^{-1} = \{0\}$, while in case (I) again $NX = 0$ means $z = 0$, but $X - (14)(23)X = (2x - z, 2y - z, z - 2y, z - 2x)$, hence $\hat{H}^{-1} = \mathbb{Z}/2$. But for our integral evaluations we need to choose representatives for $\mathbb{Z}/2 = F^\times/NE^\times$, not only to know their cardinality.

A standard integration formula from the group to a Levi subgroup containing the torus in question, reduces the study of orbital integrals of regular elements to that of the study in the case of elliptic elements, and their centralizers, the elliptic tori. These are the cases (I – IV).

C. Explicit representatives.

We proceed to describe a set of representatives for $t \in T_H$ and for their stably conjugate but not conjugate elements.

Example. Case of $SL(2)$. As a preliminary example, let us consider the case of an elliptic torus $\mathbf{T}$ in $\mathbf{G} = SL(2)/F$ which splits over the quadratic extension $E = F(\sqrt{D})$ of F. If $\mathbf{T}^*$ is the diagonal torus, then a representative of such $\mathbf{T}$ is $\mathbf{T} = h_D^{-1}\mathbf{T}^*h_D, h_D = \left(\begin{smallmatrix} 1 & \sqrt{D} \\ 1 & -\sqrt{D}\end{smallmatrix}\right)$. Note that $h'_D = \mathrm{diag}(\|h_D\|^{-1}, 1)h_D$, where $\|h_D\| = \det h_D$, lies in $SL(2, E)$. If σ is the generator of $\mathrm{Gal}(E/F)$, then $\sigma(h_D) = h_D\boldsymbol{\varepsilon} = wh_D, \boldsymbol{\varepsilon} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right), w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)$. The elements of $\mathbf{T}$ are $t = h_D^{-1}ah_D (a \in \mathbf{T}^*)$, and we have $\sigma t = h_D^{-1}w\sigma(a)wh_D$, hence the action of σ on $\mathbf{T}$ induces the action $\sigma^*(a) = \mathrm{Int}(w)\big(\sigma(a)\big)$ on $\mathbf{T}^*$.

If $t, t_1 \in G$ are stably conjugate then $t_1 = g^{-1}tg = \sigma g^{-1} \cdot t \cdot \sigma g$, hence $g_\sigma = g\sigma(g)^{-1} = h_D^{-1} a_\sigma h_D$ lies in $\mathbf{T}$ $(= Z_{\mathbf{G}}(t); \sigma t = t$ and $\sigma t_1 = t_1$ since $t, t_1 \in G)$. Now $1 = g_\sigma \sigma(g_\sigma) = \mathrm{Int}(h_D^{-1})\big(a_\sigma w \sigma(a_\sigma) w\big) = a_\sigma \sigma(a_\sigma)^{-1}$, thus $a_\sigma = \mathrm{diag}(R, R^{-1})$ with $R = \sigma R \in F^\times$. Of course the cocycle g_σ or $a_\sigma \in \mathbf{T}^*$, can be modified by $c\sigma^*(c)^{-1} = (\gamma, \gamma^{-1})(\sigma\gamma, \sigma\gamma^{-1})$, hence R ranges over $F^\times / N_{E/F} E^\times$. The relation $g\sigma(g)^{-1} = h_D^{-1} a_\sigma h_D = h_D^{-1} a_\sigma w \sigma(h_D)$ implies

$$h_D g = a_\sigma w \sigma(h_D g) = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & R \\ R^{-1} & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \overline{x} & \overline{y} \\ \overline{z} & \overline{t} \end{smallmatrix}\right) = \left(\begin{smallmatrix} R\overline{z} & R\overline{t} \\ \overline{x}R^{-1} & \overline{y}R^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} R\overline{z} & R\overline{t} \\ z & t \end{smallmatrix}\right)$$

where we wrote $\overline{x}$ for σx. To have g of determinant 1 we note that $1 = ||g|| = -R(\overline{z}t - z\overline{t})/2\sqrt{D}$ has the solution $z = 1$ and $t = -\sqrt{D}/R$. Then

$$g = g_R = \frac{1}{2\sqrt{D}}\left(\begin{smallmatrix} \sqrt{D} & \sqrt{D} \\ 1 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} R & \sqrt{D} \\ 1 & -\sqrt{D}/R \end{smallmatrix}\right) = \frac{1}{2}\left(\begin{smallmatrix} R+1 & (R-1)\sqrt{D} \\ \frac{R-1}{\sqrt{D}} & R+1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & R^{-1} \end{smallmatrix}\right) \in SL(2, E).$$

Moreover,

$$t = \left(\begin{smallmatrix} a & bD \\ b & a \end{smallmatrix}\right), \quad t_1 = g^{-1}tg = \left(\begin{smallmatrix} 1 & 0 \\ 0 & R \end{smallmatrix}\right)\left(\begin{smallmatrix} a & bD \\ b & a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & R^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & bD/R \\ Rb & a \end{smallmatrix}\right)$$

make a complete set of representatives for the conjugacy classes within the stable conjugacy class of $t \in T \subset G$.

We shall next similarly describe representatives for the elliptic elements in $H = GSp(2, F)$, and for elements stably conjugate but not conjugate to these representatives.

Notation. Write $\left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)\right]$ for $\begin{pmatrix} a & 0 & 0 & b \\ 0 & \alpha & \beta & 0 \\ 0 & \gamma & \delta & 0 \\ c & 0 & 0 & \delta \end{pmatrix}$.

The tori $\mathbf{T}_H$ of $\mathbf{H} = GSp(2)$ of type (I) split over a quadratic extension $E = F(\sqrt{D})$ of F, whose Galois group is generated by σ.

1. Lemma. *A torus $\mathbf{T}_H$ of type* (I) *is given by*

$$\mathbf{T}_H = \widetilde{h}_D'^{-1} \mathbf{T}_H^* \widetilde{h}_D' = \{t = [\mathbf{a}, \mathbf{b}] = \widetilde{h}_D'^{-1}(a, b, \sigma b, \sigma a)\widetilde{h}_D';$$
$$\mathbf{a} = \left(\begin{smallmatrix} a_1 & a_2 D \\ a_2 & a_1 \end{smallmatrix}\right), \mathbf{b} = \left(\begin{smallmatrix} b_1 & b_2 D \\ b_2 & b_1 \end{smallmatrix}\right), \|\mathbf{a}\| = \|\mathbf{b}\|\},$$

where $a = a_1 + a_2\sqrt{D}, b = b_1 + b_2\sqrt{D}$, and $\widetilde{h}_D' = [h_D', h_D']$. Moreover $t_1 = \mathrm{Int}(\widetilde{g}^{-1})t = \mathrm{Int}\big([I, \left(\begin{smallmatrix} 1 & 0 \\ 0 & R \end{smallmatrix}\right)]\big)t, R \in F - N_{E/F}E$, is stably conjugate but not conjugate to t in H, where $\widetilde{g} = [I, g]$, and $g = g_R$ is as described in the example of $SL(2)$ above.

Proof. In the proof of Lemma B.2, case (I), we saw that if $t_1 = \widetilde{g}^{-1}t\widetilde{g}$ and t are stably conjugate then $\widetilde{g}_\sigma = \widetilde{g}\sigma(\widetilde{g})^{-1} = \widetilde{h}_D^{-1} a_\sigma \widetilde{h}_D$, with $\widetilde{h}_D = [h_D, h_D]$ and $a_\sigma = (1, R, R^{-1}, 1), R \in F^\times / N_{E/F} E^\times$. Since $\sigma(\widetilde{h}_D)\widetilde{h}_D^{-1} = \left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right)$, we need to solve the equation $\widetilde{h}_D \widetilde{g} = a_\sigma \left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right) \sigma(\widetilde{h}_D \widetilde{g})$ in $\widetilde{g} \in \mathbf{H}(E)$. Using the $g \in SL(2, E)$ found in the discussion of $SL(2)$ above, clearly $\widetilde{g} = \mathrm{diag}(1, g, 1)$ is a solution. □

The $\mathbf{H}$-tori $\mathbf{T}_H$ of type (II) and (III) split over a biquadratic extension $E = E_1 E_2$, $E_3 = F(\sqrt{A})$ is the fixed field of σ in E, $E_1 = F(\sqrt{D})$ is the fixed field of τ in E; $E_2 = F(\sqrt{AD})$ is assumed to be ramified over F, and A, D are normalized to be integral of minimal order such that E_1, E_2, E_3 are the three quadratic extensions of F.

2. Lemma. *A torus* $\mathbf{T}_H$ *of type* (III) *is given by*

$$\mathbf{T}_H = h^{-1}\mathbf{T}_H^* h = \{t = [\mathbf{a}, \mathbf{b}] = h^{-1}(a, b, \tau b, \sigma a)h;$$
$$\mathbf{a} = \left(\begin{smallmatrix} a_1 & a_2 D \\ a_2 & a_1 \end{smallmatrix}\right), \mathbf{b} = \left(\begin{smallmatrix} b_1 & b_2 AD \\ b_2 & b_1 \end{smallmatrix}\right), \|\mathbf{a}\| = \|\mathbf{b}\|\},$$

$a = a_1 + a_2\sqrt{D}, b = b_1 + b_2\sqrt{AD}, h = [h'_D, h'_{AD}]$.

Proof. By Lemma B.2, case III, the stable conjugacy class of such t consists of a single conjugacy class. □

3. Lemma. *A torus* $\mathbf{T}_H$ *of type* (II) *is given by* $\mathbf{T}_H = h^{-1}\mathbf{T}_H^* h$, *where* $h = \left(\begin{smallmatrix} h_A & 0 \\ 0 & \varepsilon h_A \varepsilon \end{smallmatrix}\right)\left(\begin{smallmatrix} I & \sqrt{D} \\ I & -\sqrt{D} \end{smallmatrix}\right)$. *It consists of* $t = \left(\begin{smallmatrix} \mathbf{a} & \mathbf{b}D \\ \mathbf{b} & \mathbf{a} \end{smallmatrix}\right), \mathbf{a} = \left(\begin{smallmatrix} a_1 & a_2 A \\ a_2 & a_1 \end{smallmatrix}\right), \mathbf{b} = \left(\begin{smallmatrix} b_1 & b_2 A \\ b_2 & b_1 \end{smallmatrix}\right)$, *and* $t = h^{-1}(t, \tau t, \sigma\tau t, \sigma t)h$, *where as a scalar* $t = a + b\sqrt{D}, \tau t = \tau a + \tau b\sqrt{D}, \sigma t = a - b\sqrt{D}$, *and* $t\sigma t \in F^\times$. *Take* $R \in E_3^\times$ *such that* ($N_{E_3/F}R \notin N_{E_1/F}E_1^\times$ *namely*) $R \notin N_{E/E_3}E^\times$. *If* $R = R_1 + R_2\sqrt{A}$, *put* $\mathbf{R} = \left(\begin{smallmatrix} R_1 & R_2 A \\ R_2 & R_1 \end{smallmatrix}\right)$. *Put* $g = g_R = \frac{1}{2}\left(\begin{smallmatrix} \mathbf{R}+I & (\mathbf{R}-I)\sqrt{D} \\ (\mathbf{R}-I)/\sqrt{D} & \mathbf{R}+I \end{smallmatrix}\right)\left(\begin{smallmatrix} I & 0 \\ 0 & \mathbf{R}^{-1} \end{smallmatrix}\right)$. *Then* g *lies in* $Sp(2, E)$, *and* $g^{-1}tg = \left(\begin{smallmatrix} \mathbf{a} & \mathbf{b}D\mathbf{R}^{-1} \\ \mathbf{R}\mathbf{b} & \mathbf{a} \end{smallmatrix}\right)$ *is stably conjugate but not conjugate to* t.

Proof. Since $\sigma(h)h^{-1} = \left(\begin{smallmatrix} h_A & 0 \\ 0 & \varepsilon h_A\varepsilon \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} h_A^{-1} & 0 \\ 0 & \varepsilon h_A^{-1}\varepsilon \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & w\varepsilon \\ \varepsilon w & 0 \end{smallmatrix}\right)$, we have $\rho(\sigma) = \sigma^* = (14)(23)$, indeed $\sigma(h^{-1}th) = h^{-1}h\sigma(h)^{-1}\sigma(t)\sigma(h)h^{-1}h$. Similarly, since $\tau(h)h^{-1} =$ $\left(\begin{smallmatrix} h_A\varepsilon h_A^{-1} & 0 \\ 0 & \varepsilon h_A\varepsilon h_A^{-1}\varepsilon \end{smallmatrix}\right) = \left(\begin{smallmatrix} w & 0 \\ 0 & -w \end{smallmatrix}\right)$, τ acts on $\mathbf{T}_H^*$ as $(12)(34)$. Then $\mathbf{T}_H = h^{-1}\mathbf{T}_H^* h, \mathbf{T}_H^* =$ diagonal subgroup, is indeed of type (II), and it consists of

$$h^{-1}(t, \tau t, \sigma\tau t, \sigma t)h = h_D^{-1}\left(\begin{smallmatrix} h_A^{-1}(t,\tau t)h_A & 0 \\ 0 & h_A^{-1}(\sigma t, \sigma\tau t)h_A \end{smallmatrix}\right)h_D = \left(\begin{smallmatrix} \mathbf{a} & \mathbf{b}D \\ \mathbf{b} & \mathbf{a} \end{smallmatrix}\right),$$

where

$$h_A^{-1}\left(\begin{smallmatrix} t & 0 \\ 0 & \tau t \end{smallmatrix}\right)h_A = \frac{1}{2}\left(\begin{smallmatrix} t+\tau t & (t-\sigma t)\sqrt{A} \\ \frac{t-\tau t}{\sqrt{A}} & t+\tau t \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1+b_1\sqrt{D} & (a_2+b_2\sqrt{D})A \\ a_2+b_2\sqrt{D} & a_1+b_1\sqrt{D} \end{smallmatrix}\right).$$

If $t_1 = g^{-1}tg$ is stably conjugate to t then $g_\sigma = g\sigma(g^{-1}) = h^{-1}a_\sigma h$ defines a cocycle which was analyzed in Lemma B.2, proof of case (II). Thus we can take $a_\tau = I$, and so $g_\tau = I$ and $\tau(g) = g$, $\tau(g_\sigma) = g_\sigma$, while $a_\tau = \tau^*(a_\sigma) = (R, \tau R, 1/\tau R, 1/R)$, with R ranging over $R = \sigma R \in E_3^\times/N_{E/E_3}E^\times$ (thus $N_{E_3/F}R$ does not lie in $N_{E_1/F}E_1^\times$ unless $R \in N_{E/E_3}E^\times$). Since $h = \left(\begin{smallmatrix} 0 & w\varepsilon \\ w\varepsilon & 0 \end{smallmatrix}\right)\sigma h$, we then need to solve the equation $g\sigma(g)^{-1} = h^{-1}a_\sigma\left(\begin{smallmatrix} 0 & w\varepsilon \\ \varepsilon w & 0 \end{smallmatrix}\right)\sigma(h)$, or $hg = a_\sigma\left(\begin{smallmatrix} 0 & w\varepsilon \\ \varepsilon w & 0 \end{smallmatrix}\right)\sigma(hg)$. The g in the statement of the lemma is a solution:

$$\frac{1}{2}\left(\begin{smallmatrix} h_A & 0 \\ 0 & \varepsilon h_A\varepsilon \end{smallmatrix}\right)h_D\left(\begin{smallmatrix} \mathbf{R}+I & (\mathbf{R}-I)\sqrt{D} \\ (\mathbf{R}-I)/\sqrt{D} & \mathbf{R}+I \end{smallmatrix}\right)$$
$$= \left(\begin{smallmatrix} 0 & (R,\tau R)w\varepsilon \\ \varepsilon(\tau R^{-1}, R^{-1})w & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} h_A & 0 \\ 0 & \varepsilon h_A\varepsilon \end{smallmatrix}\right)h_D\left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)\frac{1}{2}\left(\begin{smallmatrix} \mathbf{R}+I & -(\mathbf{R}+I)\sqrt{D} \\ -(\mathbf{R}-I)/\sqrt{D} & \mathbf{R}+I \end{smallmatrix}\right)$$

since

$$h_A^{-1}\left(\begin{smallmatrix} R & 0 \\ 0 & \tau R \end{smallmatrix}\right)wh_A = \mathbf{R}\varepsilon, \text{ and } \left(\begin{smallmatrix} \mathbf{R} & \mathbf{R}\sqrt{D} \\ I & -\sqrt{D} \end{smallmatrix}\right) = \left(\begin{smallmatrix} I & 0 \\ 0 & \varepsilon \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & \mathbf{R}\varepsilon \\ \tau\mathbf{R}^{-1}\varepsilon & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} I & 0 \\ 0 & \varepsilon \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathbf{R} & \mathbf{R}\sqrt{D} \\ I & -\sqrt{D} \end{smallmatrix}\right)\left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$$

(and $\boldsymbol{\varepsilon}\mathbf{R}^{-1} = \tau\mathbf{R}^{-1}\cdot\boldsymbol{\varepsilon}$). Finally note that t lies in $GSp(2,F)$ when

$$t\theta(t^{-1}) = \begin{pmatrix} \mathbf{a} & \mathbf{b}D \\ \mathbf{b} & \mathbf{a} \end{pmatrix}\begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix}\begin{pmatrix} {}^t\mathbf{a} & {}^t\mathbf{b} \\ D{}^t\mathbf{b} & {}^t\mathbf{a} \end{pmatrix}\begin{pmatrix} 0 & -w \\ w & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{a}^2-\mathbf{b}^2D & (\mathbf{ba}-\mathbf{ab})D \\ \mathbf{ba}-\mathbf{ab} & \mathbf{a}^2-\mathbf{b}^2D \end{pmatrix}$$

is a scalar in $F^\times$ (note that $w{}^t\mathbf{a}w = \mathbf{a}$), thus $t\sigma(t) \in F^\times$. □

A torus $\mathbf{T}_H$ of type (IV) is associated with a quadratic extension $F(\sqrt{D}) = E_3(\sqrt{D})$ of $E_3 = F(\sqrt{A})$, where $D = \alpha + \beta\sqrt{A} \in E_3$ and $A \in F - F^2$. The extension $E_3(\sqrt{D})/F$ is cyclic or non Galois, and the group of field homomorphisms $E_3(\sqrt{D}) \to \overline{F}$ over F is generated by σ, which maps $\sigma\sqrt{A} = -\sqrt{A}$, and $\sigma\sqrt{D} = \sqrt{\sigma D}, \sigma^2\sqrt{D} = -\sqrt{D}, \sigma^3\sqrt{D} = -\sqrt{\sigma D}$. Then E_3 is the fixed field of σ^2 in $E_3(\sqrt{D})$.

4. Lemma. *A torus $\mathbf{T}_H$ of type (IV) is given by $\mathbf{T}_H = h^{-1}\mathbf{T}_H^*h$, $h = (-4\sqrt{AD}, 4\sqrt{A\sigma D}, w)^{-1}$ $\widetilde{h}_D\begin{pmatrix} h_A & 0 \\ 0 & h_A \end{pmatrix}, \widetilde{h}_D = (23)\left(\begin{pmatrix} h_D & 0 \\ 0 & \sigma h_D \end{pmatrix}\right)$. It consists of $\begin{pmatrix} \mathbf{a} & \mathbf{bD} \\ \mathbf{b} & \mathbf{a} \end{pmatrix} = h^{-1}(t, \sigma t, \sigma^3 t, \sigma^2 t)h$, $t \in F(\sqrt{D})$ with $t\sigma^2 t = \sigma t\sigma^3 t$. Here, if $t = a + b\sqrt{D}, a = a_1 + a_2\sqrt{A}$, then $\mathbf{a} = \begin{pmatrix} a_1 & a_2A \\ a_2 & a_1 \end{pmatrix}\big($ and $\mathbf{b} = \begin{pmatrix} b_1 & b_2A \\ b_2 & b_1 \end{pmatrix}$ if $b = b_1 + b_2\sqrt{A}\big)$, and $\sigma t = \sigma a + \sigma b\sqrt{\sigma D}$, $\sigma^2 t = a - b\sqrt{D}, \sigma^3 t = \sigma a - \sigma b\sqrt{\sigma D}$.*

Proof. Note that $\sigma(h_A) = wh_A$, hence $\sigma(h_A)h_A^{-1} = w$. Then $\sigma(h)h^{-1}$ is equal to

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/4\sqrt{AD} \\ -4\sqrt{AD} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

If $t = h^{-1}t^*h$, then $\sigma(t) = h^{-1}\cdot h\sigma(h)^{-1}\cdot\sigma(t^*)\cdot\sigma(h)h^{-1}\cdot h$, and so the induced action on the diagonal subgroup $\mathbf{T}_H^*$ is $\sigma^*(a,b,c,d) = (\sigma c, \sigma a, \sigma d, \sigma b)$, thus $\sigma = (3421)$, and $\mathbf{T}_H^*(F) = \{(t, \sigma t, \sigma^3 t, \sigma^2 t)\}$. Stable conjugacy reduces to conjugacy in case (IV). □

D. Stable θ-conjugacy.

Similarly, we describe the (F-rational) θ-conjugacy classes within the stable ($\overline{F}$-) θ-conjugacy class of a strongly θ-regular element t in G. Fix a θ-invariant F-torus $\mathbf{T}^*$; in fact we take $\mathbf{T}^*$ to be the diagonal subgroup. The stable θ-conjugacy class of t in G intersects $\mathbf{T}^*$ ([KS, Lemma 3.2.A]). Hence there is $h \in \mathbf{G}$ and $t^* \in \mathbf{T}^*$, such that $t = h^{-1}t^*\theta(h)$. The centralizers are related by $Z_{\mathbf{G}}(t\theta) = h^{-1}Z_{\mathbf{G}}(t^*\theta)h$. Further $Z_{\mathbf{G}}(t^*\theta) = \mathbf{T}^{*\theta}$, the centralizer of $Z_{\mathbf{G}}(t\theta)$ in $\mathbf{G}$ is an F-torus $\mathbf{T}$ which is $\theta_t = \mathrm{Int}(t)\circ\theta$ invariant, and $Z_{\mathbf{G}}(t\theta) = \mathbf{T}^{\theta_t}$. The θ-conjugacy classes within the stable θ-conjugacy class of t can be classified as follows.

(1) Suppose that $t_1 = g^{-1}t\theta(g)$ and t are stably θ-conjugate in G. Then $g_\sigma = g\sigma(g)^{-1} \in Z_G(t\theta) = T^{\theta_t}$. The set $D(F,\theta,t) = \ker[H^1(F,\mathbf{T}^{\theta_t}) \to H^1(F,\mathbf{G})]$ parametrizes, via $(t_1,t) \mapsto \{\sigma \mapsto g_\sigma\}$, the θ-conjugacy classes within the stable θ-conjugacy class of t. The Galois action on $\mathbf{T}, \sigma(t) = \sigma\big(h^{-1}t^*\theta(h)\big) = h^{-1}\cdot h\sigma(h)^{-1}\cdot\sigma(t^*)\cdot\theta\big(\sigma(h)h^{-1}\big)\theta(h)$ induces a Galois action σ^* on $\mathbf{T}^*$, given by $\sigma^*(t^*) = h\sigma(h)^{-1}\sigma(t^*)\theta\big(\sigma(h)h^{-1}\big)$, and $H^1(F,\mathbf{T}^{\theta_t}) = H^1(F,\mathbf{T}^{*\theta})$.

(2) The norm map $N : \mathbf{T}^* \to \mathbf{T}_H^*$ factorizes via the projection $\mathbf{T}^* \to \mathbf{T}^*/\mathbf{V}, \mathbf{V} = (1-\theta)\mathbf{T}^*$, and the isomorphism $\mathbf{U} = \mathbf{T}_\theta^* = \mathbf{T}^*/\mathbf{V} \tilde{\to} \mathbf{T}_H^*$. Suppose that the norm Nt^* of $t^* \in \mathbf{T}^*$ is defined over F. Then for each $\sigma \in$, there is $\ell \in \mathbf{T}^*$ such that $\sigma^*(t^*) = \ell t^*\theta(\ell)^{-1}$. Then

$$h^{-1}t^*\theta(h) = t = \sigma(t) = \sigma h^{-1}\cdot\sigma t^*\cdot\theta(\sigma h) = \sigma(h)^{-1}\ell t^*\theta\big(\ell^{-1}\sigma(h)\big),$$

hence

$$t^* = h_\sigma \ell \cdot t^* \cdot \theta(h_\sigma \ell)^{-1}, \;\; h_\sigma = h\sigma(h)^{-1},$$

and $h_\sigma \ell \in Z_{\mathbf{G}}(t^*\theta) = \mathbf{T}^{*\theta}$, so that $h_\sigma \in \mathbf{T}^*$. Moreover, $(1-\theta)(h_\sigma) = t^*\sigma(t^*)^{-1}$. Hence (h_σ, t^*) lies in $H^1(F, \mathbf{T}^* \xrightarrow{1-\theta} \mathbf{T}^*)$, in a subset isomorphic to $H^1(F, \mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V})$; this invariant parametrizes the (strongly θ-regular) θ-conjugacy classes which have the same norm. See [KS, Appendix A], or Section G below, for a definition and properties of these hypercohomology groups; the lines preceding [KS, Lemma 6.3.A], for the definition of $\mathrm{obs}(\delta)$; [KS, 6.2], for the definition of $\mathrm{inv}'(\delta, \delta')$; and [KS, page prior to Theorem 5.1D], for the definition of $\mathrm{inv}(\delta, \delta')$: if $t_1 = g^{-1}t\theta(g)$ as in (1) above, then $\mathbf{T}_t = Z_{\mathbf{G}}(Z_{\mathbf{G}}(t\theta)^0)$ is a maximal torus in $\mathbf{G}$. Denote its inverse image under the natural homomorphism $\pi : \mathbf{G}_{sc} \to \mathbf{G}$ by $\mathbf{T}_t^{sc}$ ($\mathbf{G}_{sc}$ is the simply connected covering F-group of the derived group of $\mathbf{G}$), and write $g = \pi(g_1)z$, g_1 in $\mathbf{G}_{sc}$, z in $Z(\mathbf{G})$. Then $\sigma(g_1)g_1^{-1}$ lies in $\mathbf{T}_t^{sc}$, $(1-\theta_t)\pi(\sigma(g_1)g_1^{-1}) = \sigma(b)b^{-1}$, where $b = \theta(z)z^{-1} = (1-\theta_t)(z^{-1}) \in \mathbf{V}_t = (1-\theta_t)(\mathbf{T}_t)$. Hence $(\sigma \mapsto \sigma(g_1)g_1^{-1}, b)$ defines the element $\mathrm{inv}(t, t_1)$ of $H^1(F, \mathbf{T}_t^{sc} \xrightarrow{(1-\theta_t)\circ\pi} \mathbf{V}_t)$. It parametrizes the θ-conjugacy classes under G_{sc} within the stable θ-conjugacy class of t. The image in $H^1(F, \mathbf{T}_t \xrightarrow{1-\theta_t} \mathbf{V}_t)$, under the map $[\mathbf{T}_t^{sc} \to \mathbf{V}_t] \to [\mathbf{T}_t \to \mathbf{V}_t]$ (induced by $\pi : \mathbf{T}_t^{sc} \to \mathbf{T}_t$), is denoted $\mathrm{inv}'(t, t_1)$. It parametrizes the θ-conjugacy classes within the stable θ-conjugacy class of t, as noted in (1) above.

Note that there is an exact sequence

$$H^0(F, \mathbf{T}^*) = \mathbf{T}^{*\Gamma} = T^* \xrightarrow{1-\theta} H^0(F, \mathbf{V}) = V \to H^1(F, \mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V}) \to H^1(F, \mathbf{T}^*) \xrightarrow{1-\theta} H^1(F, \mathbf{V}).$$

Moreover, the exact sequence $1 \to \mathbf{T}^{*\theta} \to \mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V} \to 1$ induces the exact sequence

$$H^0(F, \mathbf{T}^*) \xrightarrow{1-\theta} H^0(F, \mathbf{V}) \to H^1(F, \mathbf{T}^{*\theta}) \to H^1(F, \mathbf{T}^*) \xrightarrow{1-\theta} H^1(F, \mathbf{V}).$$

Hence, $H^1(F, \mathbf{T}^{*\theta}) = H^1(F, \mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V})$ and $D(F, \theta, t)$ is $\ker[H^1(F, \mathbf{T}^{*\theta}) \to H^1(F, \mathbf{G})] \simeq \ker[H^1(F, \mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V}) \to H^1(F, \mathbf{G})]$.

In our case the group $H^1(F, \mathbf{G})$ is trivial $\big(\mathbf{G} = GL(4) \times GL(1)\big)$, and so is $H^1(F, \mathbf{T}^*)$. Hence $D(F, \theta, t) = H^1(F, \mathbf{T}^{*\theta}) = H^1(F, \mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V}) = V/(1-\theta)T^*$. The θ-invariant F-tori $\mathbf{T}$ determine homomorphisms $\rho : , \to W(\mathbf{T}^{*\theta}, \mathbf{G}^\theta) = W(\mathbf{T}^*, \mathbf{G})^\theta$. We proceed to describe a set of representatives for the F-tori $\mathbf{T}$ in $\mathbf{G}$, and the groups $H^1(F, \mathbf{T}^* \to \mathbf{V}) = H^1(F, \mathbf{T}^{*\theta})$ which parametrize the θ-conjugacy classes within the stable θ-conjugacy classes of strongly θ-regular elements in G, which are represented by elements of T. Since $W(\mathbf{T}^*, \mathbf{G})^\theta = W(\mathbf{T}_H^*, \mathbf{H})$, our list of θ-invariant tori $\mathbf{T}$ is obtained from the list of tori $\mathbf{T}_H$, where $\mathbf{T}$ is the centralizer of $\mathbf{T}_H$.

A useful fact would be that we can choose $h \in \mathbf{G}$ such that $\theta(h) = h$. Then the stable θ-conjugacy classes of strongly θ-regular elements are represented by $t = h^{-1}t^*\theta(h) = h^{-1}t^*h, t^* \in \mathbf{T}^*$, and we also exhibit a complete list of representatives for the θ-conjugacy classes within the stable θ-conjugacy class of such a strongly θ-regular element t.

The following is a list of the θ-invariant F-tori in $\mathbf{G}$ up to F-isomorphism; they are parametrized by the homomorphisms $\rho : , \to W = W(\mathbf{T}^{*\theta}, \mathbf{G}^\theta) = W(\mathbf{T}^*, \mathbf{G})^\theta$. Note that $\mathbf{G}^\theta = Sp(2)$. Further we compute $H^1(F, \mathbf{T}^{*\theta}) = H^1(F, \mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V})$, we give an explicit realization of $\mathbf{T} = h^{-1}\mathbf{T}^*h$ (and $h = \theta(h)$), and for $t \in T$, strongly θ-regular, a set of representatives in G for the θ-conjugacy classes in the stable θ-conjugacy class of t. Note that the only significant difference from the non twisted case is that we work with $\mathbf{G}^\theta = Sp(2)$ instead of with $\mathbf{H} = GSp(2)$.

Let us clarify that $t \in G$ is strongly θ-regular means that $t = h^{-1}t^*\theta(h), h \in \mathbf{G}$, where t^* is such that $Z_{\mathbf{G}}(t^*\theta)$ is $\mathbf{T}^{*\theta}$. Then $Z_{\mathbf{G}}(t\theta) = h^{-1}Z_{\mathbf{G}}(t^*\theta)h$ is the torus $\mathbf{T}^{\mathrm{Int}(t)\circ\theta}$, where $\mathbf{T}$ is $Z_{\mathbf{G}}\big(Z_{\mathbf{G}}(t\theta)\big)$, an $\mathrm{Int}(t)\circ\theta$-invariant maximal torus in $\mathbf{G}$. If $u = h^{-1}u^*h \in T$, where $u^* \in T^{*\theta}_{\mathrm{reg}}$, then $h_\sigma\sigma(u^*)h_\sigma^{-1} = u^* = \theta(u^*) = \theta(h_\sigma)\sigma(u^*)\theta(h_\sigma)^{-1}$ implies that $h_\sigma = h\sigma(h^{-1})$ is a θ-invariant element in the Weyl group $W(\mathbf{T}^*, \mathbf{G})$ of $\mathbf{T}^*$, hence it can be represented by an element of $W = W(\mathbf{T}^{*\theta}, \mathbf{G}^{*\theta})$, and the tori $\mathbf{T}$ in $\mathbf{G}$ so obtained define $\rho : \,, \to W$. Hence we consider the centralizers of the tori in $\mathbf{G}^{*\theta}$.

As in the case of $\mathbf{H} = GSp(2)$, we denote by $\tilde{E}$ a minimal splitting field for the torus $\mathbf{T}$ in $\mathbf{G}$. The torus $\mathbf{T}$ is associated with a homomorphism $\rho : \,, = \mathrm{Gal}(\tilde{E}/F) \to W$. Usually $\tilde{E}$ is E. Recall: $\mathbf{V} = \{(\alpha, \beta, \beta, \alpha; 1/\alpha\beta)\}$.

(1) When $\rho(,\,) = \langle(12)(34)\rangle, [E : F] = 2, T^* = \mathbf{T}^*(F)$ consists of $\{(a, \sigma a, b, \sigma b; e); a, b \in E^\times, e \in F^\times\}$, where σ generates $\mathrm{Gal}(E/F)$. Then $V = \mathbf{V}(F)$ consists of $\{(\alpha, \sigma\alpha, \sigma\alpha, \alpha;\, 1/\alpha\sigma\alpha); \alpha \in E^\times\}$, and $(1-\theta)T^* = \{(a\sigma b, b\sigma a, b\sigma a, a\sigma b; 1/a\sigma ab\sigma b); a, b \in E^\times\}$. Hence $H^1(T^* \to V) = V/(1-\theta)T^*$ is $\{1\}$. Further, $T^{*\theta} = \{(a, \sigma a, 1/\sigma a, 1/a; e); a \in E^\times, e \in F^\times\}$. Hence $H^1(T^{*\theta}) = \hat{H}^{-1}(T^{*\theta})$ is

$$\{X = (x, y, -y, -x; z); X + \sigma X = 0, \text{i.e.}: x + y = 0 = z\}/\langle X - \sigma X = (x - y, y - x, \dots; 0)\rangle = \{0\}.$$

Similarly, if $\rho(,\,) = \langle(13)(24)\rangle$, then $T^* = \{(a, b, \sigma a, \sigma b; e); a, b \in E^\times, e \in F^\times\}, V = \{(a, b, \sigma a = b, \sigma b = a; 1/a\sigma a)\}$, and $(1-\theta)T^* = \{(a\sigma b, b\sigma a, b\sigma a, a\sigma b; 1/ab\sigma a\sigma b)\}$, so that $H^1(T^* \to V) = \{1\}$. Further, $T^{*\theta} = \{(a, 1/\sigma a, \sigma a, 1/a; e)\}$ and $H^1(T^{*\theta}) = \hat{H}^{-1}(T^{*\theta})$ consists of

$$\{X = (x, y, -y, -x; z); x - y = 0 = z\}/\langle(x, y, -y, -x; z) - (-y, -x, x, y; z)\rangle = \{0\}.$$

(2) $\rho(,\,) = \langle(23)\rangle, [E : F] = 2, T^* = \{(a, b, \sigma b, d; e); b \in E^\times, a, d, e \in F^\times\}, V = \{(a, b, b, a; 1/ab)\}, (1-\theta)T^* = \{(ad, b\sigma b, b\sigma b, ad; 1/adb\sigma b)\}$, so that $H^1(T^* \to V) = F^\times/N_{E/F}E^\times$. Further, $T^{*\theta}$ is $\{(a, b, \sigma b = 1/b, 1/a; e); b \in E^\times, a, e \in F^\times\}$, and $H^1(T^{*\theta})$ is the quotient of $\{x = (a, b, 1/b, 1/a; e); x\sigma x = 1$, i.e., $a\sigma a = 1, e\sigma e = 1, \sigma b = b; a, b, e \in E^\times\}$ by $\{x\sigma(x)^{-1}\}$, thus it is $F^\times/NE^\times$, by Hilbert Theorem 90.

(3) The analogous result holds when $\rho(,\,) = \langle(14)\rangle : H^1(T^{*\theta}) = \{x = (a, 1, 1, 1/a; 1); a \in F^\times/NE^\times\}$.

The tori T of (1), (2), (3) are not θ-anisotropic, namely T^θ contains the split torus $\{(z, z, 1/z, 1/z; 1)\}$ $\big($ and $\{(z, 1/z, z, 1/z; 1)\}\big)$, $\{(z, 1, 1, 1/z; 1)\}$ and $\{(1, z, 1/z, 1; 1)\}$, and $Z(G^\theta) = \{(\pm I, t); t \in F^\times\}$, as the center of $Sp(2, F)$ is $\{\pm I\}$.

In case (2) the torus $\mathbf{T}$ can be presented as $\mathbf{T} = h^{-1}\mathbf{T}^*h, h = [I, h'_D]$, if $E = F(\sqrt{D}), D \in F - F^2$, and $h'_D = \left(\begin{smallmatrix} \|h_D\|^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)h_D, h_D = \left(\begin{smallmatrix} 1 & \sqrt{D} \\ 1 & -\sqrt{D} \end{smallmatrix}\right)$. Then $h = \theta(h)$, and $\mathrm{Int}\,\big(h\sigma(h)^{-1}\big) = (23), T$ consists of $t = \left(\left[\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right), \mathbf{b}\right], e\right), \mathbf{b} = \left(\begin{smallmatrix} b_1 & b_2D \\ b_2 & b_1 \end{smallmatrix}\right)$ if $b = b_1 + b_2\sqrt{D}$, and a stably θ-conjugate but not θ-conjugate element to t is given by

$$g^{-1}tg, \; g = [I, g_R], \; g_R = \frac{1}{2}\begin{pmatrix} R+1 & (R-1)\sqrt{D} \\ (R-1)/\sqrt{D} & R+1 \end{pmatrix}\left(\begin{smallmatrix} 1 & 0 \\ 0 & R^{-1} \end{smallmatrix}\right),$$

thus $\mathbf{b}$ of t is replaced by $\left(\begin{smallmatrix} b_1 & b_2D/R \\ b_2R & b_1 \end{smallmatrix}\right)$.

In case (3), $g=[g_R,I]$, where $R\in F-N_{E/F}E$, and $T=\{\left(\left[\mathbf{b},\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)\right],e\right);ad\cdot b\sigma b=1\}$.

The θ-elliptic tori are the following.

(I) $\rho(,)=\langle(14)(23)\rangle,[E:F]=2,T^*=\{(a,b,\sigma b,\sigma a;e);a,b\in E^\times,e\in F^\times\},(1-\theta)T^*=\{(a\sigma a,b\sigma b,b\sigma b,a\sigma a;1/a\sigma ab\sigma b)\}$, and $V=\{(a,b,\sigma b=b,\sigma a=a;1/ab)\}$. Hence $H^1(T^*\to V)=F^\times/NF^\times\times F^\times/NF^\times$. Further, $T^{*\theta}=\{(a,b,\sigma b=1/b,\sigma a=1/a;e);a,b\in E^\times,e\in F^\times\}$, and $H^1(T^{*\theta})$ is the quotient of $\{x=(a,b,1/b,1/a;e);x\sigma x=1$, thus $e\sigma e=1$, and $a=\sigma a,b=\sigma b$, in $F^\times\}$ by $\{x\sigma(x)^{-1}=(a\sigma a,b\sigma b,\dots;e/\sigma e)\}$, thus it is $(F^\times/NE^\times)^2$.

In case (I), $\mathbf{T}=h^{-1}\mathbf{T}^*h$, where $h=[h'_D,h'_D]$, consists of $([\mathbf{a},\mathbf{b}],e),\mathbf{a}=\left(\begin{smallmatrix} a_1 & a_2D \\ a_2 & a_1 \end{smallmatrix}\right)$ if $a=a_1+a_2\sqrt{D}$ in $E^\times$, and $a\sigma a\cdot b\sigma b=1$, and representatives for the θ-conjugacy classes within the stable θ-conjugacy class of t are given by $t_1=g^{-1}tg,g=[g_R,g_S]$, where R,S range over $F^\times/N_{E/F}E^\times$. Then t_1 is obtained from t on replacing $\mathbf{a}$ by $\left(\begin{smallmatrix} a_1 & a_2DR \\ a_2/R & a_1 \end{smallmatrix}\right)$ and $\mathbf{b}$ by $\left(\begin{smallmatrix} b_1 & b_2DS \\ b_2/S & b_1 \end{smallmatrix}\right)$.

(II) $\rho(,)=\langle\rho(\sigma\tau)=(14),\rho(\tau)=(23)\rangle$, the splitting field of T is $E=E_1E_2$, where $E_1=F(\sqrt{D}),E_2=F(\sqrt{AD}),E_3=F(\sqrt{A})$ are the different quadratic extensions of F. The extension E_2/F is assumed to be ramified, and $\mathrm{Gal}(E/F)$ is generated by $\sigma,\tau,\sigma\tau$ whose fixed fields are $E_1=E^{\langle\tau\rangle}$, $E_2=E^{\langle\sigma\tau\rangle}$, $E_3=E^{\langle\sigma\rangle}$. Then $T^*=\{(a,b,\tau b,\sigma a;e);a\in E_1^\times,b\in E_2^\times,e\in F^\times\},V=\{(a,b,\tau b=b,\sigma a=a;1/ab);a,b\in F^\times\}$, and $(1-\theta)T^*=\{(a\sigma a,b\tau b,b\tau b,a\sigma a;1/a\sigma ab\tau b)\}$. Hence $H^1(\mathbf{T}^*\to\mathbf{V})=F^\times/N_{E_1/F}E_1^\times\times F^\times/N_{E_2/F}E_2^\times$. Further, $T^{*\theta}=\{(a,b,\tau b=1/b,\sigma a=1/a;e);a\in E_1^\times,b\in E_2^\times,e\in F^\times\}$, and additively, $H^1(\mathbf{T}^{*\theta})$ is the quotient of $\{(x,y,-y,-x;0)\}$ by $\langle X-\sigma X=(2x,0,0,-2x;0),X-\tau X=(0,2y,-2y,0;0)\rangle$, namely it is $\mathbb{Z}/2\times\mathbb{Z}/2$.

Let us compute $H^1(T^{*\theta})$ explicitly. Consider a cocycle $\{a_\sigma\}$. If $a_{\sigma\tau}=(a,b,1/b,1/a;e)$ and $a_\tau=(c,d,1/d,1/c;f)$, then $a_\sigma\sigma^*(a_\sigma)=1$ implies $b\sigma\tau b=1,e\sigma\tau e=1,c\tau c=1$, hence $b=\beta/\sigma\tau\beta,e=\varepsilon/\sigma\tau\varepsilon,c=\gamma/\tau\gamma$, and $g=(\gamma,\beta,\beta^{-1},\gamma^{-1};\varepsilon)$ has the property that the cocycle $\{a_\sigma g^{-1}\sigma(g)\}$, renamed $\{a_\sigma\}$, has $a_{\sigma\tau}=(a,1,1,a^{-1};1)$ and $a_\tau=(1,b,b^{-1},1;e)$, where $a=\sigma\tau a,b=\tau b,e\tau e=1$. The relation $a_{\sigma\tau}\sigma\tau^*(a_\tau)=a_\tau\tau^*(a_{\sigma\tau})$ implies $a=\tau a,b=\sigma\tau b,e=\sigma\tau e$. Hence $e=\varepsilon/\tau\varepsilon,\varepsilon=\sigma\tau\varepsilon\in E_2^\times$, and $a,b\in F^\times$. If $g=(\alpha,\beta,\beta^{-1},\alpha^{-1};\varepsilon)$, with $\alpha\in E_1^\times$, $\beta\in E_2^\times$, then $g\sigma\tau g^{-1}=(\alpha\sigma\alpha,1,1,1/\alpha\sigma\alpha;1)$ and $g\tau g^{-1}=(1,\beta\tau\beta,1/\beta\tau\beta,1;e)$. Hence the class of the cocycle $\{a_\sigma\}$ is determined by $a\in F^\times/N_{E_1/F}E_1^\times$, $b\in F^\times/N_{E_2/F}E_2^\times$, and $H^1(\mathbf{T}^{*\theta})=F^\times/N_{E_1/F}E_1^\times\times F^\times/N_{E_2/F}E_2^\times$.

The torus $\mathbf{T}$ is $\mathbf{T}=h^{-1}\mathbf{T}^*h,h=[h'_D,h'_{AD}]$ if $E_1=F(\sqrt{D}),E_2=F(\sqrt{AD})$, and T consists of $t=h^{-1}t^*h=([\mathbf{a},\mathbf{b}],e),\mathbf{a}=\left(\begin{smallmatrix} a_1 & a_2D \\ a_2 & a_1 \end{smallmatrix}\right),\mathbf{b}=\left(\begin{smallmatrix} b_1 & b_2AD \\ b_2 & b_1 \end{smallmatrix}\right)$, if $a=a_1+a_2\sqrt{D},b=b_1+b_2\sqrt{AD}$. Here $t^*=(a,b,\tau b,\sigma a;e),a\in E_1^\times,b\in E_2^\times,e\in F^\times$. A complete set of representatives for the θ-conjugacy classes within the stable θ-conjugacy class of t is given by $g^{-1}tg$, where $g=[g_R,g_S],R\in F^\times/N_{E_1/F}E_1^\times,S\in F^\times/N_{E_2/F}E_2^\times$. Note that $g=\theta(g)$, and that $g^{-1}tg$ is obtained from t on replacing $\mathbf{a}$ by $\left(\begin{smallmatrix} a_1 & a_2DR \\ a_2/R & a_1 \end{smallmatrix}\right)$ and $\mathbf{b}$ by $\left(\begin{smallmatrix} b_1 & b_2ADS \\ b_2/S & b_1 \end{smallmatrix}\right)$.

(III) $\rho(,)=\langle\rho(\tau)=(12)(34),\rho(\sigma)=(14)(23)\rangle$, the splitting field of T is $E=E_1E_2$, where E_1,E_2,E_3 are the three quadratic extensions of F, $\mathrm{Gal}(E/F)$ is generated by σ and τ, of order two, with $E_3=E^{\langle\sigma\rangle}=F(\sqrt{A})$, $E_1=E^{\langle\tau\rangle}=F(\sqrt{D})$. Then $T^*=\{(a,\tau a,\tau\sigma a,\sigma a;e);a\in E^\times,e\in F^\times\},V=\{(a,\tau a,\tau\sigma a=\tau a,\sigma a=a;1/a\tau a);a\in E_3^\times\}$ and $(1-\theta)T^*=\{(a\sigma a,\tau a\tau\sigma a,\dots;$

$1/a\tau a\sigma a\sigma\tau a); a\in E^\times\}$. Then $H^1(\mathbf{T}^*\to\mathbf{V})=E_3^\times/N_{E/E_3}E^\times=F^\times/N_{E_1/F}E_1^\times$ (see Remark in Section B). Further, $T^{*\theta}$ is $\{(a,\tau a,\tau\sigma a=1/\tau a,\tau a=1/a;e); a\in E^\times\}$, and additively, $H^1(\mathbf{T}^{*\theta})$ is the quotient of $\{(x,y,-y,-x;0)\}$ by $\langle(x-y,y-x,\dots),(2x,2y,\dots)\rangle=\langle(x,y,\dots); x\equiv y \bmod 2\rangle$, namely it is $\mathbb{Z}/2$.

To compute $H^1(\mathbf{T}^{*\theta})$ directly, let $\{a_\sigma\}$ be a cocycle. Then $a_\sigma=(a_1,a_2,a_2^{-1},a_1^{-1};e)$, $a_\tau=(b_1,b_2,b_2^{-1},b_1^{-1};f)$. The relation $1=a_\tau\tau^*(a_\tau)$ implies that $b_1\tau b_2=1$, and $f\tau f=1$, thus $f=\varepsilon/\tau\varepsilon$, and $a_\tau=b^{-1}\tau^*(b)$, where $b=(b_1^{-1},1,\alpha,b_1;\varepsilon^{-1})$. We replace a_σ by $a_\sigma b\sigma^*(b^{-1})$, to get $a_\tau=I$. Then $a_\sigma=\tau^*(a_\sigma)$, so $a_\sigma=(a_1,\tau a_1,\tau a_1^{-1},a_1^{-1};e)$, $e=\tau e$. The relation $I=a_\sigma\sigma^*(a_\sigma)$, implies that $a_1=\sigma a_1\in E_3^\times$ and $e\sigma e=1$. Replacing a_σ by $a_\sigma c\sigma^*(c^{-1})$ with $c=\tau^*(c)=(\alpha,\tau\alpha,\tau\alpha^{-1},\alpha^{-1};\varepsilon),\varepsilon\in E_1^\times$ with $e=\varepsilon/\sigma\varepsilon$, we see that the class of $\{a_\sigma\}$ is determined by $a_1\in E_3^\times/N_{E/E_3}E^\times$.

The torus $\mathbf{T}=h^{-1}\mathbf{T}^*h$ is defined by $h=\left(\begin{smallmatrix}\gamma&0\\0&1\end{smallmatrix}\right)h'$, where $\gamma=1/4\sqrt{AD}$ and

$$h'=\left(\begin{smallmatrix}h_A&0\\0&\boldsymbol{\varepsilon}h_A\boldsymbol{\varepsilon}\end{smallmatrix}\right)\left(\begin{smallmatrix}I&\sqrt{D}\\I&-\sqrt{D}\end{smallmatrix}\right)$$

is the h used in the Lemma C.3 which deals with the torus $\mathbf{T}_H$ of type (II). Again $\sigma^*=\operatorname{Int}\left(\sigma(h)h^{-1}\right)=(14)(23)$ and $\tau^*=\operatorname{Int}\left(\tau(h)h^{-1}\right)=(12)(34)$. The advantage of our h over h' is that $\theta(h)=h$. Then $T=h^{-1}T^*h$ consists of $t=h^{-1}(t,\tau t,\sigma\tau t,\sigma t;e)h=\left(\left(\begin{smallmatrix}\mathbf{a}&\mathbf{b}D\\\mathbf{b}&\mathbf{a}\end{smallmatrix}\right),e\right)$, in the notations of that Lemma. To find an element $t_1=g^{-1}t\theta(g)$ which is stably θ-conjugate but not θ-conjugate to t, we need to solve $g_\sigma=g\sigma(g^{-1})=h'^{-1}a_\sigma h'=h'^{-1}a_\sigma\left(\begin{smallmatrix}0&w\boldsymbol{\varepsilon}\\\boldsymbol{\varepsilon}w&0\end{smallmatrix}\right)\sigma h'$, namely $h'g=a_\sigma\left(\begin{smallmatrix}0&w\boldsymbol{\varepsilon}\\\boldsymbol{\varepsilon}w&0\end{smallmatrix}\right)\sigma(h'g)$, where $a_\sigma=(R,\tau R,\tau R^{-1},R^{-1})$. Here $R\in E_3^\times/N_{E/E_3}E^\times$. A solution is given by the g_R of Lemma C.3, as verified there. Note that $\theta(g_R)=g_R$, and that t_1 is given by $\left(\begin{smallmatrix}\mathbf{a}&\mathbf{b}D\mathbf{R}^{-1}\\\mathbf{R}\mathbf{b}&\mathbf{a}\end{smallmatrix}\right)$ (and that $\mathbf{bR}=\mathbf{Rb}$).

(IV) $\rho(,)$ contains $\rho(\sigma)=(3421)$, and T is isomorphic to the multiplicative group $E^\times$ of an extension $E=F(\sqrt{D})=E_3(\sqrt{D})$ of F of degree 4, where $E_3=F(\sqrt{A})$ is a quadratic extension of $F(A\in F-F^2, D=\alpha+\beta\sqrt{A}\in E_3)$. The Galois closure $\widetilde{E}/F$ of $F(\sqrt{D})/F$ is $E=F(\sqrt{D})$ when $F(\sqrt{D})/F$ is cyclic, and $\widetilde{E}=F(\sqrt{D},\zeta)$ when $F(\sqrt{D})/F$ is not Galois; here $\zeta^2=-1$, and $\operatorname{Gal}(\widetilde{E}/F)$ is the dihedral group D_4. We have $\sigma\sqrt{D}=\sqrt{\sigma D},\sigma^2\sqrt{D}=-\sqrt{D},\sigma^3\sqrt{D}=-\sqrt{\sigma D},\sigma\sqrt{A}=-\sqrt{A}$.

In the D_4-case, the group $\operatorname{Gal}(\widetilde{E}/F)$ contains also the element τ of order two with $\tau\zeta=-\zeta,\tau\sqrt{A}=\sqrt{A}$. In this case we take $D=\sqrt{A}$, and so $\sigma\tau\sigma=\tau$. Further, if $\sigma\tau\sigma=\tau$, then $x=\sigma\tau,\tau\sigma$ and $\sigma^2\tau$ solve the equation $\sigma x\sigma=x$ too, and they are all of order 2. Renaming τ we may assume that $\rho(\tau)=(23)$ (the other possibilities are $(43)(21),(42)(13),(14)$). In all cases E_3 is the fixed field of σ^2 in $E=E_3(\sqrt{D})$, and $T^*=\{(a,\sigma a,\sigma^3a,\sigma^2a;e); a\in E^\times, e\in F^\times\}$. Further $V=\{(a,\sigma a,\sigma^3a=\sigma a,\sigma^2a=a;1/a\sigma a); a\in E_3^\times\}$, and $(1-\theta)T^*=\{(a\sigma^2a,\sigma a\sigma^3a,\sigma a\sigma^3a,a\sigma^2a;1/a\sigma a\sigma^2a\sigma^3a)\}$. Hence $H^1(\mathbf{T}^*\to\mathbf{V})=E_3^\times/N_{E/E_3}E^\times$.

Further $T^{*\theta}=\{(a,\sigma a,\sigma^3a=1/\sigma a,\sigma^2a=1/a;e); a\in E^\times, e\in F^\times\}$, and additively $H^1(\mathbf{T}^{*\theta})$ is the quotient of $\{(x,y,-y,-x;0)\}$ by $\langle(x-y,y-x,\dots),(0,2y,-2y,0;0)\rangle$, namely $\mathbb{Z}/2$. Explicitly, a cocycle in $H^1(\mathbf{T}^{*\theta})$ is $a_\sigma=(e,f,f^{-1},e^{-1})$ with $1=a_{\sigma^4}=a_\sigma\sigma^*(a_\tau)\sigma^{*2}(a_\sigma)\sigma^{*3}(a_\sigma)$, namely $e/\sigma^2e=\sigma f/\sigma^3f$, thus $e\sigma^3f\in E_3^\times$. If $b_\sigma=(c,d,d^{-1},c^{-1})$ then $b_\sigma\sigma^*(b_\sigma^{-1})=(c\sigma d,d/\sigma c,\sigma c/d,1/c\sigma d)$. We can first assume that $f=1$, then the choice of $d=\sigma c$ shows that the class of a_σ depends on e, which we now denote by R, in $E_3^\times/N_{E/E_3}E^\times$.

The torus $\mathbf{T}$ takes the form $h^{-1}\mathbf{T}^*h$, where – as in the Lemma C.4 which dealt with tori $\mathbf{T}_H$ of type (IV) – h is $\operatorname{diag}(-1/4\sqrt{AD}, 1/4\sqrt{A\sigma D}, w)\widetilde{h}_D \operatorname{diag}(h_A, h_A)$, where $\widetilde{h}_D = \operatorname{Int}\big((1,w,1)\big)(h_D, \sigma h_D)$. Then $\sigma(h)\cdot h^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/4\sqrt{AD} \\ -1/4\sqrt{AD} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, and $\theta(h) = h$. To find $t_1 = g^{-1}t\theta(g)$ which is stable θ-conjugate but not θ-conjugate to t, as usual we need to solve: $hg = a_\sigma h\sigma(h)^{-1}\sigma(hg)$. A solution is given by

$$g = g_R = \frac{1}{2}\begin{pmatrix} \mathbf{R}+I & (\mathbf{R}-I)\sqrt{\mathbf{D}} \\ (\mathbf{R}-I)(\sqrt{\mathbf{D}}^{-1}) & \mathbf{R}+I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & \mathbf{R}^{-1} \end{pmatrix},$$

where $\mathbf{R} = h_A^{-1}\begin{pmatrix} R & 0 \\ 0 & \sigma R \end{pmatrix}h_A = \begin{pmatrix} R_1 & R_2A \\ R_2 & R_1 \end{pmatrix}$ if $R = R_1 + R_2\sqrt{A}$ in $E_3^\times$, and $\sqrt{\mathbf{D}} = h_A^{-1}\begin{pmatrix} \sqrt{D} & 0 \\ 0 & \sqrt{\sigma D} \end{pmatrix}h_A$ has inverse $(\sqrt{D}^{-1})$. Further, $t_1 = g^{-1}t\theta(g) = \begin{pmatrix} I & 0 \\ 0 & \mathbf{R} \end{pmatrix}\begin{pmatrix} \mathbf{a} & \mathbf{bD} \\ \mathbf{b} & \mathbf{a} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & \mathbf{R}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{a} & \mathbf{bDR}^{-1} \\ \mathbf{Rb} & \mathbf{a} \end{pmatrix}$, and $\theta(g_R) = g_R$. When solving our equation it is convenient to rewrite it as:

$$\begin{aligned}
\widetilde{g} &= \begin{pmatrix} h_A & 0 \\ 0 & h_A \end{pmatrix} g \begin{pmatrix} h_A & 0 \\ 0 & h_A \end{pmatrix}^{-1} \\
&= \widetilde{h}_D^{-1}\begin{pmatrix} -4\sqrt{AD} & & 0 \\ & 4\sqrt{A\sigma D} & \\ 0 & & w \end{pmatrix}\begin{pmatrix} -R/4\sqrt{AD} & & 0 \\ & I & \\ 0 & & 4\sqrt{AD}/R \end{pmatrix}\begin{pmatrix} 0&0&1&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&1&0&0 \end{pmatrix} \\
&\quad \begin{pmatrix} 1/4\sqrt{A\sigma D} & & 0 \\ & 1/4\sqrt{AD} & \\ 0 & & w \end{pmatrix}\sigma(\widetilde{h}_D)\sigma(\widetilde{g})\begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix} \\
&= \left[\widetilde{h}_D^{-1}\begin{pmatrix} R & & & 0 \\ & 1 & & \\ & & R^{-1} & \\ 0 & & & 1 \end{pmatrix}\widetilde{h}_D\right]\begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}\sigma(\widetilde{g})\begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix},
\end{aligned}$$

and further as ((23) stands for $(1,w,1)$):

$$\begin{aligned}
(23)\tilde{g}(23) &= \begin{pmatrix} \frac{1}{2}(R+R^{-1}) & \frac{1}{2}(R-R^{-1})\sqrt{D} & 0 \\ \frac{1}{2}(R-R^{-1})/\sqrt{D} & \frac{1}{2}(R+R^{-1}) & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\sigma\left((23)\tilde{g}(23)\right)\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \\
&= \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \sigma T & \sigma Z \\ \sigma Y & \sigma X \end{pmatrix} = \begin{pmatrix} X & Y \\ \sigma Y & \sigma X \end{pmatrix}.
\end{aligned}$$

Then $X = E\sigma^2 X$. As $E = t^{-1}\frac{1}{2}\begin{pmatrix} R+R^{-1} & R-R^{-1} \\ R-R^{-1} & R+R^{-1} \end{pmatrix}t = t^{-1}\rho^{-1}\begin{pmatrix} R & 0 \\ 0 & R^{-1} \end{pmatrix}\rho t$, $\rho = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $t = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{D} \end{pmatrix}$, we need to solve

$$\begin{aligned}
\rho t X &= \begin{pmatrix} R & 0 \\ 0 & R^{-1} \end{pmatrix}\rho t\sigma^2 X = \begin{pmatrix} R & 0 \\ 0 & R^{-1} \end{pmatrix}(-w)\sigma^2(\rho t X) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} 0 & -R \\ -R^{-1} & 0 \end{pmatrix}\begin{pmatrix} \sigma^2 a & \sigma^2 b \\ \sigma^2 c & \sigma^2 d \end{pmatrix} = \begin{pmatrix} -R\sigma^2 c & -R\sigma^2 d \\ -\sigma^2 a/R & -\sigma^2 b/R \end{pmatrix}.
\end{aligned}$$

Choosing $a = e$ and $b = \sqrt{D}$, we get $X = \frac{1}{2}\begin{pmatrix} R+1 & (1-R^{-1})\sqrt{D} \\ (R-1)/\sqrt{D} & 1+R^{-1} \end{pmatrix}$. Note that $g\theta(g^{-1}) = \operatorname{diag}(\|X\|, \|\sigma X\|)$. We choose X to have determinant 1, so that $\theta(g) = g$ lies in $Sp(2,E)$. Also we take $Y = 0$. Then $g = \begin{pmatrix} h_A & 0 \\ 0 & h_A \end{pmatrix}^{-1}(23)\begin{pmatrix} X & 0 \\ 0 & \sigma X \end{pmatrix}(23)\begin{pmatrix} h_A & 0 \\ 0 & h_A \end{pmatrix}$ is as asserted.

E. Useful facts.

We collect here the following observations, used below.

Remark. For $A \in F - F^2$, we introduce the subgroup $\mathbf{C}_A$ of $\left(\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d}\end{smallmatrix}\right) \in \mathbf{H} = GSp(2)$, where $\mathbf{a} = \left(\begin{smallmatrix} a_1 & a_2 \\ a_2 A & a_1\end{smallmatrix}\right)$, $\mathbf{b} = \dots$. We shall use below the observation that the tori T_H of type (II) and (IV) embed in C_A. Moreover, C_A is naturally isomorphic to $GL\big(2, F(\sqrt{A})\big)'$, the group of $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in GL\big(2, F(\sqrt{A})\big)$ with $ad - bc \in F^\times$. The isomorphism is given by $\mathbf{a} \mapsto a = a_1 + a_2\sqrt{A}$. Also let $\mathbf{C}_0$ be the group of $[\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)] = \begin{pmatrix} a & 0 & 0 & b \\ 0 & \alpha & \beta & 0 \\ 0 & \gamma & \delta & 0 \\ c & 0 & 0 & d\end{pmatrix} \in \mathbf{H}$. The group C_0 is isomorphic to $GL(2, F \oplus F)' = \{(g, g') = \big(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)\big); \det g = \det g'\}$. The tori T_H of type (I) and (III) (and (2), (3)) naturally embed in C_0.

Remark. The norm map $N : T^* \to T_H^*$ is defined by $X = (x, y, z, t; w) \mapsto (xyw, xzw, tyw, tzw; xyztw^2) = NX$. If $\sigma = (23)$, then σX has the norm $(xzw, xyw, ztw, tyw; xyztw^2) = \tau NX$, where $\tau = (12)(34)$. If $\sigma = (14)$ then $\tau = (13)(24)$, if $\sigma = (12)(34)$ then $\tau = (23)$, if $\sigma = (13)(24)$ then $\tau = (14)$, if $\sigma = (14)(23)$ then $\tau = (14)(23)$, if $\sigma = (3421)$ then $\tau = (2431)$. Our numbering of the tori $\mathbf{T}_H$ and $\mathbf{T}$ is such that the norm preserves the type, thus the norm of $\mathbf{T}^*$ of type (II) is $\mathbf{T}_H^*$ of type (II), and not of type (III), although the centralizer in $\mathbf{G} = GL(4) \times GL(1)$ of a torus of type (III) in $\mathbf{H} = GSp(2)$ is a torus of type (II).

For tori of type (IV) it will be useful to note the following. Assume the residual characteristic is odd.

Lemma. *If E is an extension of F of degree 4 which is not a compositum of two quadratic extensions, then $E = F(\sqrt{D})$, $D = \alpha + \beta\sqrt{A}$, $\alpha, \beta \in F$, $A \in F - F^2$, $D \in E_3 - E_3^2$, $E_3 = F(\sqrt{A})$, and we have the following possibilities. If $A = \boldsymbol{\pi}$ then $D = \sqrt{\boldsymbol{\pi}}$. If $-1 \in R^{\times 2}$ and $A \in R^\times$, then $D = \sqrt{A}$ or $\boldsymbol{\pi}\sqrt{A}$. If $A = -1 \in R^\times - R^{\times 2}$, then $\alpha, \beta \in R^\times$ or $\alpha, \beta \in \pi R^\times$. The extension $F(\sqrt{D})/F$ is Galois, cyclic with Galois group $\mathbb{Z}/4$, unless it is completely ramified ($A = \boldsymbol{\pi}$) and $-1 \notin R^{\times 2}$.*

Proof. Denote by ζ a fourth root of 1. An extension E_3 of degree two of F is given by $E_3 = F(\sqrt{A})$ for some $A \in F - F^2$. An extension of degree two of E_3 is given by $E_3(\sqrt{D})$ with $D \in E_3 - E_3^2$. Denote by $\boldsymbol{\pi}$ a generator of the maximal ideal in the (local) ring of integers R of F, and by ε a non square unit ($\varepsilon \in R^\times - R^{\times 2}$). There are three quadratic extensions of F: two are ramified, namely $F(\sqrt{\boldsymbol{\pi}})/F$, $F(\sqrt{\varepsilon\boldsymbol{\pi}})/F$, and one is unramified: $F(\sqrt{\varepsilon})/F$. The extensions of degree four of F are as follows.

(i) Suppose that $A = \boldsymbol{\pi}$ and $E_3 = F(\sqrt{\boldsymbol{\pi}}) = F(\sqrt{\varepsilon^2\boldsymbol{\pi}})$ is ramified over F. A quadratic ramified extension of E_3 is defined by $D = \sqrt{\boldsymbol{\pi}}$ or $\varepsilon\sqrt{\boldsymbol{\pi}}$; indeed $R^\times/(1 + \boldsymbol{\pi}R) \simeq R_3^\times/(1 + \boldsymbol{\pi}_3 R_3)$, where R_3 is the ring of integers in E_3, and $\boldsymbol{\pi}_3$ is a uniformizer. In particular -1 is a square in $R^\times$ if and only if it is a square in $F(\sqrt{D})$. The field homomorphisms of $F(\sqrt{D})$ into a Galois closure, which fix F, are generated by σ which maps $\sqrt{A}$ to $-\sqrt{A}$, and $\sqrt{D}$ to $\zeta\sqrt{D}$. Then $F(\sqrt{D})/F$ is Galois, cyclic with Galois group $\mathbb{Z}/4$, when $\zeta \in R^\times$, and it is non Galois when $\zeta \notin R^\times$. In this case $F(\zeta, \sqrt{D})/F$ is Galois with group D_4, generated by $\sigma\big(\sigma(\sqrt{D}) = \zeta\sqrt{D}\big)$ and an endomorphism τ which fixes $\sqrt{D}$ and maps ζ to $-\zeta$.

(ii) If $A=\boldsymbol{\pi}$, thus $E_3=F(\sqrt{\boldsymbol{\pi}})$ is ramified, but $E_3(\sqrt{D})/E_3$ is unramified, we can take D to be a non square unit in $E_3^\times$, namely a non square unit ε in $R^\times$. Hence $F(\sqrt{\boldsymbol{\pi}},\sqrt{\varepsilon})$ is the compositum of two quadratic extensions of F, and its Galois group is $\mathbb{Z}/2\times\mathbb{Z}/2$.

(iii) Suppose that $A=\varepsilon$, so that $E_3=F(\sqrt{A})$ is unramified over F. The ramified quadratic extensions of E_3 are $E_3(\sqrt{\boldsymbol{\pi}})$ (in which case $\mathrm{Gal}\big(E_3(\sqrt{\boldsymbol{\pi}})/F\big)=\mathbb{Z}/2\times\mathbb{Z}/2$) and $E_3(\sqrt{\boldsymbol{\pi}\varepsilon_3})$, where $\varepsilon_3\in R_3^\times-R_3^{\times 2}$. Indeed, $\boldsymbol{\pi}$ generates the maximal ideal in the ring R_3 of integers of the unramified extension E_3 of F. The extension $E_3(\sqrt{\varepsilon_3\boldsymbol{\pi}})$ of F is cyclic with Galois group $\mathbb{Z}/4$, generated by σ, described as follows.

If $\zeta\in R^\times$ then $\varepsilon_3=\sqrt{\varepsilon}$, and $\sigma(\sqrt{\varepsilon_3\boldsymbol{\pi}})=\zeta\sqrt{\varepsilon_3\boldsymbol{\pi}}$. Then $\sigma(\boldsymbol{\pi}\varepsilon_3)=-\boldsymbol{\pi}\varepsilon_3$, $\sigma^2(\sqrt{\boldsymbol{\pi}\varepsilon_3})=-\sqrt{\boldsymbol{\pi}\varepsilon_3}$. Note that $\sqrt{\varepsilon}$ is not a square in $E_3^\times$. Indeed, if $\sqrt{\varepsilon}=(a+b\sqrt{\varepsilon})^2=a^2+b^2\varepsilon+2ab\sqrt{\varepsilon}$ with $a,b\in F$, then $b=1/2a$, and $-a^2=b^2\varepsilon=\varepsilon/4a^2$, so that $\sqrt{\varepsilon}=2\zeta a^2$ would lie in $F^\times$.

If $\zeta\notin R^\times$ take $\varepsilon=-1$, then $\zeta\in R_3^\times$ and $\sigma\zeta=-\zeta$, but $\zeta\in R_3^{\times 2}$. Indeed, since $-1\notin R^{\times 2}$, either 2 or -2 lies in $R^{\times 2}$, and $\zeta=\big((1\pm\zeta)/\sqrt{\pm 2}\big)^2$. Take $\varepsilon_3=a+b\zeta\in R_3-R_3^2$, and put $\overline{\varepsilon}_3=a-b\zeta$. Then $\sigma\sqrt{\boldsymbol{\pi}\varepsilon_3}=\sqrt{\boldsymbol{\pi}\overline{\varepsilon}_3},\sigma\sqrt{\boldsymbol{\pi}\overline{\varepsilon}_3}=-\sqrt{\boldsymbol{\pi}\varepsilon_3},\sigma^2\sqrt{\boldsymbol{\pi}\varepsilon_3}=-\sqrt{\boldsymbol{\pi}\varepsilon_3},\sigma^3\sqrt{\boldsymbol{\pi}\varepsilon_3}=-\sqrt{\boldsymbol{\pi}\overline{\varepsilon}_3}$. Note that $\varepsilon_3/\overline{\varepsilon}_3$ lies in $R_3^{\times 2}$.

(iv) If $A=\varepsilon$ and $E_3=F(\sqrt{A})$ is unramified over F, and $D=\varepsilon_3\in R_3-R_3^2$ so that $E_3(\sqrt{D})/E_3$ is unramified, then $E_3(\sqrt{D})/F$ is Galois with cyclic group $\mathbb{Z}/4$. It is the unique unramified extension of F of degree 4. If $-1\in R^{\times 2}$, $\varepsilon\in R^\times-R^{\times 2}$ and $\varepsilon_3=\sqrt{\varepsilon}$ is then in $R_3-R_3^\times$, and $\sigma\sqrt{\varepsilon_3}=\zeta\sqrt{\varepsilon_3}$ generates the Galois group. If $-1\notin R^{\times 2}$ take $\varepsilon=-1$, and $\varepsilon_3\in R_3^\times-R_3^{\times 2}$. Then the Galois group is generated by $\sigma\sqrt{\varepsilon_3}=\sqrt{\overline{\varepsilon}_3}$ and $\sigma\sqrt{\overline{\varepsilon}_3}=-\sqrt{\varepsilon_3}$, where $\overline{\varepsilon}_3=a-b\zeta$ if $\varepsilon_3=a+b\zeta$; $a,b\in R$. $\square$

Remark. Twisted endoscopic groups are defined in [KS, 2.1]. Let us recall this definition.

Let us begin with a review of L-groups. Let G be a connected reductive group over a local field F of characteristic 0. Write , for $\mathrm{Gal}(\overline{F}/F)$ and W_F for the absolute Weil group of F. Denote by $\hat{G}$ the Langlands dual group of G. By definition there is an identification $\eta_G:\Psi^\vee(G)\xrightarrow{\sim}\Psi(\hat{G})$, where $\Psi^\vee(G)$ (resp. $\Psi(\hat{G})$) is the dual based root datum of G (resp. based root datum of $\hat{G}$).

An action ρ : , $\to\mathrm{Aut}(\hat{G})$ of , on $\hat{G}$ is called an *L-action* if it preserves some splitting $\mathrm{spl}_{\hat{G}}=(\hat{B},\hat{T},\{X_{\alpha^\vee}\}_\alpha)$ of $\hat{G}$. If this is the case then we call $\mathrm{spl}_{\hat{G}}$ a *, -splitting* for ρ and form the L-group by $\hat{G}\rtimes_\rho W_F$, where W_F acts through , via ρ. If the composition , $\xrightarrow{\rho}\mathrm{Aut}(\hat{G})\twoheadrightarrow\mathrm{Out}(\hat{G})$ coincides (under the identification η_G) with the , -action on $\Psi^\vee(G)$, then this L-group is that of G. This ρ is usually denoted by ρ_G. A triple $(\mathrm{spl}_{\hat{G}},\rho_G,\eta_G)$ of this type is called an *L-group data* for G (sometimes $(\hat{G},\rho_G,\eta_G)$ is refered to as L-group data, but the inclusion of a , -splitting in the data is convenient).

A tuple $(H,\mathcal{H},s,\xi)$ is said in [KS, 2.1] to be an endoscopic data for G and $\theta(\in\mathrm{Aut}\,G)$ if [KS, 2.1.i] $(1\le i\le 4)$ hold. Here [KS, 2.1.1] is: H is a quasisplit F-group. Fix L-group data $(\mathrm{spl}_{\hat{H}},\rho_H,\eta_H)$ for H. The second ingredient $\mathcal{H}$ is a split extension $1\to\hat{H}\to\mathcal{H}\to W_F\to 1$. Hence we can choose a section $c:W_F\hookrightarrow\mathcal{H}$ of this extension. Consider $\hat{H}$ as a closed subgroup of $\mathcal{H}$. Then we have a W_F-action ρ_c on $\hat{H}$: define $\rho_c(w)$ to be $\mathrm{Int}(c(w))|_{\hat{H}}$. Of course this is not necessarily an L-action (i.e. it might not preserve any splitting of $\hat{H}$). But we have a unique family $\{h_w \in \hat{H}_{ad}; w \in W_F\}$ such that $\rho_c(w)(\mathrm{spl}_{\hat{H}}) = \mathrm{Int}(h_w)(\mathrm{spl}_{\hat{H}})$ for all $w \in W_F$. This gives the L-action $\rho_{\mathcal{H}} : W_F \ni w \mapsto \mathrm{Int}(h_w^{-1}) \circ \rho_c(w) \in \mathrm{Aut}(\hat{H})$, which does not depend on the choice of $c : W_F \hookrightarrow \mathcal{H}$. Then [KS, 2.1.2] is: $\rho_{\mathcal{H}}$ coincides with ρ_H.

Let us clarify (I wish to thank Takuya Kon-no for this explanation) that $\mathcal{H}$ need not be isomorphic to ${}^L H$ under this requirement. Note that for $w, w' \in W_F$ we have

$$\begin{aligned}&\mathrm{Int}(h_{ww'}) \circ \rho_H(ww') = \rho_c(ww') = \rho_c(w) \circ \rho_c(w')\\ &= \mathrm{Int}(h_w) \circ \rho_H(w) \circ \mathrm{Int}(h_{w'}) \circ \rho_H(w^{-1})\rho_H(w)\rho_H(w') = \mathrm{Int}(h_w \rho_H(w)(h_{w'})) \circ \rho_H(ww').\end{aligned}$$

That is, $\{h_w; w \in W_F\}$ is a $\hat{H}_{ad}$-valued 1-cocycle. It defines a class in $H^1(F, \hat{H}_{ad})$. This class is trivial if and only if there exists some $h \in \hat{H}_{ad}$ such that $h_w = h^{-1}w(h)$ for all $w \in W_F$. Equivalently, $\rho_c(w) = \mathrm{Int}(h^{-1}w(h)) \circ \rho_H(w) = \mathrm{Int}(h^{-1}) \circ \rho_H(w) \circ \mathrm{Int}(h)$ for all $w \in W_F$. But this amounts to the fact that ρ_c is an L-action (it preserves the splitting $\mathrm{Int}(h^{-1})(\mathrm{spl}_{\hat{H}})$). In this case we have (from $\rho_{\mathcal{H}} = \rho_H$) $\mathcal{H} \simeq {}^L H$. Of course one can find examples for the situation $\mathcal{H} \not\simeq {}^L H$ when $H^1(F, \hat{H}_{ad})$ is non-trivial.

Finally, [KS, 2.1.3] requires that the element $s \in \hat{G}$ is such that $s\hat{\theta}$ be semi-simple in $\hat{G} \rtimes \hat{\theta}$, and [KS, 2.1.4] that $\xi : \mathcal{H} \to {}^L G$ be an L-homomorphism, whose image $\xi(\mathcal{H})$ is contained in the group of fixed points $Z_{{}^L G}(s{}^L\theta)$ in ${}^L G$ of $\mathrm{Int}(s) \circ {}^L\theta$, where ${}^L\theta(g \times w) = \hat{\theta}(g) \times w$, and that ξ map $\hat{H}$ isomorphically onto the identity component $Z_{\hat{G}}(s\hat{\theta})^0$ of the group $Z_{\hat{G}}(s\hat{\theta})$ of fixed points of $\mathrm{Int}(s) \circ \hat{\theta}$ in $\hat{G}$.

F. Endoscopic groups.

Our Theorem is the "fundamental lemma" for the lifting of representations from $GSp(2)$ to $GL(4)$. It is compatible with a dual group situation, which we proceed to describe.

Let $\mathbf{G}$ be the F-group $\mathbf{G}_1 \times \mathbf{G}_m$, where $\mathbf{G}_1 = GL(4)$ and $\mathbf{G}_m = GL(1)$. Let $\hat{G} = \hat{G}_1 \times \hat{G}_m = GL(4, \mathbb{C}) \times GL(1, \mathbb{C})$ be its connected dual group. Put $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $J = \left(\begin{smallmatrix} 0 & w \\ -w & 0 \end{smallmatrix}\right)$, and $\hat{\theta}(g_1) = \theta(g_1) = J{}^t g_1^{-1} J^{-1}$ for $g_1 \in \mathbf{G}_1$, where ${}^t g_1$ is the transpose of g_1. For $g = (g_1, t)$ in $\hat{G}$, write $\hat{\theta}(g) = \hat{\theta}(g_1, t) = \big(t\theta(g_1), t\big)$. This is an automorphism of $\hat{G}$ of order 2. We often attach a subscript 1 to indicate the $GL(4)$-component of an object in $\mathbf{G} = GL(4) \times GL(1)$, and sometimes abuse notations and ignore the $GL(1)$-component.

Denote by $\hat{T}$ the diagonal subgroup in $\hat{G}$ (thus $\hat{T} = \hat{T}_1 \times \mathbb{C}^\times$), and by $\mathbf{T}^*$ the diagonal subgroup of $\mathbf{G}$. Let $\hat{B}$ and $\mathbf{B}$ be the upper triangular subgroups in $\hat{G}$ and $\mathbf{G}$. Then the group $X_*(\hat{T}) = \mathrm{Hom}(\mathbf{G}_m, \hat{T}) = \{(a, b, c, d; e)\}$ is isomorphic to $X^*(\mathbf{T}^*) = \mathrm{Hom}(\mathbf{T}^*, \mathbf{G}_m)$, and $X^*(\hat{T}) = \{(x, y, z, t; u)\} = X_*(\mathbf{T}^*)$. The automorphism $\hat{\theta}$ induces an automorphism θ on $\mathbf{G}$ (fixing $\mathbf{B}$), given on $\mathbf{T}^*$ as follows.

$$\begin{aligned}&\big(\theta(x, y, z, t; u)\big)(a, b, c, d; e) = (x, y, z, t; u)\big(\hat{\theta}(a, b, c, d; e)\big)\\ &\quad= (x, y, z, t; u)(e/d, e/c, e/b, e/a; e) = a^{-t}b^{-z}c^{-y}d^{-x}e^{x+y+z+t+u}\\ &\quad= (-t, -z, -y, -x, x + y + z + t + u)(a, b, c, d; e).\end{aligned}$$

Then for $(g, t) \in \mathbf{G}, \theta(g, t) = \big(\theta(g), t\|g\|\big)$, where $\|g\|$ denotes the determinant of g.

We are concerned with lifting of representations and transfer of orbital integrals between $\mathbf{G}$ and its endoscopic groups, in fact its twisted (by θ) such groups. The twisted endoscopic groups of $(\hat{G},\hat{\theta})$ are determined by $\hat{H}=Z_{\hat{G}}(\hat{s}\hat{\theta})^0$ (superscript zero for "connected component of the identity"), where this centralizer is

$$Z_{\hat{G}}(\hat{s}\hat{\theta})=\{(x,t)\in\hat{G}; x\hat{s}\theta(x)^{-1}=t\hat{s}\}\subset Z_{GL(4,\mathbb{C})}\big(\hat{s}\hat{\theta}(\hat{s})\big)\times GL(1,\mathbb{C}),$$

and by a Galois action $\rho : \Gamma = \mathrm{Gal}(\overline{F}/F)\to Z_{\hat{G}}(\hat{s}\hat{\theta})$. Here $\hat{s}$ is a semi-simple element in $\hat{G}$ (which can and will be taken to be $\hat{s}=(\hat{s}_1,1)\big)$, which can and will be taken to be diagonal, chosen up to $\hat{\theta}$-conjugacy, namely $\hat{T}\ni\hat{s}\equiv g\hat{s}\hat{\theta}(g^{-1})$. Using a diagonal g we conclude that $\hat{s}=\mathrm{diag}(1,1,c,d)$. Taking g to be a representative in $\hat{G}$ of the reflections $(23),(14),(12)(34)$ in the Weyl group of $\hat{G}$ (these elements are fixed by $\hat{\theta}$), we conclude that the $\hat{\theta}$-conjugacy class of $\hat{s}$ does not change if c is replaced by c^{-1}, d by d^{-1}, and (c,d) by (d,c). Let us list the possibilities. Recall ([KS, 2.1]) that an endoscopic group $\mathbf{H}$ is called *elliptic* if $\big(Z(\hat{H})^\Gamma\big)^0$ is contained in the center $Z(\hat{G})$ of $\hat{G}$.

A list of the twisted endoscopic groups of $(\hat{G},\hat{\theta})$ is as follows.

1. $\hat{s}=I$, $Z_{\hat{G}}(\hat{\theta})=GSp(2,\mathbb{C})$ is connected, hence equal to $\hat{H}$, the Galois action is trivial, and the endoscopic group is $\mathbf{H}=GSp(2)$ over F. Since $Z(\hat{H})=\mathbb{C}^\times=Z(\hat{G})$, $\mathbf{H}$ is elliptic.

An endoscopic group $\mathbf{C}$ of $\mathbf{H}$ is determined by a semi-simple (diagonal, up to conjugacy) element s in $\hat{H}$. The only proper elliptic endoscopic group of $\mathbf{H}$ is determined by $s=\mathrm{diag}(1,-1,-1,1)$, whose centralizer in $\hat{H}$ is $\hat{C}_0=\begin{pmatrix}\bullet&0&0&\bullet\\0&\bullet&\bullet&0\\0&\bullet&\bullet&0\\\bullet&0&0&\bullet\end{pmatrix}=\{(a,b)\in GL(2,\mathbb{C})^2;$ $\det a=\det b\}$. Note that the connected component of $Z(\hat{C}_0)=\langle Z(\hat{H}),s\rangle$ is $Z(\hat{H})$, so that $\mathbf{C}_0$ is elliptic. Also, $X_*(\hat{T}_0)=\{(a,b,c,d);a+d=b+c\}=X^*(T_0^*)$ has dual $X_*(T_0^*)=X^*(\hat{T}_0)=\{(x,y,z,t)/(u,-u,-u,u)\}$, hence $\mathbf{C}_0=GL(2)\times GL(2)/GL(1)$, where $GL(1)$ embeds via $u\mapsto(u,u^{-1})$.

The dual group of $\mathbf{H}_0=Sp(2)$ is $\hat{H}_0=PGSp(2,\mathbb{C})$. Its proper elliptic endoscopic groups are obtained as follows. (i) The centralizer of $s=\mathrm{diag}(1,-1,-1,1)$ in $\hat{H}_0$ is generated by the reflection $\mathrm{diag}(w,w)$ and its connected component $\hat{C}_0/\hat{Z}=(GL(2,\mathbb{C})\times GL(2,\mathbb{C}))'/\mathbb{C}^\times$, the prime indicates equal determinants. The corresponding endoscopic group is $(GL(2)\times GL(2))'/GL(1)$, unless there is a quadratic extension E/F whose Galois group permutes the two factors, in which case $\mathrm{Res}_{E/F}\,GL(2)'/GL(1)$ is obtained (its group of F-points is $GL(2,E)'/F^\times$, where the prime indicates here determinant in $F^\times$). (ii) The centralizer of $s_1=\mathrm{diag}(1,1,-1,-1)$ in $\hat{H}_0$ is generated by $\left(\begin{smallmatrix}0&\boldsymbol{\varepsilon}\\-\boldsymbol{\varepsilon}&0\end{smallmatrix}\right)$ (where $\boldsymbol{\varepsilon}=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$) and its connected component $\hat{C}_1^0=\{\mathrm{diag}(x,\lambda\boldsymbol{\varepsilon}x\boldsymbol{\varepsilon});\,x\in PGL(2,\mathbb{C}),\,\lambda\in\mathbb{C}^\times\}$. The endoscopic group is elliptic only when there is a quadratic extension E/F such that $\mathrm{Gal}(E/F)$ acts via $\mathrm{Int}\left(\begin{smallmatrix}0&\boldsymbol{\varepsilon}\\-\boldsymbol{\varepsilon}&0\end{smallmatrix}\right)$ on this connected component, thus by $\sigma(x,\lambda)=(x,\lambda^{-1})$ on $(x,\lambda)\in PGL(2,\mathbb{C})\times\mathbb{C}^\times$, and then the endoscopic group is $SL(2)\times U(1,E/F)$, where $U(1,E/F)$ is the unitary group with F-points $E^1=\{x\in E^\times;x\overline{x}=1\}$.

2. $\hat{s}=\left(\begin{smallmatrix}I&0\\0&-I\end{smallmatrix}\right)$, $Z_{\hat{G}}(\hat{s}\hat{\theta})=GO\big(\left(\begin{smallmatrix}0&w\\w&0\end{smallmatrix}\right);\mathbb{C}\big)$ is $\{(x,t)\in\hat{G};x\left(\begin{smallmatrix}0&w\\w&0\end{smallmatrix}\right){}^tx\left(\begin{smallmatrix}0&w\\w&0\end{smallmatrix}\right)=t\}$. It is isomorphic

to

$$\langle\big(A, B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\big) = \big(\left(\begin{smallmatrix} aA & bA\boldsymbol{\varepsilon} \\ c\boldsymbol{\varepsilon}A & d\boldsymbol{\varepsilon}A\boldsymbol{\varepsilon} \end{smallmatrix}\right),\ \|AB\|\big), (\mathrm{diag}(1,w,1),1)\rangle = \langle\big(\left(\begin{smallmatrix} A & 0 \\ 0 & t\boldsymbol{\varepsilon}A\boldsymbol{\varepsilon} \end{smallmatrix}\right), t\|A\|\big), (\mathrm{diag}(1,w,1),1)\rangle,$$

which has connected component $\hat{C} = GL(2,\mathbb{C})^2/\mathbb{C}^\times$, with $\mathbb{C}^\times$ embedding via $z \mapsto (z, z^{-1})$. Note that $Z(\hat{C}) = \mathbb{C}^\times$ is $Z(\hat{G})$, hence $\mathbf{C}$ is elliptic. Now

$$X^*(T_C^*) = X_*(\hat{T}_C) = \{(a,b;c,d)/(u,u;u^{-1},u^{-1})\}$$

has dual $X_*(T_C^*) = X^*(\hat{T}_C) = \{(x,y;z,t); x+y = z+t\}$, thus $\mathbf{C} = \big(GL(2)\times GL(2)\big)'$, where the prime means the subgroup of (A,B) with $\|A\| = \|B\|$, when , acts trivially. If there is a quadratic field extension E/F and $\rho(\sigma) \in \mathrm{diag}(1,w,1)\hat{C}$ for σ in $\mathrm{Gal}(E/F)$, then σ acts on $\mathbf{C} = \mathbf{C}_E = \mathrm{Res}_{E/F}\, GL(2)'$ by permuting the two factors. In particular, $C_E = \mathbf{C}_E(F) = GL(2,E)'$, the prime indicating determinant in $F^\times$. Note that the centralizer of $(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon})$ in $\hat{C} = GL(2,\mathbb{C})^2/\mathbb{C}^\times$ is generated by the diagonals and (w,w), hence C has no elliptic endoscopic groups.

3. $\hat{s} = \mathrm{diag}(1,1,1,-1)$, $Z_{\hat{G}}(\hat{s}\hat{\theta}) = \langle(\mathrm{diag}(a,B,b), \|B\|), (\iota,1);\ \iota = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{pmatrix},\ B \in GL(2,\mathbb{C}),$ $a,b \in \mathbb{C}^\times, ab = \|B\|\rangle$ has connected component $\hat{C}_+ = (GL(2,\mathbb{C})\times GL(1,\mathbb{C})^2)'$ (the prime indicates (a,B,b) with $ab = \|B\|$), with center $Z(\hat{C}_+) = \mathbb{C}^{\times 2}$, which will not be elliptic unless the Galois action is non trivial, namely there is a quadratic extension E/F with $\rho(\sigma) = \iota$, $\langle\sigma\rangle = \mathrm{Gal}(E/F)$. In this case $\big(Z(\hat{C}_+)^\Gamma\big)^0 = \mathbb{C}^\times$ is $Z(\hat{G})$. We have $X_*(\hat{T}_+) = \{(a,b,c,b+c-a; b+c)\} = X^*(T_+^*)$, with dual $X^*(\hat{T}_+) = \{(x,y,z,t;w)\}/\{(u,v,v,u;-u-v)\} = \{(x,y,z,t)\}/\{(u,-u,-u,u)\} = X_*(T_+^*)$. We conclude that $\mathbf{C}_+ = \mathbf{C}_+^E = (GL(2)\times \mathrm{Res}_{E/F}\, GL(1))/GL(1)$, $GL(1)$ embeds as (z,z^{-1}), and $C_+ = \mathbf{C}_+(F) = GL(2,F)\times E^\times/F^\times \simeq GL(2,F)\times E^1$.

4. $\hat{s} = \left(\begin{smallmatrix} I & 0 \\ 0 & cI \end{smallmatrix}\right)$, $c \neq \pm 1$, $Z_{\hat{G}}(\hat{s}\hat{\theta}) = \langle\big(\left(\begin{smallmatrix} A & 0 \\ 0 & t\boldsymbol{\varepsilon}A\boldsymbol{\varepsilon} \end{smallmatrix}\right), t\|A\|\big)\rangle$ is connected but not elliptic.

5. $\hat{s} = \mathrm{diag}(1,1,1,d), d \neq \pm 1, Z_{\hat{G}}(\hat{s}\hat{\theta}) = \langle\big(\mathrm{diag}(a, A, \|A\|/a; \|A\|\big)\rangle$ is connected but not elliptic.

6. $\hat{s} = \mathrm{diag}(1,1,-1,d), d \neq \pm 1, Z_{\hat{G}}(\hat{s}\hat{\theta}) = \langle\big(\mathrm{diag}(a,b,t/b,t/a),t\big), \big(\mathrm{diag}(1,w,1),1\big)\rangle$ is not elliptic.

7. $\hat{s} = \mathrm{diag}(1,1,c,d), c^2 \neq 1 \neq d^2, c \neq d, d^{-1}, Z_{\hat{G}}(\hat{s}\hat{\theta}) = \langle\big(\mathrm{diag}(a,b,t/b,t/a),t\big)\rangle$ is connected but not elliptic.

The **norm map** is defined as follows. Put $\mathbf{V} = (1-\theta)\mathbf{T}^*$ and $\mathbf{U} = \mathbf{T}_\theta^* = \mathbf{T}^*/\mathbf{V}$. Since $\mathbf{T}^*$ consists of $(a,b,c,d;e)$ and $\theta(a,b,c,d;e) = (d^{-1},c^{-1},b^{-1},a^{-1}; eabcd)$, we have that $\mathbf{V}$ consists of $(\alpha,\beta,\beta,\alpha; 1/\alpha\beta)$. The isomorphism $\hat{U} = \hat{T}^{\hat{\theta}} \simeq \hat{T}_H$, where $\mathbf{T}_H^*$ is the diagonal torus in $\mathbf{H} = GSp(2)$, defines a morphism

$$X_*(\mathbf{T}^*) \to X_*(\mathbf{T}^*)/X_*(\mathbf{V}) = X^*(\hat{T})/X^*(\hat{V}) = X^*(\hat{U} = \hat{T}^{\hat{\theta}}) = X^*(\hat{\mathbf{T}}_H) \xrightarrow{\sim} X_*(\mathbf{T}_H^*),$$

the last arrow being defined by

$$(x,y,z,t;w) \mapsto (x+y+w, x+z+w, t+y+w, t+z+w; x+y+z+t+2w),$$

and a norm map $N:\mathbf{T}^*\to\mathbf{T}^*_H$, given by

$$(x,y,z,t;w)\bmod(\alpha,\beta,\beta,\alpha;1/\alpha\beta)\mapsto(xyw,xzw,tyw,tzw;xyztw^2)=(a,b,e/b,e/a;e),$$

which is surjective since $(b,a/b,1,e/a;1)\mapsto(a,b,e/b,e/a;e)$.

To describe the norm for the twisted endoscopic group $\mathbf{C}$ (of (2) above), note that $\hat{T}_C\xrightarrow{\sim}\hat{T}_H$ by $((a,d),(b,c))\mapsto(ab,ac,bd,cd)$. Hence $X^*(\hat{T}_H)\xrightarrow{\sim}X^*(\hat{T}_C)$ via $(x,y,z,t)\bmod\{(\alpha,\beta,\beta,\alpha)\}$ $\mapsto\big((x+y,z+t),(x+z,y+t)\big)$, and the composition $X_*(\mathbf{T}^*)\to X^*(\hat{T}_H)\simeq X^*(\hat{T}_C)$ defines the norm map

$$N_C:\mathbf{T}^*\to\mathbf{T}^*_C,\ (x,y,z,t;w)\mapsto\big((xyw,ztw);(xzw,ytw)\big)\Big(=\big(\left(\begin{smallmatrix}xyw&0\\0&ztw\end{smallmatrix}\right),\left(\begin{smallmatrix}xzw&0\\0&ytw\end{smallmatrix}\right)\big)\Big).$$

Let us also describe the norm map for the twisted endoscopic group $\mathbf{C}_+$ of (3) above. Since the map $X^*(\hat{T}^{\hat{\theta}})\xrightarrow{\sim}X_*(T^*_+)$ is the identity, the norm is defined by

$$N:X_*(\mathbf{T}^*)\to X_*(\mathbf{T}^*)/X_*(\mathbf{V})=X^*(\hat{T})/X^*(\hat{V})=X^*(\hat{U}=\hat{T}^{\theta}=\hat{T}_+)=X_*(\mathbf{T}^*_+),$$

$N(x,y,z,t)=(x,y,z,t)\bmod(u,u^{-1},u^{-1},u)$.

G. Instability.

Recall that the set of θ-conjugacy classes within the stable θ-conjugacy class of a strongly θ-regular element t in G is parametrized by the set $D(F,\theta,t)=\ker[H^1(F,\mathbf{T}^*\xrightarrow{1-\theta}\mathbf{V})\to H^1(F,\mathbf{G})]=\ker[H^1(F,\mathbf{T}^{*\theta})\to H^1(F,\mathbf{G})]$, which is a group in our case, as $H^1(F,\mathbf{G})$ is trivial. There is an exact sequence

$$H^0(F,\mathbf{T}^*)=T^*\xrightarrow{1-\theta}H^0(F,\mathbf{V})=V\to D(F,\theta,t)\to H^1(F,\mathbf{T}^*)\xrightarrow{1-\theta}H^1(F,\mathbf{V}).$$

In our case of $\mathbf{G}=GL(4)\times GL(1)$, we have $H^1(F,\mathbf{T}^*)=\{1\}$ for all tori (or Galois actions, namely subgroups of the symmetric group S_4 on four letters), hence $D(F,\theta,t)=V/(1-\theta)T^*$.

There is a dual five term exact sequence, useful when stabilizing the twisted trace formula. Let $\phi:\hat{V}\to\hat{T}$ be the homomorphism dual to $\mathbf{T}^*\xrightarrow{1-\theta}\mathbf{V}$. Thus $\phi:X_*(\hat{V})=X^*(\mathbf{V})\to X^*(\mathbf{T}^*)=X_*(\hat{T})$ takes $\chi=(x,y,z,t;w)$ to $\big(\phi(\chi)\big)(a,b,c,d;e)=\chi(ad,bc,bc,ad;1/abcd)=(ad)^{x+t-w}(bc)^{y+z-w}$. Namely, ϕ takes $(x,y,z,t;w)$ in $\hat{V}=\hat{T}/\hat{U}=\hat{T}/\hat{T}^{\hat{\theta}}$ to $(xt/w,yz/w,yz/w,xt/w;1)$ in $\hat{T}$. Recall that $\hat{T}^{\hat{\theta}}=\{(a,b,e/b,e/a;e)\}$.

To obtain the dual sequence recall the Langlands isomorphism $H^1(W_F,\hat{T})=\mathrm{Hom}_{cts}(T,\mathbb{C}^\times)$ ($T=\mathbf{T}(F)$; [KS, about a page after Lemma A.3.A]), and its hypercohomology analogue ([KS, Lemma A.3.B]): $H^1(W_F,\hat{V}\xrightarrow{\phi}\hat{T})$ is isomorphic to the group $\mathfrak{K}(F,\theta,T^*)$ of characters $\mathrm{Hom}_{cts}\big(H^1(F,\mathbf{T}^*\xrightarrow{1-\theta}\mathbf{V}),\mathbb{C}^\times\big)$. Since the Weil group W_F of F acts on $\hat{T}$ and $\hat{V}$ via the Galois group , $=\mathrm{Gal}(\overline{F}/F)$, one has

$$H^0(W_F,\hat{V})=\hat{V}^{\Gamma}\xrightarrow{\phi}H^0(W_F,\hat{T})=\hat{T}^{\Gamma}\to\mathfrak{K}(F,\theta,T^*)\to H^1(W_F,\hat{V})\xrightarrow{\phi}H^1(W_F,\hat{T}).$$

Here $\mathfrak{K}(F,\theta,T^*)=H^1(W_F,\hat{V}\xrightarrow{\phi}\hat{T})$. This is the exact sequence [KS, A.1.1], for $\phi:\hat{V}\to\hat{T}$, which is dual to the previous five terms exact sequence for $1-\theta:\mathbf{T}^*\to\mathbf{V}$.

For each F-torus $\mathbf{T}$ in $\mathbf{G}$, and a strongly θ-regular element t in T, we can make the:

Definition. The *stable θ-orbital integral* Φ^{st} is the sum of the θ-orbital integrals on the θ-conjugacy classes within the stable θ-conjugacy class of t.

The set of such θ-conjugacy classes (for some t) is parametrized by the group $H^1(F, \mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V}) = H^1(F, \mathbf{T}^{*\theta})$ computed above. For each character κ of this group (into the group of roots of unity in $\mathbb{C}^\times$), we can also make the:

Definition. The *κ-orbital integral* is the linear combination of the θ-orbital integrals weighted by the values of κ at the element of $H^1(F, \mathbf{T}^* \to \mathbf{V})$ parametrizing the θ-conjugacy class.

These weighted (by κ) combinations of the θ-orbital integrals are to be compared with stable orbital integrals on the θ-endoscopic groups $\mathbf{H}$ of $(\mathbf{G}, \theta)$. The θ-endoscopic group $\mathbf{H}$ is determined from κ, by [KS, Lemma 7.2.A], via the surjection $H^1(W_F, \hat{V} \xrightarrow{\phi} \hat{T}) \to \mathrm{Hom}_{cts}\big(H^1(F, \mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V}), \mathbb{C}^\times\big)$ (see [KS, Lemma A.3.B]). Recall ([KS, A.1]) that:

Definition. The *first hypercohomology group* $H^1(G, A \xrightarrow{f} B)$ of the short complex $A \xrightarrow{f} B$ of abelian G-modules in degrees 0 and 1, is the quotient of the group of 1-hypercocycles, by the subgroup of 1-hypercoboundaries. A 1-*hypercocycle* is a pair (a, b) with a being a 1-cocycle of G in A, and $b \in B$ such that $f(a) = \partial b$ (∂b is the 1-cocycle $\sigma \mapsto b^{-1}\sigma(b)$ of G in B). A 1-hypercoboundary is a pair $\big(\partial a, f(a)\big), a \in A$.

Thus $H^1(W_F, \hat{V} \xrightarrow{\phi} \hat{T})$ consists of elements represented by pairs $(a, b), a \in H^1(W_{K/F}, \hat{V})$, where K/F is a Galois extension over which T splits and $\hat{V} = \hat{T}/\hat{U}, \hat{U} = (\hat{T}^{\hat{\theta}})^0$. Here $\phi : \hat{V} \to \hat{T}$ is the map dual to $1 - \theta : \mathbf{T}^* \to \mathbf{V}$, thus $\phi(x, y, z, t; w) = (xt/w, yz/w, yz/w, xt/w; 1)$, and $b \in \hat{T}$ satisfies $\phi(a) = \partial b$. The θ-endoscopic group $\mathbf{H}$ has a dual group whose connected component $\hat{H}$ is $Z_{\hat{G}}(b\hat{\theta})^0$, the connected centralizer of $b\hat{\theta}$ in $\hat{G}$ ([KS, Lemma 7.2.A]).

We proceed to describe the 1-**hypercocycles representing the non trivial characters** κ on $H^1(F, \mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V})$. The listing is as above, except that $H^1(\mathbf{T}^* \to \mathbf{V})$ is trivial in the case (1). Since $\mathbf{V}$ embeds in $\mathbf{T}$, we have that $H^1(F, \mathbf{T}^* \xrightarrow{1-\theta} \mathbf{V})$ embeds in $H^1(F, \mathbf{T}^* \xrightarrow{1-\theta} \mathbf{T})$, and we extend κ to a character of the bigger group.

(2) Here $\rho(\,,\,) = \langle \rho(\sigma) = (23) \rangle$, $\mathbf{T}$ splits over the quadratic extension E/F, $V = \{x = (\alpha, \beta, \beta, \alpha; 1/\alpha\beta); \alpha, \beta \in F^\times\}$, $(1 - \theta)T^* = \{x \in V; \alpha \in F^\times, \beta \in N_{E/F}E^\times\}$, then $\kappa \neq 1$ on $H^1(\mathbf{T}^* \to \mathbf{V})$ is given by $\kappa(x) = \chi_{E/F}(\beta)$, where $\chi_{E/F}$ is the non trivial character on $F^\times$ which is trivial on $NE^\times$. Extend $\chi_{E/F}$ to a character χ on $E^\times$. Then κ extends to $H^1(\mathbf{T}^* \to \mathbf{T}^*)$ by $(\alpha, \beta, \sigma\beta, \delta; e) \mapsto \chi(\beta)$. Recall that we have an exact sequence $1 \to E^\times \to W_{E/F} \to \langle \sigma \rangle \to 1$, in fact $W_{E/F} = \langle z \in E^\times, \sigma; \sigma^2 \in F - NE, \sigma z = \overline{z}\sigma \rangle$. Now $a \in H^1(W_{E/F}, \hat{T})$ is given by a function $a : W_{E/F} \to \hat{T}$ satisfying in particular $a_\sigma \sigma(a_z) = a_{\sigma z} = a_{\overline{z}\sigma} = a_{\overline{z}} a_\sigma$, thus $\sigma(a_z) = a_{\overline{z}}$. Take $a_z = \big(1, \chi(z), \chi(\overline{z}), 1\big)$. Then $a_{\sigma^2} = (1, -1, -1, 1)$. Take $a_\sigma = (1, 1, -1, 1) \in \hat{T}$ (then $a_{\sigma^2} = a_\sigma \sigma(a_\sigma)$, as $\sigma = (23)$). By definition of ϕ, we have $\phi(a_\sigma) = (1, -1, -1, 1)$ (the 5th entry is 1 if it is not explicitly written out). For $b = (1, 1, -1, 1) \in \hat{T}$, $\sigma b = (1, -1, 1, 1)$, and $\partial b(\sigma) = b^{-1}\sigma b$ is equal to $\phi(a_\sigma)$. Then $(a, b) \in H^1(\hat{T} \xrightarrow{\phi} \hat{T})$ represents κ. Now $Z_{\hat{G}}(b\hat{\theta}) \subset Z_{\hat{G}}\big(b\hat{\theta}(b)\big)$,

and $b\hat{\theta}b = (1,-1,-1,1)$, hence $\hat{H} = Z_{\hat{G}}(b\hat{\theta})^0$ is $\left\{\left(\left(\begin{smallmatrix}\alpha & 0 & 0 & \beta\\ 0 & a & 0 & 0\\ 0 & 0 & b & 0\\ \gamma & 0 & 0 & \delta\end{smallmatrix}\right), t\right); ab = t = \alpha\delta - \beta\gamma\right\}$. Note that $\mathbf{T}$ is not elliptic, so $\mathbf{H}$ is contained in a Levi subgroup of a (maximal) parabolic subgroup of $\mathbf{G}$.

(3) The case of $\rho(,) = \langle\sigma = \rho(14)\rangle$ is similarly handled.

(I) Here $\rho(,) = \langle\rho(\sigma) = (14)(23)\rangle, V = \{x = (\alpha,\beta,\sigma\beta = \beta,\sigma\alpha = \alpha; 1/\alpha\beta)\}$ and since $V/(1-\theta)T^* = (F^\times/NE^\times)^2$, there are 4 κ's, 3 non trivial. Two of these can be dealt with as in case (2) above (i.e. when κ is $x \mapsto \chi_{E/F}(\beta)$; the case when κ is $x \mapsto \chi_{E/F}(\alpha)$ is analogous to the case where $\sigma = (14)$ as in (3)). But now σ acts (non trivially) by permuting the two one parameter multiplicative entries in $\hat{H}$, thus we obtain the elliptic θ-endoscopic group $H = \mathbf{C}_+^E$ of type (3) in Section F.

The remaining κ on $V/(1-\theta)T^*$ is given by $x \mapsto \chi_{E/F}(\alpha\beta)$. Choosing extensions χ_1, χ_2 of $\chi_{E/F}$ to $E^\times$, we extend κ to T^* by $x = (\alpha,\beta,\sigma\beta,\sigma\alpha; e) \in T^* \mapsto \chi_1(\alpha)\chi_2(\beta)$. As in case (2), we define a 1-cocycle a of $W_{E/F}$ in $\hat{T}$ by $a_z = \big(\chi_1(z),\chi_2(z),\chi_2(\sigma z),\chi_1(\sigma z)\big)(z \in E^\times)$, then $a_{\sigma^2} = -I$, since $\chi_1(\sigma^2) = -1$. An a_σ which satisfies $a_\sigma\sigma(a_\sigma) = a_{\sigma^2} = -I$ is given by $a_\sigma = (1,1,-1,-1) \in \hat{T}$, and so $\phi(a_\sigma) = -I$. Choosing $b = (1,1,-1,-1) \in \hat{T}$, we have $\sigma b = (-1,-1,1,1)$, and $\partial b(\sigma) = b^{-1}\sigma b = -I$. Note that the norm N maps (x,y,z,t) in T to $((xy,zt),(xz,yt))$ in T_C, and $\sigma = (14)(23)$ then acts on T_C by $\sigma((a,b),(c,d)) = ((b,a),(d,c))$. Then σ does not permute the two factors in $\mathbf{C} = (GL(2)\times GL(2))'$, and we obtain the endoscopic group $\mathbf{C}$ of type (2) (see Section F). The other two κ correspond to the elliptic θ-endoscopic groups of type (3), as noted above.

(II) Here $\rho(,) = \langle\rho(\sigma\tau) = (14), \rho(\tau) = (23)\rangle$, and there are three non trivial characters κ of $V/(1-\theta)T^*$, given at $x = (\alpha,\beta,\tau\beta = \beta,\sigma\alpha = \alpha; 1/\alpha\beta)$ in V by $\chi_{E/E_1}(\alpha), \chi_{E/E_2}(\beta)$, $\chi_{E/E_1}(\alpha)\chi_{E/E_2}(\beta)$, where T splits over E/F, and $E_1 = E^{\langle\tau\rangle}, E_2 = E^{\langle\sigma\tau\rangle}$. The first two characters are dealt with as in case (I) ((2), and (3)). To deal with the last case, extend χ_{E/E_1} to a character χ_1 on $E^\times$, and χ_{E/E_2} to a character χ_2 on $E^\times$. We get a character $(\alpha,\beta,\tau\beta,\sigma\alpha; e) \mapsto \chi_1(\alpha)\chi_2(\beta)$ of T. A 1-cocycle of $W_{E/F}$ in $\hat{T}$ is given by

$$a_z = \big(\chi_1(z),\chi_2(z),\chi_2(\tau z),\chi_1(\sigma z)\big), \quad a_\sigma = (1,1,1,-1) \in \hat{T}, \quad a_\tau = (1,1,-1,1) \in \hat{T},$$

and $b = (1,1,-1,-1) \in \hat{T}$ satisfies $\phi(a_\sigma) = (-1,1,1,-1) = \partial b(\sigma)$, $\phi(a_\tau) = (1,-1,-1,1) = \partial b(\tau)$; the θ-endoscopic group of type (2) is obtained.

Note that τ acts on $\hat{T}_C$ by mapping $((u,v);(x,y)) = ((\begin{smallmatrix}u & 0\\ 0 & v\end{smallmatrix}),(\begin{smallmatrix}x & 0\\ 0 & y\end{smallmatrix})) = \operatorname{diag}(ux,vx,uy,vy)$ to $((x,y);(u,v))$, hence $(A,B) \in Z_{\hat{G}}(\hat{s}\hat{\theta})$ to (B,A); σ maps it to $((v,u);(y,x))$, and $\sigma\tau$ to $((y,x);(v,u))$. Thus the endoscopic group is $\mathbf{C}_{E_3}$, as $E_3 = E^\sigma$. Its group of rational points is $GL(2,E_3)'$.

(III) Here $\rho(,) = \langle\rho(\tau) = (12)(34), \rho(\sigma) = (14)(23)\rangle$, and the non trivial character κ of $V/(1-\theta)T^*$ is given by $x = (\alpha,\tau\alpha,\tau\sigma\alpha = \tau\alpha,\sigma\alpha = \alpha; 1/\alpha\tau\alpha) \mapsto \chi_{E/E_3}(\alpha), \alpha \in E_3 = E^{\langle\sigma\rangle}$. It extends to a character $x = (\alpha,\tau\alpha,\tau\sigma\alpha,\sigma\alpha; e) \mapsto \chi(x)$, if χ extends χ_{E/E_3} from $E_3^\times$ to $E^\times$. A corresponding $(a,b) \in H^1(\hat{T} \to \hat{T})$ is given by $a_z = \big(\chi(z),\chi(\tau z),\chi(\tau\sigma z),\chi(\sigma z)\big), z \in E^\times$. Since $\sigma^2 \in E_3 - N_{E/E_3}E$, we have $a_{\sigma^2} = (-1,-1,-1,-1)$, and $a_\sigma = (1,1,-1,-1) \in \hat{T}$

solves $a_\sigma\sigma(a_\sigma) = a_{\sigma^2}$. Then $\phi(a_\sigma) = -I = \partial b(\sigma)$ for $b = (1,1,-1,-1) \in \hat{T}$. Further, $\tau^2 \in E_1(-N_{E/E_1}E), \tau(\tau^2) = \tau^2, \tau^2\sigma(\tau^2) \in N_{E/E_3}E^\times$, hence $a_{\tau^2} = \Big(\chi(\tau^2), \chi\big(\tau(\tau^2)\big), 1/\chi\big(\tau(\tau^2)\big), 1/\chi(\tau^2)\Big)$, and $a_\tau = \big(\chi(\tau^2), 1, 1, 1/\chi(\tau^2)\big)$ satisfies $a_\tau\tau(a_\tau) = a_{\tau^2}\Big(\chi\big(\tau(\tau^2)\big) = \chi(\tau^2)\Big)$ and $a_\tau\tau(a_\sigma) = a_{\tau\sigma} = a_{\sigma\tau} = a_\sigma\sigma(a_\tau)$. Moreover, $\phi(a_\tau) = I = \partial b(\tau)$. The θ-endoscopic group defined by $b = (1,1,-1,-1)$ is of type (2).

Now τ acts on $\hat{T}_C$ by mapping $((u,v);(x,y))$ to $((v,u);(x,y))$; $\sigma\tau$ maps it to $((u,v);(y,x))$, and σ to $((v,u);(y,x))$, thus the endoscopic group is $\mathbf{C} = (GL(2) \times GL(2))'$.

(IV) Suppose that $\rho(,) = \langle\rho(\sigma) = (3421)\rangle$, $T^* = \{x = (\alpha, \sigma\alpha, \sigma^3\alpha, \sigma^2\alpha; e); \alpha \in E^\times, e \in F^\times\}$, $V = \{x = (\alpha, \sigma\alpha, \sigma^3\alpha = \sigma\alpha, \sigma^2\alpha = \alpha; 1/\alpha\sigma\alpha); \alpha \in E_3^\times)\}$, and $\kappa \neq 1$ is given by $x (\in V/(1-\theta)T^*) \mapsto \chi_{E/E_3}(\alpha), \chi_{E/E_3}$ being the non trivial character of $E_3^\times$ which is trivial on $N_{E/E_3}E^\times$. Choosing an extension χ of χ_{E/E_3} to $E^\times$, we can extend κ to T^* (and $H^1(T^* \to T^*)$) by $x \mapsto \chi(\alpha)$. A corresponding element of $H^1(\hat{T} \xrightarrow{\phi} \hat{T})$ is a pair (a,b), where a is a 1-cocycle of $W_{E/F}$ in $\hat{T}$. Note that $1 \to E^\times \to W_{E/E_3} \to \langle\sigma^2\rangle \to 1$, where $(\sigma^2)^2 \in E_3 - N_{E/E_3}E$. Put $a_z = \big(\chi(z), \chi(\sigma z), \chi(\sigma^3 z), \chi(\sigma^2 z); 1\big)$. As $\sigma^4 \in E_3 - N_{E/E_3}E$, and $\sigma(\sigma^4)\cdot\sigma^4 \in N_{E/E_3}E^\times$, we have $\chi(\sigma^4) = -1$ and $\chi\big(\sigma(\sigma^4)\big)\chi(\sigma^4) = 1$. Then $a_{\sigma^4} = (-1,-1,-1,-1)$. From $a_{\sigma^4} = a_{\sigma^2}\sigma^2(a_{\sigma^2})$, if $a_{\sigma^2} = (a,b,c,d)$, then $ad = -1 = bc$. Then $a_{\sigma^2} = (1,1,-1,-1) = a_\sigma\sigma(a_\sigma) = (ac, ba, cd, db)$ has the solution $a_\sigma = (1,1,1,-1) \in \hat{T}$. Also $\phi(a_\sigma) = (-1,1,1,-1)$. If $b = (a',b',c',d'), \sigma b = (c',a',d',b')$, and $\partial b(\sigma) = (c'/a', a'/b', d'/c', b'/d')$ has to be $\phi(a_\sigma)$, then a solution is given by $b = (1,1,-1,-1) \in \hat{T}$.

The centralizer $Z_{\hat{G}}(b\hat{\theta})$ is the group $GO(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix})$ of orthogonal similitudes of the symmetric matrix $(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix})$. This group is isomorphic to $\big(GL(2,\mathbb{C}) \times GL(2,\mathbb{C})\big)/\mathbb{C}^\times$ via

$$\Big(g, g_1 = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})\Big) \mapsto \begin{pmatrix} ag & bg\boldsymbol{\varepsilon} \\ c\boldsymbol{\varepsilon}g & d\boldsymbol{\varepsilon}g\boldsymbol{\varepsilon} \end{pmatrix}, \boldsymbol{\varepsilon} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, w = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}),$$

where the similitude factor is $\det g_1 \cdot \det g$. This is the θ-endoscopic group (2).

Note that σ acts on $\hat{T}_C$ by mapping $((u,v);(x,y))$ to $((y,x);(u,v))$, and σ^2 then maps it to $((v,u);(y,x))$. The endoscopic group is then $\mathbf{C}_{E_3}$, $E_3 = E^{\sigma^2}$.

Suppose that $\rho(,) = W = \langle\rho(\sigma) = (12)(34), \rho(\tau) = (23)\rangle$. Then $\sigma\tau\sigma = (14)$, a splitting field of T is E/F, and E_1 denotes the fixed field of $\sigma\tau\sigma$ in E. The non trivial character κ of $V/(1-\theta)T^*$ is given by $x = (\alpha, \sigma\alpha, \tau\sigma\alpha = \sigma\alpha, \sigma\tau\sigma\alpha = \alpha; 1/\alpha\sigma\alpha) \mapsto \chi_{E/E_1}(\alpha)$. It extends to T^* by $x = (\alpha, \sigma\alpha, \tau\sigma\alpha, \sigma\tau\sigma\alpha; e) \mapsto \chi(\alpha)$, where χ extends χ_{E/E_1} from $E_1^\times$ to $E^\times$. A 1-cocycle of $W_{E/F}$ in $\hat{T}$ is given as follows. At $z \in E^\times$, put $a_z = \Big(\chi(z), \chi\big(\sigma(z)\big), \chi\big(\sigma\tau(z)\big), \chi\big(\sigma\tau\sigma(z)\big)\Big)$. Then $a_{(\sigma\tau\sigma)^2} = (-1,1,1,-1)$. Hence $a_{\sigma\tau\sigma} = (1,1,1,-1) = a_\sigma\sigma(a_{\tau\sigma}) = a_\sigma\sigma(a_\tau)\sigma\tau(a_\sigma) = (a,b,c,d)(\beta,\alpha,\delta,\gamma)(c,a,d,b)$ has a solution $a_\sigma = I, a_\tau = (1,1,-1,1) \in \hat{T}$, and $\phi(a_\tau) = (1,-1,-1,1)$. Then $b = (1,1,-1,-1) \in \hat{T}$ satisfies $\partial b(\sigma) = I, \partial b(\tau) = (1,-1,-1,1), \partial b(\sigma\tau\sigma) = \phi(a_{\sigma\tau\sigma}) = (-1,1,1,-1)$, and the corresponding θ-endoscopic group is of type (2).

In the comparison of the unstable (κ-)θ-orbital integral at a strongly θ-regular element t, and the stable orbital integral on the endoscopic group H_κ determined by κ, a **transfer factor** appears. It is a product of a sign and of a Jacobian factor $\Delta_{G,H_\kappa} = \Delta_G/\Delta_{H_\kappa}$, denoted Δ_{IV}

in [KS, 4.5], which we proceed to describe in the main cases. Thus

$$\Delta_G(t^*\theta) = |\det\big(1 - \mathrm{Ad}(t^*)\theta\big)|\,\mathrm{Lie}\,\mathbf{G}/\,\mathrm{Lie}\,\mathbf{T}^*|_F^{1/2}$$

is $= \Delta_H(Nt^*) = |\det\big(1 - \mathrm{Ad}(Nt^*)\big)|\,\mathrm{Lie}\,\mathbf{H}/\,\mathrm{Lie}\,Z_{\mathbf{H}}(Nt^*)|_F^{1/2}$. If $t^* = \mathrm{diag}(x,y,z,t), Nt^* = N_H t^* = (xy, xz, yt, zt)$. Here $\mathbf{H} = GSp(2), Z_{\mathbf{H}}(Nt^*)$ is the diagonal, and $\mathrm{Lie}(\mathbf{H})/\,\mathrm{Lie}\,Z_{\mathbf{H}}(Nt^*)$ is the direct sum $\mathrm{Lie}(\mathbf{N}) \oplus \mathrm{Lie}(\overline{\mathbf{N}})$, the upper and lower nilpotent subgroups. On $\mathrm{Lie}(\mathbf{N}) = \begin{pmatrix} 0 & x_1 & y_1 & z_1 \\ 0 & 0 & t_1 & y_1 \\ 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\det\big(1 - \mathrm{Ad}(a,b,c,d)\big)$ is $(1-a/b)(1-a/c)(1-a/d)(1-b/c)$. On $\mathrm{Lie}(\overline{\mathbf{N}})$ the same factor, but with (a,b,c,d) replaced by $(a^{-1}, b^{-1}, c^{-1}, d^{-1})$, is obtained. Hence

$$\Delta_G(t^*\theta) = |(x-t)(y-z)(xy-zt)(xz-yt)|_F/|xyzt|_F^{3/2}.$$

For $\kappa = 1$, we have $\Delta_G(t^*\theta) = \Delta_H(Nt^*)$. For $\kappa \neq 1$ which defines the endoscopic group $\mathbf{C}$, the norm $N_G t^*$ is $\Big(\big(\begin{smallmatrix} xy & 0 \\ 0 & zt \end{smallmatrix}\big), \big(\begin{smallmatrix} xz & 0 \\ 0 & yt \end{smallmatrix}\big)\Big)$, and

$$\Delta_C(N_C t^*) = |(1 - \frac{xy}{zt})(1 - \frac{zt}{xy})(1 - \frac{xz}{yt})(1 - \frac{yt}{xz})|_F^{1/2} = |(xy-zt)(xz-yt)|_F/|xyzt|_F.$$

Then

$$\Delta_{G,C}(t^*) = \Delta_G(t^*\theta)/\Delta_C(N_C t^*) = |(x-t)(y-z)|_F/|xyzt|_F^{1/2}.$$

For $\kappa \neq 1$ which defines the endoscopic group $\mathbf{C}_+$, the norm $N_{C_+}t^*$ is

$$(x,y,z,t)\,\mathrm{mod}(u, 1/u, 1/u, u), \qquad \text{and} \qquad \Delta_{C_+}(N_{C_+}t^*) = |(y-z)^2/yz|_F^{1/2},$$

so that

$$\Delta_{G,C_+}(t^*) = |(x-t)(xy-zt)(xz-yt)|_F/|xt|_F^{3/2}|yz|_F.$$

H. Kazhdan's decomposition.

A main ingredient in our proof of the matching is the (twisted analogue [F7] of) Kazhdan's decomposition [K, p. 226], which we now recall. Let $\mathbf{H}$ be a connected reductive R-group, where R is the ring of integers of F, and put $H = \mathbf{H}(F), K_H = \mathbf{H}(R)$.

Definition ([K]). An element $k \in H$ is called *absolutely semi simple* if $k^a = 1$ for some positive integer a which is prime to the residual characteristic p of R. A $k \in H$ is called *topologically unipotent* if $k^{q^N} \to 1$ as $N \to \infty$, $q = \#(R/\boldsymbol{\pi} R)$, $\boldsymbol{\pi}$ generates the maximal ideal in R.

1. Proposition ([K]). *Any element $k \in K_H$ has a unique decomposition $k = su = us$, where s is absolutely semi simple, u is topologically unipotent, and s, u lie in K_H. For any $k \in K_H$ and $x \in H$, if $\mathrm{Int}(x)k (= xkx^{-1})$ lies in K_H, then x is in $K_H Z_H(s)$, where $Z_H(s)$ denotes the centralizer of s in H.* □

In fact [K] proves this only for $\mathbf{H} = GL(n)$, but since s is defined as a limit of a sequence of the form k^{q^m}, both s and u lie in K_H.

The twisted analogue which we need is reproduced next (from [F7]). Let $\mathbf{G}$ be a reductive connected R-group and θ an automorphism of $G = \mathbf{G}(F)$ of order ℓ $\big((\ell, p) = 1\big)$, whose restriction to $K = \mathbf{G}(R)$ is an automorphism of K of order ℓ. Denote by $\langle K, \theta\rangle$ the group generated by K and θ.

Definition. The element $k\theta$ of $G\theta \subset \langle G, \theta\rangle$ is called *absolutely semi-simple* if $(k\theta)^a = 1$ for some positive integer a indivisible by p.

2. Proposition ([F7, Proposition 2]). *Any $k\theta \in K\theta$ has a unique decomposition $k\theta = s\theta \cdot u = u \cdot s\theta$ with absolutely semi simple $s\theta$ (called the absolutely semi simple part of $k\theta$) and topologically unipotent u (named the topologically unipotent part of $k\theta$). Both s and u lie in K. In particular, $Z_G(s\theta \cdot u)$ lies in $Z_G(s\theta)$.* □

3. Proposition ([F7, Proposition 3]). *Given $k \in K$, put $\tilde\theta(h) = s\theta(h)s^{-1}$, where $k\theta = s\theta \cdot u$. This $\tilde\theta$ is an automorphism of order ℓ on $Z_K\big((s\theta)^\ell\big)$. Suppose that the first cohomology set $H^1\Big(\langle\tilde\theta\rangle, Z_K\big((s\theta)^\ell\big)\Big)$, of the group $\langle\tilde\theta\rangle$ generated by $\tilde\theta$, with coefficients in the centralizer $Z_K\big((s\theta)^\ell\big)$ in K, injects in $H^1\Big(\langle\tilde\theta\rangle, Z_G\big((s\theta)^\ell\big)\Big)$. Then any $x \in G$ such that $\mathrm{Int}(x)(k\theta)$ is in $K\theta$, must lie in $KZ_G(s\theta)$.* □

The supposition of this proposition can be verified for our group $G = GL(4,F) \times GL(1,F)$ and our automorphism θ in the same way it is verified in [F7] for $GL(3,F)$. Note also (see [F7]) that if the elements $k\theta = s\theta \cdot u$ and $k'\theta = s'\theta \cdot u'$ of $K\theta$ are conjugate by $\mathbf{G}(\overline{F})$ ($\overline{F}$ is a separable closure of F) then so are $s\theta$ and $s'\theta$, and if $s = s'$ then u, u' are conjugate in $Z_{\mathbf{G}(\overline{F})}(s\theta)$.

Our argument uses the function

$$1_{s\theta}(u) = |K/K \cap Z_G(s\theta)| 1_K(s\theta \cdot u) = \int_{K/K\cap Z_G(s\theta)} 1_K\big(\mathrm{Int}(x)(s\theta \cdot u)\big)dx.$$

Then the orbital integral $\Phi_{1_K}(k\theta) = \int_{G/Z_G(k\theta)} 1_K(\mathrm{Int}(x)(k\theta))dx$ is equal – by Proposition 3 – to $\int_{Z_G(s\theta)/Z_G(s\theta\cdot u)} 1_{s\theta}(\mathrm{Int}(x)u)dx = \Phi_{1_{s\theta}}(u)$, the orbital integral of the characteristic function $1_{s\theta}$ of the compact subgroup $Z_K(s\theta) = K \cap Z_G(s\theta)$ of $Z_G(s\theta)$ (multiplied by $|K/Z_K(s\theta)|$) at the topologically unipotent element u in $Z_K(s\theta)$.

Since $(k\theta)^2 = s\theta(s) \cdot u^2$, where in our case $\theta(g,t) = \big(\theta(g), t\det g\big), g \in GL(4,F), t \in F^\times$, $\theta(g) = J\,{}^tg^{-1}J^{-1}$, we shall deal with various cases according to the values of $s\theta(s)$ (s denotes also the $GL(4,F)$-component of s).

4. Lemma. *If $x = s\theta(s)$ has the eigenvalue λ, then it has the eigenvalue λ^{-1} too.*

Proof. If $\xi \neq 0$ is a vector with ${}^tx\xi = \lambda\xi$, then $\theta(x)J\xi = J\,{}^tx^{-1}J^{-1} \cdot J\xi = \lambda^{-1}J\xi$, and $s\theta(x)s^{-1} = x$. □

Then $s\theta(s)$ has eigenvalues $(\lambda, \lambda^{-1}, \mu, \mu^{-1})$. The main case to be considered is when $s\theta(s) = I$. Then $sJ\,{}^ts^{-1}J^{-1} = I$ implies $sJ = J\,{}^ts = -{}^t(sJ)$ is anti symmetric, and

$$\begin{aligned} Z_G(s\theta) = \{(g,t);\ &(g,t)(s,1)(\theta(g)^{-1}, t^{-1}\det g^{-1}) = (s,1),\\ &\text{thus } \det g = 1 \text{ and } gsJ\,{}^tg = sJ\} = Sp(sJ) \times GL(1). \end{aligned}$$

Any anti symmetric matrix sJ is similar to J, namely there exists some h in $GL(4,F)$ with $sJ = hJ\,{}^th$, thus $s = h\theta(h)^{-1}$, and $Sp(sJ) = hSp(J)h^{-1}$, thus we may assume $s = I$.

I. Decompositions for $GL(2)$.

Before we start computing the orbital integrals of 1_K on $GSp(2,F)$ and the θ-orbital integrals of 1_K on $GL(4,F)$, let us compute the analogous integral for $GL(2,F)$. Let $D \in F - F^2$ with $|D| = 1$ or $|\boldsymbol{\pi}|$. Denote by T the torus $T = \{(\begin{smallmatrix} a & bD \\ b & a \end{smallmatrix}) \in GL(2,F)\}$; put $K = GL(2,R)$, $w = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}), \boldsymbol{\varepsilon} = (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})$, $\mathbf{D} = (\begin{smallmatrix} 0 & D \\ 1 & 0 \end{smallmatrix})$, $\|g\|$ denotes det g.

1. Lemma. *We have a disjoint decomposition* $G = GL(2,F) = \underset{m\geq 0}{\cup} T(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^m \end{smallmatrix})K$.

Proof. Consider the embedding $T\backslash G \hookrightarrow X(D) = \{x \in G; \|x\| = -D, (w\boldsymbol{\varepsilon}x)^2 = D$ (equivalently: ${}^t x = x)\}$, by $g \mapsto \boldsymbol{\varepsilon}wg^{-1}\mathbf{D}g = \|g\|^{-1}\ {}^t g\boldsymbol{\varepsilon}w\mathbf{D}g$. Any $x \in X(D)$ has the form $x = k(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix})k'$ with $|\alpha| \leq |\beta|, k, k' \in K$. If $|\alpha| = |\beta|$ then $|\alpha| = 1, x \in K, w\boldsymbol{\varepsilon}x$ is semi simple $\big((w\boldsymbol{\varepsilon}x)^2 = D\big)$ with eigenvalues $\alpha_1, \alpha_2, \alpha_1\alpha_2 = -D$ and $\alpha_1^2 = \alpha_2^2 = D$, thus $\alpha_1 = -\alpha_2 = \sqrt{D}$. Then there exists k_1 in K with $w\boldsymbol{\varepsilon}x = k_1^{-2}\mathbf{D}k_1$, and $Tk_1 \mapsto x = \boldsymbol{\varepsilon}wk_1^{-1}\mathbf{D}k_1$. If $|\alpha| < |\beta|$ then $x = k(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix})k' = {}^t k'(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}){}^t k$ implies ${}^t({}^t kk'^{-1}) = (\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}){}^t kk'^{-1}(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix})^{-1}$, thus ${}^t kk'^{-1} = (\begin{smallmatrix} k_1 & k_2 \\ k_2\alpha/\beta & k_4 \end{smallmatrix})$, where k_1, k_4 are units, hence – putting $\alpha' = k_1\alpha\|k'\|$ and $\beta' = (k_4\beta - k_1\alpha(k_2/k_1)^2)\|k'\|$ – we have

$$k(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix})k' = {}^t k'(\begin{smallmatrix} k_1 & k_2\alpha/\beta \\ k_2 & k_4 \end{smallmatrix})(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix})k' = \|k''\|^{-1}\ {}^t k''(\begin{smallmatrix} \alpha' & 0 \\ 0 & \beta' \end{smallmatrix})k'',$$

where $k'' = (\begin{smallmatrix} 1 & k_2/k_1 \\ 0 & 1 \end{smallmatrix})k'$, and $|\alpha'| = |\alpha|$, $|\beta'| = |\beta|$. Since

$$(-ab)^{-1}(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix})(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})(\begin{smallmatrix} 0 & D \\ 1 & 0 \end{smallmatrix})(\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix}) = (\begin{smallmatrix} aD/b & 0 \\ 0 & -b/a \end{smallmatrix}),$$

any (α,β) with $|\alpha| < |\beta|, \alpha\beta = -D$, is obtained from (a,b) with $|a| < |b|$. As $T(\begin{smallmatrix} 0 & b \\ a & 0 \end{smallmatrix})K = T(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix})K = T(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^m \end{smallmatrix})K$ for some $m \geq 1$, we are done. $\square$

Denote by 1_K the (quotient by the volume $|K|$ of K of the) characteristic function of K in G, by e the ramification index of $E = F(\sqrt{D})$ over F, by q the cardinality of $R/\boldsymbol{\pi}R$, and by $q_E = q^{2/e}$ the cardinality of $R_E/\boldsymbol{\pi}_E R_E, \boldsymbol{\pi} = \boldsymbol{\pi}_E^e$. Put $\mathrm{ord}(\varepsilon\boldsymbol{\pi}^n) = n, |\varepsilon| = 1, |\varepsilon\boldsymbol{\pi}^n| = q^{-n}$. Fix $\gamma = \alpha + \beta\sqrt{D}$ with $\beta \neq 0$ in $E, \alpha, \beta \in F$. Write $\boldsymbol{\gamma} = (\begin{smallmatrix} \alpha & \beta D \\ \beta & \alpha \end{smallmatrix}) \in T$.

2. Lemma. *The integral* $\int_{T\backslash G} 1_K(g^{-1}\boldsymbol{\gamma}g)dg$ *is equal to* $\frac{q-1+2/e}{q-1}|\beta|^{-1} - \frac{2/e}{q-1}$.

Proof. If $f \in C_c^\infty(T\backslash G)$, then

$$\int_{T\backslash G} f(g)dg = \sum_{m\geq 0}\int_{K\cap(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{-m} \end{smallmatrix})T(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^m \end{smallmatrix})\backslash K} f\big((\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^m \end{smallmatrix})k\big)dk.$$

But $K\cap(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{-m} \end{smallmatrix})T(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^m \end{smallmatrix}) = \{(\begin{smallmatrix} a & bD\boldsymbol{\pi}^m \\ \boldsymbol{\pi}^{-m}b & a \end{smallmatrix}) \in K\} = \mathrm{Int}(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{-m} \end{smallmatrix})\{(\begin{smallmatrix} a & bD \\ b & a \end{smallmatrix}); |b| \leq |\boldsymbol{\pi}|^m\}$ is $R_E(m)^\times$, where $R_E(m) = \{a + b\sqrt{D}; |a| \leq 1, |b| \leq |\boldsymbol{\pi}^m|\} = R + \boldsymbol{\pi}^m R_E = R + R\boldsymbol{\pi}^m\sqrt{D}$. Put also $R_E^\times = \{a + b\sqrt{D}; a^2 - b^2D \in R^\times\}$. Then

$$\int_{T\backslash G} 1_K(g^{-1}\boldsymbol{\gamma}g)dg = \sum_{m\geq 0}[R_E^\times : R_E(m)^\times]1_K(\begin{smallmatrix} \alpha & \beta D\boldsymbol{\pi}^m \\ \beta\boldsymbol{\pi}^{-m} & \alpha \end{smallmatrix}).$$

This sum ranges over $0 \le m \le \operatorname{ord}(\beta) = B$. The index is computed as follows:

$$\begin{aligned}[R_E^\times : R_E(m)^\times] &= [R_E^\times : 1 + \boldsymbol{\pi}^m R_E]/[R_E(m)^\times : 1 + \boldsymbol{\pi}^m R_E] \\ &= \frac{(q_E - 1)q_E^{em-1}}{(q-1)q^{m-1}} = \begin{cases} 1, & \text{if } e = 1, m = 0, \\ (q+1)q^{m-1}, & \text{if } e = 1, m \ge 1, \\ q^m, & \text{if } e = 2, \end{cases}\end{aligned}$$

since $R_E(m)^\times/(1 + \boldsymbol{\pi}^m R_E) \simeq R^\times/R^\times \cap (1 + \boldsymbol{\pi}^m R_E), q_E = q^{2/e}, \boldsymbol{\pi} = \boldsymbol{\pi}_E^e$. Then the integral is equal (when $e = 2$) to:

$$= \sum_{0 \le m \le B} q^m = \frac{q^{B+1} - 1}{q - 1} = \frac{q|\beta|^{-1} - 1}{q - 1} = \frac{q}{q-1}|\beta|^{-1} - \frac{1}{q-1},$$

and to

$$= 1 + (q+1) \sum_{1 \le m \le B} q^{m-1} = 1 + (q+1)\frac{q^m - 1}{q - 1} = \frac{q+1}{q-1}|\beta|^{-1} - \frac{2}{q-1}$$

when $e = 1$. □

We shall also need an analogous decomposition for $SL(2, F)$. For $D \in F - F^2$ put $E = F(\sqrt{D})$. The torus $T = \{(\begin{smallmatrix} a & bD \\ b & a \end{smallmatrix}) \in GL(2, F)\}$ is isomorphic to $E^\times$. For $\rho \in F^\times$ put $T^\rho = (\begin{smallmatrix} 1 & 0 \\ 0 & \rho \end{smallmatrix})^{-1} T (\begin{smallmatrix} 1 & 0 \\ 0 & \rho \end{smallmatrix}) = \{(\begin{smallmatrix} a & bD\rho \\ b/\rho & a \end{smallmatrix})\}$. Write $\phi_\rho^D(a + b\sqrt{D}) = (\begin{smallmatrix} a & bD\rho \\ b/\rho & a \end{smallmatrix})$. Put $T_0^\rho = T^\rho \cap SL(2, F), K_0 = K \cap SL(2, F)$. As usual, $\boldsymbol{\pi}$ is a generator of the maximal ideal in the ring R of integers in F, and ε is a unit, in $R^\times$. Write $\overline{\rho} = \operatorname{ord}(\rho)$ thus $|\rho| = |\boldsymbol{\pi}|^{\overline{\rho}}$. Fix $\rho \in \{1, \boldsymbol{\pi}\}$ if E/F is unramified, and $\rho \in \{1, \varepsilon\} = R^\times/R^{\times 2}$ if E/F is ramified.

3. Lemma. *If E/F is unramified then $SL(2, F)$ is the disjoint union over the set of $j \ge 0$ such that 2 divides $j - \overline{\rho}$, and over $\varepsilon \in R^\times/R^{\times 2}$ if $j > 0$ and $\varepsilon = 1$ if $j = 0$, of the sets $T_0^\rho r_{j,\varepsilon} K_0$, where $r_{j,\varepsilon} = t_\varepsilon \operatorname{diag}(\boldsymbol{\pi}^{-(j-\overline{\rho})/2}, \varepsilon\boldsymbol{\pi}^{(j-\overline{\rho})/2})$, and where t_ε is an element of T^ρ with determinant $\|t_\varepsilon\| = \varepsilon^{-1}$. If E/F is ramified then the union $SL(2, F) = \cup_{j \ge 0} T_0^\rho r_j K_0$ is disjoint, where $r_j = \phi_\rho^D(\boldsymbol{\pi}_E^{-j}) \operatorname{diag}(1, \boldsymbol{\pi}^j), \boldsymbol{\pi}_E = \sqrt{-\boldsymbol{\pi}}, D = -\boldsymbol{\pi}$.*

Proof. We have a disjoint union $GL(2, F) = \underset{j \ge 0}{\cup} T(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^j \end{smallmatrix})K = \underset{j \ge 0}{\cup} T^\rho(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{j-\overline{\rho}} \end{smallmatrix})K$, for any $\rho = u\boldsymbol{\pi}^{\overline{\rho}}$ in $F^\times (u \in R^\times)$. When E/F is unramified, $\boldsymbol{\pi}^{j-\overline{\rho}}$ lies in $R^\times N_{E/F} E^\times \big(= \det(T^\rho K)\big)$ precisely when 2 divides $j - \overline{\rho}$. In this case, $r_j = \operatorname{diag}(\boldsymbol{\pi}^{-(j-\overline{\rho})/2}, \boldsymbol{\pi}^{(j-\overline{\rho})/2})$ lies in $T^\rho \operatorname{diag}(1, \boldsymbol{\pi}^{j-\overline{\rho}}) \cap SL(2, F)$. If tr_jk lies in $SL(2, F)$, then $\|t\|$ lies in $R^\times$, in fact multiplying t by $\varepsilon \in R^\times$ we may assume that $\|t\|$ ranges over $R^\times/R^{\times 2}$. Note that $\|t\| = N_{E/F}\big((\phi_\rho^D)^{-1}(t)\big)$. Since E/F is unramified, we have $N_{E/F} R_E^\times = R^\times$, where R_E is the ring of integers in E. Hence for any ε in $R^\times$ there is t_ε in T^ρ with $\|t_\varepsilon\| = \varepsilon^{-1}$, and we may assume that $t = t_0 t_\varepsilon \in T_0^\rho t_\varepsilon$. Then tr_jk lies in $T_0^\rho t_\varepsilon r_j \operatorname{diag}(1, \varepsilon) K_0$.

If $j = 0$, then $T_0^\rho t_\varepsilon (\begin{smallmatrix} 1 & 0 \\ 0 & \varepsilon \end{smallmatrix}) K_0 = T_0^\rho K_0$. Otherwise the cosets $T_0^\rho t_\varepsilon r_j (\begin{smallmatrix} 1 & 0 \\ 0 & \varepsilon \end{smallmatrix}) K_0$ and $T_0^\rho r_j K_0$ are disjoint, since $r_j^{-1} t t_\varepsilon r_j \in K_0 (\begin{smallmatrix} 1 & 0 \\ 0 & \varepsilon \end{smallmatrix})$ implies that $\varepsilon = \|tt_\varepsilon\| \in R^{\times 2}$ when $j > 0$. In particular, when $\overline{\rho} = 1$, and $\varepsilon \in R^\times - R^{\times 2}$, t_ε is not in K.

If E/F is ramified we can choose the uniformizer $\boldsymbol{\pi}_E$ in $R_E \subset E$ to be $\sqrt{-\boldsymbol{\pi}}$, and D to be $-\boldsymbol{\pi}$, so that $N_{E/F}\boldsymbol{\pi}_E$ is $\boldsymbol{\pi}$. Then $GL(2,F) = \underset{j\geq 0}{\cup} T^\rho \phi_\rho^D(\boldsymbol{\pi}_E^{-j})(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^j \end{smallmatrix})K$. If tr_jk lies in $SL(2,F)$ then $\|t\| \in R^\times \cap N_{E/F}E^\times = R^{\times 2}$. Hence $t = t_0\boldsymbol{\varepsilon}$ with $\|t_0\| = 1$ and $\boldsymbol{\varepsilon} \in R^\times$, and $SL(2,F) = \underset{j\geq 0}{\cup} T_0^\rho r_j K_0$. □

4. Corollary. *For $f \in C_c^\infty\big(SL(2,F)\big)$, since $SL(2,F) = \underset{r\in R}{\cup} T_0 r K_0$, we have*

$$\begin{aligned}\int_{SL(2,F)} f(h)dh &= \sum_{r\in R} |T_0 \cap rK_0r^{-1}|_{T_0}^{-1} \int_{T_0} dt \int_{K_0} f(trk)dk \\ &= \sum_{r\in R} |R_T^\times|^{-1}[R_T^\times : T_0 \cap rK_0r^{-1}] \int_{T_0} dt \int_{K_0} f(trk)dk,\end{aligned}$$

where $R_T = T_0(R) = T_0 \cap K_0$. □

Yet another analogue is when $E_1 = F(\sqrt{D})$ and $E_3 = F(\sqrt{A})$ are two quadratic extensions of F, one of which is ramified while the other is not. A prime indicates determinant in $F^\times$, for $GL(2,E_3)', K'\big(K = GL(2,R_3)\big), T'_\rho$ (T_ρ is the torus $(\begin{smallmatrix} a & bD\rho \\ b/\rho & a \end{smallmatrix})$ in $GL(2,E_3)$ which is isomorphic to $E^\times, E = E_1E_3$). We normalize A, D, ρ to be integral of minimal order, ρ represents $E_3^\times/N_{E/E_3}E^\times$, and we write $\rho = u\boldsymbol{\pi}_3^{\overline{\rho}}$, $\overline{\rho} = \operatorname{ord}_3 \rho$, $u \in R_3^\times$. Of course, R_3 is the ring R_{E_3} of integers in E_3, and $\boldsymbol{\pi}_3$ denotes $\boldsymbol{\pi}_{E_3}$.

5. Lemma. *We have a disjoint decomposition $GL(2,E_3)' = \cup T'_\rho r_j K'$, where $j \geq 0$ and $r_j \in T_\rho(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}_3^j \end{smallmatrix})$ if E/E_3 is ramified (E_1/F is ramified), while when E/E_3 is unramified, the summation ranges over $j \geq 0$ such that $j - \overline{\rho}$ is even, and $r_j = \boldsymbol{\pi}_3^{-(j-\overline{\rho})/2}\begin{pmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}_3^{j-\overline{\rho}} \end{pmatrix}$.*

Proof. We use $GL(2,E_3) = \cup_{j\geq 0} T_\rho(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}_3^{j-\overline{\rho}} \end{smallmatrix})K$. When E/E_3 is ramified, $\overline{\rho} = 0, \boldsymbol{\pi}_3 = -D \in F^\times$ and $\boldsymbol{\pi}_E = \sqrt{D}$, so that $N_{E/E_3}(\boldsymbol{\pi}_E) = \boldsymbol{\pi}_3$. Hence if $h = trk \in GL(2,E_3)'$ we may assume that $\|h\| \in R^\times$, and rewrite h as $h = tt_0rk$ for some $t_0 \in T_\rho$ with $\|t_0r\| = 1$. Then $\|t\| \in R_3^\times \cap N_{E/E_3}E^\times = R_3^{\times 2}$, so there is $\varepsilon \in R_3^\times$ with $\|t\| = \varepsilon^2$, and we can write $h = \varepsilon^{-1}t \cdot t_0r \cdot \varepsilon k$, with $\|\varepsilon^{-1}t\| = 1$ and $\|\varepsilon k\| \in R^\times$.

When E/E_3 is unramified, and $h = trk \in GL(2,E_3)', r = (\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}_3^{j-\overline{\rho}} \end{smallmatrix})$, since $\|t\| \in N_{E/E_3}E^\times = \boldsymbol{\pi}_3^{2\mathbb{Z}}R_3^\times$, and $\boldsymbol{\pi}_F = \boldsymbol{\pi}_3^2$ (since E_3/F is ramified), we must have that $j - \overline{\rho}$ is even. We may assume that $\|h\|$ lies in $R^\times$, take r_j as in the lemma, and modify k by a scalar in $R_3^\times$. Then $\|k\|$ is represented by $R_3^\times/R_3^{\times 2}$, namely by $R^\times$, since $R_3^\times = R_3^{\times 2}R^\times$ when E_3/F is ramified $(a + b\sqrt{\boldsymbol{\pi}} = a(1 + \frac{b}{a}\sqrt{\boldsymbol{\pi}}) \in R^\times R_3^{\times 2})$. □

J. Decomposition for $Sp(2)$.

In computing the orbital integrals of 1_K on $H = GSp(2,F)$, we shall use the following decomposition.

1. Lemma. *We have a disjoint decomposition* $H = GSp(2,F) = \underset{n\geq 0}{\cup} Ku_n\mathbf{C}_A = \underset{n\geq 0}{\cup}\mathbf{C}_A u_n K$, *where* $A \in F - F^2$, $u_n = \begin{pmatrix} 1&0&0&\boldsymbol{\pi}^{-n}/A \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$, $\mathbf{C}_A = \left\{\left(\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right) \in H; \mathbf{a} = \left(\begin{smallmatrix} a_1 & a_2 \\ Aa_2 & a_1 \end{smallmatrix}\right), \mathbf{b} = \dots\right\}$, $K = GSp(2,R)$, *and* $|A| = 1$ *or* $= |\boldsymbol{\pi}|$.

Proof. It suffices to show one of these decompositions, since $u_n^{-1} = \left(\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right) u_n \left(\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right)$. Write $g_1 \equiv g$ if $g_1 \in KgC_A$. Using $GL(2,F) = \underset{m\geq 0}{\cup} \left\{\left(\begin{smallmatrix} a & bA \\ b & a \end{smallmatrix}\right)\right\} \left(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^m \end{smallmatrix}\right) GL(2,R)$ $= \underset{m\geq 0}{\cup} GL(2,R) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^m \end{smallmatrix}\right) \left\{\left(\begin{smallmatrix} a & b \\ bA & a \end{smallmatrix}\right)\right\}$, we conclude that any $g \in H = KP$, where P is the Siegel parabolic, of type $(2,2)$, has

$$g \equiv \begin{pmatrix} Y & 0 \\ 0 & w^tY^{-1}w \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \equiv \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^n \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} \boldsymbol{\pi}^{-n} & 0 \\ 0 & 1 \end{smallmatrix} \end{pmatrix} \begin{pmatrix} 1&0&0&\boldsymbol{\pi}^{-i}/A \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$$

$$\in K \begin{pmatrix} 1&0&0&\boldsymbol{\pi}^{-j}/A \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} C_A.$$

The last relation ($\in$) is clear when $n=0$, where $j=i$. If $i = 0 < n$ then $j = n$, since

$$\begin{pmatrix} 1 &&& 1/A \\ & \boldsymbol{\pi}^n & 0 & \\ & 0 & \boldsymbol{\pi}^{-n} & \\ 0 &&& 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & (1+\boldsymbol{\pi}^{-n})/A \\ 0 & 1 & 1+\boldsymbol{\pi}^{-n} & 0 \\ 0 & \boldsymbol{\pi}^n & 0 & 0 \\ A\boldsymbol{\pi}^n & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 &&& -\boldsymbol{\pi}^{-n}/A \\ & 1 & 0 & \\ & 0 & 1 & \\ 0 &&& 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1+\boldsymbol{\pi}^n & 0 & 0 & 0 \\ 0 & \boldsymbol{\pi}^n & \boldsymbol{\pi}^n+1 & 0 \\ 0 & 1 & 0 & 0 \\ \boldsymbol{\pi}^n A & 0 & 0 & -1 \end{pmatrix}.$$

Note that $\begin{pmatrix} 1 &&& A^{-1} \\ & 1 & 0 & \\ & 0 & 1 & \\ 0 &&& 1 \end{pmatrix} = \begin{pmatrix} 1 &&& 0 \\ & 1 & -1 & \\ & 0 & 1 & \\ 0 &&& 1 \end{pmatrix} \begin{pmatrix} 1 &&& A^{-1} \\ & 1 & 1 & \\ & 0 & 1 & \\ 0 &&& 1 \end{pmatrix} \in KC_A$. If $i \leq 2n$ we reduce to $i = 0 < n$ to get $j = n$, since

$$\begin{pmatrix} 1 &&& 0 \\ & \boldsymbol{\pi}^n & 0 & \\ & 0 & \boldsymbol{\pi}^{-n} & \\ 0 &&& 1 \end{pmatrix} \begin{pmatrix} 1&0&0&\boldsymbol{\pi}^{-i}/A \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} \equiv \begin{pmatrix} 1 &&& 0 \\ & \boldsymbol{\pi}^n && \\ && \boldsymbol{\pi}^{-n} & \\ &&& 1 \end{pmatrix} \begin{pmatrix} 1 &&& \\ & 1 & -\boldsymbol{\pi}^{-i} & \\ & 0 & 1 & \\ 0 &&& 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 &&& 0 \\ & 1 & -\boldsymbol{\pi}^{2n-i} & \\ & 0 & 1 & \\ 0 &&& 1 \end{pmatrix} \begin{pmatrix} 1 &&& 0 \\ & \boldsymbol{\pi}^n & 0 & \\ & 0 & \boldsymbol{\pi}^{-n} & \\ 0 &&& 1 \end{pmatrix} \equiv \begin{pmatrix} 1 &&& 0 \\ & \boldsymbol{\pi}^n & 0 & \\ & 0 & \boldsymbol{\pi}^{-n} & \\ 0 &&& 1 \end{pmatrix}.$$

When $i > 2n$ we obtain $j = i - n$, since

$$\begin{pmatrix} 1 & 0 & 0 & \boldsymbol{\pi}^{-i}/A \\ & \boldsymbol{\pi}^n & 0 & 0 \\ & 0 & \boldsymbol{\pi}^{-n} & 0 \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\pi}^{-n} & 0 & 0 & 1 \\ 0 & \boldsymbol{\pi}^{-n} & A & 0 \\ 0 & \boldsymbol{\pi}^i(1-\boldsymbol{\pi}^{-n}) & -\boldsymbol{\pi}^n & 0 \\ \boldsymbol{\pi}^i(1-\boldsymbol{\pi}^{-n})A & 0 & 0 & -\boldsymbol{\pi}^n \end{pmatrix} \begin{pmatrix} 1 & & \boldsymbol{\pi}^{n-i}/A \\ & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & \\ 0 & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1-\boldsymbol{\pi}^{n-i}/A \\ 0 & 1 & \boldsymbol{\pi}^n A & 0 \\ 0 & \boldsymbol{\pi}^{i-n}(1-\boldsymbol{\pi}^{-n}) & -1 & 0 \\ \boldsymbol{\pi}^i(1-\boldsymbol{\pi}^{-n})A & 0 & 0 & -\boldsymbol{\pi}^n \end{pmatrix} \begin{pmatrix} 1 & & \boldsymbol{\pi}^{n-i}/A \\ & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & \\ 0 & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & \boldsymbol{\pi}^n A & 0 \\ 0 & \boldsymbol{\pi}^{i-n}(1-\boldsymbol{\pi}^{-n}) & -1 & 0 \\ A\boldsymbol{\pi}^i(1-\boldsymbol{\pi}^{-n}) & 0 & 0 & -1 \end{pmatrix} \in K.$$

In order to verify that the union is disjoint, we need to show that $u_n h u_m^{-1} \in K$ for $h \in C_A$ implies that $m = n$. Thus

$$K \ni \begin{pmatrix} 1 & & \boldsymbol{\pi}^{-n}/A \\ & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_2 A & a_1 & b_2 A & b_1 \\ c_1 & c_2 & d_1 & d_2 \\ c_2 A & c_1 & d_2 A & d_1 \end{pmatrix} \begin{pmatrix} 1 & & -\boldsymbol{\pi}^{-m}/A \\ & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & \\ 0 & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + c_2\boldsymbol{\pi}^{-n} & a_2 + c_1\boldsymbol{\pi}^{-n}/A & b_1 + d_2\boldsymbol{\pi}^{-n} & b_2 + d_1\boldsymbol{\pi}^{-n}/A - \boldsymbol{\pi}^{-m}(a_1 + c_2\boldsymbol{\pi}^{-n})/A \\ Aa_2 & a_1 & Ab_2 & b_1 - a_2\boldsymbol{\pi}^{-m} \\ c_1 & c_2 & d_1 & d_2 - c_1\boldsymbol{\pi}^{-m}/A \\ Ac_2 & c_1 & Ad_2 & d_1 - c_2\boldsymbol{\pi}^{-m} \end{pmatrix}.$$

If $m = 0$ and $n \geq 1$, using the top row we see that $|c_2| < 1$, $|c_1| < 1$, $|d_2| < 1$, $|d_1| < 1$, but then considering the bottom two rows we see that the last matrix is not in K, hence $n = m$ if $m = 0$.

If $n \neq m$, without loss of generality $1 \leq n < m$. Using the right column: $c_2 \in \boldsymbol{\pi}^m R$, $c_1 \in \boldsymbol{\pi}^m R$, $a_2 \in \boldsymbol{\pi}^{m-n}A^{-1}R$ (since $b_1 \in \boldsymbol{\pi}^{-n}A^{-1}R$ by top row, third entry, and third entry of bottom row). Then a_1 is a unit (the last three entries in the first column are in $\boldsymbol{\pi}R$), hence the fourth entry of top row, x, has absolute value $|a_1\boldsymbol{\pi}^{-m}A^{-1}| > 1$, contradiction. Then $n = m$ as asserted. □

There is an analogous decomposition for $Sp(2,F)$.

2. Lemma. *We have a disjoint decomposition* $Sp(2,F) = \bigcup_{m\geq 0} \mathbf{C}_A^1 u_m K^1$, *where the superscript* 1 *stands for the subgroup of elements with determinant one.*

Proof. We can write $K = \cup \left(\begin{smallmatrix} I & 0 \\ 0 & r \end{smallmatrix}\right) K^1$, union over $r \in R^\times$, and $\mathbf{C}_A = \cup \mathbf{C}_A^1 \left(\begin{smallmatrix} I & 0 \\ 0 & \lambda \end{smallmatrix}\right)$, $\lambda \in F^\times$. Then $\mathbf{C}_A u_m K$ is a union of $\mathbf{C}_A^1 \left(\begin{smallmatrix} I & 0 \\ 0 & \lambda \end{smallmatrix}\right) u_m \left(\begin{smallmatrix} I & 0 \\ 0 & r \end{smallmatrix}\right) K^1$, and such a coset lies in $Sp(2,F)$ (thus $\left(\begin{smallmatrix} I & 0 \\ 0 & \lambda \end{smallmatrix}\right) u_m \left(\begin{smallmatrix} I & 0 \\ 0 & r \end{smallmatrix}\right)$ lies in $Sp(2,F)$) only when $\lambda r = 1$. Writing $\left(\begin{smallmatrix} I & 0 \\ 0 & r^2 \end{smallmatrix}\right) = r\left(\begin{smallmatrix} r^{-1} & 0 \\ 0 & r \end{smallmatrix}\right)$, and noting that $\left(\begin{smallmatrix} r^{-1} & 0 \\ 0 & r \end{smallmatrix}\right)$ lies in

K^1 and $\mathbf{C}_A^1$, we have $Sp(2,F)=\cup\mathbf{C}_A^1\left(\begin{smallmatrix} I & 0\\ 0 & r^{-1}\end{smallmatrix}\right)u_m\left(\begin{smallmatrix} I & 0\\ 0 & r\end{smallmatrix}\right)K^1$, where r ranges over $R^\times/R^{\times 2}$. Note that instead of u_m we can work with $u_m'=\left(\begin{smallmatrix} I & X\\ 0 & I\end{smallmatrix}\right)$, $X=\left(\begin{smallmatrix} 0 & 0\\ \boldsymbol{\pi}^{-m} & 0\end{smallmatrix}\right)$, since the quotient of the two elements lies in $\mathbf{C}_A^1$. It remains to note that

$$\begin{pmatrix} 1 & & & 0\\ & 1 & -\boldsymbol{\pi}^{-m} & \\ & 0 & 1 & \\ 0 & & & 1\end{pmatrix}\begin{pmatrix} 0 & 1 & -\boldsymbol{\pi}^{-m}r & 0\\ A & 0 & 0 & -\boldsymbol{\pi}^{-m}r\\ -\boldsymbol{\pi}^m c & 0 & 0 & -r\\ 0 & -\boldsymbol{\pi}^m c & -rA & 0\end{pmatrix}\begin{pmatrix} 1 & & & 0\\ & 1 & r\boldsymbol{\pi}^{-m} & \\ & 0 & 1 & \\ 0 & & & 1\end{pmatrix}$$

$$=\begin{pmatrix} 0 & 1 & 0 & 0\\ A+c & 0 & 0 & 0\\ -c\boldsymbol{\pi}^m & 0 & 0 & -r\\ 0 & -c\boldsymbol{\pi}^m & -cr-rA & 0\end{pmatrix}$$

lies in K^1 when $c=-r^{-1}-A$ (we choose $r\in R^\times$ with $c\neq 0$). Hence $\mathbf{C}_A^1\left(\begin{smallmatrix} I & 0\\ 0 & r^{-1}\end{smallmatrix}\right)u_m\left(\begin{smallmatrix} I & 0\\ 0 & r\end{smallmatrix}\right)K^1=\mathbf{C}_A^1u_mK^1$, and the lemma follows. □

Put $K=GSp(2,R)$, $K_m^A=\mathbf{C}_A\cap u_mKu_m^{-1}$, and $C_A=GL\big(2,F(\sqrt{A})\big)'$ for the group of $g\in GL\big(2,F(\sqrt{A})\big)$ with determinant in $F^\times$. We write $a=a_1+a_2/\sqrt{A}$ for an element of $F(\sqrt{A})$, $a_1,a_2\in F$, and define

$$\phi\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} a_1 & a_2/A & b_2 & b_1\\ a_2 & a_1 & b_1A & b_2\\ c_2/A & c_1/A & d_1 & d_2/A\\ c_1 & c_2/A & d_2 & d_1\end{pmatrix}.$$

3. Lemma. *The map* $\phi_m: C_A=GL\big(2,F(\sqrt{A})\big)'\to\mathbf{C}_A$,

$$\phi_m:\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\mapsto\phi\Big(\left(\begin{smallmatrix} 1 & 0\\ 0 & A\boldsymbol{\pi}^m\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0\\ -1 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0\\ 0 & \sqrt{A}\end{smallmatrix}\right)\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0\\ 0 & 1/\sqrt{A}\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0\\ 1 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0\\ 0 & 1/A\boldsymbol{\pi}^m\end{smallmatrix}\right)\Big),$$

is an isomorphism which maps $K_m=GL\big(2,R_{F(\sqrt{A})}(m)\big)'$ *onto* K_m^A. *Here* $R_{F(\sqrt{A})}(m)=R+\boldsymbol{\pi}^m\sqrt{A}R=R+\boldsymbol{\pi}^mR_{F(\sqrt{A})}$, *and as usual, prime indicates "determinant in* $F^\times$*".*

Proof. Note that

$$u_m\phi\left(\begin{pmatrix} 1 & 0\\ 0 & A\boldsymbol{\pi}^m\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & \boldsymbol{\pi}^{-m}A^{-1}\end{pmatrix}\right)u_m^{-1}=$$

$$\begin{pmatrix} 1 & & & \boldsymbol{\pi}^{-m}A^{-1}\\ & 1 & 0 & \\ & 0 & 1 & \\ 0 & & & 1\end{pmatrix}\begin{pmatrix} a_1 & a_2/A & b_2/A\boldsymbol{\pi}^m & b_1/A\boldsymbol{\pi}^m\\ a_2 & a_1 & b_1/\boldsymbol{\pi}^m & b_2/A\boldsymbol{\pi}^m\\ c_2\boldsymbol{\pi}^m & c_1\boldsymbol{\pi}^m & d_1 & d_2/A\\ c_1A\boldsymbol{\pi}^m & c_2\boldsymbol{\pi}^m & d_2 & d_1\end{pmatrix}\begin{pmatrix} 1 & & & -A^{-1}\boldsymbol{\pi}^{-m}\\ & 1 & 0 & \\ & 0 & 1 & \\ 0 & & & 1\end{pmatrix}$$

$$=\begin{pmatrix} a_1+c_1 & (a_2+c_2)/A & (b_2+d_2)/A\boldsymbol{\pi}^m & (b_1+d_1-a_1-c_1)/A\boldsymbol{\pi}^m\\ a_2 & a_1 & b_1/\boldsymbol{\pi}^m & (b_2-a_2)/A\boldsymbol{\pi}^m\\ c_2\boldsymbol{\pi}^m & c_1\boldsymbol{\pi}^m & d_1 & (d_2-c_2)/A\\ c_1A\boldsymbol{\pi}^m & c_2\boldsymbol{\pi}^m & d_2 & d_1-c_1\end{pmatrix}$$

lies in K precisely when $a_1,a_2,d_1,d_2,c_1\in R; a_2+c_2, c_2-d_2\in AR; b_2+d_2, a_2-b_2, Ab_1, b_1+d_1-a_1-c_1\in A\boldsymbol{\pi}^m R$, in particular $a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2\in R$. Replacing $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ by

$$\begin{pmatrix} 1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 1 & 0\\ 1 & 1\end{pmatrix}=\begin{pmatrix} a+b & b\\ c-a+d-b & d-b\end{pmatrix},$$

namely replacing a by $a+b$, d by $d-b$, c by $c-a+d-b$ (so $a+c$ becomes $c+d$, $c-d$ becomes $c-a$), this condition becomes: $a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2\in R; c_2+d_2, a_2-c_2\in AR; d_2,a_2,Ab_1,c_1\in A\boldsymbol{\pi}^m R$, and in fact the conditions $c_2+d_2, a_2-c_2\in AR$ can then be replaced by $c_2\in AR$. Next we further replace $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ by $\left(\begin{smallmatrix} 1 & 0\\ 0 & \sqrt{A}\end{smallmatrix}\right)\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0\\ 0 & 1/\sqrt{A}\end{smallmatrix}\right)=\left(\begin{smallmatrix} a & b/\sqrt{A}\\ c\sqrt{A} & d\end{smallmatrix}\right)$, where $b/\sqrt{A}=b_2/A+b_1/\sqrt{A}$, $c\sqrt{A}=c_2+c_1A/\sqrt{A}$. Thus we replace b_1 by $b_2/A, b_2$ by b_1, c_1 by c_2, c_2 by c_1A. Then the condition changes to: $a_1,a_2,b_1,c_2,d_1,d_2\in R; c_1\in R; d_2,a_2,b_2,c_2\in A\boldsymbol{\pi}^m R$. This is the claim of the lemma. □

Denote by $T_\rho=\{\left(\begin{smallmatrix} a & bD\rho\\ b/\rho & a\end{smallmatrix}\right)\}$ an elliptic torus in $GL(2,E_3)$. Thus $a,b\in E_3, D\in E_3-E_3^2$ will be assumed to lie in R_3 and to have minimal order in $R_3=R_{E_3}$, and ρ is taken in a set of (two) representatives (including 1) for $E_3^\times/N_{E/E_3}E^\times$, again $\rho\neq 1$ will be taken in R_3 to have minimal order. Here $E=E_3(\sqrt{D})$, and $E_3=F(\sqrt{A})$. Write $\tilde{C}_A$ for $GL\big(2,F(\sqrt{A})\big)$, and recall that $C_A=\{g\in\tilde{C}_A;\|g\|\in F^\times\}$. Also $\overline{\rho}=\operatorname{ord}\rho$.

4. Lemma. *When E/E_3 is ramified we have $C_A=\bigcup_{j\geq 0}T'_\rho r_jK'$, where $r_j\in T_\rho\left(\begin{smallmatrix} 1 & 0\\ 0 & \boldsymbol{\pi}_3^j\end{smallmatrix}\right)$ has $\|r_j\|=1$. When E/E_3 is unramified, $C_A=\bigcup_j T'_\rho r_jK'$ ($j\geq 0, j-\overline{\rho}$ is even), where $r_j=\boldsymbol{\pi}_3^{-(j-\overline{\rho})/2}\left(\begin{smallmatrix} 1 & 0\\ 0 & \boldsymbol{\pi}_3^{j-\overline{\rho}}\end{smallmatrix}\right)$.*

Proof. We have $\tilde{C}_A=GL(2,E_3)=\bigcup_{j\geq 0}T_\rho\left(\begin{smallmatrix} 1 & 0\\ 0 & \boldsymbol{\pi}_3^{j-\overline{\rho}}\end{smallmatrix}\right)K$, $K=GL(2,R_3)$, $\boldsymbol{\pi}_3=\boldsymbol{\pi}_{E_3}$. When E/E_3 is ramified we choose $\boldsymbol{\pi}_3=-D=N_{E/E_3}(\boldsymbol{\pi}_E), \boldsymbol{\pi}_E=\sqrt{D}$. If $h=trk\in\tilde{C}_A$, $r=\left(\begin{smallmatrix} 1 & 0\\ 0 & \boldsymbol{\pi}_3^{j-\overline{\rho}}\end{smallmatrix}\right)$, changing t in T_ρ we may assume $\|h\|\in R^\times$, and that there is $t_0\in T_\rho$ with $\|t_0r\|=1$. Then $h=tt_0rk$, so that $\|t\|\in R_3^\times\cap N_{E/E_3}E^\times=R_3^{\times 2}$, and $\|t\|=\varepsilon^{-2}$ for some $\varepsilon\in R_3^\times$. Then $h=\varepsilon t\cdot t_0r\cdot\varepsilon^{-1}k$, $\|\varepsilon t\|=1$, $\|\varepsilon^{-1}k\|\in R^\times$, as required.

When E/E_3 is unramified, if $h=trk\in C_A$ where $r=\left(\begin{smallmatrix} 1 & 0\\ 0 & \pi_3^{j-\overline{\rho}}\end{smallmatrix}\right)$, since $N_{E/E_3}E^\times$ is $\boldsymbol{\pi}_3^{2\mathbb{Z}}R_3^\times$, we have that $j-\overline{\rho}$ must be even (note that $\boldsymbol{\pi}_F=\boldsymbol{\pi}_3^2$). We can then assume that $\|h\|\in R^\times$. Further, changing t in $h=trk$, where $r=\boldsymbol{\pi}_3^{-(j-\overline{\rho})/2}\left(\begin{smallmatrix} 1 & 0\\ 0 & \boldsymbol{\pi}_3^{j-\overline{\rho}}\end{smallmatrix}\right)$ and $\|h\|\in R^\times$, we may change $\|k\|\in R_3^\times$ by an element of $R_3^{\times 2}$ (= scalar in T_ρ). But $R_3^\times=R_3^{\times 2}R^\times$, since $a+b\sqrt{A}=a(1+\frac{b}{a}\sqrt{A})\in R^\times R_3^{\times 2}$ when $A=\boldsymbol{\pi}_F$ ($E_3=F(\sqrt{A})$ is ramified over F if E/E_3 is unramified, $E=E_3(\sqrt{D})$, since A,D are non squares in R, and AD has order 1). Hence we may assume that $\|k\|\in R^\times$ so that $k\in K'$, as required. □

We need an analogous result for $A\in F^{\times 2}$. Note that for $A\in F-F^2$, the subgroup C_A of H is isomorphic to $GL(2,E)'$, $E=F(\sqrt{A})$, where the prime indicates elements with determinant in $F^\times$. The isomorphism is given by $\mathbf{a}\mapsto\tilde{a}=a_1+a_2\sqrt{A}$. Let

$$C_0=\left\{\left[\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right),\left(\begin{smallmatrix} \alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\right]=\begin{pmatrix} a & 0 & 0 & b\\ 0 & \alpha & \beta & 0\\ 0 & \gamma & \delta & 0\\ c & 0 & 0 & d\end{pmatrix}\in H\right\};$$

it is isomorphic to the group $GL(2,F\oplus F)'=\{(g,g')=\left(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right),\left(\begin{smallmatrix}\alpha&\beta\\\gamma&\delta\end{smallmatrix}\right)\right);\det g=\det g'\}$. Put

$$z(m)=\begin{pmatrix}1&0&\boldsymbol{\pi}^{-m}&0\\0&1&0&\boldsymbol{\pi}^{-m}\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

5. Lemma. *We have a disjoint decomposition $H=GSp(2,F)=\cup_{m\geq 0}Kz(m)C_0$.*

Proof. Using the decomposition $H=KNM$ where NM is the Heisenberg parabolic, of type $(1,2,1)$, we have $H=K\begin{pmatrix}1&*&*&*\\&1&0&*\\&0&1&*\\0&&&1\end{pmatrix}C_0$, and representatives for $K\backslash H/C_0$ are given by

$$\begin{pmatrix}1&\boldsymbol{\pi}^{-n}&\boldsymbol{\pi}^{-m}&\boldsymbol{\pi}^{-n-m}\\0&1&0&\boldsymbol{\pi}^{-m}\\0&0&1&-\boldsymbol{\pi}^{-n}\\0&0&0&1\end{pmatrix},m\geq n.$$

But this is equal to

$$\begin{pmatrix}1&&&0\\&1&0&\\&-\boldsymbol{\pi}^{m-n}&-1&\\0&&&-1\end{pmatrix}\begin{pmatrix}1&0&-\boldsymbol{\pi}^{-m}&0\\0&1&0&-\boldsymbol{\pi}^{-m}\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&&&\boldsymbol{\pi}^{-n-m}\\&1&0&\\&-\boldsymbol{\pi}^{m-n}&-1&\\0&&&-1\end{pmatrix}.$$

To verify that the union is disjoint, it suffices to show that if

$$\begin{pmatrix}1&0&\boldsymbol{\pi}^{-m}&0\\0&1&0&\boldsymbol{\pi}^{-m}\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}a&0&0&b\\0&\alpha&\beta&0\\0&\gamma&\delta&0\\c&0&0&d\end{pmatrix}\begin{pmatrix}1&0&-\boldsymbol{\pi}^{-n}&0\\0&1&0&-\boldsymbol{\pi}^{-n}\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}a&\gamma\boldsymbol{\pi}^{-m}&\delta\boldsymbol{\pi}^{-m}-a\boldsymbol{\pi}^{-n}&b-\gamma\boldsymbol{\pi}^{-m-n}\\c\boldsymbol{\pi}^{-m}&\alpha&\beta-c\boldsymbol{\pi}^{-m-n}&d\boldsymbol{\pi}^{-m}-\alpha\boldsymbol{\pi}^{-n}\\0&\gamma&\delta&-\gamma\boldsymbol{\pi}^{-n}\\c&0&-c\boldsymbol{\pi}^{-n}&d\end{pmatrix}$$

lies in K, then $m=n$. If $n=0<m$, then $\gamma,\delta,c,d\in\boldsymbol{\pi}^mR$, but this is impossible (bottom row in $\boldsymbol{\pi}R$). Without loss of generality $0<n<m$. Then $c\in\boldsymbol{\pi}^mR$ implies $d\in R^\times$ (bottom row). Since $\alpha\in R$, the last entry on the second row, $d\boldsymbol{\pi}^{-m}-\alpha\boldsymbol{\pi}^{-n}$, is not in R, contradiction. We conclude that $m=n$, and the union is indeed disjoint. □

Put $H^1=Sp(2,F)$, $C_0^1=C_0\cap H^1\simeq C_1=SL(2,F)\times SL(2,F)$, $K^1=K\cap H^1$.

6. Lemma. $H^1=\underset{m\geq 0}{\cup}C_0^1z(m)K^1$, *where the union is disjoint.*

Proof. We have $H=\underset{m\geq 0}{\cup}C_0z(m)K$. Then $hz(m)k\in H^1$ implies that $h=[a,b]$ with $\|a\|=\|b\|\in R^\times$, and $\|k\|=\|a\|^{-2}\in R^{\times 2}$. Multiplying a,b by $\varepsilon\in R^\times$ and k by ε^{-1}, we have that $\|a\| \in R^\times/R^{\times 2}$. Then $H^1 = \bigcup_{\substack{m\geq 0\\ \varepsilon\in R^\times/R^{\times 2}}} C_0^1 x^{-1}z(m)xK^1$. where $x = \mathrm{diag}(1,\varepsilon,1,\varepsilon)$. But $x^{-1}z(m)x = z(m)$. The lemma follows. $\square$

Denote by $\phi_m : \big(GL(2,F)\times GL(2,F)\big)' \to C_0$, where the prime indicates the subgroup of pairs (A,B) with $\|A\| = \|B\|$, the isomorphism $\phi_m\big((A,B)\big) = \left(\begin{smallmatrix} I & 0\\ 0 & \boldsymbol{\pi}^m\end{smallmatrix}\right)[A, \boldsymbol{\varepsilon} wBw\boldsymbol{\varepsilon}]\left(\begin{smallmatrix} I & 0\\ 0 & \boldsymbol{\pi}^{-m}\end{smallmatrix}\right)$. It maps $C_1 = SL(2,F)\times SL(2,F)$ onto C_0^1.

7. Lemma. *ϕ_m maps $K_m^1 = \{(A,B)\in SL(2,R)\times SL(2,R); A-\boldsymbol{\varepsilon}B\boldsymbol{\varepsilon}\in \boldsymbol{\pi}^m M_2(R)\}$ isomorphically to $K_m^{C_0^1} = C_0^1\cap z(m)K^1z(m)^{-1}$, and $K_m = \{(A,B)\in (GL(2,R)\times GL(2,R))'; A-\boldsymbol{\varepsilon}B\boldsymbol{\varepsilon}\in \boldsymbol{\pi}^m M_2(R)\}$ onto $K_m^{C_0} = C_0\cap z(m)Kz(m)^{-1}$. Note that $K_m^1 = K_m\cap C_0^1$.*

Proof. Multiply:

$$z(m)^{-1}\begin{pmatrix} I & 0\\ 0 & \boldsymbol{\pi}^m\end{pmatrix}\begin{pmatrix} a & 0 & 0 & b\\ 0 & \delta & -\gamma & 0\\ 0 & -\beta & \alpha & 0\\ c & 0 & 0 & d\end{pmatrix}\begin{pmatrix} I & 0\\ 0 & \boldsymbol{\pi}^{-m}\end{pmatrix}z(m)$$

$$= \begin{pmatrix} 1 & 0 & -\boldsymbol{\pi}^{-m} & 0\\ 0 & 1 & 0 & -\boldsymbol{\pi}^{-m}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} a & 0 & 0 & b\boldsymbol{\pi}^{-m}\\ 0 & \delta & -\gamma\boldsymbol{\pi}^{-m} & 0\\ 0 & -\beta\boldsymbol{\pi}^m & \alpha & 0\\ c\boldsymbol{\pi}^m & 0 & 0 & d\end{pmatrix}\begin{pmatrix} 1 & 0 & \boldsymbol{\pi}^{-m} & 0\\ 0 & 1 & 0 & \boldsymbol{\pi}^{-m}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix} a & \beta & (a-\alpha)\boldsymbol{\pi}^{-m} & (b+\beta)\boldsymbol{\pi}^{-m}\\ -c & \delta & -(\gamma+c)\boldsymbol{\pi}^{-m} & (\delta-d)\boldsymbol{\pi}^{-m}\\ 0 & -\beta\boldsymbol{\pi}^m & \alpha & -\beta\\ c\boldsymbol{\pi}^m & 0 & c & d\end{pmatrix}.$$

This lies in K^1 precisely when $a,b,c,d,\alpha,\beta,\gamma,\delta$ lie in R, $a-\alpha, b+\beta, c+\gamma, d-\delta$, lie in $\boldsymbol{\pi}^m R$, and $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, have determinant 1. $\square$

PART II. Main comparison.

A. Strategy.

Let us review our strategy in computing the θ-orbital integrals of 1_K. It is based on the twisted Kazhdan decomposition. Given a semi-simple $t\theta \in K \rtimes \langle\theta\rangle$, $G = GL(4,F) \times GL(1,F)$, $K = GL(4,R) \times GL(1,R)$, it has the decomposition $t\theta = u \cdot s\theta = s\theta \cdot u$, where $s\theta$ is absolutely semi simple, and u is topologically unipotent. Then $\Phi^G_{1_K}(t\theta) = \Phi^{Z_G(s\theta)}_{1_{Z_K(s\theta)}}(u)$. The associated stable θ-orbital integral we wish to relate to the stable orbital integral $\Phi^{H,st}_{1_{K_H}}(Nt)$, where H is the endoscopic group $GSp(2,F)$, and Nt is the stable orbit of the norm of t. To compute the norm we write $t = h^{-1}t^*\theta(h)$, where $h \in \mathbf{G}\ \big(= \mathbf{G}(\overline{F})\big)$, and $t^* \in T^*$, where $\mathbf{T}^*$ is the diagonal subgroup and $T^* = \mathbf{T}^{*\Gamma}$. On T^* the norm is defined by $T^* \to T^*/(1-\theta)T^* \simeq T^*_H$, thus $N(a,b,c,d;e) = (abe, ace, bde, cde; e^2abcd)$. A θ-semi-simple t ($t\theta$ is semi simple in $G \rtimes \langle\theta\rangle$) is called *strongly θ-regular* if $Z_G(t\theta)$ is abelian, in which case the centralizer $Z_G(Z_G(t\theta)^0)$ of $Z_G(t\theta)^0$ in G is an F-torus T in G which is invariant under $\mathrm{Int}(t) \circ \theta$, and $Z_G(t\theta) = T^{\mathrm{Int}(t)\circ\theta}$. The θ-orbit of t intersects T^*, thus there is $h \in \mathbf{G}$ and $t^* \in T^*$ with $t = h^{-1}t^*\theta(h)$, and $Z_G(t\theta) = h^{-1}Z_G(t^*\theta)h = h^{-1}T^{*\theta}h$. Then $T = Z_G(h^{-1}T^{*\theta}h) = h^{-1}T^*h$, and $Z_G(t\theta) = T^{\mathrm{Int}(t)\circ\theta}$ consists of the $x \in T$ with $t\theta(x)t^{-1} = x$, thus $x^{-1}t\theta(x) = t$.

An F-torus T in G is determined by $h \in \mathbf{G}$ and the Galois action on $\mathbf{T}^*$. Namely, for $t = h^{-1}t^*h \in T = h^{-1}T^*h$ we have $h^{-1}t^*h = t = \sigma t = \sigma h^{-1}\sigma t^* \sigma h$, and so $\sigma t^* = h_\sigma^{-1}t^*h_\sigma$, where $\mathrm{Int}(h_\sigma^{-1}) \in \mathrm{Norm}(T^*,G)$ has the image w_σ in $W = W(T^*,G)$
$= \mathrm{Norm}(T^*,G)/\mathrm{Cent}(T^*,G)$. If T^* is a θ-invariant F-torus, taking $t^* \in T^{*\theta}$ we conclude that $\mathrm{Int}(h_\sigma^{-1}) = \mathrm{Int}\big(\theta(h_\sigma)^{-1}\big)$, thus $w_\sigma \in W^\theta$, and the torus determines a cocycle $\langle w_\sigma\rangle$ in $H^1(F, W^\theta)$. We denoted the homomorphism , $\to W^\theta$, $\sigma \mapsto w_\sigma$, by ρ, and classified the tori according to the image of $\rho : \mathrm{Gal}(\overline{F}/F) \to W^\theta$, as types (1) – (3) and (I) – (IV). We explicitly realized T in the form $T = h^{-1}T^*h$, with $h = \theta(h)$. Thus in each stable θ-conjugacy class of strongly θ-regular elements we have a representative $t = h^{-1}t^*h$, and further we found representatives for the θ-conjugacy classes within its stable θ-conjugacy class, of the form $g^{-1}tg$, $g = g_R$ with $g = \theta(g)$.

A θ-semi-simple $t \in G$ is called *θ-elliptic* if $Z_G(t\theta)^0/Z(G)^0$ is anisotropic. The associated tori $T = Z_G\big(Z_G(t\theta)^0\big)$ are called θ-elliptic. A complete set of representatives for the θ-elliptic tori is given by the tori of type (I)-(IV). The computations of θ-orbital integrals of non θ-elliptic strongly θ-regular elements can be reduced – using a standard integration formula – to the case of the θ-elliptic elements, so we deal only with t in tori T of types (I) - (IV).

B. Twisted orbital integrals of type (I).

Let $u = \theta(u)$ be a topologically unipotent element in $GL(4,R) \times GL(1,R)$. Then $\Phi^G_{1_K}(u\theta) = \Phi^{Z_G(\theta)}_{1_{Z_K(\theta)}}(u)$, where $Z_G(\theta) = H^1 = Sp(2,F)$ and $Z_K(\theta) = K^1 = K \cap H^1$. We compute the value of this integral at u in a torus of type (I). To consider also the integrals at stably θ-conjugate but not θ-conjugate elements, we look at a complete set of representatives, parametrized by ρ_1, ρ_2. Here $\rho_i \in \{1, \boldsymbol{\pi}\}$ if E/F is unramified, and $\rho_i \in \{1, \varepsilon\} = R^\times/R^{\times 2}$ if E/F is ramified. Thus take t_ρ in the torus $T^1_\rho = \{t_\rho = [\phi^D_{\rho_1}(a_1 + b_1\sqrt{D}), \phi^D_{\rho_2}(a_2 + b_2\sqrt{D})] \in C^1_0\}$, where $\phi^D_\rho(a + b\sqrt{D}) = \left(\begin{smallmatrix} a & bD\rho \\ b/\rho & a \end{smallmatrix}\right)$. If $E^1 = \{x \in E^\times; N_{E/F}x = 1\}$, then T^1_ρ is isomorphic to $E^1 \times E^1$.

By Lemma I.J.6 we have

$$\Phi^{H^1}_{1_{K^1}}(t_\rho) = \int_{T^1_\rho\backslash H^1} 1_{K^1}(g^{-1}t_\rho g)dg$$
$$= \sum_{m\geq 0} |K^1|_{H^1} \int_{T^1_\rho\backslash C^1_0/C^1_0\cap z(m)K^1z(m)^{-1}} 1_{K^1}\big(z(m)^{-1}h^{-1}t_\rho hz(m)\big)dh.$$

The integrand in the last integral is non zero precisely when $h^{-1}t_\rho h$ lies in $z(m)K^1z(m)^{-1}$ $\cap C^1_0 = K_m^{C^1_0}$. Hence we get

$$= \sum_{m\geq 0} |K^1|_{H^1} \int_{T^1_\rho\backslash C^1_0/K_m^{C^1_0}} 1_{K_m^{C^1_0}}(h_0^{-1}t_\rho h_0)dh_0.$$

Using Lemma I.J.7 we have an isomorphism $\phi_m : C_1 \to C^1_0$ $(\phi_m(h) = h_0)$, $\phi_m(K^1_m) = K_m^{C^1_0}$. Define x_ρ by $\phi_m(x_\rho) = t_\rho$, and note that $T^1_\rho = Z_{C^1_0}(t_\rho)$. Hence our expression is

$$= \sum_{m\geq 0} |K^1|_{H^1} \int_{Z_{C_1}(x_\rho)\backslash C_1/K^1_m} 1_{\phi_m(K^1_m)}(\phi_m(h)^{-1}\phi_m(x_\rho)\phi_m(h))dh$$

$$= \sum_{m\geq 0} [K^1_0 : K^1_m] \int_{Z_{C_1}(x_\rho)\backslash C_1} 1_{K^1_m}(h^{-1}x_\rho h)dh.$$

Next we change variables on $C_1 = SL(2,F)\times SL(2,F)$. If m is even,

$$h \mapsto (I, w\boldsymbol{\varepsilon})\big(\big(\begin{smallmatrix} \boldsymbol{\pi}^{m/2} & 0 \\ 0 & \boldsymbol{\pi}^{-m/2}\end{smallmatrix}\big), \big(\begin{smallmatrix} \boldsymbol{\pi}^{m/2} & 0 \\ 0 & \boldsymbol{\pi}^{-m/2}\end{smallmatrix}\big)\big)h$$

sends $h^{-1}x_\rho h$ to $h^{-1}((\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{m}\end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{m}\end{smallmatrix}))(I,\boldsymbol{\varepsilon}w)x_\rho(I,w\boldsymbol{\varepsilon})((\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{-m}\end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{-m}\end{smallmatrix}))h = h^{-1}t'_\rho h$, where $t'_\rho = (t_{\rho_1}, t_{\rho_2}) \in C_1$, $t_{\rho_i} = \phi^D_{\rho_i}(a_i + b_i\sqrt{D})$.

If m is odd, and E/F is unramified, $h \mapsto (I, w\boldsymbol{\varepsilon})\big(\big(\begin{smallmatrix} \boldsymbol{\pi}^{(m+i)/2} & 0 \\ 0 & \boldsymbol{\pi}^{-(m+i)/2}\end{smallmatrix}\big), \big(\begin{smallmatrix} \boldsymbol{\pi}^{(m+j)/2} & 0 \\ 0 & \boldsymbol{\pi}^{-(m+j)/2}\end{smallmatrix}\big)\big)h$ sends $h^{-1}x_\rho h$ to $h^{-1}((\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{i}\end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{j}\end{smallmatrix}))t'_\rho((\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{-i}\end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{-j}\end{smallmatrix}))h$, where $i,\ j \in \{\pm 1\}$, i is taken to be 1 if $\rho_1 = \boldsymbol{\pi}$ and -1 if $\rho_1 = 1$ ($j = 1$ if $\rho_2 = \boldsymbol{\pi}$ and $j = -1$ if $\rho_2 = 1$). Then $h^{-1}x_\rho h$ is mapped to $h^{-1}t'_{\tilde\rho}h$, where if $\rho = (\rho_1, \rho_2)$ then $\tilde\rho = (\tilde\rho_1, \tilde\rho_2)$, and $\tilde\rho_i$ is defined by $\{\rho_i, \tilde\rho_i\} = \{1, \boldsymbol{\pi}\}$.

If m is odd, and E/F is ramified, we take

$$h \mapsto (I, w\boldsymbol{\varepsilon})\big(\big(\begin{smallmatrix} \boldsymbol{\pi}^{(m+1)/2} & 0 \\ 0 & \boldsymbol{\pi}^{-(m+1)/2}\end{smallmatrix}\big), \big(\begin{smallmatrix} \boldsymbol{\pi}^{(m+1)/2} & 0 \\ 0 & \boldsymbol{\pi}^{-(m+1)/2}\end{smallmatrix}\big)\big)(w\boldsymbol{\varepsilon}, w\boldsymbol{\varepsilon})h,$$

which maps $h^{-1}x_\rho h$ to $h^{-1}t'_{\tilde\rho}h$, where $\tilde\rho_i = -1/\rho_i$ ($\rho_i \mapsto \tilde\rho_i$ is a permutation, trivial if $-1 \in R^{\times 2}$, of $R^\times/R^{\times 2}$).

Put $\rho_m = \rho$ if m is even, and $\rho_m = \tilde\rho$ if m is odd. We get

$$= \sum_{m\geq 0} [K^1_0 : K^1_m] \int_{T^1_{\rho_m}\backslash C_1} 1_{K^1_m}(h^{-1}t_{\rho_m}h)dh.$$

Using the double coset decomposition for $SL(2,F)$ of Lemma I.I.3 we get

$$=\sum_{m\geq 0}\sum_{r\in R_{\rho_m}}[R_T^1 : T^1_{\rho_m}\cap rK_0^1r^{-1}][K_0^1 : K_m^1]\int_{K_0^1}1_{K_m^1}(k^{-1}r^{-1}t_{\rho_m}rk)dk.$$

Here $R_T^1=T^1_{\rho_m}\cap K_0^1=T^1_{\rho_m}(R)$. Let $\mathbf{j}$ signify (j_1,j_2). To simplify the notations we write ρ for ρ_m until the index m is explicitly needed.

By Lemma I.I.3, the representatives $r\in R_\rho$ have the form (when E/F is unramified)

$$r=r_{\mathbf{j}}=t_{\varepsilon_1}\operatorname{diag}(\boldsymbol{\pi}^{-(j_1-\overline{\rho}_1)/2},\varepsilon_1\boldsymbol{\pi}^{(j_1-\overline{\rho}_1)/2})\times t_{\varepsilon_2}\operatorname{diag}(\boldsymbol{\pi}^{-(j_2-\overline{\rho}_2)/2},\varepsilon_2\boldsymbol{\pi}^{(j_2-\overline{\rho}_2)/2}),$$

$j_1,j_2\geq 0, j_1-\overline{\rho}_1$ and $j_2-\overline{\rho}_2$ are even, $t_{\varepsilon_i}\in\phi^D_{\rho_i}(E^\times)$ has determinant ε_i^{-1}, and ε_i ranges over $R^\times/R^{\times 2}$ if $j_i>0$, it is $\varepsilon_i=1$ if $j_i=0$. When E/F is ramified the representatives take the form

$$r=r_{\mathbf{j}}=\phi^D_{\rho_1}(\boldsymbol{\pi}_E^{-j_1})\left(\begin{smallmatrix}1&0\\0&\boldsymbol{\pi}^{j_1}\end{smallmatrix}\right)\times\phi^D_{\rho_1}(\boldsymbol{\pi}_E^{-j_2})\left(\begin{smallmatrix}1&0\\0&\boldsymbol{\pi}^{j_2}\end{smallmatrix}\right)\qquad(j_1,j_2\geq 0).$$

1. Lemma. *The index $[R_T^1 : T_\rho^1\cap r_{\mathbf{j}}K_0^1r_{\mathbf{j}}^{-1}]$ is the product of $q^{j_1+j_2}$ and : 1 if E/F is ramified or $j_1=j_2=0$, $\frac{q+1}{2q}$ if E/F is unramified and either $j_1=0$ or $j_2=0$, $(\frac{q+1}{2q})^2$ if E/F is unramified and $j_1j_2\geq 1$.*

Proof. The intersection $T_\rho^1\cap rK_0^1r^{-1}$ consists of $t_\rho\in T_\rho^1$ such that $r^{-1}t_\rho r$ lies in K_0^1. But

$$\begin{aligned}r_{\mathbf{j}}^{-1}t_\rho r_{\mathbf{j}}&=\left(\begin{smallmatrix}1&0\\0&\varepsilon_1^{-1}\boldsymbol{\pi}^{-(j_1-\overline{\rho}_1)}\end{smallmatrix}\right)\left(\begin{smallmatrix}a_1&b_1D\rho_1\\b_1/\rho_1&a_1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\0&\varepsilon_1\boldsymbol{\pi}^{j_1-\overline{\rho}_1}\end{smallmatrix}\right)\times\dots\\&=\left(\begin{smallmatrix}a_1&b_1D\rho_1\varepsilon_1\boldsymbol{\pi}^{j_1-\overline{\rho}_1}\\\varepsilon_1^{-1}b_1\rho_1^{-1}\boldsymbol{\pi}^{-(j_1-\overline{\rho}_1)}&a_1\end{smallmatrix}\right)\times\dots.\end{aligned}$$

Then $r^{-1}t_\rho r\in K_0^1$ means that $b_i\in\boldsymbol{\pi}^{j_i}R$. Hence $T_\rho^1\cap rK_0^1r^{-1}$ is isomorphic to $R_E(j_1)^1\times R_E(j_2)^1$. Here $R_E(j)=R+\boldsymbol{\pi}^jR_E=R+\boldsymbol{\pi}^j\sqrt{D}R\subset R_E=R+\sqrt{D}R$, and the superscript 1 indicates the subgroup of elements with norm $N_{E/F}$ equal to 1.

To compute the index we use the exact sequence

$$1\to R_E^1/R_E(j)^1\to R_E^\times/R_E(j)^\times\to R_E^\times/R_E^1R_E(j)^\times\to 1.$$

Via the norm $N=N_{E/F}$, we have the isomorphism $R_E^\times/R_E^1R_E(j)^\times\tilde{\to}NR_E^\times/NR_E(j)^\times$. The last group is $R^\times/R^{\times 2}$ if E/F is unramified and $j\geq 1$; it is trivial if E/F is ramified or $j=0$. Further, we have

$$[R_E(j)^\times : 1+\boldsymbol{\pi}^jR_E]=[R^\times : R^\times\cap(1+\boldsymbol{\pi}^jR_E)]=[R^\times : 1+\boldsymbol{\pi}^jR]=(q-1)q^{j-1}.$$

Hence $[R_E^\times : R_E(j)^\times]=[R_E^\times : 1+\boldsymbol{\pi}^jR_E]/[R_E(j)^\times : 1+\boldsymbol{\pi}^jR_E]$ is

$$=(q^2-1)q^{2(j-1)}/(q-1)q^{j-1}=(q+1)q^{j-1}$$

when E/F is unramified and $j\geq 1$, since $\boldsymbol{\pi}_E=\boldsymbol{\pi}$ and $q_E=q^2$. When E/F is ramified it is

$$=(q-1)q^{2j-1}/(q-1)q^{j-1}=q^j$$

$(j \geq 0)$, since $\boldsymbol{\pi}_E^2 = \boldsymbol{\pi}$ and $q_E = q$. Then $[R_T^1 : T_\rho^1 \cap r_{\mathbf{j}} K_0^1 r_{\mathbf{j}}^{-1}]$ is the product of $[R_E^\times : R_E(j_1)^\times], [R_E^\times : R_E(j_2)^\times]$, and 1 (if E/F is ramified or $j_1 = j_2 = 0$), $\frac{1}{2}$ (if E/F is unramified and either $j_1 = 0$ or $j_2 = 0$), or $\frac{1}{4}$ (if E/F is unramified and $j_1 j_2 \geq 1$). □

Put $R_m = R/\boldsymbol{\pi}^m R$, $\overline{K}_m^1 = K_m^1/K(\boldsymbol{\pi}^m)$, where again $K_m^1 = \{(A,B) \in SL(2,R) \times SL(2,R); A \equiv \boldsymbol{\varepsilon} B \boldsymbol{\varepsilon} (\mathrm{mod}\, \boldsymbol{\pi}^m)\}$, and $K(\boldsymbol{\pi}^m) = \{(A,B) \in SL(2,R)^2; A \equiv I, B \equiv I (\mathrm{mod}\, \boldsymbol{\pi}^m)\}$. Then $\overline{K}_m^1 = \{(A, \boldsymbol{\varepsilon} A \boldsymbol{\varepsilon}); A \in SL(2,R_m)\}$. Here $m \geq 1$. Also put $\overline{K}_0^1 = K_0^1/K(\boldsymbol{\pi}^m) = SL(2,R_m) \times SL(2,R_m)$.

2. Lemma. *We have that* $[K_0^1 : K_m^1] \int_{K_0^1} 1_{K_m^1}(k^{-1} r^{-1} t_{\rho_m} r k) dk$ *is equal to the cardinality of*

$$L_m^1 = L_{m,\rho_m}^1 = \{y \in \overline{K}_0^1/\overline{K}_m^1;\ y^{-1} r^{-1} t_{\rho_m} r y \in \overline{K}_m^1\}.$$

Proof. The integral can be expressed as

$$\int_{K_0^1/K_m^1} 1_{K_m^1}(k^{-1} r^{-1} t_{\rho_m} r k) dk = \#\{k K_m^1 \in K_0^1/K_m^1;\ k^{-1} r^{-1} t_{\rho_m} r k \in K_m^1\} = \# L_m^1.$$

□

To compute the cardinality $\# L_m^1$ of L_m^1, introduce $N_i = \mathrm{ord}(b_i)$ and a unit B_i with $b_i = B_i \boldsymbol{\pi}^{N_i} (i = 1,2)$, $\nu_i = N_i - j_i$, $b_i' = (B_i/\varepsilon_i u_i) \boldsymbol{\pi}^{\nu_i}$ (where $\rho_i = u_i \boldsymbol{\pi}^{\overline{\rho}_i}$), and $D_i = D \varepsilon_i^2 u_i^2 \boldsymbol{\pi}^{2j_i}$. Further, put $X = \mathrm{ord}(a_1 - a_2)$, and write $\overline{a}$ for the image of a in R_m. Then $t_{\rho,r} = r^{-1} t_\rho r = \left(\begin{smallmatrix} a_1 & b_1' D_1 \\ b_1' & a_1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} a_2 & b_2' D_2 \\ b_2' & a_2 \end{smallmatrix}\right)$. Also put $d(A)$ for $(A, \boldsymbol{\varepsilon} A \boldsymbol{\varepsilon})$. When $\nu_1 = \nu_2$, we write ν for this value.

3. Lemma. *The set* L_m^1 *is non empty precisely when (1)* $0 \leq m \leq X$, *(2)* $\nu_i \geq 0$, *(3)* $\nu_1 < m$ *if and only if* $\nu_2 < m$, *in which case* $\nu_1 = \nu_2$ *and we write* ν *for the common value, (4) if* $\nu < m$, *and* $\nu_1 < N_1$ *or* $\nu_2 < N_2$ *or* E/F *is ramified, then* $u_1/u_2 \in \frac{B_1 \varepsilon_1}{B_2 \varepsilon_2} R^{\times 2}$, *(5) if* $m > 2N_i - \nu_i + \mathrm{ord}\, D$ $(\geq \nu_i$, *thus* $\nu_1 = \nu_2)$ *for some* $i (= 1,2)$, *then* $N_1 = N_2$ *(the common value is denoted by* N*), and* $m + \nu \leq X$.

If the set L_m^1 *is non empty then its cardinality is:* 1, *if* $m = 0$; $(q^2-1) q^{3m-2}$, *if* $1 \leq m \leq \min(\nu_1, \nu_2)$ *(thus* $\overline{b}_i' = 0$*);* $2q^{m+2\nu}$, *if* $\nu < m$, *and* E/F *is ramified or* $\nu_1 < N_1$ *or* $\nu_2 < N_2$*;* $(q+1) q^{m+2\nu-1}$, *if* $\nu < m$ *and* E/F *is unramified and* $\nu_1 = N_1$ *and* $\nu_2 = N_2$.

Proof. The set L_m^1 is isomorphic to the set of y in $SL(2,R_m) \times SL(2,R_m)/d\big(SL(2,R_m)\big)$, such that $y^{-1} r^{-1} t_\rho r y$ lies in $d\big(SL(2,R_m)\big)$. This is isomorphic to the set of $\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)$ in $SL(2,R_m)$ with

$$(*) \qquad \begin{pmatrix} \overline{a}_1 & \overline{b}_1' \overline{D}_1 \\ \overline{b}_1' & \overline{a}_1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} \overline{a}_2 & \overline{b}_2' \overline{D}_2 \\ \overline{b}_2' & \overline{a}_2 \end{pmatrix}.$$

If L_m^1 is non empty, comparing the traces of the two components of $r^{-1} t_\rho r$, we obtain $\overline{a}_1 = \overline{a}_2$, thus $0 \leq m \leq X = \mathrm{ord}(a_1 - a_2)$. Consequently $(*)$ holds with $\overline{a}_1$ and $\overline{a}_2$ replaced by 0. Then $\overline{b}_1' = 0$ if and only if $\overline{b}_2' = 0$, namely $\nu_1 \geq m$ if and only if $\nu_2 \geq m$.

Multiplying out the matrices in $(*)$, we see that L_m^1 is then the set of $(x_1, x_2, x_3, x_4) \in R_m^4$ with $x_1x_4 - x_2x_3 = 1$, which satisfy

$$\overline{b}_1'\overline{D}_1x_3 = \overline{b}_2'x_2, \quad \overline{b}_1'\overline{D}_1x_4 = x_1\overline{b}_2'\overline{D}_2$$
$$x_1\overline{b}_1' = x_4\overline{b}_2', \quad x_2\overline{b}_1' = x_3\overline{b}_2'\overline{D}_2.$$

If $\nu_1 < m$, thus $\overline{b}_1' \neq 0$, and $|b_2'| < |b_1'|$, then the last relation implies that $|x_2| < 1$, while the third relation implies that $|x_1| < 1$. Here we write $|x| < 1$ if a representative in R of x in R_m has this property. This contradicts $x_1x_4 - x_2x_3 = 1$. Hence, when $\nu_1 < m$ or $\nu_2 < m, \nu_1 = \nu_2$.

The quantitative part of the lemma is clear when $m = 0$. When $\overline{b}_i' = 0$ we simply have that $L_m^1 = SL(2, R_m)$. The cardinality of this group is $(q^2-1)q$ when $m = 1$ and so $R/\boldsymbol{\pi}$ is a field. For $m \geq 1$, apply induction on m using the natural surjection $SL(2, R_m) \to SL(2, R_{m-1})$. Suppose then that $\nu = \nu_1 = \nu_2 < m$. Now for each solution x of $(*)$ there are a_2, a_2', a_4 in R_ν, such that on putting $\eta = (B_1/\varepsilon_1u_1)/(B_2/\varepsilon_2u_2)$, we have

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1 & (\overline{D}_1x_3 + \boldsymbol{\pi}^{m-\nu}a_2)\eta \\ x_3 & (x_1 + \boldsymbol{\pi}^{m-\nu}a_4)\eta \end{pmatrix}$$
$$= \begin{pmatrix} x_1 & x_3\overline{D}_1 \\ x_3 & x_1 \end{pmatrix} \begin{pmatrix} 1 & \boldsymbol{\pi}^{m-\nu}A_2 \\ 0 & 1 + \boldsymbol{\pi}^{m-\nu}A_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}$$

on using the first and third relations in $(*)$, and

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1 & (\overline{D}_2x_3 + \boldsymbol{\pi}^{m-\nu}a_2')\eta^{-1} \\ x_3 & (x_1 + \boldsymbol{\pi}^{m-\nu}a_4)\eta \end{pmatrix}$$
$$= \begin{pmatrix} x_1 & \overline{D}_2x_3\eta^{-2} \\ x_3 & x_1 \end{pmatrix} \begin{pmatrix} 1 & \boldsymbol{\pi}^{m-\nu}A_2' \\ 0 & 1 + \boldsymbol{\pi}^{m-\nu}A_4' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}$$

on using the fourth and third relations in $(*)$, where $A_2, A_4, A_2', A_4' \in R_\nu$ are defined by

$$\begin{pmatrix} x_1 & x_3\overline{D}_1 \\ x_3 & x_1 \end{pmatrix} \begin{pmatrix} A_2 \\ A_4 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_4 \end{pmatrix}, \quad \begin{pmatrix} x_1 & x_3\overline{D}_2\eta^{-2} \\ x_3 & x_1 \end{pmatrix} \begin{pmatrix} A_2' \\ A_4' \end{pmatrix} = \begin{pmatrix} a_2'\eta^{-2} \\ a_4 \end{pmatrix}.$$

Since $x_1x_4 - x_2x_3 = 1$ and $1 + \boldsymbol{\pi}^{m-\nu}A_4 \in R_m^{\times 2}$, η lies in each of the groups

$$N_{D_1} = \{y \in R_m^\times; y = x_1^2 - x_3^2D\varepsilon_1^2u_1^2\boldsymbol{\pi}^{2j_1}\}, \quad N_{D_2} = \{y \in R_m^\times; y = x_1^2 - x_3^2D\varepsilon_2^2\boldsymbol{\pi}^{2j_2}\eta^{-2}\}.$$

The intersection $N_D = N_{D_1} \cap N_{D_2}$ is $R_m^{\times 2}$ if $j_1 > 0$ or $j_2 > 0$ or $D \notin R^\times$, it is $R_m^\times$ if $j_1 = 0, j_2 = 0$, and $D \in R^\times$. Since $\eta = (B_1/\varepsilon_1u_1)/(B_2/\varepsilon_2u_2)$, the 4th qualitative claim of the lemma follows.

When $\nu < m$, the cardinality of L_m^1 is the product of the cardinalities of the sets $\{A_2 \in R_m/\boldsymbol{\pi}^\nu R_m \simeq R/\boldsymbol{\pi}^\nu R\}$ and $\{x_1, x_3 \in R_m; x_1^2 - \overline{D}_1x_3^2 \text{ and } x_1^2 - \overline{D}_2\overline{\eta}^{-2}x_3^2 \in 1 + \boldsymbol{\pi}^{m-\nu}R_m\}$. The first set has cardinality q^ν. The second has cardinality

$$\#\{x_1, x_3 \in R_m; x_1^2 - \overline{D}_1x_3^2 \quad \text{and} \quad x_1^2 - \overline{D}_2\overline{\eta}^{-2}x_3^2 \in N_D\}/[N_D : 1 + \boldsymbol{\pi}^{m-\nu}R_m].$$

The denominator here is

$$[R^\times : 1+\boldsymbol{\pi}^{m-\nu}R]/[R_m^\times : N_D] = (q-1)q^{m-\nu-1}/[R_m^\times : N_D].$$

Hence the cardinality of L_m^1 is

$$[R_m^\times : N_D]\frac{q^{2\nu-m+1}}{q-1}\cdot\begin{cases} q^{2m}-q^{2(m-1)}, & \text{if } D_1\in R^\times \text{ and } D_2\in R^\times,\\ (q-1)q^{m-1}\cdot q^m, & \text{if } D_1\in\boldsymbol{\pi}R \text{ or } D_2\in\boldsymbol{\pi}R.\end{cases}$$

Hence, when $\nu<m$, if E/F is ramified ($D\in\boldsymbol{\pi}R$) or $j_1>0(\nu_1<N_1)$ or $j_2>0$, this is $2q^{m+2\nu}$, while if E/F is unramified ($D\in R^\times$) and $j_1=0$, $j_2=0$, we have $N_D=R_m^\times$, and the cardinality of L_m^1 is $(q+1)q^{m+2\nu-1}$. This completes the quantitative part of the lemma.

If x_1 or x_4 are not units, then $x_1x_4-x_2x_3=1$ implies that $x_2,x_3\in R^\times$. When $\nu_1=\nu_2=\nu<m$, the relations $(*)$ imply that $\overline{D}_1,\overline{D}_2$ are units, hence $j_1=j_2=0$, namely $N_1=\nu_1=\nu_2=N_2$, and that $\operatorname{mod}\boldsymbol{\pi}^{m-\nu}$, we have $\eta=\overline{b}_1'/\overline{b}_2'=(\overline{b}_2'\overline{D}_2)/(\overline{b}_1'\overline{D}_1)$, or $(D_1/D_2)\eta^2=1$, or $(B_1/B_2)^2\equiv 1 \pmod{\boldsymbol{\pi}^{m-\nu}}$.

If x_1 and x_4 are units then we have $\eta=x_4/x_1=\overline{b}_1'/\overline{b}_2' \pmod{\boldsymbol{\pi}^{m-\nu}}$ and $\eta\overline{b}_1'\overline{D}_1=\overline{b}_2'\overline{D}_2$. This last relation implies that: $m>2N_1-\nu_1+\operatorname{ord}D(\geq\nu_1$, so $\nu_1=\nu_2)$ if and only if $2N_2-\nu_2+\operatorname{ord}D<m$, and if this happens then $N_1=N_2$; the common value is denoted by N. Further, if $2N-\nu+\operatorname{ord}D<m$, then $m>0$ and $X(\geq m)>0$. The relation $\eta\overline{b}_1'\overline{D}_1=\overline{b}_2'\overline{D}_2$ can now be rewritten as asserting that

$$\eta\equiv\frac{B_2\varepsilon_2u_2}{B_1\varepsilon_1u_1}(\operatorname{mod}\boldsymbol{\pi}^{m-2N+\nu-\operatorname{ord}D}).$$

Together with $\eta=(B_1/\varepsilon_1u_1)/(B_2/\varepsilon_2u_2)$, we obtain that $(B_2/B_1)^2\equiv 1(\operatorname{mod}\boldsymbol{\pi}^{m-2N+\nu-\operatorname{ord}D})$.

Thus we have this last relation when x_1,x_4 are units, and when they are not. Since B_i are units, we rewrite the relation as $m-2N+\nu-\operatorname{ord}D\leq\operatorname{ord}(B_1^2-B_2^2)$, namely as $m+\nu\leq\operatorname{ord}\big(D(b_1^2-b_2^2)\big)=\operatorname{ord}(a_1^2-a_2^2)=X$. Indeed, since t_ρ is topologically unipotent, we cannot have $|a_1+a_2|<1$. Finally note that $|a_1-a_2|=|a_1^2-a_2^2|=|Db_1^2-Db_2^2|\leq\max(|Db_1^2|,|Db_2^2|)$, hence $X\geq\operatorname{ord}D+2\min(N_1,N_2)$. □

C. Orbital integrals of type (I).

We computed above the orbital integrals on the twisted conjugacy classes within the stable θ-conjugacy class of a strongly θ-regular element (which is topologically unipotent and θ-fixed) $u=t_\rho=h^{-1}t^*h$, where $t^*=(t_1,t_2,\sigma t_2,\sigma t_1;e)$, $t_1=a_1+b_1\sqrt{D}$, $t_2=a_2+b_2\sqrt{D}$. The norm Nu of u is the stable conjugacy class of $(t_1t_2e,t_1\sigma t_2e,t_2\sigma t_1e,\sigma t_1\sigma t_2e;e^2t_1t_2\sigma t_1\sigma t_2)$, or Nt^*, in H. This stable conjugacy class is of type (I). Put $x^*=Nt^*$. Consider $x_\rho=h^{-1}x^*h$ of type (I), with $x^*=(x_1,x_2,\sigma x_2,\sigma x_1;e)$, in H. Its stable class consists of two conjugacy classes, parametrized by $\rho(\in\{1,\boldsymbol{\pi}\}$ if E/F is unramified, $\in\{1,\varepsilon\}=R^\times/R^{\times 2}$ if E/F is ramified), in the torus

$$T_\rho=\{x_\rho=[\phi^D(\alpha_1+\beta_1\sqrt{D}),\phi_\rho^D(\alpha_2+\beta_2\sqrt{D})]\in C_0\}$$

in $H = GSp(2,F)$. We write $x_1 = \alpha_1 + \beta_1\sqrt{D}$, $x_2 = \alpha_2 + \beta_2\sqrt{D}$ ($\alpha_i, \beta_i \in F$). Then we have to compute $\Phi^H_{1_K}(x_\rho)$

$$= \int_{T_\rho\backslash H} 1_K(g^{-1}x_\rho g)dg = \sum_{m\geq 0} |K|_H \int_{T_\rho\backslash C_0/C_0\cap z(m)Kz(m)^{-1}} 1_K\big(z(m)^{-1}h^{-1}x_\rho hz(m)\big)dh.$$

The last equality follows from the disjoint decomposition $H = \underset{m\geq 0}{\cup} C_0 z(m)K$ of Lemma I.J.5.

The integrand in the last integral is non zero precisely when $h^{-1}x_\rho h$ lies in $z(m)Kz(m)^{-1}\cap C_0 = K_m^{C_0}$. Since $[K : K_m^{C_0}] = [K_0 : K_m]$ (by Lemma I.J.7), we get

$$= \sum_{m\geq 0}[K_0 : K_m]\int_{T_\rho\backslash C_0} 1_{K_m}(h^{-1}x_\rho h)dh.$$

In contrast to the case considered in the last section, where we worked in $SL(2)\times SL(2)$, the change of variables (which led to the introduction of $\tilde{\rho}$ and t_m there) does not change our x_ρ.

Using a partition $C_0 = \big(GL(2,F)\times GL(2,F)\big)' = \underset{r\in R}{\cup} T_\rho r K_0$, this can be written as

$$= \sum_{m\geq 0}\sum_{r\in R_\rho}[R_T : T_\rho\cap rK_0r^{-1}][K_0 : K_m]\int_{K_0} 1_{K_m}(k^{-1}r^{-1}x_\rho rk)dk,$$

where $R_T = T_\rho\cap K_0 = T_\rho(R)$. Recall that $\rho = u\boldsymbol{\pi}^{\overline{\rho}}$, thus $\overline{\rho} = \mathrm{ord}(\rho)$ is 0 when E/F is ramified, and it is 0 or 1 when E/F is unramified.

1. Lemma. *A set of representatives R_ρ for $T_\rho\backslash C_0/K_0$ is given by $[r_{j_1}, r_{j_2}], j_i \geq 0, r_{j_1} = \phi^D(\sqrt{-\boldsymbol{\pi}}^{-j_1})(\begin{smallmatrix}1&0\\0&\boldsymbol{\pi}^{j_1}\end{smallmatrix}), r_{j_2} = \phi^D_\rho(\sqrt{-\boldsymbol{\pi}}^{-j_2})(\begin{smallmatrix}1&0\\0&\boldsymbol{\pi}^{j_2}\end{smallmatrix})$, when E/F is ramified. When E/F is unramified, it is given by $I\times\boldsymbol{\pi}^{-(j_2-\overline{\rho})/2}(\begin{smallmatrix}1&0\\0&\boldsymbol{\pi}^{j_2-\overline{\rho}}\end{smallmatrix})(j_2\geq 0, j_2-\overline{\rho}$ even), $\boldsymbol{\pi}^{-j_1/2}(\begin{smallmatrix}1&0\\0&\boldsymbol{\pi}^{j_1}\end{smallmatrix})\times I(j_1 > 0$, even j_1), $\boldsymbol{\pi}^{-[j_1/2]}(\begin{smallmatrix}1&0\\0&\boldsymbol{\pi}^{j_1}\end{smallmatrix})\times\phi^D_\rho(\varepsilon')\boldsymbol{\pi}^{-[(j_2-\overline{\rho})/2]}(\begin{smallmatrix}1&0\\0&\varepsilon\pi^{j_2-\overline{\rho}}\end{smallmatrix})$ $(j_1, j_2 > 0$, even $j_1 - j_2 + \overline{\rho}$; ε ranges over $R^\times/R^{\times 2}, \varepsilon'\in R_E^\times$ with norm $N_{E/F}\varepsilon' = \varepsilon^{-1}$). Here $[*]$ denotes the maximal integer bounded by $*$.*

Proof. Using the double coset decomposition (Lemma I.I.1) for $T_\rho\backslash GL(2,F)/K$, we can write

$$C_0 = \big(GL(2,F)\times GL(2,F)\big)' = \cup_{j_1,j_2\geq 0}\big(T_1(\begin{smallmatrix}1&0\\0&\boldsymbol{\pi}^{j_1}\end{smallmatrix})K\times T_\rho(\begin{smallmatrix}1&0\\0&\boldsymbol{\pi}^{j_2-\overline{\rho}}\end{smallmatrix})K\big)'.$$

If E/F is ramified then $\overline{\rho} = 0$, $E = F(\sqrt{-\boldsymbol{\pi}})$, and $\|\phi^D_\rho(\sqrt{-\boldsymbol{\pi}})\| = N_{E/F}(\sqrt{-\boldsymbol{\pi}}) = \boldsymbol{\pi}$, so that r_{j_1}, r_{j_2} have determinant one, and $C_0 = \underset{j_1,j_2\geq 0}{\cup}(T_1 r_{j_1}K\times T_\rho r_{j_2}K)'$. We naturally denote $T_\rho\subset C_0$ also as $(T_1\times T_\rho)'\subset\big(GL(2,F)\times GL(2,F)\big)'$. We still have to show that $C_0 = \cup T_\rho\cdot r_{j_1}\times r_{j_2}\cdot K_0$. For that, note that if $\|t_1 r_{j_1}k_1\| = \|t_2 r_{j_2}k_2\|$, then $\|k_1k_2^{-1}\|$ lies in $R^\times\cap N_{E/F}E^\times = R^{\times 2}$. Then t_1 can be multiplied by a scalar in $R^\times$, so that $\|k_1\| = \|k_2\|$, namely $[k_1, k_2]$ lies in K_0, and so also $[t_1, t_2]$ lies in $T_\rho\subset C_0\subset H$, as asserted.

If E/F is unramified, we need to consider the conditions implied by the equation $\|t_1(\begin{smallmatrix}1&0\\0&\boldsymbol{\pi}^{j_1}\end{smallmatrix})k_1\| = \|t_2(\begin{smallmatrix}1&0\\0&\boldsymbol{\pi}^{j_2-\overline{\rho}}\end{smallmatrix})k_2\|$. These are: $|\boldsymbol{\pi}^{j_1-j_2+\overline{\rho}}|\in R^\times N_{E/F}E^\times$, thus $j_1 - j_2 + \overline{\rho}$ is even.

We would like $k = k_1 k_2^{-1}$ to have determinant 1, and we can modify k by multiplication by $\varepsilon \in R^\times$ (thus $\|k\|$ ranges over $R^\times/R^{\times 2}$), or by $\left(\begin{smallmatrix}1 & 0\\ 0 & \boldsymbol{\pi}^{-j_1}\end{smallmatrix}\right)t\left(\begin{smallmatrix}1 & 0\\ 0 & \boldsymbol{\pi}^{j_1}\end{smallmatrix}\right) \in K$ or $\left(\begin{smallmatrix}1 & 0\\ 0 & \rho^{-1}\boldsymbol{\pi}^{-(j_2-\overline{\rho})}\end{smallmatrix}\right)t\left(\begin{smallmatrix}1 & 0\\ 0 & \rho\boldsymbol{\pi}^{j_2-\overline{\rho}}\end{smallmatrix}\right) \in K$, $t \in T_1$, whose determinants are in $R^{\times 2}$ if $j_1 > 0$ (resp. $j_2 > 0$), or in $N_{E/F} R_E^\times = R^\times$ otherwise. We then obtain the representatives of the lemma, which lie in C_0. To repeat, if $j_1 j_2 = 0$ then $\varepsilon = 1$, if $j_1 j_2 \neq 0$ then ε ranges over $R^\times/R^{\times 2}$ and $j_1 - j_2 + \overline{\rho}$ is even. $\square$

2. Lemma. *The index $[R_T : T_\rho \cap rK_0r^{-1}]$ is the product of $q^{j_1+j_2}$ and: 1 if E/F is ramified or $j_1 = 0 = j_2$; $(q+1)/q$ if E/F is unramified, and either $j_1 = 0$ or $j_2 = 0$; $\frac{1}{2}(\frac{q+1}{q})^2$ if E/F is unramified and $j_1 j_2 \neq 0$.*

Proof. The intersection $T_\rho \cap rK_0r^{-1}$ consists of x_ρ such that $r^{-1}x_\rho r$ lies in K_0. Since

$$r^{-1}x_\rho r = \begin{pmatrix}\alpha_1 & \beta_1 D\boldsymbol{\pi}^{j_1}\\ \beta_1/\boldsymbol{\pi}^{j_1} & \alpha_1\end{pmatrix} \times \begin{pmatrix}\alpha_2 & \beta_2 D\rho\cdot\varepsilon\boldsymbol{\pi}^{j_2-\overline{\rho}}\\ \beta_2/(\rho\boldsymbol{\pi}^{j_2-\overline{\rho}}\varepsilon) & \alpha_2\end{pmatrix},$$

it follows that $T_\rho \cap rK_0r^{-1}$ is isomorphic to $\big(R_E(j_1)^\times \times R_E(j_2)^\times\big)'$, where $R_E(j) = R + \boldsymbol{\pi}^j R_E$, and the prime indicates (x, y) with $N_{E/F}x = N_{E/F}y$. Since R_T is $(R_E^\times \times R_E^\times)'$ under the same isomorphism, we are to compute the cardinality of the kernel in the exact sequence

$$\begin{aligned}1 &\to (R_E^\times \times R_E^\times)'/\big(R_E(j_1)^\times \times R_E(j_2)^\times\big)' \to R_E^\times \times R_E^\times/R_E(j_1)^\times \times R_E(j_2)^\times\\ &\to R_E^\times \times R_E^\times/(R_E^\times \times R_E^\times)'\big(R_E(j_1)^\times \times R_E(j_2)^\times\big) \to 1.\end{aligned}$$

For the middle term, note that $[R_E^\times : R_E(j)^\times]$ is 1 if $j = 0$ and it is the quotient of $[R_E^\times : 1 + \boldsymbol{\pi}^j R_E]$ by $[R^\times : 1 + \boldsymbol{\pi}^j R] = (q-1)q^{j-1}$ when $j \geq 1$. When E/F is ramified then $\boldsymbol{\pi}_E^2 = \boldsymbol{\pi}$ and $q_E = q$ so that the quotient is $(q-1)q^{2j-1}/(q-1)q^{j-1} = q^j$. When E/F is unramified, $\boldsymbol{\pi}_E = \boldsymbol{\pi}$ and $q_E = q^2$, so that the quotient is $(q^2-1)q^{2(j-1)}/(q-1)q^{j-1} = (q+1)q^{j-1}$.

It remains to compute the cardinality of the image in the short exact sequence. This set is isomorphic to its image under the norm map $N = N_{E/F}$. The cardinality of $NR_E^\times \times NR_E^\times/\{(x,x)\} \cdot NR_E(j_1)^\times \times NR_E(j_2)^\times$ is 1 if E/F is ramified or $j_1 j_2 = 0$, and it is $[NR_E^\times : R_E^{\times 2}] = 2$ if E/F is unramified and $j_1 j_2 \geq 1$. $\square$

As usual write $R_m = R/\boldsymbol{\pi}^m R, \beta_i = B_i' \boldsymbol{\pi}^{n_i}$ (B_i' in $R^\times$, integral n_i), $\nu_i = n_i - j_i (i = 1, 2), \beta_1' = B_1' \boldsymbol{\pi}^{\nu_1}, \beta_2' = (B_2'/\varepsilon u)\boldsymbol{\pi}^{\nu_2}$ (where $\rho = u\boldsymbol{\pi}^{\overline{\rho}}$), $D_1 = D\boldsymbol{\pi}^{2j_1}, D_2 = Du^2\varepsilon^2\boldsymbol{\pi}^{2j_2}, \chi = \operatorname{ord}(\alpha_1 - \alpha_2), \overline{\alpha}$ for the image of $\alpha \in R$ in R_m, $d(A)$ for $(A, \boldsymbol{\varepsilon}A\boldsymbol{\varepsilon})$. If $\nu_1 = \nu_2$, put ν for the common value.

3. Lemma. *The integral $[K_0 : K_m]\int_{K_0} 1_{K_m}(k^{-1}r^{-1}x_\rho rk)dk$ is equal to the cardinality of*

$$L_m = \{y \in \big(GL(2, R_m) \times GL(2, R_m)\big)'/d\big(GL(2, R_m)\big); y^{-1}r^{-1}x_\rho ry \in d\big(GL(2, R_m)\big)\}.$$

If this set is non empty then $0 \leq m \leq \chi$, and $\nu_1 \geq m$ if and only if $\nu_2 \geq m$. If $\nu_1 < m$ or $\nu_2 < m$ then $\nu_1 = \nu_2$.

Proof. Since

$$L_m = \{\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix} \in SL(2, R_m); \begin{pmatrix}\overline{\alpha}_1 & \overline{\beta}_1'\overline{D}_1\\ \overline{\beta}_1' & \overline{\alpha}_1\end{pmatrix}\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix} = \begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}\begin{pmatrix}\overline{\alpha}_2 & \overline{\beta}_2'\overline{D}_2\\ \overline{\beta}_2' & \overline{\alpha}_2\end{pmatrix}\},$$

the proof is exactly the same as in Section B, where the group was $Sp(2, F)$ rather than $GSp(2, F)$. $\square$

4. Lemma. *If L_m is non empty, then its cardinality is:* 1 *if* $m = 0$; $(q^2 - 1)q^{3m-2}$ *if* $1 \leq m \leq \min(\nu_1, \nu_2)$ *(thus* $\overline{\beta}_i' = 0$*);* $2q^{m+2\nu}$*, if* $\nu < m$*, and* E/F *is ramified or* $\nu_1 < n_1$ *or* $\nu_2 < n_2$*;* $(q+1)q^{m+2\nu-1}$ *if* $\nu < m, \nu_1 = n_1, \nu_2 = n_2$*, and* E/F *is unramified.*

Proof. Since $L_m \simeq L_m^1$, the proof is the same as in the case of $Sp(2, F)$. ☐

5. Lemma. *Suppose that* $\nu < m$*. If* $2n_i - \nu + \operatorname{ord} D < m$ *for some* $i (= 1, 2)$*, then* $n_1 = n_2$ *(the common value is then denoted by* n*), and (*$0 \leq \nu < m \leq \chi$ *and)* $m + \nu \leq \chi$*. Further,* $B_1'/B_2' \in \varepsilon u R^{\times 2}$ *unless* $\nu_1 = n_1, \nu_2 = n_2$*, and* E/F *is unramified. Note that when* E/F *is ramified,* $\varepsilon = 1$ *and* $\rho = u$*.*

Proof. The proof proceeds exactly the same as in the case of $Sp(2, F)$, to show that $m + \nu \leq \operatorname{ord}\big(D(\beta_1^2 - \beta_2^2)\big) = \operatorname{ord}(\alpha_1^2 - \alpha_2^2)$. It remains to show that $\chi = \operatorname{ord}(\alpha_1 - \alpha_2)$ is equal to $\operatorname{ord}\big(D(\beta_1^2 - \beta_2^2)\big)$. For this, recall that $x_1 = t_1 t_2 = \alpha_1 + \beta_1\sqrt{D}, x_2 = t_1 \sigma t_2 = \alpha_2 + \beta_2\sqrt{D}$, t_1 and t_2 are units, and so if $\operatorname{tr} = 1 + \sigma$, then

$$\begin{aligned}
|\alpha_1 - \alpha_2|^2 &= |\operatorname{tr} x_1 - \operatorname{tr} x_2|^2 = |\operatorname{tr}(t_1 t_2) - \operatorname{tr}(t_1 \sigma t_2)|^2 = |(t_1 - \sigma t_1)(t_2 - \sigma t_2)|^2 |t_1 t_2|^2 \\
&= |(x_1 - \sigma x_2)(x_1 - x_2)|^2 = |\big((\alpha_1 - \alpha_2) + (\beta_1 + \beta_2)\sqrt{D}\big)\big((\alpha_1 - \alpha_2) + (\beta_1 - \beta_2)\sqrt{D}\big)|^2 \\
&= |\big((\alpha_1 - \alpha_2)^2 - (\beta_1 + \beta_2)^2 D\big)\big((\alpha_1 - \alpha_2)^2 - (\beta_1 - \beta_2)^2 D\big)|.
\end{aligned}$$

Note that $0 \leq \nu < m \leq \chi$, so that $|\alpha_1 - \alpha_2| < 1$. If $|\beta_1 - \beta_2| \leq |\alpha_1 - \alpha_2| < 1$ then $|(\alpha_1 - \alpha_2)^2 - (\beta_1 + \beta_2)^2 D| = 1$, and so $|D| = 1$ and $|\beta_1 + \beta_2| = 1$, hence $|\beta_i| = 1$ and $n = 0$, so $\nu = 0$ and $\chi \geq m$ is our claim. The last sentence is valid with β_2 replaced by $-\beta_2$. It remains to deal with the case where $|\beta_1 \pm \beta_2| > |\alpha_1 - \alpha_2|$. Then $|\alpha_1 - \alpha_2| = |D(\beta_1^2 - \beta_2^2)|$, as was to be shown. ☐

D. Comparison in stable case (I), E/F unramified.

Let us summarize the result of the computation of the stable twisted orbital integral in Section B. It is

$$\Phi_{1_K}^{G,st}(u\theta) = \Phi_{1_{Z_K(\theta)}}^{Z_G(\theta),st}(u) = \sum_{\rho} \sum_{m \geq 0} \sum_{r \in R_{\rho m}} [R_T^1 : T_{\rho_m}^1 \cap r K_0^1 r^{-1}] \# L_{m,\rho_m}^1,$$

where $u = t_\rho = h^{-1} t^* h$ is topologically unipotent. Recall that L_{m,ρ_m}^1 depends on m and ρ_m, but for each m, the set $\{\rho_m\}$ is the same as the set of ρ. Hence we replace ρ_m by ρ in the triple sum above.

Put $N = \min(N_1, N_2)$, where $N_i = \operatorname{ord}(b_i)$. In the case where E/F is unramified, $\rho = (\rho_1, \rho_2), \rho_i \in \{1, \boldsymbol{\pi}\}, u_i = 1$, and the sum over r is a sum over $j_1, j_2 \geq 0$ such that $j_1 - \overline{\rho}_1, j_2 - \overline{\rho}_2$ are even, and over ε_i in $R^\times/R^{\times 2}$ if $j_i > 0$. When $j_1 > 0$ or $j_2 > 0$, and $\nu = \nu_1 = \nu_2 < m$, we have $\varepsilon_1 \varepsilon_2 \in B_1 B_2 R^{\times 2}$. In other words, we have a sum over $\nu_i = N_i - j_i$ $(i = 1, 2)$, $0 \leq \nu_i \leq N_i$, and over $\varepsilon_i \in R^\times/R^{\times 2}$ if $\nu_i < N_i$ for $i = 1, 2$. (If $\nu_i = N_i$ for some i, then $\varepsilon_i \in R^\times/R^\times$).

Then we need to sum over m. We have the range $0 \leq m \leq \min(\nu_1, \nu_2)$, then the range $\nu (= \nu_1 = \nu_2) < m \leq 2N - \nu$ (since $\operatorname{ord} D = 0$ when E/F is unramified ($\nu_i < m$ implies $\nu_1 = \nu_2$)), and the range $2N - \nu < m \leq X - \nu$ ($2N - \nu < m$ implies $\nu < m, N_1 = N_2$, and

$m \leq X - \nu$). Let $\delta(m=0) = \delta(m,0)$ be 0 if $m \neq 0$ and 1 if $m \neq 0$, and $\delta(m \geq 1)$ be 0 if $m < 1$ and 1 if $m \geq 1$. Thus we get the sum of three expressions:

$$\sum_{0\leq\nu_1\leq N_1}\ \sum_{0\leq\nu_2\leq N_2}\ \sum_{0\leq m\leq\min(\nu_1,\nu_2)}$$
$$= \sum_{0\leq m\leq N} \big(\delta(m=0) + \delta(m\geq 1)(1-q^{-2})q^{3m}\big)\big[\sum_{m\leq\nu_1<N_1}\ \sum_{m\leq\nu_2<N_2}$$
$$4(\frac{q+1}{2q})^2 q^{N_1-\nu_1+N_2-\nu_2} + \sum_{m\leq\nu_1<N_1} 2\frac{q+1}{2q}q^{N_1-\nu_1} + \sum_{m\leq\nu_2<N_2} 2\frac{q+1}{2q}q^{N_2-\nu_2} + 1\big];$$

$$\sum_{0\leq\nu\leq N}\ \sum_{\nu<m\leq 2N-\nu} 2(\frac{q+1}{2q})^2 q^{N_1+N_2-2\nu}\cdot 2q^{m+2\nu};$$

$$\delta(N_1,N_2)\sum_{0\leq\nu\leq N}\ \sum_{2N-\nu<m\leq X-\nu} = \delta(N_1,N_2)\big[\sum_{N<m\leq X-N}\frac{q+1}{q}q^{2N+m}$$
$$+ \sum_{0\leq\nu<N}\ \sum_{2N-\nu<m\leq X-\nu} 2(\frac{q+1}{2q})^2 q^{2N-2\nu}\cdot 2q^{m+2\nu}\big].$$

To compute the first expression, note that

$$\frac{q+1}{q}\sum_{m\leq\nu_1<N_1} q^{N_1-\nu_1} + 1 = 1 + (q+1)\sum_{\nu_1=0}^{N_1-1-m} q^{\nu_1} = 1 + \frac{q+1}{q-1}(q^{N_1-m}-1)$$
$$= \frac{q+1}{q-1}q^{N_1-m} - \frac{2}{q-1}.$$

Hence $[\dots]$ is

$$\big[\big(\frac{q+1}{q-1}q^{N_1-m} - \frac{2}{q-1}\big)\big(\frac{q+1}{q-1}q^{N_2-m} - \frac{2}{q-1}\big)\big].$$

So the first expression is

$$\big(\frac{q+1}{q-1}q^{N_1} - \frac{2}{q-1}\big)\big(\frac{q+1}{q-1}q^{N_2} - \frac{2}{q-1}\big)$$
$$+ \sum_{1\leq m\leq N}\frac{q^{-2}(q^2-1)}{(q-1)^2}\big((q+1)^2 q^{N_1+N_2+m} - 2(q+1)(q^{N_1}+q^{N_2})q^{2m} + 4q^{3m}\big)$$

$$= (q-1)^{-2}\big[\big((q+1)q^{N_1}-2\big)\big((q+1)q^{N_2}-2\big)\big] + \frac{q^{-1}(q+1)}{q-1}\big[\frac{(q+1)^2}{q-1}q^{N_1+N_2}(q^N-1)$$
$$- \frac{2q(q+1)}{q^2-1}(q^{N_1}+q^{N_2})(q^{2N}-1) + \frac{4q^2}{q^3-1}(q^{3N}-1)\big].$$

The second expression is

$$q^{N_1+N_2-1}(q+1)^2 \sum_{0\le\nu\le N} \sum_{\nu<m\le 2N-\nu} q^{m-1} = q^{N_1+N_2-1}(q+1)^2 \sum_{0\le\nu\le N} q^\nu \sum_{0\le m<2N-2\nu} q^m$$

$$= q^{N_1+N_2-1}\frac{(q+1)^2}{q-1} \sum_{0\le\nu\le N} (q^N-1)q^\nu = q^{N_1+N_2-1}(\frac{q+1}{q-1})^2(q^N-1)(q^{N+1}-1).$$

The third expression is the product of $\delta(N_1,N_2)$ and the sum of

$$(q+1)^2 q^{2N-1} \sum_{0\le\nu<N} \sum_{2N-\nu<m\le X-\nu} q^{m-1} = \frac{(q+1)^2}{q-1} q^{2N-1} \sum_{0\le\nu<N} (q^X - q^{2N})q^{-\nu}$$

$$= \frac{(q+1)^2}{(q-1)^2} q^{2N}(q^X-q^{2N})(1-q^{-N}),$$

and of

$$\frac{(q+1)}{q-1} q^{2N}(q^{X-N}-q^N),$$

namely it is

$$\delta(N_1,N_2)(q-1)^{-2}[(q^{X-N}-q^N)q^{N_1+N_2}(q+1)\big(q-1+(q+1)(q^N-1)\big)].$$

A pleasant surprise is that the stable orbital integral $\Phi_{1_K}^{GSp(2,F),st}(Nt_\rho)$ takes precisely the same form. Indeed, we have in this case a sum over $\rho \in \{1, \boldsymbol{\pi}\}$, a sum over $j_1, j_2 \ge 0$ such that $j_1 - (j_2 - \overline{\rho})$ is even, and a sum over $\varepsilon \in R^\times/R^{\times 2}$ when $j_1 j_2 \ge 1$. When $\nu = \nu_1 = \nu_2 < m$, and $j_1 j_2 \ge 1$, there is a condition $\varepsilon \in (B_1'/B_2')R^{\times 2}$. In other words we have a sum over $\nu_i = n_i - j_i (i=1,2), 0 \le \nu_i \le n_i$, and over $\varepsilon \in R^\times/R^{\times 2}$ if $m \le \nu_i < n_i (i=1,2)$. The sum over m is cut into three ranges, as in the twisted case. Exactly the same expressions are obtained, but for slightly different reasons. In the first range, the coefficient $4 \cdot (\frac{q+1}{2q})^2$ of the twisted case becomes $2 \cdot \frac{1}{2}(\frac{q+1}{q})^2$; and $2 \cdot \frac{q+1}{2q}$ is the index $\frac{q+1}{q}$. Similarly in the second and third ranges, $2 \cdot (\frac{q+1}{2q})^2$ is $\frac{1}{2}(\frac{q+1}{q})^2$. Writing in the non twisted case n_1, n_2, n and χ for the integers denoted by N_1, N_2, N, X in the twisted case, we obtain

$$(q-1)^{-2}\big\{\big((q+1)q^{n_1}-2\big)\big((q+1)q^{n_2}-2\big) + (q+1)^3 q^{n_1+n_2-1}(q^n-1)$$
$$-2(q+1)(q^{n_1}+q^{n_2})(q^{2n}-1) + \frac{4q(q+1)}{q^2+q+1}(q^{3n}-1) + (q+1)^2 q^{n_1+n_2-1}(q^n-1)(q^{n+1}-1)$$
$$+\delta(n_1,n_2)(q+1)q^{n_1+n_2}(q^{\chi-n}-q^n)\big(q-1+(q+1)(q^n-1)\big)\big\}.$$

Notations. For the actual comparison, we use the following notations: $t^* = (t_1, t_2, \sigma t_2, \sigma t_1)$ (the last – fifth – component e, has to be a unit in $R^\times$, and will not affect otherwise the value of the integral), and $Nt^* = (x_1 = t_1 t_2, x_2 = t_1 \sigma t_2, \sigma x_2, \sigma x_1)$. Further, $t_1 = a_1 + b_1\sqrt{D}, t_2 = a_2 + b_2\sqrt{D}, N_i = \mathrm{ord}(b_i)$, and $n_i = \mathrm{ord}(\beta_i)$, where

$$x_1 = \alpha_1 + \beta_1\sqrt{D} = t_1 t_2 = a_1 a_2 + D b_1 b_2 + \sqrt{D}(a_2 b_1 + a_1 b_2)$$
$$x_2 = \alpha_2 + \beta_2\sqrt{D} = t_1 \sigma t_2 = a_1 a_2 - D b_1 b_2 + \sqrt{D}(a_2 b_1 - a_1 b_2).$$

Also, $\chi = \operatorname{ord}(\alpha_1 - \alpha_2) = \operatorname{ord}(2Db_1b_2) = \operatorname{ord} D + N_1 + N_2$. Note that t^* is topologically unipotent, hence a_1, a_2 are units. Since the value of the θ-orbital integral is not changed if in t^* the entry t_2 is multiplied by -1, (so is σt_2), we may assume that $|a_1 - a_2| \leq |a_1 + a_2|$, namely that $|a_1 + a_2| = 1$. Then

$$\begin{aligned} X &= \operatorname{ord}(a_1 - a_2) = \operatorname{ord}[(a_1^2 - a_2^2)(a_2^2 - b_2^2 D)] = \operatorname{ord}\{D[(b_1^2 - b_2^2)a_2^2 - b_2^2(a_2^2 - a_1^2)]\} \\ &= \operatorname{ord} D(b_1^2 a_2^2 - a_1^2 b_2^2) = \operatorname{ord} D\beta_1\beta_2 = \operatorname{ord} D + n_1 + n_2. \end{aligned}$$

Further, if $N_1 < N_2$, since a_1, a_2 are units, we have $n_1 = n_2 = n = N_1$. If $n_1 < n_2$ then $N_1 = N_2 = N = n_1$. Otherwise $n_1 = n_2 = n = N = N_1 = N_2$ and $X = \chi$, in which case the two expressions to be compared are obviously equal. By symmetry, it suffices to perform the comparison when $n_1 < n_2$, thus $n_2 > n_1 = n = N_1 = N_2$, $\chi = 2N$ and $X = n_1 + n_2$.

The first, "twisted", expression, multiplied by $(q-1)^2$, is equal to

$$\begin{aligned} A = &\big((q+1)q^n - 2\big)\big((q+1)q^n - 2\big) + (1 + q^{-1})[(q+1)^2 q^{2n}(q^n - 1) - 4q^{n+1}(q^{2n} - 1) \\ &+ \frac{4q^2(q-1)}{q^3 - 1}(q^{3n} - 1)] + (q+1)^2 q^{2n-1}(q^n - 1)(q^{n+1} - 1) \\ &+ (q+1)q^{2n}(q^{n_2} - q^n)\big((q+1)q^n - 2\big). \end{aligned}$$

The last summand appears since $N_1 = N_2 (= n)$.

This we compare with the second, untwisted integral, which, multiplied by $(q-1)^2$, is

$$\begin{aligned} a = &\big((q+1)q^n - 2\big)\big((q+1)q^{n_2} - 2\big) + (1 + q^{-1})[(q+1)^2 q^{n+n_2}(q^n - 1) \\ &- 2q(q^n + q^{n_2})(q^{2n} - 1) + \frac{4q^2(q-1)}{q^3 - 1}(q^{3n} - 1)] \\ &+ (q+1)^2 q^{n+n_2-1}(q^n - 1)(q^{n+1} - 1). \end{aligned}$$

The contribution from the third range is zero since $n_2 \neq n_1 (= n)$.

A simple subtraction yields

$$\begin{aligned} A - a &= (q^n - q^{n_2})[(q+1)\big((q+1)q^n - 2\big) + (1 + q^{-1})[(q+1)^2(q^n - 1)q^n - 2q(q^{2n} - 1)] \\ &\quad + (q+1)^2 q^{n-1}(q^n - 1)(q^{n+1} - 1) - (q+1)q^{2n}\big((q+1)q^n - 2\big)] \\ &= (q^n - q^{n_2})(q+1)[\big((q+1)q^n - 2\big)(1 - q^{2n}) + (q+1)^2 q^{n-1}(q^n - 1) - 2q^{2n} \\ &\quad + 2 + (q+1)q^{n-1}(q^n - 1)(q^{n+1} - 1)], \end{aligned}$$

and this is 0 (on opening parenthesis in $[\dots]$). This completes the comparison in case (I), when E/F is unramified, once we show that the measure factor which appears in the statement of the theorem is 1 in our case, of type (I), E/F unramified.

Lemma. *In the case of tori of type (I), the measure factor*

$$[T^{*\theta}(R) : (1+\theta)\big(T^*(R)\big)]/[T_H^*(R) : N\big(T^*(R)\big)]$$

is equal to the ramification index $e(E/F)$ of E over F.

Proof. The norm map N takes $(a,b,\sigma b,\sigma a)\in T^*(R)$ (thus $a,b\in R_E^\times$) to $(ab,a\sigma b,b\sigma a,\sigma a\sigma b)$ in $T_H^*(R)$. To measure the index of the image in $T_H^*(R)=\{(x,y,\sigma y,\sigma x);x,y\in R_E^\times,x\sigma x=y\sigma y\}$, we need to solve $x=ab,y=a\sigma b$ in $a,b\in R_E^\times$, given $x,y\in R_E^\times,x\sigma x=y\sigma y$. It suffices to solve in $b\in R_E^\times$ the equation $b/\sigma b=x/y$, where $(x/y)\sigma(x/y)=1$. By Hilbert theorem 90, there is a solution b in $E^\times$. If E/F is unramified, $\boldsymbol{\pi}_E=\boldsymbol{\pi}$, and if $b=B\boldsymbol{\pi}^n$ is a solution ($B\in R_E^\times$), then so is $B\in R_E^\times$. However, if E/F is ramified, $\sigma\boldsymbol{\pi}_E=-\boldsymbol{\pi}_E$, hence $z=u\boldsymbol{\pi}_E^n(u\in R_E^\times)$ has $z/\sigma z=(-1)^n u/\sigma u$. Writing $u=\alpha+\beta\boldsymbol{\pi}_E$ in $R_E^\times$, we have $\alpha\in R^\times$ and $\beta\in R$, hence $u/\sigma u\equiv 1(\operatorname{mod}\boldsymbol{\pi}_E)$, and the index of $R_E^1=\{u/\sigma u;u\in R_E^\times\}$ in $E^1=\{z/\sigma z;z\in E^\times\}$ is $2=e(E/F)$. Hence $[T_H^*(R):N\big(T^*(R)\big)]$ is $e(E/F)$.

Similarly we need to compute the index in $T^{*\theta}(R)=\{(x,y,\sigma y,\sigma x);x,y\in R_E^\times,x\sigma x=1=y\sigma y\}$ of the image under $(1+\theta)$ of $T^*(R)$, thus of $(1+\theta)(a,b,\sigma b,\sigma a)=(a/\sigma a,b/\sigma b,\sigma b/b,\sigma a/a)$. Again $[E^1:R_E^1]=e(E/F)$, hence $[T^{*\theta}(R):(1+\theta)T^*(R)]=e(E/F)^2$, and the measure factor is $e(E/F)$. □

This computation is naturally used also in the case where E/F is ramified, which we consider next.

E. Comparison in stable case (I), E/F ramified.

Here $\operatorname{ord}(D)=1$. The twisted orbital integral is a sum over $\rho=(\rho_1,\rho_2),\rho_i\in R^\times/R^{\times 2},(\rho_i=u_i\in\{1,\varepsilon\}$ and $\overline{\rho}_i=0)$, and over $j_1,j_2\geq 0$ (these parametrize the representatives $r\in R_\rho$), of the product of the index $q^{j_1+j_2}$, and the quantity: 1 if $m=0,(q^2-1)q^{3m-2}$ if $1\leq m\leq\min(\nu_1,\nu_2),2q^{m+2\nu}$ if $\nu_i<m$ (for some i, but then $\nu=\nu_1=\nu_2$, and $m+\nu\leq X$, and $\rho_1/\rho_2=u_1/u_2\in\big(B_1/B_2)R^{\times 2}\big)$. In this last range: $\nu<m\leq X-\nu$. Note that when $2N_i-\nu+1<m$ for some i, we have $N_1=N_2$. Without loss of generality assume that $N_1\leq N_2$. Thus we get $4q^{N_1+N_2}$ times the sum of

$$
\begin{aligned}
A&=\sum_{0\leq\nu_1\leq N_1}\sum_{0\leq\nu_2\leq N_2}q^{-\nu_1-\nu_2}=\frac{1-q^{-N_1-1}}{1-q^{-1}}\cdot\frac{1-q^{-N_2-1}}{1-q^{-1}}\\
&=\frac{q^2}{(q-1)^2}(1-q^{-N_1-1})(1-q^{-N_2-1});\\
B&=(1-q^{-2})\sum_{0\leq\nu_1\leq N_1}\sum_{0\leq\nu_2\leq N_2}q^{-\nu_1-\nu_2}\sum_{1\leq m\leq\min(\nu_1,\nu_2)}q^{3m}\\
&=\frac{(1-q^{-2})q^3}{q^3-1}\sum_{\substack{0\leq\nu_1\leq N_1\\0\leq\nu_2\leq N_2}}q^{-\nu_1-\nu_2}(q^{3\min(\nu_1,\nu_2)}-1)\\
&=\frac{q(q^2-1)}{q^3-1}\Big[\sum_{0\leq\nu_1\leq N_1}q^{-\nu_1}\sum_{0\leq\nu_2\leq\nu_1}(q^{2\nu_2}-q^{-\nu_2})+\sum_{0\leq\nu_1\leq N_1}(q^{2\nu_1}-q^{-\nu_1})\sum_{\nu_1<\nu_2\leq N_2}q^{-\nu_2}\Big]
\end{aligned}
$$

(here we used $N_1 \le N_2$)

$$= \frac{q(q^2-1)}{q^3-1} \sum_{0\le\nu_1\le N_1} \big[q^{-\nu_1}(\frac{q^{2\nu_1+2}-1}{q^2-1} - \frac{q^{-\nu_1-1}}{q^{-1}-1}) + (q^{2\nu_1}-q^{-\nu_1})\frac{q^{-N_2-1}-q^{-\nu_1-1}}{q^{-1}-1}\big]$$

$$= \frac{q(q^2-1)}{q^3-1} \sum_{0\le\nu_1\le N_1} \big[\frac{q^{\nu_1+2}-q^{-\nu_1}}{q^2-1} - \frac{q^{1-\nu_1}-q^{-2\nu_1}}{q-1} - \frac{q^{-N_2}}{q-1}(q^{2\nu_1}-q^{-\nu_1}) + \frac{q^{\nu_1}-q^{-2\nu_1}}{q-1}\big]$$

$$= \frac{q(q^2-1)}{q^3-1} \sum_{0\le\nu_1\le N_1} \big[q^{\nu_1}(\frac{q^2}{q^2-1} + \frac{1}{q-1}) - \frac{q^{-N_2}}{q-1}q^{2\nu_1} - q^{-\nu_1}(\frac{1}{q^2-1} + \frac{q}{q-1} - \frac{q^{-N_2}}{q-1})\big]$$

$$= \frac{q(q^2-1)}{q^3-1}\big[\frac{q^{N_1+1}}{q-1}\cdot\frac{q^2+q+1}{q^2-1} - \frac{q^{2N_1+2}-1}{q^2-1}\cdot\frac{q^{-N_2}}{q-1} - \frac{1-q^{-N_1-1}}{1-q^{-1}}(\frac{q^2+q+1}{q^2-1} - \frac{q^{-N_2}}{q-1})\big]$$

$$= \frac{q(q^2-1)}{q^3-1}\cdot\frac{1-q^{-N_1-1}}{(q-1)^2}(\frac{q^3-1}{q^2-1}q^{N_1+1} - \frac{q^{2N_1-N_2+2}(1+q^{-N_1-1})}{q+1} - \frac{q(q^3-1)}{q^2-1} + q^{-N_2+1});$$

and (since $4q^{m+2\nu}q^{j_1+j_2} = 4q^{N_1+N_2}q^m$)

$$C = \sum_{0\le\nu\le\min(N_1,N_2)} \sum_{\nu<m\le X-\nu} q^m = \frac{q}{q-1}\sum_{0\le\nu\le N_1}(q^{X-\nu}-q^\nu)$$

$$= \frac{q}{q-1}(q^X\frac{1-q^{-N_1-1}}{1-q^{-1}} - \frac{q^{N_1+1}-1}{q-1}).$$

Then $A+C+B$ is

$$\frac{q^2}{(q-1)^2}(1-q^{-N_1-1})\big[(1-q^{-N_2-1}) + (q^X-q^{N_1}) +$$
$$\big(q^{N_1} - \frac{q-1}{q^3-1}q^{2N_1-N_2+1}(1+q^{-N_1-1}) - 1 + \frac{q^2-1}{q^3-1}q^{-N_2}\big)\big]$$
$$= \frac{q^2}{(q-1)^2}(1-q^{-N_1-1})\Big(q^X - q^{-N_2}(\frac{1}{q} - \frac{q^2-1}{q^3-1} + \frac{q-1}{q^3-1}(q^{2N_1+1}+q^{N_1}))\Big)$$
$$= \frac{q^2}{(q-1)^2}(1-q^{-N_1-1})\Big(q^X - \frac{q-1}{q^3-1}q^{-N_2-1}(1+q^{N_1+1}+q^{2N_1+2})\Big).$$

The product of this with $4q^{N_1+N_2}$ is the product of $4q^2/(q-1)^2$ and

$$q^{N_1+N_2}(1-q^{-N_1-1})(q^X - q^{-N_2-1}\frac{1+q^{1+N_1}+q^{2+2N_1}}{1+q+q^2}).$$

This is the stable twisted orbital integral of 1_K at the strongly θ-regular topologically unipotent element $t_\rho = h^{-1}t^*h$ under consideration. The stable orbital integral of 1_K in $GSp(2,F)$ at its norm is computed similarly. The only differences are that there are only two conjugacy classes in the stable class of the norm, parametrized by ρ which ranges over a set $\{1,\varepsilon\}$ of representatives for $R^\times/R^{\times 2}$. The constraint $\rho_1/\rho_2 = u_1/u_2 \in B_1B_2R^{\times 2}$ in the twisted case (of $Sp(2,F)$) is now replaced by $\rho = u \in B_1'B_2'R^{\times 2}$. Hence we obtain $\frac{1}{2}$ of the expression which was computed in the evaluation of the stable orbital integral of type (I) of 1_K on $Sp(2,F)$. Hence we obtain $\frac{1}{2}$ of exactly the same expression obtained in the twisted case, except that the parameters N_1, N_2, X of $t_\rho = h^{-1}t^*h$ will be denoted by n_1, n_2, χ in the case of its norm. As in the unramified case we have $\chi = \operatorname{ord} D + N_1 + N_2, \operatorname{ord} D = 1$, and $X = \operatorname{ord} D + n_1 + n_2$. If $N_1 < N_2$ then $n_1 = n_2 = N_1$; if $n_1 < n_2$ then $N_1 = N_2 = n_1$. When $n_1 = n_2$ and $N_1 = N_2$ we have $n_1 = n_2 = N_1 = N_2$ and $X = \chi$, then the comparison follows at once. When $N_1 < N_2$ the twisted expression is $4q^2/(q-1)^2$ times

$$q^{N_1+N_2}(1-q^{-N_1-1})(q^{1+2N_1} - q^{-N_2-1}\frac{1+q^{1+N_1}+q^{2+2N_1}}{1+q+q^2}).$$

The expression for the non twisted integral at the norm is the product of $2q^2/(q-1)^2$ and

$$q^{2N_1}(1-q^{-N_1-1})(q^{1+N_1+N_2} - q^{-N_1-1}\frac{1+q^{1+N_1}+q^{2+2N_1}}{1+q+q^2}).$$

Multiplying the last expressions by the measure factor $2 = e(E/F)$, as computed in the Lemma of Section D, we conclude that these expressions are equal. The case where $n_1 < n_2$ follows (e.g. on interchanging n's and N's). The comparison is then complete in Case (I).

F. Endoscopy for $H = GSp(2)$, type (I).

The computations of the orbital integrals of 1_K can be used to compare the unstable orbital integral of 1_K at an element of type (I) or (II), where there are two conjugacy classes in the stable conjugacy class, with the orbital integral of 1_K on the proper endoscopic group $\mathbf{C}_0$ of $\mathbf{H}$. The unstable orbital integral is a difference of the two orbital integrals, multiplied by a transfer factor. These objects are as follows. The dual group $\hat{H}$ of $\mathbf{H} = GSp(2)$ is $GSp(2,\mathbb{C})$, and its principal endoscopic group has dual which is the centralizer $\hat{C}_0 = Z_{\hat{H}}\big(\operatorname{diag}(1,-1,-1,1)\big) \simeq \big(GL(2,\mathbb{C}) \times GL(2,\mathbb{C})\big)'$. Thus $C_0 = \big(GL(2) \times GL(2)\big)/\{(z,z^{-1})\}$.

Let $\mathbf{T}_H$ be a maximal torus in $\mathbf{H}$. Its group of cocharacters is $X_*(\mathbf{T}_H) = \{(x_1,y_1,y_2,x_2); x_1+x_2 = y_1+y_2\}$. Its dual group is $X^*(\mathbf{T}_H) = X_*(\hat{T}_H) = \{(z_1,t_1,t_2,z_2)\}/\langle(z,-z,-z,z)\rangle$. The x_i, y_i, z_i, t_i are in $\mathbb{Z}$; $\hat{T}_H$ denotes a maximal torus in $\hat{H}, \hat{T}_0$ in $\hat{C}_0$, $\mathbf{T}_0$ in $\mathbf{C}_0$. The group $X_*(\hat{T}_0) = \{(x_1,y_1,y_2,x_2); x_1+x_2 = y_1+y_2\}$ is isomorphic to $X_*(\hat{T}_H)$, via $X_*(\hat{T}_H) \tilde{\to} X_*(\hat{T}_0)$, $(z_1,t_1,t_2,z_2) \mapsto (z_1+t_1, z_1+t_2, t_1+z_2, t_2+z_2)$. The dual map, from $X_*(\mathbf{T}_0) = \{(u_1,v_1,v_2,u_2)\}$ $/\langle(z,-z,-z,z)\rangle$ to $X_*(\mathbf{T}_H)$, is given by $(u_1,v_1,\dots) \mapsto (u_1+v_1, u_1+v_2, v_1+u_2, v_2+u_2)$. The tori $\mathbf{T}_0$ and $\mathbf{T}_H$ are determined by their cocharacter groups, thus we obtain an isomorphism, $\mathbf{T}_0 \to \mathbf{T}_H$, $\big(\left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} v_1 & 0 \\ 0 & v_2 \end{smallmatrix}\right)\big)/(z,z^{-1}) \mapsto \operatorname{diag}(x_1 = u_1v_1, x_2 = u_1v_2, x_2' = u_2v_1, x_1' = u_2v_2)$. The dual group data includes a choice of a set of positive roots $\alpha > 0$, so that we have a discriminant $D(t) = \prod_{\alpha>0} |1-\alpha(t)|$ on $t \in \mathbf{T}$. In particular, on $\mathbf{T}_0$ we have $D_0\big(\left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} v_1 & 0 \\ 0 & v_2 \end{smallmatrix}\right)\big) = |1-u_1/u_2||1-v_1/v_2|$, and on $\mathbf{T}_H$ we have

$$D_H\big(\operatorname{diag}(x_1,x_2,x_2',x_1')\big) = |1-x_1/x_2||1-x_1/x_2'||1-x_2/x_2'||1-x_1/x_1'|.$$

The quotient is

$$D_H(u_1v_1, u_1v_2, u_2v_1, u_2v_2)/D_0\big(\left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} v_1 & 0 \\ 0 & v_2 \end{smallmatrix}\right)\big) = |1 - x_1/x_1'||1 - x_2/x_2'|.$$

In the case of tori of type (I) the isomorphism $\mathbf{T}_0 \tilde{\to} \mathbf{T}_H$ yields a map of F-rational points $\lambda : T_0 \to T_H$, induced from $\big(\left(\begin{smallmatrix} t_1 & 0 \\ 0 & \sigma t_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} t_2 & 0 \\ 0 & \sigma t_2 \end{smallmatrix}\right)\big)/(z, z^{-1}) \mapsto x^* = \operatorname{diag}(x_1 = t_1t_2, x_2 = t_1\sigma t_2, \sigma x_2, \sigma x_1)$. If $x_i = \alpha_i + \beta_i\sqrt{D}$, then $x = [\mathbf{x}_1, \mathbf{x}_2] = h^{-1}x^*h$, where $h = [h'_D, h'_D]$, lies in $T_H\big(= \{h^{-1}(y_1, y_2, \sigma y_2, \sigma y_1)h; y_i \in E^\times\}\big)$, and a stably conjugate but non conjugate element is given by $x_R = [\mathbf{x}_{1R}, \mathbf{x}_2]$, where $\mathbf{x}_{1R} = \left(\begin{smallmatrix} \alpha_1 & \beta_1 DR \\ \beta_1 R^{-1} & \alpha_1 \end{smallmatrix}\right)$, $R \in F^\times - N_{E/F}E^\times$. Then the unstable orbital integral is $\Phi^{us}_{1_K}(x) = \Phi_{1_K}(x) - \Phi_{1_K}(x_R)$. For emphasis, we sometimes write K_H for K of H, and K_0 for the standard maximal compact of C_0.

The orbital integrals on H and C_0 depend on a choice of Haar measures, which we choose in a compatible way, as follows. Denote by t_0 a regular element in $T_0 \subset C_0$, and $x = \lambda(t_0)$ for its image under $\lambda : T_0 \to T_H \subset H$. We have $\Phi^{C_0}_{1_{K_0}}(t_0) = \int_{T_0\backslash C_0} 1_{K_0}(g^{-1}t_0g)d_{C_0}(g)/d_{T_0}$. Here d_{C_0} is a Haar measure on C_0, while d_{T_0} is one on T_0. A Haar measure is unique up to a scalar, determined by the volume of the maximal compact subgroup. The function 1_{K_0} is the unit element in the Hecke algebra $C_c(K_0\backslash C_0/K_0)$, thus it is the quotient of the characteristic function of K_0 in C_0 by the volume $|K_0|$ of K_0 according to d_{C_0}. In particular, the measure $1_{K_0}d_{C_0}$ is independent of the choice of $|K_0|$: the integral $\int_{C_0} 1_{K_0}d_{C_0}$ is 1. We can then assume that $|K_0| = 1$, so that 1_{K_0} is the characteristic function of K_0 in C_0. This is used in all of our computations above, to simplify the notations. Similarly $\Phi^H_{1_{K_H}}(x)$ is $\int_{T_H\backslash H} 1_{K_H}(h^{-1}xh)d_H(h)/d_{T_H}$, and we may assume that $|K_H|_{1_H} = 1$ and 1_{K_H} is the characteristic function of K_H in H. The problem is to relate the measures d_{T_H} and d_{T_0}. This we do by means of the morphism $\lambda : \mathbf{T}_0 \to \mathbf{T}_H$. Given a measure d_{T_H} on T_H, we can define a measure $\lambda^*(d_{T_H}) = d_{T_H} \circ \lambda$ on T_0. Then there is $\mu > 0$ such that d_{T_0} is $\mu\lambda^*(d_{T_H})$. The factor μ is given by the following computation, in which R_{T_0}, R_{T_H}, denote the maximal compact subgroups in T_0, T_H, and $|R_{T_0}|, |R_{T_H}|$ their volumes. Thus

$$|R_{T_0}| = d_{T_0}(R_{T_0}) = \mu d_{T_H}\big(\lambda(R_{T_0})\big) = \mu|R_{T_H}|/[R_{T_H} : \lambda(R_{T_0})],$$

and $\mu = [R_{T_H} : \lambda(R_{T_0})]|R_{T_0}|/|R_{T_H}|$, or $[R_{T_H} : \lambda(R_{T_0})]$, if we take – as we do – d_{T_0} and d_{T_H} to be normalized by $|R_{T_0}| = 1, |R_{T_H}| = 1$. Then $d_{T_0} = [R_{T_H} : \lambda(R_{T_0})]\lambda^*(d_{T_H})$, and we relate $\Phi^H_{1_{K_H}}(x; d_H/d_{T_H})$ with $[R_{T_H} : \lambda(R_{T_0})]\Phi^{C_0}_{1_{K_0}}(t_0; d_{C_0}/d_{T_0})$.

1. Theorem. *Let E/F be a quadratic extension, and $x = h^{-1}(x_1, x_2, \sigma x_2, \sigma x_1)h$ a regular element of type (I) (thus $x_1\sigma x_1 = x_2\sigma x_2$) in $GSp(2, F)$. Introduce $t_1, t_2 \in E^\times$ by $t_1/\sigma t_1 = x_1/\sigma x_2, t_2/\sigma t_2 = x_1/x_2$. Suppose that t_1, t_2 are units, in $R_E^\times$. Let $\chi_{E/F}$ be the non trivial character on $F^\times/N_{E/F}E^\times$. Then*

$$\begin{aligned}
\chi_{E/F}&\big((x_1 - \sigma x_1)(x_2 - \sigma x_2)/D\big)|1 - x_1/\sigma x_1||1 - x_2/\sigma x_2|\Phi^{H,us}_{1_{K_H}}(x; d_H/d_{T_H}) \\
&= [R_{T_H} : \lambda(R_{T_0})]\Phi^{C_0}_{1_{K_0}}\Big(\big(\left(\begin{smallmatrix} t_1 & 0 \\ 0 & \sigma t_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} t_2 & 0 \\ 0 & \sigma t_2 \end{smallmatrix}\right)\big); d_{C_0}/d_{T_0}\Big) \\
&= \Phi^{C_0}_{1_{K_0}}\Big(\big(\left(\begin{smallmatrix} t_1 & 0 \\ 0 & \sigma t_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} t_2 & 0 \\ 0 & \sigma t_2 \end{smallmatrix}\right)\big); d_{C_0}/\lambda^*(d_{T_H})\Big).
\end{aligned}$$

Proof. To compute the right side recall that if $t = a + b\sqrt{D}$, $\mathbf{t} = \left(\begin{smallmatrix} a & bD \\ b & a \end{smallmatrix}\right)$, then on $G = GL(2, F)$ we have

$$\int_{T\backslash G} 1_K(g^{-1}\mathbf{t}g)dg = 1_{R_E^\times}(t)(q-1)^{-1}\begin{cases} q|(t-\sigma t)/\sqrt{D}|^{-1} - 1, & D \in \boldsymbol{\pi} R^\times, \\ (q+1)|(t-\sigma t)/\sqrt{D}|^{-1} - 2, & D \in R^\times. \end{cases}$$

Recall that $x_1 = t_1 t_2$, $x_2 = t_1 \sigma t_2$, $x_i = \alpha_i + \beta_i\sqrt{D}$, $\beta_i = B_i' \boldsymbol{\pi}^{n_i}$, $B_i' \in R^\times$, put $n = \min(n_1, n_2)$, $|\alpha_1 - \alpha_2| = q^{-\chi}$. Suppose that x is absolutely unipotent. Then $\chi > 0$, and we have:

2. Lemma. *The unordered pair* $\{|(t_1 - \sigma t_1)/\sqrt{D}|^{-1}, |(t_2 - \sigma t_2)/\sqrt{D}|^{-1}\}$ *equals* $\{q^n, q^{\chi - n}|D|\}$.

Proof. This is the statement $n = N$ and $\chi = N_1 + N_2 + \operatorname{ord} D$, proven in "Notations" of Section D. Here is an alternative proof. The product of the two terms is indeed $q^\chi D$, since

$$\begin{aligned} q^{-\chi} &= |\alpha_1 - \alpha_2| = |x_1 + \sigma x_1 - x_2 - \sigma x_2| \\ &= |t_1 t_2 + \sigma t_1 \sigma t_2 - t_1 \sigma t_2 - t_2 \sigma t_1| = |t_1 - \sigma t_1||t_2 - \sigma t_2|. \end{aligned}$$

This is also equal to

$$= |x_1 - \sigma x_2||x_1 - x_2| = |(\alpha_1 - \alpha_2)^2 - (\beta_1 + \beta_2)^2 D|^{1/2} |(\alpha_1 - \alpha_2)^2 - (\beta_1 - \beta_2)^2 D|^{1/2}.$$

If $|\alpha_1 - \alpha_2| < |\beta_1 \pm \beta_2|$ for both choices of sign then the two factors are $|(\beta_1 + \beta_2)\sqrt{D}|$ and $|(\beta_1 - \beta_2)\sqrt{D}|$, one of which has to be $q^{-n}|\sqrt{D}|$, as required. Note that $n_1 < n_2$ implies $2n_1 + \operatorname{ord} D = \chi$. If $|\beta_1 \pm \beta_2| \le |\alpha_1 - \alpha_2| < 1$ for some choice of sign, then the identity displayed above implies that $|\beta_1 \mp \beta_2| = 1$ and $|D| = 1$, thus $|\beta_1| = |\beta_2| = 1$, so $n_1 = n_2 = 0$, and one of the two factors is equal to 1. The lemma follows. $\square$

In conclusion, the orbital integral on C_0 is the product of $1_{R_E^\times}(t_1) 1_{R_E^\times}(t_2)$ and

$$\begin{aligned} (q-1)^{-2}(q^{n+1} - 1)(q^{\chi - n} - 1), &\quad D \in \boldsymbol{\pi} R^\times, \\ (q-1)^{-2}\big((q+1)q^n - 2\big)\big((q+1)q^{\chi - n} - 2\big), &\quad D \in R^\times. \end{aligned}$$

Let us consider first the case where E/F is *unramified*, thus $D \in R^\times$. Here the factor $|1 - x_1/\sigma x_1||1 - x_2/\sigma x_2| = |x_1 - \sigma x_1||x_2 - \sigma x_2| = |\beta_1 \beta_2 D|$ is $q^{-n_1 - n_2}$. Further, $N_{E/F} R_E^\times = R^\times$, and $N_{E/F} E^\times = R^\times \boldsymbol{\pi}^{2\mathbb{Z}}$, hence $\chi_{E/F}$ is the character on $E^\times$ which is trivial on $R^\times$, and takes the value -1 at $\boldsymbol{\pi}$. Then $\chi_{E/F}\big((x_1 - \sigma x_1)(x_2 - \sigma x_2)/D\big) = \chi_{E/F}(\beta_1 \beta_2) = (-1)^{n_1 + n_2}$, and the transfer factor is $(-q)^{-n_1 - n_2}$. The unstable orbital integral $\Phi_{1_K}^{H,us}(x)$ is a difference of two sums, each of which was computed in the case of the stable orbital integral, which is the sum of the two integrals in question. These two orbital integrals are parametrized by ρ, ranging over the set $\{1, \boldsymbol{\pi}\}$ of representatives for $F^\times / N_{E/F} E^\times$, with $\rho = \boldsymbol{\pi}^{\overline{\rho}}$, $\overline{\rho} \in \{0, 1\}$. The sum (over $j_1, j_2 \ge 0$, with 2 dividing $j_1 - (j_2 - \overline{\rho})$) has now the coefficient $(-1)^{\overline{\rho}}$ (the coefficient was 1 in the stable case), which is equal to $(-1)^{j_1 + j_2} = (-1)^{n_1 - \nu_1 + n_2 - \nu_2}$.

Consequently the unstable orbital integral is the sum of the following three sums.

$$\sum_{0 \le \nu_1 \le n_1} \sum_{0 \le \nu_2 \le n_2} \sum_{0 \le m \le \min(\nu_1, \nu_2)} = \sum_{0 \le m \le n} \big(\delta(m = 0) + \delta(m \ge 1)(1 - q^{-2})q^{3m}\big)$$

$$[\sum_{\substack{m\le\nu_1<n_1\\ m\le\nu_2<n_2}} 2\cdot\frac{1}{2}\cdot(\frac{q+1}{q})^2(-q)^{n_1-\nu_1+n_2-\nu_2}+\sum_{m\le\nu_1<n_1}\frac{q+1}{q}(-q)^{n_1-\nu_1}$$
$$+\sum_{m\le\nu_2<n_2}\frac{q+1}{q}(-q)^{n_2-\nu_2}+1];$$

$$\sum_{0\le\nu\le n}\sum_{\nu<m\le 2n-\nu}\frac{1}{2}(\frac{q+1}{q})^2(-q)^{n_1+n_2-2\nu}\cdot 2q^{m+2\nu};$$

$$\delta(n_1,n_2)\sum_{0\le\nu\le n}\sum_{2n-\nu<m\le\chi-\nu}$$
$$=\delta(n_1,n_2)[\sum_{0\le\nu<n}\sum_{2n-\nu<m\le\chi-\nu}\frac{1}{2}(\frac{q+1}{q})^2q^{2n-2\nu}\cdot 2q^{m+2\nu}+\sum_{n<m\le\chi-n}\frac{q+1}{q}q^{2n+m}].$$

The $[\dots]$ in the first sum is

$$(q+1)^2\sum_{0\le j_i<n_i-m}(-q)^{j_1+j_2}-(q+1)\sum_{0\le j_1<n_1-m}(-q)^{j_1}-(q+1)\sum_{0\le j_2<n_2-m}(-q)^{j_2}+1$$
$$=(-q)^{n_1+n_2-2m}.$$

Hence the first sum is

$$(-q)^{n_1+n_2}\big(1+(1-q^{-2})\sum_{0<m\le n}q^m\big)=(-q)^{n_1+n_2}\big(1+(1+q^{-1})(q^n-1)\big).$$

The second sum is

$$(-q)^{n_1+n_2}(\frac{q+1}{q})^2\sum_{0\le\nu\le n}[q^{\nu+1}\sum_{0\le m<2n-2\nu}q^m].$$

Here $[\dots]$ is $q^{\nu+1}(q^{2n-2\nu}-1)/(q-1)=\big(q/(q-1)\big)(q^{2n-\nu}-q^\nu)$. Hence $\sum_{0\le\nu\le n}[\dots]$ is $\big(q/(q-1)\big)(q^n-1)$ times $\sum_{0\le\nu\le n}q^\nu=(q^{n+1}-1)/(q-1)$, and we get

$$(-q)^{n_1+n_2}\frac{(q+1)^2}{(q-1)^2q}(q^{2n+1}-q^{n+1}-q^n+1).$$

In the third sum $n=n_1=n_2$. It is the sum of two terms, namely

$$(\frac{q+1}{q})^2q^{2n}\sum_{0\le\nu<n}q^{2n-\nu+1}(q^{\chi-2n}-1)/(q-1)=(\frac{q+1}{q})^2q^{2n}q^{n+2}\frac{q^n-1}{(q-1)^2}(q^{\chi-2n}-1)$$

and

$$\frac{q+1}{q}q^{2n}\cdot q^{n+1}\frac{q^{\chi-2n}-1}{q-1}.$$

The third sum is then

$$(-q)^{n_1+n_2}\frac{q+1}{(q-1)^2q}q^{n+1}(q^{\chi-2n}-1)(q^{n+1}+q^n-2).$$

When $n_1<n_2$ we have $n=n_1$ and $\chi=2n$, and the sum of the three sums is

$$(-1)^{n_1+n_2}(q-1)^{-2}[(q-1)^2+q^{-1}(q+1)(q^n-1)\{(q-1)^2+(q+1)(q^{n+1}-1)\}],$$

and $[\dots]$ is $\big((q+1)q^n-2\big)^2$. If $(n=)n_1=n_2$, we need to add the third sum (which is zero when $n_1\neq n_2$), thus to $[\dots]$ we add $\big((q+1)q^n-2\big)(q+1)q^n(q^{\chi-2n}-1)$. Hence in all cases $(n_1=n_2$ or $n_1\neq n_2)$, the unstable orbital integral adds up to

$$(-q)^{n_1+n_2}(q-1)^{-2}\big((q+1)q^n-2\big)\big((q+1)q^{\chi-n}-2\big).$$

Since the transfer factor is $(-q)^{-n_1-n_2}$, our comparison is complete in the case where E/F is unramified.

Next we consider the case where E/F is *ramified*, thus $D\in\boldsymbol{\pi}R^\times$.

The factor $|1-x_1/\sigma x_1||1-x_2/\sigma x_2|=|\beta_1\beta_2 D|$ is $q^{-n_1-n_2-1}$. Further $N_{E/F}E^\times=R^{\times 2}\boldsymbol{\pi}^{\mathbb{Z}}$, so that $\chi_{E/F}$ is trivial at $\boldsymbol{\pi}(=n_{E/F}\boldsymbol{\pi}_E,\boldsymbol{\pi}_E=\sqrt{-\boldsymbol{\pi}}$, thus we take $D=-\boldsymbol{\pi})$ and its restriction to $R^\times$ has the kernel $R^{\times 2}$. Since $(x_i-\sigma x_i)/\sqrt{D}=\beta_i=B_i'\boldsymbol{\pi}^{n_i}$, the transfer factor is $\chi_{E/F}(B_1'B_2')q^{-n_1-n_2-1}$. The unstable orbital integral is a difference of two integrals, indexed by ρ which ranges over a set of representatives $\{1,u\}$ for $R^\times/R^{\times 2}(=F^\times/N_{E/F}E^\times)$. The stable orbital integral was a sum, over ρ, of the two integrals. We expressed each of these two integrals as sums, of terms denoted above by A,B,C, which are also sums, over different domains of summation. Over the domains of summation of A and B, the contributions associated to $\rho=1$ and $\rho=u$ are equal, yielding a factor 2 in the computation of the stable integral, and a factor 0 in the case of the unstable integral. Over the domain of summation of C, namely $0\le\nu\le n=\min(n_1,n_2)$ and $\nu<m\le\chi-\nu$, we have the condition $\rho\in B_1'B_2'R^{\times 2}$. In the computation of the stable integral we obtained in C a coefficient 1: precisely one of the $\rho\in\{1,u\}$ satisfies $\rho\in B_1'B_2'R^{\times 2}$. In the unstable case the contribution appears in the positive (resp. negative) integral if $\chi_{E/F}(B_1'B_2')$ is 1 (resp. -1). Hence the unstable orbital integral is $\chi_{E/F}(B_1'B_2')\cdot 2q^{n_1+n_2}\cdot C$, where we recall that

$$C=q(q-1)^{-2}(q^{\chi-n}-1)(q^{n+1}-1).$$

Multiplying by the transfer factor $\chi_{E/F}(B_1'B_2')q^{-n_1-n_2-1}$ we are left with $2(q-1)^{-2}(q^{\chi-n}-1)(q^{n+1}-1)$, which is the orbital integral of 1_K on C_0 in the case where E/F is ramified, using the following.

3. Lemma. *The index $[R_{T_H} : \lambda(R_{T_0})]$ is 1 if E/F is unramified, and 2 if E/F is ramified.*

Proof. Recall that $\lambda\big((t_1, \sigma t_1), (t_2, \sigma t_2)\big) = (x_1 = t_1 t_2, x_2 = t_1 \sigma t_2, \sigma x_2, \sigma x_1)$. Thus given x_1, x_2 in $R_E^\times$, we look for solutions t_1, t_2 in $R_E^\times$ for the equations $x_1 = t_1 t_2, x_2 = t_1 \sigma t_2$. It suffices to solve $x_1/x_2 = t_2/\sigma t_2$ in $t_2 \in R_E^\times$. Denote by E^1 the group $\{x/\sigma x; x \in E^\times\}$. When E/F is unramified, E^1 is equal to $\{x/\sigma x; x \in R_E^\times\}$, so t_2 exists. When E/F is ramified, write $x = t\boldsymbol{\pi}_E^n, t \in R_E^\times$. Then $x/\sigma x = u/\sigma u(-1)^n$, and since $u/\sigma u \equiv 1(\operatorname{mod} \boldsymbol{\pi}_E)$, the group $\{x/\sigma x; x \in R_E^\times\}$ has index 2 in E^1, and $x_1/x_2 = t_2/\sigma t_2$ has a solution in $t_2 \in R_E^\times$ if $x_1 \equiv x_2$, but not when $x_1 \equiv -x_2(\operatorname{mod} \boldsymbol{\pi}_E)$. Note that $x_1/x_2 \equiv \pm 1(\operatorname{mod} \boldsymbol{\pi}_E)$, since $x_1 \sigma x_1 = x_2 \sigma x_2$ implies that $x_1/x_2 \cdot \sigma(x_1/x_2) = 1$; if $x_1/x_2 = a + b\sqrt{D}$ then $a^2 - b^2 D = 1$, and $a^2 \equiv 1(\operatorname{mod} \boldsymbol{\pi})$, so $x_1/x_2 \equiv a(\operatorname{mod} \boldsymbol{\pi}_E) \equiv \pm 1(\operatorname{mod} \boldsymbol{\pi}_E)$. The lemma follows. □

Unstable twisted case. Twisted endoscopic group of type I.F.2.

The explicit computation of the θ-orbital integrals can be used to compute the unstable κ-θ-orbital integrals, at a strongly θ-regular topologically θ-unipotent element $t^* = (t_1, t_2, \sigma t_2, \sigma t_1)$ (thus $t^*\theta$ is topologically unipotent) of type (I). The character κ is defined on the group $(F^\times/N_{E/F}E^\times)^2$ of θ-conjugacy classes within the stable θ-conjugacy class of t^*. Thus $\kappa = \kappa_1 \times \kappa_2, \kappa_i$ on $F^\times/N_{E/F}E^\times$. The stable case is that where $\kappa_i = 1, i = 1, 2$. The endoscopic group associated with κ with $\kappa_i \neq 1 (i = 1, 2)$ is $\mathbf{C} = \big(GL(2) \times GL(2)\big)'$. We deal with this case now. The norm $N_C t^*$ is $\left(\left(\begin{smallmatrix} t_1 t_2 & 0 \\ 0 & \sigma(t_1 t_2) \end{smallmatrix}\right), \left(\begin{smallmatrix} t_1 \sigma t_2 & 0 \\ 0 & t_2 \sigma t_1 \end{smallmatrix}\right)\right)$. If $t_i = a_i + b_i\sqrt{D}$, then $\Delta_{G,C}(t^*) = |(t_1 - \sigma t_1)(t_2 - \sigma t_2)|_F / |t_1 \sigma t_1 \cdot t_2 \sigma t_2|_F^{1/2} = |b_1 b_2 D|_F$. If $N_i = \operatorname{ord}(b_i), n_i = \operatorname{ord}(\beta_i)$, where $x_1 = t_1 t_2 = \alpha_1 + \beta_1\sqrt{D}, x_2 = t_1 \sigma t_2 = \alpha_2 + \beta_2\sqrt{D}$, then the orbital integral $\Phi_{1_{K_C}}(N_C t^*)$ of 1_{K_C} on C at the norm $N_C t^*$ is a product of two integrals of 1_K on $GL(2, F)$ at the conjugacy classes with eigenvalues $(x_1, \sigma x_1)$ and $(x_2, \sigma x_2)$. By Lemma F.2, this integral is the product of $(q^{N_1+1} - 1)(q-1)^{-1}$ and $(q^{N_2+1} - 1)(q-1)^{-1}$ when E/F is ramified, and of $\big((q-1)q^{N_1} - 2\big)(q-1)^{-1}$ and $\big((q-1)q^{N_2} - 2\big)(q-1)^{-1}$ when E/F is unramified.

Theorem. *Let t^* be a topologically θ-unipotent strongly θ-regular element of type (I). Then*

$$\kappa_1\big((t_1 - \sigma t_1)/2\sqrt{D}\big)\kappa_2\big((t_2 - \sigma t_2)/2\sqrt{D}\big)\Delta_{G,C}(t^*)\Phi_{1_K}^\kappa(t^*\theta) = \Phi_{1_{K_C}}^C(N_C t^*).$$

Proof. When E/F is unramified, ρ_i ranges over $\{1, \boldsymbol{\pi}\}$, which represents $F^\times/N_{E/F}E^\times$, and then $\kappa_i\big((t_i - \sigma t_i)/2\sqrt{D}\big) = \kappa_i(b_i) = (-1)^{N_i}$. When E/F is ramified, ρ_i ranges over a set $\{1, \varepsilon\}$ of representatives for $R^\times/R^{\times 2} (= F^\times/N_{E/F}E^\times), \kappa_i(\boldsymbol{\pi}) = 1$, and since $b_i = B_i \boldsymbol{\pi}^{N_i}$, the factor $\kappa_i\big((t_i - \sigma t_i)/2\sqrt{D}\big) = \kappa_i(b_i)$ is $\kappa_i(B_i)$. The κ-θ-orbital integral is a sum no different than the stable orbital integral, except that the summation over ρ_1 and ρ_2 in $F^\times/N_{E/F}E^\times$ is now weighted by the sign $\kappa_1(\rho_1)\kappa_2(\rho_2)$. Indeed, recall that ρ_m is ρ if m is even, but it is $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2)$ if m is odd, where $\{\rho_i, \tilde{\rho}_i\} = \{1, \boldsymbol{\pi}\}$ if E/F is unramified, and $\rho_i \mapsto \tilde{\rho}_i = -1/\rho_i$ is a permutation of $R^\times/R^{\times 2}$ if E/F is ramified. Hence in our sum

$$\sum_\rho \kappa_1(\rho_1)\kappa_2(\rho_2) \sum_{m \geq 0} \sum_{r \in R_{\rho_m}} [R_T^1 : T_{\rho_m}^1 \cap r K_0 r^{-1}] \# L_{m,\rho_m}^1,$$

replacing ρ_m by ρ changes neither the factor $\#L^1_{m,\rho_m}$ nor the index $[\dots]$. The indexing set R_{ρ_m} is not changed either, when E/F is ramified. However, when E/F is unramified, R_ρ, defined by $j_i \equiv \overline{\rho}_i (\bmod 2)$, is changed when ρ is replaced by ρ_m. In this unramified case we may replace ρ_m by ρ provided we multiply each summand by $(-1)^m(-1)^m = 1$. The weighted sum thus obtained is precisely the same as that obtained in the proof of Theorem F.1, which deals with endoscopy for $H = GSp(2)$, type (I), and computes the unstable orbital integral of type (I). The theorem follows. $\square$

Twisted endoscopic group of type I.F.3, E/F unramified.

When E/F is unramified, the orbital integral of 1_K on the twisted endoscopic group of type (3) of Section I.F is $((q+1)|b_2|^{-1} - 2)/(q-1)$, $|b_i| = q^{-N_i}$. It has to be divided by the factor $\Delta_{G,C_+}(t^*) = |(x-t)(xy-zt)(xz-yt)|/(|xt|^{3/2}|yz|) = |x-\overline{x}||xy-\overline{xy}||x\overline{y}-y\overline{x}|$ (see the last lines of Sections I.F and I.G). Here $x = a_1 + b_1\sqrt{D}$ and $y = a_2 + b_2\sqrt{D}$ are topologically unipotent, which means that they lie in $1+\boldsymbol{\pi} R_E$. Then $\mathrm{ord}((xy-\overline{xy})(x\overline{y}-y\overline{x})) = \mathrm{ord}(a_1^2b_2^2 - a_2^2b_1^2) = \mathrm{ord}(b_2^2 - b_1^2) = \mathrm{ord}(a_1^2 - a_2^2) = X$. Hence the inverse of the Δ-factor is q^{N_1+X}. We show below that the κ-orbital integral is $(-q)^{N_1+X}((q+1)q^{N_2} - 2)/(q-1)$. Put $\kappa_{G,C_+}(u) = \kappa_E((x-\overline{x})(xy-\overline{xy})(x\overline{y}-y\overline{x}))$, where $\kappa_E(R_E^\times \boldsymbol{\pi}_E^n) = (-1)^n$. We conclude the following.

Theorem. *Let u be a topologically θ-unipotent strongly θ-regular element of type (I). Then*

$$\kappa_{G,C_+}(u)\Delta_{G,C_+}(u)\Phi^{G,\kappa}_{1_K}(u\theta) = \Phi^{C_+}_{1_K}(u)$$

if E/F is unramified, while when E/F is ramified, the left side vanishes.

Proof. The computation of the twisted orbital integral is as in Section D. The κ-orbital integral is

$$\Phi^{G,\kappa}_{1_K}(u\theta) = \Phi^{Z_G(\theta),\kappa}_{1_{Z_{K(\theta)}}}(u) = \sum_\rho \kappa(\rho) \sum_{m\geq 0} \sum_{r\in R_{\rho_m}} [R^1_T : T^1_{\rho_m} \cap rK^1_0 r^{-1}]\#L^1_{m,\rho_m},$$

where $u = h^{-1}t^*h$ is topologically unipotent. Put $N = \min(N_1, N_2)$, where $N_i = \mathrm{ord}(b_i)$. The factor $\#L^1_{m,\rho_m}$ is equal to $\#L^1_{m,\rho}$, and the index $[\dots]$ is independent of ρ. When E/F is ramified we also have $R_{\rho_m} = R_\rho$, hence the sum vanishes. In the case where E/F is unramified, $\rho = (\rho_1, \rho_2), \rho_i \in \{1, \boldsymbol{\pi}\}, u_i = 1$, and ρ_m is ρ if m is even, but it is $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2)$ if m is odd, where $\{\rho_i, \tilde{\rho}_i\} = \{1, \boldsymbol{\pi}\}$. The indexing set R_{ρ_m}, defined by $j_i \equiv \overline{\rho}_i (\bmod 2)$ is changed when ρ_m is replaced by ρ. Hence we can replace ρ_m by ρ at the price of multiplying each summand by $(-1)^m$.

The sum over r is a sum over $j_1, j_2 \geq 0$ such that $j_1 - \overline{\rho}_1, j_2 - \overline{\rho}_2$ are even, and over ε_i in $R^\times/R^{\times 2}$ if $j_i > 0$. When $j_1 > 0$ or $j_2 > 0$, and $\nu = \nu_1 = \nu_2 < m$, we have $\varepsilon_1\varepsilon_2 \in B_1B_2R^{\times 2}$. In other words, we have a sum over $\nu_i = N_i - j_i$ $(i = 1, 2)$, $0 \leq \nu_i \leq N_i$, and over $\varepsilon_i \in R^\times/R^{\times 2}$ if $\nu_i < N_i$ for $i = 1, 2$. (If $\nu_i = N_i$ for some i, then $\varepsilon_i \in R^\times/R^\times$).

Then we need to sum over m. We have the range $0 \leq m \leq \min(\nu_1, \nu_2)$, then the range $\nu (= \nu_1 = \nu_2) < m \leq 2N - \nu$ (since $\mathrm{ord}\, D = 0$ when E/F is unramified ($\nu_i < m$ implies $\nu_1 = \nu_2$)), and the range $2N - \nu < m \leq X - \nu$ ($2N - \nu < m$ implies $\nu < m, N_1 = N_2$, and

$m \leq X - \nu$). Let $\delta(m = 0) = \delta(m, 0)$ be 0 if $m \neq 0$ and 1 if $m \neq 0$, and $\delta(m \geq 1)$ be 0 if $m < 1$ and 1 if $m \geq 1$.

Thus we get the sum of three expressions:

$$\sum_{0 \leq \nu_1 \leq N_1} \sum_{0 \leq \nu_2 \leq N_2} \sum_{0 \leq m \leq \min(\nu_1, \nu_2)}$$
$$= \sum_{0 \leq m \leq N} (-1)^m \big(\delta(m = 0) + \delta(m \geq 1)(1 - q^{-2})q^{3m}\big) \Big[\sum_{m \leq \nu_1 < N_1} \sum_{m \leq \nu_2 < N_2}$$
$$4(\frac{q+1}{2q})^2 (-q)^{N_1 - \nu_1} q^{N_2 - \nu_2} + \sum_{m \leq \nu_1 < N_1} 2\frac{q+1}{2q}(-q)^{N_1 - \nu_1} + \sum_{m \leq \nu_2 < N_2} 2\frac{q+1}{2q} q^{N_2 - \nu_2} + 1\Big];$$

$$\sum_{0 \leq \nu \leq N} \sum_{\nu < m \leq 2N - \nu} 2(\frac{q+1}{2q})^2 (-q)^{N_1 - \nu} q^{N_2 - \nu} \cdot 2(-q)^{m + 2\nu};$$

$$\delta(N_1, N_2) \sum_{0 \leq \nu \leq N} \sum_{2N - \nu < m \leq X - \nu} = \delta(N_1, N_2)\Big[\sum_{N < m \leq X - N} \frac{q+1}{q} q^{2N} (-q)^m$$
$$+ \sum_{0 \leq \nu < N} \sum_{2N - \nu < m \leq X - \nu} 2(\frac{q+1}{2q})^2 (-q^2)^{N - \nu} \cdot 2(-q)^{m + 2\nu}\Big].$$

To compute the first expression, note that

$$\frac{q+1}{q} \sum_{m \leq \nu_1 < N_1} (-q)^{N_1 - \nu_1} + 1 = 1 - (q+1) \sum_{0 \leq j < N_1 - m} (-q)^j = (-q)^{N_1 - m},$$

and

$$\frac{q+1}{q} \sum_{m \leq \nu_2 < N_2} q^{N_2 - \nu_2} + 1 = 1 + (q+1) \sum_{0 \leq j < N_2 - m} q^j = 1 + \frac{q+1}{q-1}(q^{N_2 - m} - 1)$$
$$= \frac{q+1}{q-1} q^{N_2 - m} - \frac{2}{q-1}.$$

Hence $(-1)^m[\dots]$ is

$$(-q)^{N_1} q^{-m} \big(\frac{q+1}{q-1} q^{N_2 - m} - \frac{2}{q-1}\big) = \frac{(-q)^{N_1}}{q-1}\big[(q+1)q^{N_2 - 2m} - 2q^{-m}\big].$$

So the first expression is

$$\frac{(-q)^{N_1}}{q-1}\big[(q+1)q^{N_2} - 2 + (1 - q^{-2}) \sum_{1 \leq m \leq N} \big((q+1)q^{N_2 + m} - 2q^{2m}\big)\big].$$

Since $\sum_{N \leq n < M} x^n = (x^M - x^N)/(x-1)$, the sum is

$$q(q+1)q^{N_2} \frac{q^N - 1}{q-1} - 2q^2 \frac{q^{2N} - 1}{q^2 - 1}.$$

We then get

$$\frac{(-q)^{N_1}}{q-1}[(q+1)q^{N_2}+(q+1)^2q^{N_2-1}(q^N-1)-2q^{2N}].$$

The second expression is the product of $(-q)^{N_1}q^{N_2-1}(q+1)^2$ and

$$-\sum_{0\le\nu\le N}(-1)^\nu\sum_{\nu<m\le 2N-\nu}(-q)^{m-1}=\sum_{0\le\nu\le N}(q^{2N-\nu}-q^\nu)/(q+1).$$

But

$$(q^N-1)\sum_{0\le\nu\le N}q^\nu=(q^N-1)(q^{N+1}-1)/(q-1),$$

hence we get

$$\frac{(-q)^{N_1}}{q-1}q^{N_2-1}(q+1)(q^{2N+1}-(1+q)q^N+1).$$

The sum of the first and second expressions is $(-q)^{N_1}q^{2N}((q+1)q^{N_2}-2)/(q-1)$.

The third expression is the product of $\delta(N_1,N_2)$ and the sum of

$$\begin{aligned}
&-q^{-1}(q+1)^2(-q^2)^N\sum_{0\le\nu<N}(-1)^\nu\sum_{2N-\nu<m\le X-\nu}(-q)^{m-1}\\
&=q^{-1}(q+1)(-q^2)^N\sum_{0\le\nu<N}((-q)^X-q^{2N})q^{-\nu}\\
&=-\frac{q+1}{q-1}(-q^2)^N((-q)^X-q^{2N})(q^{-N}-1)=\frac{q+1}{q-1}(-q)^N((-q)^X-q^{2N})(q^N-1),
\end{aligned}$$

and of $q^{2N}((-q)^{X-N}-(-q)^N)$. Since $X=2N$ when $N_1\ne N_2$, it is

$$(-q)^N((-q)^X-q^{2N})((q+1)q^N-2)/(q-1).$$

The sum of the three terms is $(-q)^{N_1}(-q)^X[(q+1)q^{N_2}-2]/(q-1)$. This completes the proof of the theorem, as noted before its statement. □

G. Twisted orbital integrals of type (II).

The stable θ-orbital integral $\Phi^{G,st}_{1_K}(u\theta)$ of a type (II) strongly θ-regular topologically unipotent element $u=\theta(u)$ in $G=GL(4,F)\times F^\times$ is equal to the stable orbital integral $\Phi^{Sp(2,F),st}_{1^1_K}(u)$ at $u\in H^1=Sp(2,F)$. We proceed to compute this integral. Let us recall our notations, in the case of type (II). There are three distinct quadratic extensions $E_1=F(\sqrt{D}),E_2=F(\sqrt{AD}),E_3=F(\sqrt{A})$ of F, two ramified and one unramified, and we take E_2 to be ramified, and normalize A,D to be integral (in R) of minimal order, thus the set $\{A,D\}$ consists of a unit and a uniformizer. The Galois group of $E=E_1E_2$ over F is $\mathbb{Z}/2\times\mathbb{Z}/2$, generated by σ,τ, such that E_1 is the fixed field of τ in E, and $E_2=E^{\langle\sigma\tau\rangle}$.

The torus $\mathbf{T}$ is defined by the Galois action ρ, thus τ acts on $\mathbf{T}^*$ as (23) and $\sigma\tau$ as (14). The torus $T = h^{-1}T^*h$ can be realized as $[\phi^D(a_1+b_1\sqrt{D}), \phi^{AD}(a_2+b_2\sqrt{AD})]$. A complete set of representatives for the θ-conjugacy classes within the stable θ-conjugacy class is given by $s_\rho = [\phi^D_{\rho_1}(a_1+b_1\sqrt{D}), \phi^{AD}_{\rho_2}(a_2+b_2\sqrt{AD})]$, where ρ_1 ranges over $F^\times/N_{E_1/F}E_1^\times$ and ρ_2 over $F^\times/N_{E_2/F}E_2^\times$. Here $\phi^D_\rho(a_1+b_1\sqrt{D}) = \left(\begin{smallmatrix} a_1 & b_1 D\rho \\ b_1/\rho & a_1\end{smallmatrix}\right)$. Then $s_\rho = h^{-1}(a,b,\tau b,\sigma a; e)h, a \in E_1^\times$, $b \in E_2^\times, e \in F^\times$. Put T^1_ρ for the torus (i.e. centralizer) containing s_ρ, in $H^1 = Sp(2,F)$.

There are two cases to consider. The ramified case is when E_1/F is ramified, namely $D = \boldsymbol{\pi}_F$ and A is a unit (in $R^\times - R^{\times 2}$), so that E_3/F is unramified. In this case $\rho_1 = u_1\boldsymbol{\pi}^{\overline{\rho}_1}$, $\overline{\rho}_1 = \operatorname{ord}\rho_1$, ranges over $R^\times/R^{\times 2}$, thus $\overline{\rho}_1 = 0$. The unramified case is when E_1/F is unramified, thus D is a non square unit in $R^\times$, and $A = \boldsymbol{\pi}_F$, so that E_3/F is ramified. In this case ρ_1 ranges over $\{1, \boldsymbol{\pi}\}$, so $\overline{\rho}_1$ over $\{0,1\}$, and $u_1 = 1$. In both cases E_2/F is ramified; so ρ_2 ranges over a set $\{1, \varepsilon\}$ of representatives for $R^\times/R^{\times 2}$, and $\overline{\rho}_2 = \operatorname{ord}\rho_2$ is 0.

The computation of the orbital integral $\Phi^{Sp(2,F)}_{1_{K^1}}(s_\rho)$ proceeds as in case (I). We use the double coset decomposition $H^1 = Sp(2,F) = \bigcup_{m\leq 0} C^1_0 z(m)K^1$, of Lemma I.J.6, to get

$$\Phi^{H^1}_{1_{K^1}}(s_\rho) = \int_{T^1_\rho\backslash H^1} 1_{K^1}(g^{-1}s_\rho g)dg$$
$$= \sum_{m\geq 0}|K^1|_{H^1}\int_{T^1_\rho\backslash C^1_0/C^1_0\cap z(m)K^1z(m)^{-1}} 1_{K^1}\big(z(m)^{-1}h^{-1}s_\rho h z(m)\big)dh.$$

The integrand in the last integral is non zero precisely when $h^{-1}t_\rho h$ lies in $z(m)K^1z(m)^{-1}$ $\cap C^1_0 = K^{C^1_0}_m$. Hence we get

$$= \sum_{m\geq 0}|K^1|_{H^1}\int_{T^1_\rho\backslash C^1_0/K^{C^1_0}_m} 1_{K^{C^1_0}_m}(h_0^{-1}s_\rho h_0)dh_0.$$

Using Lemma I.J.7 we have an isomorphism $\phi_m : C_1 \to C^1_0$ ($\phi_m(h) = h_0$), $\phi_m(K^1_m) = K^{C^1_0}_m$. Define x_ρ by $\phi_m(x_\rho) = s_\rho$, and note that $T_\rho = Z_{C^1_0}(s_\rho)$. Hence our expression is

$$= \sum_{m\geq 0}|K^1|_{H^1}\int_{Z_{C_1}(x_\rho)\backslash C_1/K^1_m} 1_{\phi_m(K^1_m)}(\phi_m(h)^{-1}\phi_m(x_\rho)\phi_m(h))dh$$

$$= \sum_{m\geq 0}[K^1_0 : K^1_m]\int_{Z_{C_1}(x_\rho)\backslash C_1} 1_{K^1_m}(h^{-1}x_\rho h)dh.$$

Next we change variables on $C_1 = SL(2,F)\times SL(2,F)$. If m is even,

$$h \mapsto (I, w\boldsymbol{\varepsilon})\Big(\left(\begin{smallmatrix}\boldsymbol{\pi}^{m/2} & 0 \\ 0 & \boldsymbol{\pi}^{-m/2}\end{smallmatrix}\right), \left(\begin{smallmatrix}\boldsymbol{\pi}^{m/2} & 0 \\ 0 & \boldsymbol{\pi}^{-m/2}\end{smallmatrix}\right)\Big)h$$

sends $h^{-1}x_\rho h$ to $h^{-1}((\begin{smallmatrix}1 & 0 \\ 0 & \boldsymbol{\pi}^m\end{smallmatrix}), (\begin{smallmatrix}1 & 0 \\ 0 & \boldsymbol{\pi}^m\end{smallmatrix}))(I, \boldsymbol{\varepsilon}w)x_\rho(I, w\boldsymbol{\varepsilon})((\begin{smallmatrix}1 & 0 \\ 0 & \boldsymbol{\pi}^{-m}\end{smallmatrix}), (\begin{smallmatrix}1 & 0 \\ 0 & \boldsymbol{\pi}^{-m}\end{smallmatrix}))h = h^{-1}s'_\rho h$, where $s'_\rho = (s_{\rho_1}, s_{\rho_2}) \in C_1$, $s_{\rho_1} = \phi^D_{\rho_1}(a_1+b_1\sqrt{D})$, $s_{\rho_2} = \phi^{AD}_{\rho_2}(a_2+b_2\sqrt{AD})$.

If m is odd, and E_1/F is unramified,

$$h \mapsto (I, w\boldsymbol{\varepsilon})\left(\left(\begin{smallmatrix} \boldsymbol{\pi}^{(m+i)/2} & 0 \\ 0 & \boldsymbol{\pi}^{-(m+i)/2} \end{smallmatrix}\right), \left(\begin{smallmatrix} \boldsymbol{\pi}^{(m+1)/2} & 0 \\ 0 & \boldsymbol{\pi}^{-(m+1)/2} \end{smallmatrix}\right)\right)(I, w\boldsymbol{\varepsilon})h$$

sends $h^{-1}x_\rho h$ to $h^{-1}((\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^i \end{smallmatrix}), \boldsymbol{\varepsilon} w(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi} \end{smallmatrix}))s'_\rho((\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{-i} \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}^{-1} \end{smallmatrix})w\boldsymbol{\varepsilon})h$, where i is taken to be 1 if $\rho_1 = \boldsymbol{\pi}$ and -1 if $\rho_1 = 1$. Then $h^{-1}x_\rho h$ is mapped to $h^{-1}s'_{\tilde{\rho}}h$, where if $\rho = (\rho_1, \rho_2)$ then $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2)$, and $\tilde{\rho}_1$ is defined by $\{\rho_1, \tilde{\rho}_1\} = \{1, \boldsymbol{\pi}\}$, and $\rho_2 \mapsto \tilde{\rho}_2 = -1/D\rho_2$ is a permutation (trivial if $-1 \notin R^{\times 2}$) of $R^\times/R^{\times 2}$.

If m is odd, and E_1/F is ramified, we take

$$h \mapsto (I, w\boldsymbol{\varepsilon})\left(\left(\begin{smallmatrix} \boldsymbol{\pi}^{(m+1)/2} & 0 \\ 0 & \boldsymbol{\pi}^{-(m+1)/2} \end{smallmatrix}\right), \left(\begin{smallmatrix} \boldsymbol{\pi}^{(m+1)/2} & 0 \\ 0 & \boldsymbol{\pi}^{-(m+1)/2} \end{smallmatrix}\right)\right)(w\boldsymbol{\varepsilon}, w\boldsymbol{\varepsilon})h,$$

which maps $h^{-1}x_\rho h$ to $h^{-1}s'_{\tilde{\rho}}h$, where $\rho_1 \mapsto \tilde{\rho}_1 = -1/\rho_1$ is a permutation, trivial if $-1 \in R^{\times 2}$, of $R^\times/R^{\times 2}$, and $\rho_2 \mapsto \tilde{\rho}_2 = -1/A\rho_2$ is a permutation (trivial if $-1 \notin R^{\times 2}$) of $R^\times/R^{\times 2}$.

Put $\rho_m = \rho$ if m is even, and $\rho_m = \tilde{\rho}$ if m is odd. We get

$$= \sum_{m \geq 0} [K_0^1 : K_m^1] \int_{T_{\rho_m} \backslash C_1} 1_{K_m^1}(h^{-1}s_{\rho_m}h)dh.$$

Using the double coset decomposition for $SL(2, F)$ of Lemma I.I.3 we get

$$= \sum_{m \geq 0} \sum_{r \in R_{\rho_m}} [R_T^1 : T_{\rho_m}^1 \cap rK_0^1 r^{-1}][K_0^1 : K_m^1] \int_{K_0^1} 1_{K_m^1}(k^{-1}r^{-1}s_{\rho_m}rk)dk.$$

Here $R_T^1 = T_{\rho_m}^1 \cap K_0^1 = T_{\rho_m}^1(R)$. Let $\mathbf{j}$ signify (j_1, j_2). To simplify the notations we write ρ for ρ_m until the index m is explicitly needed.

The decomposition of Lemma I.I.3 is $SL(2, F) = \underset{j \geq 0}{\cup} T_\rho^1 \phi_\rho^D(\boldsymbol{\pi}_E^{-j})(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}_F^j \end{smallmatrix})K^1$ if $E_1 = F(\sqrt{D})$ is ramified over F (and $\boldsymbol{\pi}_E = \sqrt{-\boldsymbol{\pi}_F}$). It is $SL(2, F) = \cup T_\rho^1 t_\varepsilon(\begin{smallmatrix} \boldsymbol{\pi}_F^{-(j-\overline{\rho})/2} & 0 \\ 0 & \varepsilon\boldsymbol{\pi}_F^{(j-\overline{\rho})/2} \end{smallmatrix})K^1$, union over $j \geq 0$ such that $j - \overline{\rho}$ is even, and over $\varepsilon \in R^\times/R^{\times 2}$ when $j \geq 1$, if E_1/F is unramified. Here $T_\rho^1 = \phi_\rho^D(E_1^1)$, $E_1 = F(\sqrt{D})$ and E_1^1 is the group of $x \in E_1^\times$ with norm $N_{E_1/F}x = 1$. Further $t_\varepsilon \in T_\rho = \phi_\rho^D(E_1^\times)$ is an element with determinant ε^{-1}. Of course, here $K^1 = SL(2, R)$. Consequently the representatives $r \in R_\rho\big(\rho = (\rho_1, \rho_2)\big)$ take the form

$$r = \phi_{\rho_1}^D(\boldsymbol{\pi}_1^{-j_1})(\begin{smallmatrix} 1 & 0 \\ 0 & (\varepsilon_0\boldsymbol{\pi}_F)^{j_1} \end{smallmatrix}) \times \phi_{\rho_2}^{AD}(\boldsymbol{\pi}_2^{-j_2})(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}_F^{j_2} \end{smallmatrix}), \qquad j_1, j_2 \geq 0,$$

when E_1/F is ramified ($\boldsymbol{\pi}_1 = \sqrt{-\varepsilon_0\boldsymbol{\pi}}$ and $\boldsymbol{\pi}_2 = \sqrt{-\boldsymbol{\pi}}$ denote uniformizers of E_1 and E_2, where $\varepsilon_0 \in R^\times - R^{\times 2}$). When E_1/F is unramified, the representatives r are $t_\varepsilon(\begin{smallmatrix} \boldsymbol{\pi}_F^{-(j_1-\overline{\rho}_1)/2} & 0 \\ 0 & \varepsilon\boldsymbol{\pi}_F^{(j_1-\overline{\rho}_1)/2} \end{smallmatrix})$ $\times\phi_{\rho_2}^{AD}(\boldsymbol{\pi}_2^{-j_2})(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}_F^{j_2} \end{smallmatrix})$, where $j_1, j_2 \geq 0, j_1 - \overline{\rho}_1$ is even, ε ranges over $R^\times/R^{\times 2}$ if $j_1 \geq 1$, and $t_\varepsilon \in \phi_{\rho_1}^D(E_1^\times)$ has determinant ε^{-1}. Write q_0 for the residual cardinality $\#R/\boldsymbol{\pi}_F R$ of F, and $q = q_3$ for $\#R_3/\boldsymbol{\pi}_3 R_3$.

1. Lemma. *The index $[R^1_{T^1_\rho} : T^1_\rho \cap rK^1_0 r^{-1}]$ is equal to $q_0^{j_1+j_2}$ if E_1/F is ramified or $j_1 = 0$, and to $q_0^{j_1+j_2}(q_0+1)/2q_0$ if E_1/F is unramified (then $q = q_0$) and $j_1 \geq 1$.*

Proof. This is proven as in the case of type (I), see Lemma B.1, on noting that $T^1_\rho \cap rK^1_0 r^{-1} = R_{E_1}(j_1)^1 \times R_{E_2}(j_2)^1, R_E(j) = R + \boldsymbol{\pi}^j_F R_E$, and $R^1_{T^1_\rho} = R^1_{E_1} \times R^1_{E_2}$. $\square$

2. Lemma. *The integral $\int_{K^1_0/K^1_m} 1_{K^1_m}(k^{-1}r^{-1}s_\rho rk)dk$ is equal to the cardinality of the set*

$$L^1_m = L^1_{m,\rho} = \{x = \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right) \in SL(2,R_m); \left(\begin{smallmatrix} \overline{a}_1 & \overline{b}'_1\overline{D}'_1 \\ \overline{b}'_1 & \overline{a}_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)\left(\begin{smallmatrix} \overline{a}_2 & \overline{b}'_2\overline{D}'_2 \\ \overline{b}'_2 & \overline{a}_2 \end{smallmatrix}\right)\}.$$

Here we put $R_m = R/\boldsymbol{\pi}^m_F R$, $\overline{a}$ denotes the image in R_m of a in R. Suppose that $b_i = B_i\boldsymbol{\pi}^{N_i}_F, \rho_i = u_i\boldsymbol{\pi}^{\overline{\rho}_i}_F$, and $\varepsilon_1 = \varepsilon_0^{j_1}$ when $e(E_1/F) = 2$ or $j_1 = 0$, and $\varepsilon_1 = \varepsilon(\in R^\times/R^{\times 2})$ when $e(E_1/F) = 1$ and $j_1 \geq 1$, so $\varepsilon_1 = \varepsilon(r), r \in R_\rho$, and $\varepsilon_2 = 1$. Then we write $b'_i = (B_i/\varepsilon_i u_i)\boldsymbol{\pi}^{\nu_i}_F$, where $\nu_i = N_i - j_i, (i = 1,2)$, and $D'_i = D_i\varepsilon_i^2 u_i^2 \boldsymbol{\pi}^{2j_i}_F$, where $D_1 = D, D_2 = AD$.

Proof. As in case (I), see Lemma B.2, recall that $d(A) = (A, \boldsymbol{\varepsilon}A\boldsymbol{\varepsilon})$, and note that K^1_0/K^1_m $=SL(2,R_m) \times SL(2,R_m)/d\big(SL(2,R_m)\big)$. $\square$

Put $X = \operatorname{ord}(a_1 - a_2)$.

3. Lemma. *The set L^1_m is non empty precisely when the following conditions are satisfied: (1) $0 \leq m \leq X$. (2) $\nu_i \geq 0$. (3) $m \leq \nu_1$ if and only if $m \leq \nu_2$. (4) If $\nu_1 < m$ or $\nu_2 < m$ then $\nu_1 = \nu_2$; we denote then the common value by ν. (5) If $\nu < m$ then $(B_1/\varepsilon_1 u_1)/(B_2/u_2) \in R^{\times 2}$. (6) Further, if $\nu < m$ then $m \leq 2N_i - \nu + \operatorname{ord} D_i (i = 1,2)$.*

If L^1_m is non empty, then its cardinality is: 1 if $m = 0$; $(q_0^2 - 1)q_0^{3m-2}$ if $1 \leq m \leq \nu_1$ (equivalently: $1 \leq m \leq \nu_2$); $2q_0^{m+2\nu}$ if $\nu < m$.

Proof. If L^1_m is not empty, then comparing the traces of $\left(\begin{smallmatrix} \overline{a}_i & \overline{b}'_i\overline{D}'_i \\ \overline{b}'_i & \overline{a}_i \end{smallmatrix}\right)$, $i = 1, 2$, we get $\overline{a}_1 = \overline{a}_2$, hence $0 \leq m \leq X = \operatorname{ord}(a_1 - a_2)$. We then replace $\overline{a}'_i$ by 0 in the equation defining L^1_m, and conclude that $\overline{b}'_1 = 0$ if and only if $\overline{b}'_2 = 0$, thus $m \leq \nu_1$ precisely when $m \leq \nu_2$.

The same equation shows that if $\overline{b}'_i \neq 0$ for some i, so $\nu_i < m$, then $|b'_1| = |b'_2|$, namely $\nu_1 = \nu_2$. The common value is denoted then by ν. Assume that $\nu(= \nu_1 = \nu_2) < m$. The set L^1_m consists of all $(x_1,x_2,x_3,x_4) \in R^4_m$ with $x_1x_4 - x_2x_3 = 1$, satisfying

$$\overline{b}'_1\overline{D}'_1 x_3 = \overline{b}'_2 x_2, \qquad \overline{b}'_1\overline{D}'_1 x_4 = \overline{b}'_2\overline{D}'_2 x_1, \qquad \overline{b}'_1 x_2 = \overline{b}'_2\overline{D}'_2 x_3, \qquad \overline{b}'_1 x_1 = \overline{b}'_2 x_4.$$

Put $\eta = (B_1/\varepsilon_1 u_1)/(B_2/u_2)$. Then for each $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ in L^1_m there are $a_2, a_4 \in R_m$ with

$$\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_1 & \eta^{-1}(D'_2x_3 + \boldsymbol{\pi}^{m-\nu}a_2) \\ x_3 & \eta(x_1 + \boldsymbol{\pi}^{m-\nu}_F a_4) \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_1 & \eta^{-2}x_3\overline{D}'_2 \\ x_3 & x_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \boldsymbol{\pi}^{m-\nu}_F A_2 \\ 0 & 1+\boldsymbol{\pi}^{m-\nu}_F A_4 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & \eta \end{smallmatrix}\right),$$

where A_2, A_4 are defined (in R_m) by $\left(\begin{smallmatrix} \eta^{-2}a_2 \\ a_4 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_1 & \eta^{-2}x_3\overline{D}'_2 \\ x_3 & x_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} A_2 \\ A_4 \end{smallmatrix}\right)$. Since the determinant is one, η lies in $N_{D_2} = \{y \in R^\times_m; y = x_1^2 - x_3^2 AD\frac{B_2^2}{B_1^2}\varepsilon_1^2 u_1^2 \boldsymbol{\pi}^{2N_2 - 2\nu_2}_F\}$, which is $R^{\times 2}_m$ since $|AD| < 1$. This is (5) of the lemma.

If x lies in L_m^1, then x_1, x_4 are units. Otherwise x_2, x_3 are units, and since we are assuming that $\nu < m$, the conditions that x satisfies imply that D_1' and D_2' are units, but AD is not a unit. Since x_1, x_4 are units, if $\overline{b}_1'\overline{D}_1$ or $\overline{b}_2'\overline{D}_2 \neq 0$, namely $m > 2N_i - \nu_i + \operatorname{ord} D_i$ for some $i = 1,2$, then $\eta\overline{b}_1'\overline{D}_1' = \overline{b}_2'\overline{D}_2' \pmod{\boldsymbol{\pi}^m}$ implies that $N_1 = N_2$ and $\operatorname{ord} AD = \operatorname{ord} D$, thus $\operatorname{ord} A = 0$, and $B_1^2 D_1 \equiv B_2^2 D_2 \pmod{(\boldsymbol{\pi}^{m-(2N_i-\nu_i)})}$. But A is not a square $(D_2/D_1 = A)$, we obtain a contradiction, and we conclude (6) of the lemma, namely that $\overline{b}_i'\overline{D}_i = 0 (i = 1,2)$.

The cardinality of L_m^1 is clearly 1 when $m = 0$, and it is $\#SL(2, R_m) = (q_0^2 - 1)q_0^{3m-2}$ when $\overline{b}_i' = 0$, namely $\nu_i \geq m (i = 1,2)$. If $\nu < m$, the cardinality of L_m^1 is the product of the cardinalities of the sets $\{A_2 \in R_m/\boldsymbol{\pi}_F^\nu R_m \simeq R/\boldsymbol{\pi}_F^\nu R\}$ and $\{x_1, x_3 \in R_m; x_1^2 - \overline{D}_1 x_3^2 \in 1 + \boldsymbol{\pi}_F^{m-\nu} R_m\}$. The cardinality of the first set is q^ν. The second has cardinality

$$\#\{x_1, x_3 \in R_m; x_1^2 - \overline{D}_1 x_3^2 \in R_m^{\times 2}\}/[R_m^{\times 2} : 1 + \boldsymbol{\pi}_F^{m-\nu} R_m].$$

The denominator is $[R^\times : 1 + \boldsymbol{\pi}^{m-\nu} R]/[R_m^\times : R_m^{\times 2}] = \frac{1}{2}(q_0 - 1)q_0^{m-\nu-1}$. Hence the cardinality of L_m^1 is $2(q_0 - 1)^{-1} q_0^{2\nu - m + 1} \cdot (q_0 - 1) q_0^{m-1} \cdot q_0^m = 2q_0^{m+2\nu}$, as asserted. □

H. Orbital integrals of type (II).

We need to compare the stable θ-orbital integral of 1_K at a topologically unipotent strongly θ-regular element $u = h^{-1}t^*h$ of type (II), computed above, with the stable orbital integral of 1_K at the norm Nu of u. We compute this integral next. This norm $Nu = h^{-1}Nt^*h$ is also of type (II) in our listing of elliptic conjugacy classes in $H = GSp(2, F)$. There are two conjugacy classes in the stable conjugacy class of a regular element of type (II), represented here by $\mathbf{s}_\rho = \left(\begin{smallmatrix} \mathbf{a} & \mathbf{b}D\boldsymbol{\rho} \\ \boldsymbol{\rho}^{-1}\mathbf{b} & \mathbf{a} \end{smallmatrix}\right)$. We write $\mathbf{a} = \left(\begin{smallmatrix} a_1 & a_2/A \\ a_2 & a_1 \end{smallmatrix}\right)$ if $a = a_1 + a_2/\sqrt{A}$ lies in $E_3 = F(\sqrt{A}) = E^{\langle\sigma\rangle}$, similarly for $\mathbf{b}, \boldsymbol{\rho}$, where ρ ranges over a set of representatives for $E_3^\times/N_{E/E_3}E^\times$, say 1 and an element of minimal order in $R_3 = R_{E_3}$. The centralizer T_ρ of $\mathbf{s}_{\boldsymbol{\rho}}$ in $H = GSp(2, F)$ lies in the subgroup

$$\mathbf{C}_A = \{\left(\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right); \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, E_3)'\},$$

the prime indicates: determinant in $F^\times$.

1. Lemma. *The integral* $\Phi_{1_K}^{GSp(2,F)}(\mathbf{s}_\rho)$ *is equal to* $\sum_{m=0}^{\infty} [K_0 : K_m] \int_{T_\rho\backslash C_A} 1_{K_m}(h^{-1}s_\rho h)dh$. *Here* $C_A = GL\big(2, F(\sqrt{A})\big)'$ *and* $K_m = GL\big(2, R_{E_3}(m)\big)'$, *where* $R_{E_3}(m) = R + \boldsymbol{\pi}^m\sqrt{A}R = R + \boldsymbol{\pi}^m R_{E_3}$, *and* $s_\rho = \left(\begin{smallmatrix} a & bD\rho \\ \rho^{-1}b & a \end{smallmatrix}\right)$.

Proof. Using the decomposition $H = GSp(2, F) = \bigcup_{m\geq 0} \mathbf{C}_A u_m K$, $K = GSp(2, R)$ of Lemma I.J.1, we deduce that

$$\int_{T_\rho\backslash GSp(2,F)} 1_K(g^{-1}\mathbf{s}_\rho g)dg = \sum_{m=0}^{\infty} |K|_H \int_{T_\rho\backslash \mathbf{C}_A/\mathbf{C}_A \cap u_m K u_m^{-1}} 1_K(u_m^{-1}h^{-1}\mathbf{s}_\rho h u_m)dh.$$

Put $K_m^A = \mathbf{C}_A \cap u_m K u_m^{-1}$. The integrand on the right is non zero precisely when $h^{-1}\mathbf{s}_\rho h \in u_m K u_m^{-1} \cap \mathbf{C}_A$, so we obtain

$$= \sum_{m\geq 0} |K|_H \int_{T_\rho\backslash \mathbf{C}_A/K_m^A} 1_{K_m^A}(h_0^{-1}\mathbf{s}_\rho h_0)dh_0.$$

Next we use the isomorphism $\phi_m : C_A \to \mathbf{C}_A$ ($\phi_m(h) = h_0$) of Lemma I.J.3, which asserts that $\phi_m(K_m) = K_m^A$. Define x_ρ by $\phi_m(x_\rho) = \mathbf{s}_\rho$. We obtain

$$= \sum_{m\geq 0} |K|_H \int_{Z_{C_A}(x_\rho)\backslash C_A/K_m} 1_{\phi_m(K_m)}(\phi_m(h)^{-1}\phi_m(x_\rho)\phi_m(h))dh,$$

in which we can erase ϕ_m everywhere. Changing variables $h \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/A\boldsymbol{\pi}^m \end{smallmatrix}\right)h$ on C_A, we obtain

$$= \sum_{m\geq 0} [K_0 : K_m] \int_{T_\rho\backslash C_A} 1_{K_m}(h^{-1}s_\rho h)dh.$$

□

Using the decomposition $C_A = \cup_r T'_\rho r K'$, our integral takes the form

$$= \sum_{m\geq 0}\sum_r [T'_0 : T'_\rho \cap rK'r^{-1}][K_0 : K_m] \int_{K_0} 1_{K_m}(k^{-1}r^{-1}s_\rho rk)dk,$$

where $T'_0 = T'_\rho \cap K' = T'_\rho(R) = R'_E$. Here $R'_E = \{x \in R_E^\times; N_{E/E_3}x \in F^\times\}$.

Recall that $q = q_3 = q_{E_3}$ denotes the residual cardinality of E_3.

2. Lemma. *The index $[T'_0 : T'_\rho \cap rK'r^{-1}]$ is equal to q^j if E/E_3 is ramified or $j = 0$, and to $(q+1)q^{j-1}$ if E/E_3 is unramified and $j \geq 1$, where $r = r_{j,\rho} = t_{j,\rho}\left(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}_3^{j-\overline{\rho}} \end{smallmatrix}\right)$.*

Proof. The intersection $T'_\rho \cap rK'r^{-1} \simeq \{t \in T'_\rho; r^{-1}tr \in K'\}$ is

$$\{a + b\sqrt{D} \in T'_\rho; \left(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}_3^{-(j-\overline{\rho})} \end{smallmatrix}\right)\left(\begin{smallmatrix} a & bD\rho \\ b/\rho & a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & \boldsymbol{\pi}_3^{j-\overline{\rho}} \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & bD\rho\boldsymbol{\pi}_3^{j-\overline{\rho}} \\ (b/\rho)\boldsymbol{\pi}_3^{-(j-\overline{\rho})} & a \end{smallmatrix}\right) \in K'\},$$

which is $R'_E \cap R_E(j)^\times$, where $R_E(j) = R_3 + \boldsymbol{\pi}_3^j R_E = R_3 + \sqrt{D}\boldsymbol{\pi}_3^j R_3$, $R_3 = R_{E_3}$, since $b \in \boldsymbol{\pi}_3^j R_3$. Put $R_E(j)'$ for $R'_E \cap R_E(j)^\times$. Consider the exact sequence

$$1 \to R'_E/R_E(j)' \to R_E^\times/R_E(j)^\times \to R_E^\times/R'_E R_E(j)^\times \to 1.$$

The last group is isomorphic, via the norm map $N = N_{E/E_3}$, to $NR_E^\times/NR_E^\times \cap R^\times \cdot NR_E(j)^\times$. Indeed, the kernel of the norm map is contained in R'_E. When E/E_3 is ramified we have $NR_E^\times = NR_E(j)^\times$. When E/E_3 is unramified, we have $NR_E^\times = R_3^\times$, and $NR_E(j)^\times = R_3^{\times 2} (j \geq 1)$. Moreover, $R_3^\times = R^\times R_3^{\times 2}$, since $a + b\sqrt{\boldsymbol{\pi}_F} = a(1 + \frac{b}{a}\sqrt{\boldsymbol{\pi}_F})$ ($a, b, \in R; E_3/F$ is ramified). Hence

$$[R'_E : R_E(j)'] = [R_E^\times : R_E(j)^\times] = [R_E^\times : 1 + \boldsymbol{\pi}_3^j R_E]/[R_E(j)^\times : 1 + \boldsymbol{\pi}_3^j R_E].$$

The denominator here is $[R_3^\times : R_3^\times \cap (1 + \boldsymbol{\pi}_3^j R_E)] = [R_3^\times : 1 + \boldsymbol{\pi}_3^j R_3] = (q-1)q^{j-1}, q = q_3$. When E/E_3 is ramified, $q_E = q$, hence the numerator is $(q-1)q^{2j-1}$ (since $\boldsymbol{\pi}_3 = \boldsymbol{\pi}_E^2$). When E/E_3 is unramified, $q_E = q^2$ and $\boldsymbol{\pi}_3 = \boldsymbol{\pi}_E$, hence the numerator is $(q^2-1)q^{2(j-1)}$. The quotient is as stated in the lemma. □

Consider the ring $S_m = R_3/\boldsymbol{\pi}_F^m R_3$, and the subring $R_m = (R + \boldsymbol{\pi}_F^m R_3)/\boldsymbol{\pi}_F^m R_3 = R/\boldsymbol{\pi}_F^m R$. If $K(\boldsymbol{\pi}_F^m) = \{k \in GL(2, R_3); k \equiv 1(\boldsymbol{\pi}_F^m)\}$, and $K_m = GL\big(2, R_3(m)\big)'$, where $R_3(m) = R + \boldsymbol{\pi}_F^m R_3$, then $K_m/K(\boldsymbol{\pi}_F^m) = GL(2, R_m)$ and $K_0/K(\boldsymbol{\pi}_F^m) = GL(2, S_m)'$. The prime indicates determinant in $R_m^\times$. We emphasize that $R_3 = R_{E_3}$ is the ring of integers in R_3, while R_m is a finite ring ($m \geq 1$); they should not be confused with each other when $m = 3$.

3. Lemma. *The integral* $\int_{K_0/K_m} 1_{K_m}(k^{-1}r_{j\rho}^{-1}s_\rho r_{j\rho}k)dk$ *is equal to the cardinality of the set*

$$L'_m = \{y \in GL(2,S_m)'/GL(2,R_m); y^{-1}r_{j\rho}^{-1}s_\rho r_{j\rho}y \in GL(2,R_m)\},$$

where $s_\rho = r_{j\rho}^{-1}s_\rho r_{j\rho} = \begin{pmatrix} a & bD\rho\boldsymbol{\pi}_3^{j-\overline{\rho}} \\ (b/\rho)\boldsymbol{\pi}_3^{-(j-\overline{\rho})} & a \end{pmatrix}$. *Consequently, if* L'_m *is not empty, then* $0 \le j \le N = \mathrm{ord}_3(b) = \mathrm{ord}_{E_3}(b)$. □

4. Lemma. *The map* $L'_m \to L_m = \{x \in SL(2,S_m); \tau x = x^{-1}, xs_{\rho,r}x^{-1} = \tau(s_{\rho,r})\}, y \mapsto x = \tau(y)y^{-1}$, *is injective. It is surjective if* E/E_3 *is ramified, while the image has index two if* E/E_3 *is unramified. In particular* $\#L'_m = \frac{1}{2}e(E/E_3)\cdot\#L_m$, *where* $e = e(E/E_3)$ *is the ramification index of* E/E_3.

Proof. For the injectivity, if $\tau(y_1)y_1^{-1} = \tau(y_2)y_2^{-1}$ then $\tau(y_1^{-1}y_2) = y_1^{-1}y_2 \in GL(2,R_m)$. If E/E_3 is ramified then E_3/F is unramified, and the map $GL(2,R_3)' \to \{x = \tau(x)^{-1} \in SL(2,R_3)\}$, $y \mapsto x = \tau(y)y^{-1}$, is onto by Hensel's Lemma. If E/E_3 is unramified then E_3/F is ramified, hence $\tau(x) \equiv x(\mathrm{mod}\,\boldsymbol{\pi}_3)$. Thus $\tau(x) = x^{-1}$ implies that $x^2 \equiv 1(\mathrm{mod}\,\boldsymbol{\pi}_3)$, and since $\|x\| = 1$, that $x \equiv \pm I(\mathrm{mod}\,\boldsymbol{\pi}_3)$. Namely $x \in L_m$ if and only if $-x \in L_m$. Further, $x \equiv I(\mathrm{mod}\,\boldsymbol{\pi}_3)$ if and only if $x = \tau(y)y^{-1}$ for some $y \in GL(2,S_m)'$. Hence L_m is the disjoint union of image(L'_m) and $-$image(L'_m). □

Remark. Put $b = B\boldsymbol{\pi}_3^N, B \in R_3^\times$, and if $\rho = u\boldsymbol{\pi}_3^{\overline{\rho}}, u \in R_3^\times$, we put $b' = (B/u)\boldsymbol{\pi}_3^\nu$, where $\nu = N - j$ (satisfies $0 \le \nu \le N$ if $\#L_m \ne 0$). Put m' for $2m/e, e = e(E/E_3)$. Then $b' \ne 0$ in $S_m = R_3/\boldsymbol{\pi}_F^m R_3 = R_3/\boldsymbol{\pi}_3^{2m/e}R_3$ precisely when $\nu < m' = 2m/e$. Let $\overline{a}$ be the image in R_m of $a \in R_3$.

5. Lemma. *The set* L_m *is non empty precisely when* $0 \le \nu \le N, 0 \le m' \le X = \mathrm{ord}_3(a - \tau a)$, *and when* $m' > \nu$ *we further have that there exists* $\varepsilon \in S_m^{\times 2}$ *such that* $\tau s_{\rho,r} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon/\tau\varepsilon \end{pmatrix} s_{\rho,r} \begin{pmatrix} 1 & 0 \\ 0 & \tau\varepsilon/\varepsilon \end{pmatrix}$, *thus* $\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} s_{\rho,r} \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}^{-1}$ *lies in* $GL(2,R_m)$ *or equivalently that* $\nu + m' \le X$, *and* $u \in BS_m^{\times 2}$ *when* E/E_3 *is ramified, and* ν *is even when* E/E_3 *is unramified.*

Proof. Suppose that $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ lies in L_m, thus $\|x\| = 1$ and $\tau x = x^{-1}$, and $xs_{\rho r}x^{-1} = \tau(s_{\rho r})$, where $s_{\rho,r} = \begin{pmatrix} \overline{a} & \overline{b}'\overline{D}' \\ \overline{b}' & \overline{a} \end{pmatrix}$, and $D' = Du^2\boldsymbol{\pi}_3^{2j}$. Taking traces we conclude that $\overline{a} = \tau\overline{a}$ lies in R_m. Hence $0 \le m' \le X = \mathrm{ord}_3\left(a - \tau a\right)$, and when $\overline{b}' = 0$ we are done. Suppose, from now on, that $\overline{b}' \ne 0$, namely $m' > \nu$. As $\tau x = x^{-1}$ and $\|x\| = 1$, we have $\begin{pmatrix} \tau x_1 & \tau x_2 \\ \tau x_3 & \tau x_4 \end{pmatrix} = \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}$. Hence there are r_2, r_3 in R_m, such that $x = \begin{pmatrix} x_1 & r_2\sqrt{A} \\ r_3\sqrt{A} & \tau x_1 \end{pmatrix}$. The relation $xs_{\rho r}x^{-1} = \tau(s_{\rho r})$, but with $\overline{a}$ replaced by 0 (since $\overline{a} = \tau\overline{a}$), is:

$$\begin{pmatrix} x_1 & r_2\sqrt{A} \\ r_3\sqrt{A} & \tau x_1 \end{pmatrix}\begin{pmatrix} 0 & \overline{b}'\overline{D}' \\ \overline{b}' & 0 \end{pmatrix} = x(s_{\rho r} - \overline{a}) = \tau(s_{\rho r} - \overline{a})x = \begin{pmatrix} 0 & \tau\overline{b}'\tau\overline{D}' \\ \tau\overline{b}' & 0 \end{pmatrix}\begin{pmatrix} x_1 & r_2\sqrt{A} \\ r_3\sqrt{A} & \tau x_1 \end{pmatrix},$$

namely

$$(*) \qquad \begin{pmatrix} \overline{b}'r_2\sqrt{A} & x_1\overline{b}'\overline{D}' \\ \overline{b}'\tau x_1 & \overline{b}'\overline{D}'r_3\sqrt{A} \end{pmatrix} = \begin{pmatrix} \tau\overline{b}\cdot\tau\overline{D}'\cdot r_3\sqrt{A} & \tau\overline{b}'\cdot\tau\overline{D}'\cdot\tau x_1 \\ \tau\overline{b}'\cdot x_1 & \tau\overline{b}'\cdot r_2\sqrt{A} \end{pmatrix}.$$

Then $x_1 \in S_m^\times$, otherwise (since $\|x\| = 1$) $A, r_2, r_3 \in R_m^\times$, hence $D \in \boldsymbol{\pi}_F R^\times$ and so $D' \in \boldsymbol{\pi}_3 S_m$, contradicting the relation obtained on comparing the entries on second row and second column. We denote this location by $(2,2)$. In fact this relation, $(2,2)$, shows that $r_2\sqrt{A} = \frac{\overline{b}'}{\tau\overline{b}'}\overline{D}' r_3\sqrt{A} + \boldsymbol{\pi}_3^{m'-\nu} S_m$. Hence

$$x = \begin{pmatrix} x_1 & r_3\overline{D}'\sqrt{A}\frac{x_1}{\tau x_1} + \boldsymbol{\pi}_3^{m'-\nu} S_m \\ r_3\sqrt{A} & \tau x_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} x_1 & \frac{x_1}{\tau x_1} r_3 \overline{D}'\sqrt{A} + \boldsymbol{\pi}_3^{m'-\nu} S_m \\ \frac{x_1}{\tau x_1} r_3\sqrt{A} & x_1 \end{pmatrix},$$

where $\varepsilon = \tau x_1/x_1$, lies in $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varepsilon \end{smallmatrix}\right) Z_{GL(2,S_m)}(s_{\rho,r})$.

Since $\nu < m'$ we have

$$\overline{b}'/\tau\overline{b}' = x_1 \cdot (\overline{b}'/\tau\overline{b}')\tau x_1 - (\overline{b}'/\tau\overline{b}) r_2\sqrt{A} r_3\sqrt{A} = x_1^2 - r_3^2 A\tau(\overline{D}') + \boldsymbol{\pi}_3^{m'-\nu} S_m = x_1^2 + \boldsymbol{\pi}_3 S_m.$$

The first equality follows from $\|\left(\begin{smallmatrix} x_1 & r_2\sqrt{A} \\ r_3\sqrt{A} & \tau x_1 \end{smallmatrix}\right)\| = 1$, the second uses the relations obtained on comparing the entries at the locations $(2,1)$ and $(1,1)$, and the last follows since E/F is ramified (thus $|A\tau(\overline{D})| < 1$). Further, from $(2,1)$ we have $\overline{b}'/\tau\overline{b}' = x_1/\tau x_1 + \boldsymbol{\pi}_3^{m'-\nu} S_m$. Hence $x_1 \tau x_1 \equiv 1 (\operatorname{mod} \boldsymbol{\pi}_3)$. If E/E_3 is unramified, then $A \notin R^\times$, hence $x_1 = \alpha + \beta\sqrt{A}, x_1\tau x_1 = \alpha^2 - \beta^2 A \equiv 1$ implies that $\alpha \equiv \pm I$ and $\beta \equiv 0 (\operatorname{mod} \boldsymbol{\pi}_3)$. Then $\alpha^{-1}x_1 \in S_m^{\times 2}$, and $\varepsilon = \tau(\alpha^{-1}x_1)/\alpha^{-1}x_1$ is as required.

If E/E_3 is ramified then E_3/F is unramified, $R_3^\times / \ker(N_{E_3/F}|R_3^\times) \simeq R^\times$ ($N_{E_3/F} : R_3^\times \to R^\times$ is surjective), hence $\ker(N_{E_3/F}|R_3^\times)$ has index $q_0 - 1$ in $R_3^\times$, and so it is contained in the index 2 subgroup $R_3^{\times 2}$ of $R_3^\times$, hence $x_1\tau x_1 \equiv 1 \ (\operatorname{mod} \boldsymbol{\pi}_3)$ implies that $x_1 \in R_3^{\times 2}$, as required. In the lemma, x_1 (or $\alpha^{-1}x_1$ when E/E_3 is unramified) is denoted by ε, as we do from now on.

Suppose again that $\overline{b}' \neq 0$ in S_m, thus $\nu < m'$. The relation $(2,1)$, and $\tau\boldsymbol{\pi}_3 = (-1)^e \boldsymbol{\pi}_3, b' = (B/u)\boldsymbol{\pi}_3^\nu$, imply that $\tau(\varepsilon)/\varepsilon \equiv \overline{b}'/\tau\overline{b}' \equiv (-1)^{\nu e}(B/u)/\tau(B/u) (\operatorname{mod} \boldsymbol{\pi}_3^{m'-\nu})$. When E/E_3 is ramified ($e = 2$), we deduce that $\varepsilon B/u \in R_m^\times$, namely $u \in B\varepsilon R_m^\times \subset BS_m^{\times 2} R_m^\times = BS_m^\times$. When E/E_3 is unramified ($\varepsilon = 1$), E_3/F is ramified, hence $S_m^\times \cap R_m\sqrt{A}$ is empty, hence $\operatorname{Re}(\varepsilon B/u) = \overline{\varepsilon B/u} + \overline{\tau(\varepsilon B/u)}$ is non zero in R_m, and $\operatorname{Re}(\varepsilon B/u) = (-1)^{\nu e} \operatorname{Re}\left(\tau(\varepsilon B/u)\right)$ implies that $(-1)^\nu = 0$, thus ν is even. Hence $\tau(\varepsilon)/\varepsilon \equiv (B/u)/\tau(B/u) \ (\operatorname{mod} \boldsymbol{\pi}_3^{m'-\nu})$, and the relation $(1,2)$ implies that

$$DBu/\tau(DBu) \equiv \varepsilon/\tau\varepsilon (\operatorname{mod} \boldsymbol{\pi}_3^{m'-(2N-\nu+\operatorname{ord} D)})$$

provided that $m' > 2N - \nu + \operatorname{ord} D (\geq \nu)$. The two relations together imply that $DB^2/\tau(DB^2) \equiv 1 (\operatorname{mod} \boldsymbol{\pi}_3^{m'-(2N-\nu+\operatorname{ord} D)})$, namely $D(B^2 - \tau B^2) \equiv 0 (\operatorname{mod} \boldsymbol{\pi}_3^{m'+\nu-2N})$. But then $|a + \tau a| = 1$, since s_ρ is topologically unipotent, and

$$(a - \tau a)(a + \tau a) = a^2 - \tau a^2 = D(b^2 - \tau b^2) \equiv 0 (\operatorname{mod} \boldsymbol{\pi}_3^{m'+\nu})$$

implies that $m' + \nu \leq X$, as required. □

As usual, q_0 is $q_F, q = q_3$ is q_{E_3}, and $e = e(E/E_3)$.

6. Lemma. *When L'_m is non empty, its cardinality is: 1 if $m = 0; q^{3m'/2}$ if $e = 1$, and $(q+1)q^{3m'/2-1}$ if $e = 2$, when $1 \le m' \le \nu$; $eq_0^m q^\nu$ when $\nu < m'$.*

Proof. This is clear when $m = 0$, and $\#L'_m$ is the cardinality of $GL(2, S_m)'/GL(2, R_m) \simeq SL(2, S_m)/SL(2, R_m)$ where $\overline{b}' = 0$, namely $1 \le m' \le \nu$. Recall that $R_m = R/\boldsymbol{\pi}_F^m R$, and $\#SL(2, R_m) = (q_0^2 - 1)q_0^{3m-2} = (q_0^2 - 1)q_0^{3em'/2-2}$. Also $S_m = R_3/\boldsymbol{\pi}_3^{m'} R_3$, and $\#SL(2, S_m) = (q^2-1)q^{3m'-2}$. When $e = 1, q = q_0$, and the quotient is $q^{3m'/2}(m' = 2m)$. When $e = 2, q = q_0^2$, and the quotient is $(q+1)q^{3m'/2-1}(m' = m)$. From now on we then assume that $\nu < m'$. In the notations of the previous proof, the set L_m consists of the

$$x = \begin{pmatrix} x_1 & \frac{\overline{b}'}{\tau\overline{b}'}\overline{D}' \cdot r_3\sqrt{A} + a \\ r_3\sqrt{A} & \tau x_1 \end{pmatrix}$$

with $\|x\| = 1$, where $r_3 \in R_m, a \in \boldsymbol{\pi}_3^{m'-\nu} S_m$ lies in $R_m\sqrt{A}$ too (since $\|x\| = 1$, and $\frac{\overline{b}'}{\tau\overline{b}'}\overline{D}'$ lies in R_m by $(1,2)$ and $(2,1)$), and $x_1 = (\overline{B/u})r_2(1+\delta)$, where $r_1 \in R_m$ and $\delta \in \boldsymbol{\pi}_3^{m'-\nu} S_m$, since by $(2,1)$ we have $x_1/(\overline{B/u}) = r_1 + \boldsymbol{\pi}_3^{m'-\nu} S_m, r_1 \in R_m$.

In other words, L_m is the set of 4-tuples $(r_1, r_3, a, \delta) \in R_m^2 \times (\boldsymbol{\pi}_3^{m'-\nu} S_m)^2$, such that $\tau a = -a$, and $r_1^2(\overline{B/u})\tau(\overline{B/u})(1+\delta)(1+\tau\delta) - r_3^2 A \cdot \frac{\overline{b}'}{\tau\overline{b}'}\overline{D}' - ar_3\sqrt{A} = 1$, subject to the equivalence relation $(r_1, \delta) \sim (r'_1, \delta')$ if $r_1(1+\delta) = r'_1(1+\delta')$. Namely we take the quotient of the set of such 4-tuples by the group $1 + R_m A \boldsymbol{\pi}_3^{m'-\nu} S_m$.

To compute the cardinality of this quotient, take $r_3 \in R_m$, $\delta \in \boldsymbol{\pi}_3^{m'-\nu} R_3/\boldsymbol{\pi}_3^{m'} R_3 = R_3/\boldsymbol{\pi}_3^\nu R_3$, $a = \alpha\sqrt{A}$, $\alpha \in R_m \cap \boldsymbol{\pi}_3^{m'-\nu-\mathrm{ord}_3\sqrt{A}} S_m \simeq R_m \cap \boldsymbol{\pi}_3^{m'-\nu} S_m$ (when $e = 2, A \in R^\times$ is a unit; when $e = 1$, ν is even, and $A = \boldsymbol{\pi}_F = \boldsymbol{\pi}_3^2$). Now put $B' = B/u$, and recall from the proof of Lemma 5 that $\overline{B}'/\tau\overline{B}' \equiv x_1^2 \pmod{\boldsymbol{\pi}_3}$, hence $\overline{B}'/\tau\overline{B}'$ is a square, in $S_m^{\times 2}$. More precisely, $x_1 \equiv \overline{B}' r_1$, so $\overline{B}'\tau\overline{B}' \equiv (\overline{B}'/\tau\overline{B}')(\tau\overline{B}')^2 = (\overline{B}'\tau\overline{B}')^2 r_1^2$, and $r_1^2\overline{B}'\tau\overline{B}' \equiv 1$, and $\overline{B}'\tau\overline{B}' \in R_m^{\times 2}$.

Since $|AD| < 1$ and $|a| < 1$, there are always two solutions in r_1. The number of α's is the same as that of the equivalence relation by which we divide. We obtain that $\#L_m$ is $2q_0^m q^\nu$ (number of r_1's, number of $r_3 \in R_m$, number of $\delta \in R_3/\boldsymbol{\pi}_3^{-\nu} R_3$). We are done since $\#L_m = 2 \cdot \#L'_m/e$. $\square$

I. Comparison in case (II), E/E_3 ramified $(e = 2)$.

We compare the stable θ-orbital integral of 1_K at $u = s_\rho = [\phi_{\rho_1}^D(\alpha_1 + \beta_1\sqrt{D}), \phi_{\rho_2}^{AD}(\alpha_2 + \beta_2\sqrt{AD}]$, a topologically unipotent θ-fixed element of the form $h_\rho^{-1} t^* h_\rho$, where $t^* = (t_1, t_2, \tau t_2, \sigma t_1; e)$ in $Sp(2)$ (the integral vanishes unless $e \in R^\times$, as we now assume, and then it is independent of e), with the stable orbital integral of 1_K at the stable orbit of the norm $Nt^* = (x_1, \tau x_1, \sigma\tau x_1, \sigma x_1)$ in $GSp(2, F)$. Here $t_1 = \alpha_1 + \beta_1\sqrt{D}$, and $t_2 = \alpha_2 + \beta_2\sqrt{AD}$.

The assumption that $e = 2$ implies that $A \in R^\times$ and $D = \boldsymbol{\pi}_F$, and we have $\alpha_1^2 - \beta_1^2 D = 1 = \alpha_2^2 - \beta_2^2 AD$. By the definition of the norm, $x_1 = t_1 t_2$ ($\tau x_1 = t_1\tau t_2, \sigma\tau x_1 = t_2\sigma t_1$, and $\sigma x_1 = \sigma t_1 \tau t_2$). Hence $x_1 = \alpha_1\alpha_2 + D\beta_1\beta_2\sqrt{A} + (\alpha_2\beta_1 + \alpha_1\beta_2\sqrt{A})\sqrt{D} = a_1 + b_1\sqrt{D}$. We denote $n_i = \mathrm{ord}_F(\beta_i)$. Hence $X = \mathrm{ord}_3(a_1 - \tau a_1) = \mathrm{ord}_3(D\sqrt{A}\beta_1\beta_2) = 1 + n_1 + n_2$ ($E_3 = F(\sqrt{A})$

is unramified over F). Further $\chi = \mathrm{ord}_F(\alpha_1 - \alpha_2) = \mathrm{ord}_F(\alpha_1^2 - \alpha_2^2) = \mathrm{ord}_F(AD\beta_2^2 - D\beta_1^2) = 1 + 2\min(n_1, n_2)$, and $N = \mathrm{ord}_3(b_1) = \mathrm{ord}_3(\alpha_2\beta_1 + \alpha_1\beta_2\sqrt{A}) = \min(n_1, n_2)$, since α_1, α_2 are units.

When $e = 1$, namely when E/E_3 is unramified, we have $D \in R^\times$ and $A = \boldsymbol{\pi}_F$, thus $\boldsymbol{\pi}_3^2 = \boldsymbol{\pi}_F$, and then we have that $\chi = \mathrm{ord}_F(\alpha_1 - \alpha_2) = \min(2n_1, 1 + 2n_2)$, $X = 1 + 2n_1 + 2n_2$, and $N = \min(2n_1, 1 + 2n_2) = \chi$.

We shall use this for the actual comparison, but let us first compute.

1. Lemma. *Put $n_1' = \min(n_1, n_2)$, $n_2' = \max(n_1, n_2)$. When E/E_3 is ramified, the stable θ-orbital integral of 1_K at a strongly θ-regular topologically unipotent element of type (II) is equal to*

$$(*) \qquad 4q_0^{n_1+n_2}\frac{q_0^2}{(q_0-1)^2}(1 - q_0^{-n_1-1})(q^\chi - q_0^{-n_2-1}\frac{1 + q_0^{1+n_1} + q_0^{2+2n_1}}{1 + q_0 + q_0^2}).$$

Proof. Let us summarize the result of the computation of the stable twisted orbital integral in Section G. It is

$$\Phi_{1_K}^{G,st}(u\theta) = \Phi_{1_{Z_{K(\theta)}}}^{Z_G(\theta),st}(u) = \sum_\rho \sum_{m\ge 0} \sum_{r \in R_{\rho_m}} [R_T^1 : T_{\rho_m}^1 \cap rK_0^1 r^{-1}] \# L_{m,\rho_m}^1,$$

where $u = t_\rho = h^{-1}t^*h$ is topologically unipotent. Recall that L_{m,ρ_m}^1 depends on m and ρ_m, but for each m, the set $\{\rho_m\}$ is the same as the set of ρ. Hence we replace ρ_m by ρ in the triple sum above.

In this case there is no ε, we have summation over $0 \le \nu_1 \le n_1$ and $0 \le \nu_2 \le n_2$, and over $0 \le m \le \chi(= 1 + \min(2n_1, 2n_2))$. Also $m \le \nu_1$ if and only if $m \le \nu_2$, and if $m > \nu_1$ or ν_2 then $\nu_1 = \nu_2$ is named ν, and m is bounded by $\min\big(2n_1 + \mathrm{ord}\, D - \nu, 2n_2 + \mathrm{ord}(AD) - \nu\big) = \chi - \nu$. On this last range we have the relation $u_1/u_2 \in (B_1/B_2)R^{\times 2}$. Then the cardinality of the ρ's is 2, instead of 4, on this range. Then the stable θ-orbital integral of 1_K at a strongly θ-regular topologically unipotent element of type (II) is:

$$\begin{aligned}
&4q_0^{n_1+n_2}[\sum_{0\le\nu_1\le n_1} q_0^{-\nu_1} \sum_{0\le\nu_2\le n_2} q_0^{-\nu_2} + (1 - q_0^{-2}) \sum_{0\le\nu_1\le n_1} q_0^{-\nu_1} \sum_{0\le\nu_2\le n_2} q^{-\nu_2} \sum_{1\le m\le\min(\nu_1,\nu_2)} q_0^{3m} \\
&\quad + \sum_{0\le\nu\le\min(n_1,n_2)} \sum_{\nu<m\le\chi-\nu} q_0^m] \\
&= 4q_0^{n_1+n_2}[\frac{1 - q_0^{-n_1-1}}{1 - q_0^{-1}} \cdot \frac{1 - q_0^{-n_2-1}}{1 - q_0^{-1}} \\
&\quad + q_0(q_0^2 - 1) \sum_{\substack{0\le\nu_1\le n_1\\0\le\nu_2\le n_2}} q_0^{-\nu_1} q_0^{-\nu_2} \frac{q_0^{3\min(\nu_1,\nu_2)} - 1}{q_0^3 - 1} + \sum_{0\le\nu\le\min(n_1,n_2)} \frac{q_0^{\chi-\nu+1} - q_0^{\nu+1}}{q_0 - 1}].
\end{aligned}$$

Assume (without loss of generality) that $n_1 \le n_2$, and note that our expression is precisely that of case (I) for ramified E/F. The lemma follows. $\square$

2. Lemma. *When E/E_3 is ramified, the stable orbital integral of 1_K at a topologically unipotent regular element of type (II) in $GSp(2,F)$, is*

$$(**)\qquad 2q_0^{2N+1}\frac{1-q_0^{-N-1}}{1-q_0^{-1}}\Big(\frac{q_0^X}{q_0-1}-\frac{q_0^{N+1}}{q_0^3-1}-\frac{1+q_0^{-N-1}}{q_0^3-1}\Big).$$

Proof. The integral is the sum over $\rho=u\in E_3^\times/N_{E/E_3}E^\times$, which can be assumed to be 1 and a (non square) unit in $R_3^\times$ in the case where E/E_3 is ramified, $e=2$. Then we have a sum over $0\le\nu\le N$ and a sum over $m(\le X-\nu)$. Note that $m'=m$ and $q=q_0^2$ when $e=2$. Also, in the range $\nu<m\le X-\nu$, we have that $u\in BR_3^{\times 2}$, namely the sum over $\rho=u$ reduces to a single term. The stable orbital integral is then

$$\sum_{0\le\nu\le N}q^{N-\nu}\big(2+2\sum_{1\le m\le\nu}(1+q^{-1})q_0^{3m}+\sum_{\nu<m\le X-\nu}eq_0^mq^\nu\big)$$

$$=2q^N\Big[\frac{1-q^{-N-1}}{1-q^{-1}}+\frac{(q+1)q_0}{q_0^3-1}\sum_{0\le\nu\le N}q^{-\nu}(q^{3\nu/2}-1)+\frac{1}{q_0-1}\sum_{0\le\nu\le N}(q_0^{X-\nu+1}-q_0^{\nu+1})\Big]$$

$$=2q^N\Big[\frac{1-q^{-N-1}}{1-q^{-1}}+\frac{(1+q)q_0}{q_0^3-1}\Big(\frac{q_0^{N+1}-1}{q_0-1}-\frac{1-q^{-N-1}}{1-q^{-1}}\Big)$$
$$+\frac{1}{q_0-1}\Big(q_0^{X+1}\frac{1-q_0^{-N-1}}{1-q_0^{-1}}-q_0\frac{q_0^{N+1}-1}{q_0-1}\Big)\Big]$$

$$=2q_0^{2N}\cdot\frac{1-q_0^{-N-1}}{1-q_0^{-1}}\Big[\frac{1+q_0^{-N-1}}{1+q_0^{-1}}+\frac{q_0^3+q_0}{q_0^3-1}\Big(q_0^N-\frac{1+q_0^{-N-1}}{1+q_0^{-1}}\Big)+\frac{1}{q_0-1}\big(q_0^{X+1}-q_0^{N+1}\big)\Big].$$

The $[\dots]$ here is

$$\frac{1}{q_0-1}q_0^{X+1}+q_0^{N+1}\Big(\frac{q_0^2+1}{q_0^3-1}-\frac{1}{q_0-1}\Big)+\frac{1+q_0^{-N-1}}{1+q_0^{-1}}\Big(1-\frac{q_0^3+q_0}{q_0^3-1}\Big).$$

Hence our stable integral is as stated in the lemma. $\square$

Since we are evaluating our stable integral at the stable orbit of Nu or Nt^*, we can take $X=1+n_1+n_2$, and $N=n_1$ if $n_1\le n_2$, as we assume. Then the stable integral is

$$=\frac{2q_0^{2n_1+2}}{(q_0-1)^2}(1-q_0^{-n_1-1})\big(q_0^{1+n_1+n_2}-\frac{q_0-1}{q_0^3-1}(q_0^{n_1+1}+1+q_0^{-n_1-1})\big)$$
$$=\frac{2q_0^{2+n_1+n_2}(1-q_0^{-n_1-1})}{(q_0-1)^2}\big(q_0^{1+2n_1}-\frac{q_0-1}{q_0^3-1}q_0^{-n_2-1}(1+q_0^{n_1+1}+q_0^{2n_1+2})\big).$$

Multiplied by 2, the stable orbital integral is equal to the stable θ-orbital integral computed above, since $\chi=1+2n_1$ as $n_1\le n_2$.

Thus it remains to show

3. Lemma. *The measure factor* $[T^{*\theta}(R) : (1+\theta)\big(T^*(R)\big)]/[T^*_H(R) : N\big(T^*(R)\big)]$ *is equal to* 2 *for tori* T *of type (II).*

Proof. The norm map N takes $(a, b, \sigma b, \sigma a)$ in $T^*(R)$, thus $a \in R_1^\times, b \in R_2^\times$ to $(ab, a\sigma b, b\sigma a, \sigma a\sigma b)$ in $T^*_H(R)$, which consists of $(x, \tau x, \sigma\tau x, \sigma x), x \in R_E^\times$ with $x\sigma x = \tau(x\sigma x) \in R_1^\times$. Thus we need to solve in $a \in R_1^\times$ the equation $a/\sigma a = x/\sigma\tau x\big(= \tau(x/\sigma\tau x), \in E_1^1 = \{y/\sigma y; y \in E_1^\times\}\big)$.

As in the proof of the corresponding Lemma for tori of type (I), we have $[E_1^1 : R_1^1] = e(E_1/F)$. Put $b = x/a$. Then $\sigma\tau(b) = \sigma\tau(x/a) = \sigma\tau(x)/\sigma a = x/a = b$ lies in $R_2^\times$. Hence $[T^*_H(R) : N\big(T^*(R)\big)] = e(E_1/F)$.

Next we compute the index in $T^{*\theta}(R) = \{(x, y, \sigma y, \sigma x); x \in R_1^1, y \in R_2^1$ (thus $y \in R_2^\times, y\sigma y = 1)\}$ of $(1+\theta)T^*(R) = \{(1+\theta)(a, b, \sigma b, \sigma a) = (a/\sigma a, b/\sigma b, \sigma b/b, \sigma a/a), a \in R_1^\times, b \in R_2^\times\}$. Since E_2/F is ramified, $[E_2^1 : R_2^1] = e(E_2/F) = 2$, we conclude that $[T^{*\theta}(R) : (1+\theta)T^*(R)] = 2e(E_1/F)$. The quotient by $e(E_1/F)$ is 2, and the lemma follows. □

Unstable twisted case. Twisted endoscopic group of type I.F.2.

The explicit computation of the θ-orbital integrals permits us to compute the unstable, κ-θ-orbital integrals, too. Let κ be the character which defines the endoscopic group $\mathbf{C}_3 = \mathbf{C}_{E_3}$. It is a character on the group of θ-conjugacy classes within the stable θ-conjugacy class of the topologically unipotent element $t^* = (t_1, t_2, \sigma t_2, \sigma t_1)$ of type (II). This group is $F^\times/N_{E_1/F}E_1^\times \times F^\times/N_{E_2/F}E_2^\times$, so κ is a product $\kappa_1 \times \kappa_2$. As $E_2 = F(\sqrt{-\boldsymbol{\pi}})$, we have $N_{E_2/F}E_2^\times = \boldsymbol{\pi}^{\mathbb{Z}}R^{\times 2}$, hence $\kappa_2(\boldsymbol{\pi}) = 1$ and $\kappa_2(\varepsilon_0) = -1$, where $\varepsilon_0 \in R^\times - R^{\times 2}$. Further, when E_1/F is ramified, $E_1 = F(\sqrt{-\varepsilon_0\boldsymbol{\pi}})$, hence $N_{E_1/F}E_1^\times = (\varepsilon_0\boldsymbol{\pi})^{\mathbb{Z}}R^{\times 2}$, and so $\kappa_1(\varepsilon_0) = \kappa_1(\boldsymbol{\pi}) = -1$. This defines the quadratic characters $\kappa_i \neq 1$, and κ. The Jacobian factor is (when $|t_1| = |t_2| = 1$, $e = 2$)

$$\Delta_{G,C_3}(t_1, t_2, \sigma t_2, \sigma t_1) = |\frac{(t_1 - \sigma t_1)^2(t_2 - \sigma t_2)^2}{t_1\sigma t_1 \cdot t_2\sigma t_2}|_F^{1/2} = |\beta_1\beta_2 D\sqrt{A}|_F = q_0^{-1-n_1-n_2}.$$

Theorem. *If* $t = h^{-1}t^*\theta(h)$ *is a strongly* θ*-regular topologically unipotent element of type (II),* E_3/F *is unramified, and* κ *is the character associated with the endoscopic group* $\mathbf{C}_3$*, then*

$$-\Delta_{G,C_3}(t^*)\kappa_1\big((t_1 - \sigma t_1)/2\sqrt{D}\big)\kappa_2\big((t_2 - \sigma t_2)/2\sqrt{AD}\big)\Phi^\kappa_{1_K}(t\theta) = \Phi^{C_3}_{1_{K_3}}(N_{C_3}t^*).$$

When E_3/F *is ramified,* $\Phi^\kappa_{1_K}(t\theta) = 0$.

Proof. The last assertion is proven in the next section. Suppose E_3/F is unramified. Then the $\kappa_1\kappa_2$ factor on the left is $\kappa_1(\beta_1)\kappa_2(\beta_2) = \kappa_1(B_1)\kappa_1(\boldsymbol{\pi}^{n_1})\kappa_2(B_2) = \kappa_0(B_1B_2)(-1)^{n_1}$, where κ_0 is the non trivial character on $R^\times/R^{\times 2}$. Recall that ρ_m is ρ if m is even, but it is $\tilde\rho = (\tilde\rho_1, \tilde\rho_2)$ if m is odd, where $\rho_i \mapsto \tilde\rho_i = -1/\rho_i$ and $\rho_2 \mapsto \tilde\rho_2 = -1/A\rho_2$ are permutations of $R^\times/R^{\times 2}$ if E/F is ramified. Hence in our sum

$$\sum_\rho \kappa_1(\rho_1)\kappa_2(\rho_2)\sum_{m\geq 0}\sum_{r\in R_{\rho_m}}[R_T^1 : T^1_{\rho_m} \cap rK_0r^{-1}]\#L^1_{m,\rho_m},$$

replacing ρ_m by ρ does not change the index $[\dots]$, but it affects the part of the factor $\#L^1_{m,\rho_m}$ described by Lemma G.3(5): the corresponding summands will have to be multiplied by $(-1)^m$.

The κ-θ-orbital integral is the sum of

$$\Big(\sum_{u_1,u_2\in R^\times/R^{\times 2}} \kappa_0(u_1u_2)\Big) \sum_{\substack{0\le\nu_1\le n_1\\ 0\le\nu_2\le n_2}} q_0^{n_1-\nu_1+n_2-\nu_2}\big(\delta(m=0)+(1-q_0^{-2})\sum_{1\le m\le\min(\nu_1,\nu_2)} q_0^{3m}\big),$$

which is zero, and

$$\sum_{0\le\nu\le n} q_0^{n_1+n_2-2\nu} \sum_{\nu<m\le\chi-\nu} 2q_0^{m+2\nu} \sum_{u_1,u_2} \kappa_0(u_1u_2)(-1)^m.$$

Here $n=\min(n_1,n_2)$, and u_1,u_2 range over $R^\times/R^{\times 2}$, subject to the relation (Lemma G.3) that $u_1u_2\in B_1B_2\varepsilon_0^{n_1-\nu}$ (there are two such pairs). The factor $(-1)^m$ comes from from changing R_{ρ_m} to R_ρ. The last displayed sum is then

$$\begin{aligned}&4\kappa_0(B_1B_2)(-q_0)^{n_1}q_0^{n_2}\sum_{0\le\nu\le n}(-1)^\nu\frac{q_0}{q_0+1}((-q_0)^{\chi-\nu}-(-q_0)^\nu)\\ &\quad=4(q_0+1)^{-1}\kappa_0(B_1B_2)(-1)^{n_1}q_0^{1+n_1+n_2}\sum_{0\le\nu\le n}[(-q_0)^\chi q_0^{-\nu}-q_0^\nu].\end{aligned}$$

The last sum is

$$\begin{aligned}&(-q_0)^\chi\big(1-q_0^{-n-1}\big)/(1-q_0^{-1})-\big(q_0^{n+1}-1\big)/(q_0-1)\\ &\quad=\big(q_0^{n+1}-1\big)\big(q_0^{\chi-n}(-1)^\chi-1\big)/(q_0-1).\end{aligned}$$

The left side of the expression of the theorem is then (note that $\chi=2n+1, q=q_0^2$) $-4(q-1)^{-1}(q^{n+1}-1)$. The measure factor is 4, and the right hand side is an orbital integral on $GL(2,E_3)$ at the elliptic element with eigenvalues $x_1,\sigma x_1$, with parameter $N=n$. Since E/E_3 is ramified, by Lemma I.I.2 this orbital integral is $(q^{N+1}-1)/(q-1)$, and we are done. $\square$

Remark. If E_3/F is unramified and precisely one of κ_1,κ_2, is non trivial, the same computation shows that the associated κ-θ-orbital integral is zero. Such κ defines the ramified twisted endoscopic group $\mathbf{C}_+$ of type (3) in Section I.F (namely $C_+=GL(2,F)\times E_1^1$, and E_1/F is ramified). This verifies the "ramified" claim of the Theorem of the next Section.

J. Comparison in case (II), E/E_3 **unramified** $(e=1)$.

In this case E_1/F is unramified, E_3/F is ramified and so $q=q_0$, and the stable θ-orbital integral is given by a summation over $0\le\nu_1\le n_1$ and $0\le\nu_2\le n_2$, over $\varepsilon\in R^\times/R^{\times 2}$ if $\nu_1<n_1$, over $\rho_2=u_2\in R^\times/R^{\times 2}$, and $\rho_1=\boldsymbol{\pi}^{\overline{\rho}_1}(u_1=1)$, $\overline{\rho}_1$ is 0 or 1, subject to the condition that $j_1-\overline{\rho}_1$ be even. Further, when $\nu_1<m$ or $\nu_2<m$ then $\nu_1=\nu_2$ is denoted by ν, and $\varepsilon/u_2\in(B_1/B_2)R^{\times 2}$, and $m\le\min\big(2n_1-\nu+\operatorname{ord}D, 2n_2-\nu+\operatorname{ord}(AD)\big)$. Here $D\in R^\times$ and $A=\boldsymbol{\pi}_F$, and as we saw $\chi=\min(2n_1,1+2n_2)$, so $m\le\chi-\nu$ when $\nu<m$. Then we obtain the following.

1. Lemma. *The stable θ-orbital integral of 1_K at a topologically unipotent strongly θ-regular element u of type (II) is given – when E/E_3 is unramified – by*

$$(*)\qquad 2(q-1)^{-2}[(q+1)q^{2+n_1+3n_2}-(q+1)q^{1+n_1+2n_2}-\frac{2(q-1)}{q^3-1}(q^{3n_2+3}-1)]$$

when $n_2<n_1$, and by

$$(**)\qquad 2(q-1)^{-2}[-2q^{1+2n_1+n_2}+(q+1)q^{1+3n_1+n_2}+\frac{q-1}{q^3-1}\big(2-(q^3+1)q^{3n_1}\big)]$$

when $n_1\le n_2$.

Proof. Recall that L^1_{m,ρ_m} depends on m and ρ_m, but for each m, the set $\{\rho_m\}$ is the same as the set of ρ. Hence we replace ρ_m by ρ in the triple sum above. Our integral is the sum of

$$2\sum_{\substack{0\le\nu_1\le n_1\\0\le\nu_2\le n_2}}q^{n_1-\nu_1+n_2-\nu_2}\big(\delta(\nu_1=n_1)+\delta(\nu_1<n_1)2\cdot\frac{q+1}{2q}\big)\big[1+\sum_{1\le m\le\min(\nu_1,\nu_2)}(1-q^{-2})q^{3m}\big]$$

and

$$2\sum_{0\le\nu\le n}q^{n_1+n_2-2\nu}\big[\delta(\nu_1=n_1)+\frac{q+1}{2q}\delta(\nu_1<n_1)\big]\sum_{\nu<m\le\chi-\nu}2q^{m+2\nu},$$

where $n=\min(n_1,n_2)$. The first sum can also be written as

$$2\sum_{0\le m\le n}\#L^1_m\big(\sum_{m\le\nu_2\le n_2}q^{n_2-\nu_2}\big)\big[\sum_{\nu_1=n_1}1+(1+q^{-1})\sum_{m\le\nu_1<n_1}q^{n_1-\nu_1}\big]$$

$$\begin{aligned}=&2[(q-1)^{-2}(q^{n_2+1}-1)\big((q+1)q^{n_1}-2\big)\\&+(1-q^{-2})\sum_{1\le m\le n}q^{3m}\cdot\frac{q^{n_2-m+1}-1}{q-1}\cdot\frac{(q+1)q^{n_1-m}-2}{q-1}]\end{aligned}$$

$$\begin{aligned}=&2(q-1)^{-2}[(q+1)q^{n_1+n_2+1}-(q+1)q^{n_1}-2q^{n_2+1}+2\\&+(1-q^{-2})\sum_{1\le m\le n}\big((q+1)q^{n_1+n_2+m+1}-(q+1)q^{n_1+2m}-2q^{n_2+2m+1}+2q^{3m}\big)].\end{aligned}$$

The inner sum here is

$$\frac{q+1}{q-1}q^{2+n_1+n_2}(q^n-1)-\frac{q+1}{q^2-1}q^{n_1+2}(q^{2n}-1)-2\frac{q^{3+n_2}}{q^2-1}(q^{2n}-1)+\frac{2q^3}{q^3-1}(q^{3n}-1),$$

so we get

$$\begin{aligned}2(q-1)^{-2}[&(q+1)q^{n_1+n_2+1}-(q+1)q^{n_1}-2q^{n_2+1}+2+(q+1)^2q^{n_1+n_2}(q^n-1)\\&-(q+1)q^{n_1}(q^{2n}-1)-2q^{n_2+1}(q^{2n}-1)+\frac{2q(q^2-1)}{q^3-1}(q^{3n}-1)].\end{aligned}$$

To compute the second sum, note that $m > \nu_1$ if and only if $m > \nu_2$, and then $\nu_1 = \nu_2$ is denoted by ν, and there are two possibilities. If $n_1 \leq n_2$ then $\chi = 2n_1$, and there is no m with $n_1 < m \leq \chi - n_1$; hence $\nu < n = n_1$ in this case. If $n_2 < n_1$ then $\chi = 1 + 2n_2$ and $n = n_2$, and the m with $n_2 < m \leq 1 + n_2$ is $m = 1 + n_2$. Hence the second sum takes the form

$$2(q+1)q^{n_1+n_2} \sum_{0 \leq \nu < n} \frac{q^{\chi-\nu} - q^\nu}{q-1} + 2\delta(n_2 < n_1) q^{n_1+n_2}(q+1)q^{n_2}$$

$$= 2\frac{q+1}{q-1} q^{n_1+n_2} \big(q^\chi \frac{1-q^{-n}}{1-q^{-1}} - \frac{q^n - 1}{q-1}\big) + 2\delta(n_2 < n_1)(q+1)q^{n_1+2n_2}$$

$$= 2\frac{q+1}{(q-1)^2} q^{n_1+n_2}(q^n - 1)(q^{\chi+1-n} - 1) + 2\delta(n_2 < n_1)(q+1)q^{n_1+2n_2}.$$

We deal separately with the two cases. When $n_2 < n_1, \chi = 1 + 2n_2$ and $n = n_2$, thus $\chi + 1 - n = 2 + n_2$, our integral is

$$\begin{gathered} 2(q-1)^{-2}[(q+1)q^{n_1+n_2+1} - (q+1)q^{n_1} - 2q^{n_2+1} + 2 \\ +(q+1)^2 q^{n_1+2n_2} - (q+1)^2 q^{n_1+n_2} - (q+1)q^{n_1+2n_2} \\ +(q+1)q^{n_1} - 2q^{1+3n_2} + 2q^{n_2+1} + \frac{2q(q^2-1)}{q^3-1}(q^{3n_2} - 1) \\ +(q+1)\big(q^{2+n_1+3n_2} - q^{2+n_1+2n_2} - q^{n_1+2n_2} + q^{n_1+n_2} + (q-1)^2 q^{n_1+2n_2}\big)]. \end{gathered}$$

Collecting the coefficients of $q^{n_1+n_2}, q^{n_1+2n_2}, q^{n_1+3n_2}$, we obtain $(*)$ of the lemma.

When $n_1 \leq n_2, \chi = 2n_1 (= N), n = n_1, \chi + 1 - n = 1 + n_1$, the integral is equal to

$$\begin{aligned} 2(q-1)^{-2}[&(q+1)q^{n_1+n_2+1} - (q+1)q^{n_1} - 2q^{n_2+1} + 2 + (q+1)^2 q^{n_1+n_2}(q^{n_1} - 1) \\ &- (q+1)q^{n_1}(q^{2n_1} - 1) - 2q^{n_2+1}(q^{2n_1} - 1) \\ &+ \frac{2q(q^2-1)}{q^3-1}(q^{3n_1} - 1) + (q+1)q^{n_1+n_2}(q^{2n_1+1} - q^{n_1+1} - q^{n_1} + 1)]. \end{aligned}$$

Collecting the coefficients of $q^{n_1+n_2}, q^{2n_1+n_2}, q^{3n_1+n_2}, q^{3n_1}$, we obtain $(**)$ of the lemma. $\square$

To complement Lemma 1, we need to compute the stable orbital integral of 1_K at the norm Nu, which is a topologically unipotent regular element in $GSp(2, F)$ of type (II), in our case $e = 1$, that is E/E_3 is unramified, $q = q_0$.

2. Lemma. *The stable orbital integral of 1_K at the topologically unipotent regular element Nu in $GSp(2, F)$ of type (II), when E/E_3 is unramified, is given by*

$$\begin{aligned} \frac{q+1}{(q-1)^2} q^N [\delta(2|N)\big(q^{\frac{X+1}{2}} - \frac{2}{q+1} q^{\frac{X+1-N}{2}} + \frac{q-1}{q+1} q^{\frac{N}{2}}\big) + \big(1 - \delta(2|N)\big) q^{\frac{X-N}{2}}(q^{\frac{N+1}{2}} - 1)] \\ - \frac{2}{q-1} \frac{q^{3([N/2]+1)} - 1}{q^3 - 1}. \end{aligned}$$

Proof. Here ρ ranges over $E_3^\times/N_{E/E_3}E^\times$, thus $\rho = \boldsymbol{\pi}_3^{\overline{\rho}}$ with $\overline{\rho} = 0, 1$. There is a sum over $\nu (0 \le \nu \le N)$ such that $N - \nu - \overline{\rho}$ is even, so the sums over $\overline{\rho}$ and ν are combined to a single sum over $\nu(0 \le \nu \le N)$. Further, we have a sum over the even $m'(= 2m)$ with $0 \le m' \le X$, but $X = 1 + 2n_1 + 2n_2$ when $e = 1$, thus $0 \le m' \le X - 1$. When $\nu < m'$ we have that $m' \le X - \nu$, and ν is even; thus $\nu < m' \le X - 1 - \nu$, as ν, m' are even and X is odd. The stable integral is then

$$\sum_{0\le\nu\le N}\ \sum_{0\le m'=2m\le X-1} q^{N-\nu}\big(\delta(\nu=N) + (1+q^{-1})\delta(\nu<N)\big)$$
$$\big(q^{3m'/2}\delta(0\le m'\le\nu) + q^{\nu+m'/2}\delta(\nu<m'\le X-1-\nu, \nu \text{ even})\big).$$

It is the sum of

$$\begin{aligned}\sum_{0\le\nu\le N}\ \sum_{0\le m'\le\nu} &= \sum_{0\le m'\le N} q^{3m'/2}\Big(\sum_{\nu=N} 1 + (1+q^{-1})\sum_{m'\le\nu<N} q^{N-\nu}\Big)\\
&= \sum_{0\le m\le N/2} q^{3m}\Big(1 + \frac{q+1}{q-1}(q^{N-2m}-1)\Big)\\
&= \frac{q+1}{(q-1)^2}q^N(q^{[N/2]+1}-1) - \frac{2}{q-1}\frac{q^{3([N/2]+1)}-1}{q^3-1},\end{aligned}$$

and, writing $\nu' = 2\nu$ for the even ν when $\nu < m' \le X - 1 - \nu$,

$$\sum_{0\le\nu'<N}\ \sum_{\nu'<m'\le X-1-\nu'} q^{\nu'+m'/2}\cdot q^{N-\nu'}(1+q^{-1}) + \delta(N \text{ is even})\cdot \sum_{N<m'\le X-1-N} q^{N+m'/2}$$

$$\begin{aligned}&= \sum_{0\le\nu<N/2}\ \sum_{\nu<m\le\frac{X-1}{2}-\nu} q^{N+m}(1+q^{-1}) + \delta(2|N)\sum_{N/2<m\le\frac{X-1}{2}-\frac{N}{2}} q^{N+m}\\
&= q^N\frac{q+1}{q-1}\sum_{0\le\nu<N/2}(q^{\frac{X-1}{2}-\nu} - q^\nu) + \delta(2|N)q^N(q-1)^{-1}\big(q^{\frac{X+1}{2}-\frac{N}{2}} - q^{\frac{N}{2}+1}\big)\\
&= \delta(2|N)\frac{q^N}{q-1}\Big[\frac{q+1}{q-1}\big(q^{\frac{X+1}{2}}(1-q^{-N/2}) - (q^{N/2}-1)\big) + q^{\frac{X+1}{2}-\frac{N}{2}} - q^{\frac{N}{2}+1}\Big]\\
&\quad + \big(1-\delta(2|N)\big)\frac{q^N}{q-1}\cdot\frac{q+1}{q-1}\big(q^{\frac{X+1}{2}}(1-q^{-\frac{N+1}{2}}) - (q^{\frac{N+1}{2}}-1)\big).\end{aligned}$$

This completes the proof of the lemma. □

We can now complete the comparison of the θ-stable and stable integrals.

When N is odd $\big(\delta(2|N) = 0\big)$, since $N = \min(2n_1, 1 + 2n_2)$, we have that $N = 1 + 2n_2$, and $n_2 < n_1$, as well as $(X - 1)/2 = n_1 + n_2$, so that we obtain

$$\frac{q+1}{(q-1)^2}q^{1+2n_2+n_1}(q^{n_2+1}-1) - \frac{2}{q-1}\frac{q^{3(n_2+1)}-1}{q^3-1},$$

which is half the expression for the stable θ-orbital integral when $n_2 < n_1$.

When N is even, $N = 2n_1, n_1 \leq n_2$, the stable orbital integral is

$$\frac{q+1}{(q-1)^2}[q^{1+3n_1+n_2} - \frac{2}{q+1}q^{1+2n_1+n_2}] + \frac{1}{q-1}[\frac{2-2q^{3n_1+3}}{q^3-1} + q^{3n_1}],$$

which is equal to half the expression for the stable θ-orbital integral when $n_1 \leq n_2$. Since the measure factor is equal to 2, the comparison is complete in the case of type (II).

Proof of Theorem I when E_3/F is ramified. The computations are the same as in the stable case of Lemma 1, except that both κ_i are now non trivial. In this case E_1/F is unramified, thus $\boldsymbol{\pi}_1 = \boldsymbol{\pi}$, hence $N_{E_1/F}E_1^\times$ is $R^\times \boldsymbol{\pi}^{2\mathbb{Z}}$, and $\kappa_1(u\boldsymbol{\pi}^{j_1}) = (-1)^{j_1}(u \in R^\times)$. The κ-θ-orbital integral is then the sum of

$$\sum_{\nu_1,\nu_2}(-q_0)^{n_1-\nu_1}q_0^{n_2-\nu_2}\sum_{u_2}\kappa_2(u_2)\big[\delta(\nu_1=n_1)+(1+q_0^{-1})\delta(\nu_1<n_1)\big]$$
$$\big[1+(1-q^{-2})\sum_{1\leq m\leq \min(\nu_1,\nu_2)}(-q_0)^{3m}\big],$$

$(0 \leq \nu_i \leq n_i)$, which is 0 since u_2 ranges over $R^\times/R^{\times 2}$, and

$$\sum_{0\leq\nu\leq n}(-1)^{\nu-n_1}q^{n_1+n_2-2\nu}\big[\sum_{u_2}\kappa_2(u_2)\cdot\delta(\nu_1=n_1)$$
$$+\sum_{u_2}\kappa_2(u_2)\sum_{\varepsilon\in u_2B_1B_2R^{\times 2}}\frac{1}{2}(1+q^{-1})\delta(\nu_1<n_1)\big]\sum_{\nu<m\leq\chi-\nu}2(-q)^{m+2\nu}$$

which is also zero (since ε is determined by u_2, leaving us with the sum $\sum_{u_2}\kappa_2(u_2)$ over $R^\times/R^{\times 2}$, which is zero). $\square$

Remark. If E_1/F is unramified, $\kappa_1 = 1$ and $\kappa_2 \neq 1$, the corresponding κ-θ-orbital integral is zero by the same argument. The only change will be that the powers of (-1) – introduced by $\kappa_1 \neq 1$ – need to be replaced by 1.

Unstable twisted case. Twisted endoscopic group of type I.F.3.

The explicit computation of the θ-orbital integrals can be used to compute the unstable κ-θ-orbital integrals, at a strongly θ-regular topologically θ-unipotent element $t^* = (t_1, t_2, \tau t_2, \sigma t_1)$ (thus $t^*\theta$ is topologically unipotent) of type (II). The character κ is defined on the group $F^\times/N_{E_1/F}E_1^\times \times F^\times/N_{E_2/F}E_2^\times$ of θ-conjugacy classes within the stable θ-conjugacy class of t^*. Thus $\kappa = \kappa_1 \times \kappa_2$, $\kappa_1 \neq 1$ on $F^\times/N_{E_1/F}E_1^\times$, and $\kappa_2 = 1$. The stable case is that where $\kappa_i = 1, i = 1, 2$. The endoscopic group associated with κ is $\mathbf{C}_+$, with $C_+ = GL(2,F) \times E_1^1$. As noted at the end of Section I.F, the $GL(2)$-part of the norm $N_{C_+}t^*$ is $\mathrm{diag}(t_2, \tau t_2)$. Recall that $t_2 \in E_2^\times$, and E_2/F is ramified. Hence by Lemma I.I.2, the orbital integral $\Phi_{1_K}(\mathrm{diag}(t_2, \sigma t_2))$ is equal to $(q|\beta_2|^{-1}-1)/(q-1) = (q^{n_2+1}-1)/(q-1)$. As usual, $t_1 = \alpha_1 + \beta_1\sqrt{D}$ and $t_2 = \alpha_2 +$

$\beta_2\sqrt{AD}$ are units, and $|\beta_i| = q^{-n_i}$. Then $\Delta_{G,C_+}(t^*) = |(t_1-\sigma t_1)(t_1t_2-\sigma t_1\tau t_2)(t_1\tau t_2-t_2\sigma t_1)|_F$ (recall that this factor is computed at the end of Section I.G) is equal to $|D\sqrt{D}\beta_1((\alpha_2\beta_1)^2 - (\alpha_1\beta_2)^2A)|$ (recall that $E_1 = F(\sqrt{D}) = E^\tau$ and $E_3 = F(\sqrt{A}) = E^\sigma$). As noted in the Remark at the end of Section I, the κ-θ-orbital integral vanishes when E_1/F is ramified. Assume that E_1/F is unramified. Then $\Delta_{G,C_+}(t^*)$ is $q^{-n_1-2n_2-1}$ if $n_2 < n_1$, but it is q^{-3n_1} if $n_2 \geq n_1$. We claim the following.

Theorem. *Let t^* be a topologically θ-unipotent strongly θ-regular element of type (II), E_1/F is unramified, $\kappa_1 \neq 1$ and $\kappa_2 = 1$. Then the κ-θ-orbital integral of 1_K is related to the orbital integral of $1_{K_{C_+}}$ on the twisted endoscopic group of type (3) of Section I.F, by*

$$\kappa_1\big(((t_1-\sigma t_1)/\sqrt{D})(t_1t_2-\sigma t_1\tau t_2)(t_1\tau t_2-t_2\sigma t_1)\big)\Delta_{G,C_+}(t^*)\Phi^{\kappa}_{1_K}(t^*\theta) = \Phi^{C_+}_{1_{K_{C_+}}}(N_{C_+}t^*).$$

When E_1/F is ramified, $\Phi^{\kappa}_{1_K}(t^*\theta) = 0$.

Proof. The computations are the same as in the stable case of Lemma 1, except that now $\kappa_1 \neq 1$ and $\kappa_2 = 1$. Recall that $q = q_0$. Our integral is the sum of

$$2\sum_{\substack{0\leq\nu_1\leq n_1\\ 0\leq\nu_2\leq n_2}}(-q)^{n_1-\nu_1}q^{n_2-\nu_2}\big(\delta(\nu_1=n_1)+\delta(\nu_1<n_1)2\cdot\frac{q+1}{2q}\big)\big[1+\sum_{1\leq m\leq\min(\nu_1,\nu_2)}(1-q^{-2})(-q)^{3m}\big]$$

and

$$2\sum_{0\leq\nu\leq n}(-q)^{n_1-\nu}q^{n_2-\nu}\big[\delta(\nu_1=n_1)+\frac{q+1}{2q}\delta(\nu_1<n_1)\big]\sum_{\nu<m\leq\chi-\nu}2(-q)^{m+2\nu},$$

where $n = \min(n_1,n_2)$. The first sum can also be written as

$$2\sum_{0\leq m\leq n}(-1)^m\#L^1_m\big(\sum_{m\leq\nu_2\leq n_2}q^{n_2-\nu_2}\big)\big[\sum_{\nu_1=n_1}1+(1+q^{-1})\sum_{m\leq\nu_1<n_1}(-q)^{n_1-\nu_1}\big].$$

Here $[\dots] = (-q)^{n_1-m}$. Hence we get

$$= 2(q-1)^{-1}(-q)^{n_1}[q^{n_2+1}-1+(q^2-1)q^{-2}\sum_{1\leq m\leq n}(q^{n_2+1}q^m-q^{2m})]$$

$$= 2(q-1)^{-1}(-q)^{n_1}[q^{n+n_2}(q+1)-q^{2n}-q^{n_2}].$$

To compute the second sum, note that $m > \nu_1$ if and only if $m > \nu_2$, and then $\nu_1 = \nu_2$ is denoted by ν, and there are two possibilities. If $n_1 \leq n_2$ then $\chi = 2n_1$, and there is no m with $n_1 < m \leq \chi - n_1$; hence $\nu < n = n_1$ in this case. If $n_2 < n_1$ then $\chi = 1 + 2n_2$ and $n = n_2$, and the m with $n_2 < m \leq 1 + n_2$ is $m = 1 + n_2$. Hence the second sum takes the form

$$-2(q+1)q^{n_1+n_2}\sum_{0\leq\nu<n}\frac{(-q)^{\chi-\nu}-(-q)^{\nu}}{-q-1}(-1)^{n_1-\nu}+2\delta(n_2<n_1)q^{n_1+n_2}(q+1)q^{n_2}(-1)^{n_1+1}$$

$$= 2(-1)^{n_1} q^{n_1+n_2}\big((-q)^{\chi}\frac{1-q^{-n}}{1-q^{-1}} - \frac{q^n-1}{q-1}\big) + 2\delta(n_2<n_1)(q+1)q^{n_1+2n_2}(-1)^{n_1+1}$$

$$= 2(-1)^{n_1} q^{n_1+n_2}(q^n-1)((-q)^{\chi}q^{1-n}-1) + 2\delta(n_2<n_1)(q+1)q^{n_1+2n_2}(-1)^{n_1+1}.$$

We deal separately with the two cases. When $n_2 < n_1, \chi = 1+2n_2$ and $n = n_2$, thus our integral is the sum of $2(q-1)^{-1}(-q)^{n_1}q^{2n_2}[q-q^{-n_2}]$ and $-2(q-1)^{-1}(-q)^{n_1}q^{2n_2}[q^{n_2+2}-q^2+1-q^{-n_2}+q^2-1]$. Namely it is $-2(q-1)^{-1}(-q)^{n_1}q^{1+2n_2}[q^{n_2+1}-1]$.

When $n_1 \le n_2, \chi = 2n_1 (= N), n = n_1, \chi+1-n = 1+n_1$, the integral is equal to $2(q-1)^{-1}(-q)^{n_1}[q^{n_1+n_2}(q+1) - q^{2n_1} - q^{n_2} + q^{n_2}(q^{n_1}-1)(q^{1+n_1}-1)]$. This is equal to $2(q-1)^{-1}(-q)^{3n_1}[q^{1+n_2}-1]$.

This κ-θ-orbital integral relates to the orbital integral of $1_{K_{C_+}}$ on the twisted endoscopic group of type (3) of Section I.F as asserted in the theorem in view of the observations stated prior to the statement of the theorem.. $\square$

K. Endoscopy for $GSp(2)$, type (II).

In the case of tori of type (II) the isomorphism $\mathbf{T}_0 \to \mathbf{T}_H$ yields a map of F-rational points $T_0 \to T_H$, determined by $\lambda : \big(\left(\begin{smallmatrix} t_1 & 0 \\ 0 & \sigma t_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} t_2 & 0 \\ 0 & \tau t_2 \end{smallmatrix}\right)\big) \mapsto \operatorname{diag}(x_1 = t_1 t_2, \tau x_1 = t_1 \tau t_2, \sigma\tau x_1 = \sigma t_1 \cdot t_2, \sigma x_1 = \sigma t_1 \cdot \tau t_2)$. Here $t_1 \in E_1^\times, E_1 = F(\sqrt{D}) = E^\tau$, and $t_2 \in E_2^\times, E_2 = F(\sqrt{AD}) = E^{\sigma\tau}$. As in the discussion of the stable orbital integrals of elements of type (II), we write

$$t_1 = \alpha_1 + \beta_1\sqrt{D}, t_2 = \alpha_2 + \beta_2\sqrt{DA}, x_1 = a_1 + b_1\sqrt{D} \in E^\times (a_1, b_1 \in E_3^\times, E_3 = F(\sqrt{A}) = E^\sigma),$$

and we recall that the numbers $N = \operatorname{ord}_3(b_1)$, $X = \operatorname{ord}_3(a_1 - \tau a_1)$, $\chi = \operatorname{ord}_F(\alpha_1 - \alpha_2)$, are equal to $\min(n_1, n_2)$, $1+n_1+n_2, 1+2N$, when $D \in \boldsymbol{\pi} R^\times$, and to χ, $1+2n_1+2n_2$, $\min(2n_1, 1+2n_2)$, when $D \in R^\times$.

A set of representatives for the conjugacy classes within the stable conjugacy class determined by $(x_1, \tau x_1, \sigma\tau x_1, \sigma x_1)$ is given by $x(R) = \left(\begin{smallmatrix} \mathbf{a}_1 & \mathbf{b}_1 D\mathbf{R} \\ \mathbf{b}_1 R^{-1} & \mathbf{a}_1 \end{smallmatrix}\right)$, where R ranges over $E_3^\times / N_{E/E_3}E^\times$. The unstable orbital integral is the difference of the orbital integral at $x(1)$ (with positive sign) and the orbital integral at $x(R), R \ne 1$ (in $E_3^\times/N_{E/E_3}E^\times$). Recall also that the norm map $N_{E_3/F}$, followed by the inclusion, induces an isomorphism $E_3^\times / N_{E/E_3}E^\times \to N_{E_3/F}E_3^\times / N_{E/F}E^\times \hookrightarrow F^\times / N_{E_1/F}E_1^\times$ (further inducing the isomorphism $R_3^\times / N_{E/E_3}R_E^\times \to N_{E_3/F}R_3^\times / N_{E/F}R_E^\times \tilde{\to} R^\times / N_{E_1/F}R_1^\times$ when E/E_3 is ramified).

1. Theorem. *Let E be the compositum of the quadratic extensions (E_1, E_2, E_3) of F, and $x = h^{-1}(x_1, \tau x_1, \sigma\tau x_1, \sigma x_1)h$ a regular element of type (II) in $GSp(2,F)$ (thus $x_1 \sigma x_1 \in E_1^\times$). Introduce $t_1 \in E_1^\times, t_2 \in E_2^\times$, by $t_1/\sigma t_1 = x_1/\sigma\tau x_1, t_2/\tau t_2 = x_1/\tau x_1$. Suppose that $t_1 \in R_1^\times, t_2 \in R_2^\times$, are units. Let $\chi_{E_1/F}$ be the non trivial character on $F^\times/N_{E_1/F}E_1^\times$. Then*

$$\chi_{E_1/F}\big((x_1 - \sigma x_1)(\tau x_1 - \tau\sigma x_1)/D\big)|1 - x_1/\sigma x_1||1 - \tau x_1/\tau\sigma x_1|\Phi_{1_K}^{H,us}(x)$$

$$= [R_{T_H} : \lambda(R_{T_0})]\Phi_{1_K}^{C_0}\big(\left(\begin{smallmatrix} t_1 & 0 \\ 0 & \sigma t_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} t_2 & 0 \\ 0 & \tau t_2 \end{smallmatrix}\right)\big).$$

The measures are related as in the case of tori of type (I).

Proof. Let us first clarify that the absolute value $|\cdot| = |\cdot|_F$ is an extension of the absolute value on $F^\times$ normalized as usual by $|\boldsymbol{\pi}_F| = q_0^{-1}, q_0 = \#(R/\boldsymbol{\pi}_F R), R = R_F$. We write q for $q_3 = q_{E_3}$, and note that

$$|\boldsymbol{\pi}_3| = |\boldsymbol{\pi}_3\tau(\boldsymbol{\pi}_3)|^{1/2} = \left\{ \begin{array}{cc} |\boldsymbol{\pi}_F|^{1/2} = q_0^{-1/2} & (E_3/F \text{ ramified }) \\ |\boldsymbol{\pi}_F| = q_0^{-1} & (E_3/F \text{ unramified }) \end{array} \right\} = q^{-1/2}.$$

As in the case of type (I) above, to compute the right side in the theorem, we use the formula for the orbital integral on $GL(2,F)$. We then compute the factors which appear in that formula.

2. Lemma. *The unordered pair* $\{|(t_1 - \sigma t_1)/\sqrt{D}|^{-1}, |(t_2 - \sigma t_2)/\sqrt{D})|^{-1}\}$ *is equal to* $\{q^{N/2},\, q^{(X-N)/2}|D|\}$.

Proof. The product of the two terms is equal to $q^{X/2}|D|$, since

$$\begin{aligned} q^{-X/2} &= |a_1 - \tau a_1| = |x_1 + \sigma x_1 - \tau x_1 - \sigma\tau x_1| = |t_1 t_2 + \sigma t_1 \sigma t_2 - t_1 \sigma t_2 - t_2 \sigma t_1| \\ &= |(t_1 - \sigma t_1)(t_2 - \sigma t_2)|. \end{aligned}$$

The last two factors are given by

$$|t_2 - \sigma t_2| = |x_1 - \tau x_1| = |(a_1 - \tau a_1)^2 - (b_1 - \tau b_1)^2 D|^{1/2}$$

and

$$|t_1 - \sigma t_1| = |x_1 - \tau\sigma x_1| = |(a_1 - \tau a_1)^2 - (b_1 + \tau b_1)^2 D|^{1/2}.$$

If $|b_1 \pm \tau b_1| > |a_1 - \tau a_1|$ for both choices of sign, then $|(t_i - \sigma t_i)/\sqrt{D}| = |b_1 \pm \tau b_1|$ (for the right choice of sign), and one of $|b_1 + \tau b_1|$ or $|b_1 - \tau b_1|$ is equal to $|b_1| = q^{-N/2}$, as required. If there is a choice of sign such that $|b_1 \pm \tau b_1| \leq |a_1 - \tau a_1|$, then $|b_1 \mp \tau b_1| = 1 = |D|, N = 0$, and $|(t_i - \sigma t_i)/\sqrt{D}| = 1$ for some i, and the lemma follows in this case too. $\square$

Remark. If $D \in R^\times$, namely E_1/F is unramified, since $t_1 \in E_1^\times$ we have $|t_1 - \sigma t_1| \in q_0^{\mathbb{Z}} = q^{\mathbb{Z}}$. Indeed $q = q_0$ as E_3/F is ramified. In this case X is odd, hence $|(t_1 - \sigma t_1)\sqrt{D}|^{-1}$ is $q^{N/2}$ if N is even, and $q^{(X-N)/2}$ if N is odd.

3. Corollary. *The integral* $\Phi_{1_K}^{C_0}\big(\left(\begin{smallmatrix} t_1 & 0 \\ 0 & \sigma t_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} t_2 & 0 \\ 0 & \tau t_2 \end{smallmatrix}\right)\big)$ *is the product of*

$$1_{R_1^\times}(t_1) 1_{R_2^\times}(t_2)(q_0 - 1)^{-2}$$

with

$$\begin{aligned} \big((q+1)q^{N/2} - 2\big)\big(q \cdot q^{(X-N-1)/2} - 1\big), &\quad \textit{if } |D| = 1, N \textit{ is even;} \\ \big((q+1)q^{(X-N)/2} - 2\big)\big(q \cdot q^{(N-1)/2} - 1\big), &\quad \textit{if } D \in R^\times, N \textit{ is odd;} \\ (q^{(1+N)/2} - 1)(q^{(X-N)/2} - 1), &\quad \textit{if } D \in \boldsymbol{\pi} R^\times\ (|D| = q^{-1/2}). \end{aligned}$$

Proof. Note that when $|D| = 1, A = \boldsymbol{\pi}$ and $|A| = q^{-1/2}$, hence $|(t_2 - \sigma t_2)/\sqrt{AD}|^{-1}$ is $q^{(X-N-1)/2}$ when N is even, and $q^{(N-1)/2}$ when N is odd. □

Remark. The transfer factor is the product of $|1 - x_1/\sigma x_1||1 - \tau x_1/\sigma\tau x_1| = |b\tau bD|$ and $\chi_{E_1/F}\big((x_1 - \sigma x_1)(\tau x_1 - \tau\sigma x_1)/D\big) = \chi_{E_1/F}(b\tau b)$ (since $|x_1| = 1$ and the residual characteristic is odd).

We now turn to the computation of the unstable orbital integral in the case where E/E_3 is ramified. As in the computation of the stable integral, we have a sum over $\rho \in \{1, u\}$, where $u \in R_3^\times - R_3^{\times 2}$. While in the stable case both terms indexed by 1 and u appeared with coefficient 1, in the unstable case the term associated with $\rho = 1$ has coefficient 1, while that associated with $\rho = u$ has coefficient -1. Only in the range $\nu < m \le X - \nu$ there appears a difference between these two terms. Namely in this range we have the condition $\rho \in BR_3^{\times 2}$, and so only one of $\{1, u\}$ makes a contribution. For m with $m \le \nu$ both of $\{1, u\}$ contribute and cancel each other. Thus the unstable orbital integral is given by the sum

$$\chi_{E_1/F}(B\tau B) 2q^N \sum_{0\le\nu\le N} \sum_{\nu<m\le X-\nu} q_0^m.$$

The double sum here is

$$(q_0 - 1)^{-1} \sum_{0\le\nu\le N} (q_0^{X+1-\nu} - q_0^{\nu+1}) = \frac{q_0}{(q_0 - 1)^2}(q_0^{X-N} - 1)(q_0^{N+1} - 1).$$

This is the product of $q_0 = q^{1/2}$ and the orbital integral $\Phi_{1_K}^{C_0}$ of the Corollary above. Since $b = B\boldsymbol{\pi}_3^N$, $|b\tau bD| = q^{-N}q_0^{-1}$, in fact $\boldsymbol{\pi}_3 = \boldsymbol{\pi}_F$ and $\boldsymbol{\pi}_F = N_{E_1/F}\boldsymbol{\pi}_1$, as $\boldsymbol{\pi}_1 = \sqrt{D}$ and $D = -\boldsymbol{\pi}_F$. Hence the transfer factor is $\chi_{E_1/F}(B\tau B)q^{-N}q_0^{-1}$, and the product of the transfer factor with the unstable integral is indeed the integral $\Phi_{1_K}^{C_0}$, as asserted.

Finally we consider the case where E/E_3 is unramified, thus $D \in R^\times$. Again we have a sum over ρ in $E_3^\times/N_{E/E_3}E^\times$, parametrizing the two integrals which make the stable and unstable orbital integrals. A set of representatives for $E_3^\times/N_{E/E_3}E^\times$ is given by $\{1, \boldsymbol{\pi}_3\}$, and as usual we write $\rho = \boldsymbol{\pi}_3^{\overline{\rho}}$, thus $\overline{\rho} \in \{0, 1\}$. The orbital integral is a sum over $\nu(0 \le \nu \le N)$ such that $N - \nu - \overline{\rho}$ is even. In the stable case, both sums were added and thus combined to a single sum over $\nu(0 \le \nu \le N)$. Now in the unstable case, we need to multiply the contribution by $(-1)^{\overline{\rho}} = (-1)^{N-\nu}$ before adding up the sum. The unstable integral is then

$$\sum_{0\le\nu\le N} \sum_{0\le m'=2m\le X-1} (-q)^{N-\nu}\big(\delta(\nu = N) + (1 + q^{-1})\delta(\nu < N)\big)$$
$$\big(q^{3m'/2}\delta(0 \le m' \le \nu) + q^{\nu+m'/2}\delta(\nu < m' \le X - 1 - \nu, \nu \text{ even})\big).$$

This is the sum of two terms. The first is

$$\sum_{0\le\nu\le N} \sum_{0\le m'\le\nu} = \sum_{0\le m'\le N} q^{3m'/2}\Big(\sum_{\nu=N} 1 + (1 + q^{-1}) \sum_{m'\le\nu<N} (-q)^{N-\nu}\Big).$$

The inner $(\dots)$ is $(-q)^{N-m'}$, so the sum is $(-q)^N \sum_{0\le m\le N/2} q^m = (-q)^N(q-1)^{-1}(q^{1+[N/2]}-1)$, where $[X]$ is the biggest integer bounded by the real number X.

Writing $\nu'=2\nu$ for the even ν when $\nu<m'\le X-1-\nu$, the second term is

$$\sum_{0\le\nu'<N}\ \sum_{\nu'<m'\le X-1-\nu'} q^{\nu'+m'/2}(-q)^{N-\nu'}(1+q^{-1})+\delta(N \text{ is even})\sum_{N<m'\le X-1-N} q^{N+m'/2}$$
$$=\sum_{0\le\nu<N/2}\ \sum_{\nu<m\le\frac12(X-1)-\nu}(1+q^{-1})(-1)^N q^{N+m}+\delta(2|N)\sum_{N/2<m\le\frac12(X-1)-\frac12 N} q^{N+m}$$
$$=\big(1-\delta(2|N)\big)\frac{(-q)^N(q+1)}{(q-1)^2}(q^{\frac12(X-N)}-1)(q^{\frac12(N+1)}-1)$$
$$+\,\delta(2|N)(q-1)^{-2}(-q)^N\big((q+1)q^{\frac12(X+1)}-2q^{\frac12(X+1-N)}-(q^2+1)q^{N/2}+q+1\big),$$

where the last equality follows at once from the corresponding computation in the stable case.

The sum of these two terms, when N is odd, is

$$\frac{(-q)^N}{(q-1)^2}(q^{\frac12(N+1)}-1)\big((q+1)q^{\frac12(X-N)}-2\big),$$

while when N is even it is

$$\frac{(-q)^N}{(q-1)^2}\big((q+1)q^{N/2}-2\big)(q^{(X-N+1)/2}-1).$$

The transfer factor is the product of $|D|=1$, $|b\tau b|=q^{-N}$(as $|\boldsymbol{\pi}_3|=q^{-1/2}$), and $\chi_{E_1/F}(b\tau b)=\chi_{E_1/F}(\boldsymbol{\pi}^N)=(-1)^N$, since $\boldsymbol{\pi}_3=\sqrt{A}$, $A=-\boldsymbol{\pi}$, and so $N_{E_3/F}\boldsymbol{\pi}_3=\boldsymbol{\pi}$. In view of the Corollary above, our comparison is complete for regular elements of type (II), once we prove:

4. Lemma. *The index $[R_{T_H}:\lambda(R_{T_0})]$ is 1 if E_1/F is unramified, and 2 if E_1/F is ramified.*

Proof. Recall that $\lambda\big((t_1,\sigma t_1),(t_2,\sigma t_2)\big)=(x_1=t_1t_2,\tau x_1=t_1\sigma t_2,\sigma\tau x_1=\sigma t_1\cdot t_2,\sigma x_1=\sigma t_1\sigma t_2)$, with $t_1\in E_1=F(\sqrt{D})=E^\tau$, and $t_2\in E_2=F(\sqrt{AD})=E^{\sigma\tau}$. Note that $x_1\sigma x_1=\tau(x_1\sigma x_1)$ implies that $(x_1/\sigma\tau x_1)\sigma(x_1/\sigma\tau x_1)=1$, hence $x_1/\sigma\tau x_1=t_1/\sigma t_1$ has a solution in $t_1\in E_1^\times$, and our index computation is the question whether there exists a solution t_1 in $R_1^\times$. Indeed, if such a unit solution t_1 is found, we can define the unit $t_2=x_1/t_1$, which satisfies $\sigma\tau(x_1/t_1)=\sigma\tau x_1/\sigma t_1=x_1/t_1=t_2\in E_2^\times$. In the proof of the analogous Lemma in the case of elements of type (I) we have seen that $t_1\in R_1^\times$ exists if E_1/F is unramified, and that $\{t_1/\sigma t_1; t_1\in R_1^\times\}$ has index 2 in $\{t_1/\sigma t_1; t_1\in E_1^\times\}$ when E_1/F is ramified. The lemma follows, as does the Theorem, transferring the orbital integrals of 1_K on $H=GSp(2,F)$ to its endoscopic group $C_0=GL(2,F)\times GL(2,F)/\{(z,z^{-1}); z\in F^\times\}$. □

L. Comparison in case (III).

In this case the norm map goes in the opposite direction than in case (II), and we shall reduce the computations here to those of case (II). Let us recall the notations. The three quadratic extensions of F are $E_1 = F(\sqrt{D})$, $E_2 = F(\sqrt{AD})$, $E_3 = F(\sqrt{A})$, E_2/F is ramified, A and D are integral of minimal order, $E = E_3(\sqrt{D})$ has Galois group $\mathbb{Z}/2 \times \mathbb{Z}/2$ generated by σ, τ such that $E_3 = E^{\langle\sigma\rangle}, E_1 = E^{\langle\tau\rangle}, E_2 = E^{\langle\sigma\tau\rangle}$. The two θ-conjugacy classes in the stable θ-conjugacy class of a strongly θ-regular element of type (III) are represented by $\left(\begin{smallmatrix} \mathbf{a} & \mathbf{b}D\boldsymbol{\rho} \\ \boldsymbol{\rho}^{-1}\mathbf{b} & \mathbf{a} \end{smallmatrix}\right)$, ρ ranges over a set of representatives for $E_3^\times/N_{E/E_3}E^\times$, including 1 and an element in R_3 of minimal order.

Our element is moreover topologically unipotent, and it commutes with θ, thus these representatives lie in $Sp(2,F)$, and they are conjugate by θ-invariant elements to the diagonal element $t^* = (t, \tau t, \sigma\tau t, \sigma t; e)$ in the diagonal torus T^*. For our integrals to be non zero, e must lie in $R^\times$, and then the integrals are independent of e, so we omit e from the notations. Now t lies in $E^\times$, and we write it as $t = a + b\sqrt{D}$, where $a = \alpha_1 + \alpha_2\sqrt{A}, b = \beta_1 + \beta_2\sqrt{A}; \alpha_i, \beta_i \in F$. Then $\tau t = \tau a + \tau b\sqrt{D}, \sigma t = a - b\sqrt{D}$, and $\tau a = \alpha_1 - \alpha_2\sqrt{A}$. The norm map maps t^* to $Nt^* = (x_1 = et\tau t, x_2 = et\sigma\tau t, \sigma x_2 = e\tau t\sigma t, \sigma x_1 = e\sigma t\tau\sigma t; e^2)$.

Note that t^* lies in $Sp(2)$, thus $t\sigma t\tau t\sigma\tau t = 1$, and we omit e from the notations. Then we have

$$x_1 = t\tau t = a\tau a + b\tau bD + (a\tau b + b\tau a)\sqrt{D} = A_1 + B_1\sqrt{D},$$

and

$$x_2 = t\sigma\tau t = a\tau a - b\tau bD + (\frac{b\tau a - a\tau b}{\sqrt{A}})\sqrt{AD} = A_2 + B_2\sqrt{AD},$$

where A_i, B_i lie in F. Further, $1 = x_1\sigma x_1 = A_1^2 - B_1^2 D$, and $1 = x_2\tau x_2 = A_2^2 - B_2^2 AD$. Since t is topologically unipotent, so is a, and $|bD| < 1$, hence α_1 is topologically unipotent, $|A\alpha_2| < 1$ and $|bD| < 1$.

We proceed to relate the numbers associated with the norm map.

1. Lemma. *If $N_i = \operatorname{ord}_F B_i (i = 1,2), n = \operatorname{ord}_3 b, \chi = \operatorname{ord}_3(a - \tau a)$ and $X = \operatorname{ord}_F(A_1 - A_2)$, then $n = \frac{1}{e}\min(2N_1, 2N_2 + \operatorname{ord} A), \chi = \frac{1}{e}(\operatorname{ord}_F A + 2\operatorname{ord}_F D + 2N_1 + 2N_2) = \frac{1}{e}(1 + \operatorname{ord}_F D + 2N_1 + 2N_2), \operatorname{ord}_F \beta_i = N_i (i = 1,2)$, and $X = \operatorname{ord}_F D + en$. Here $e = e(E/E_3) = e(E_1/F), \operatorname{ord} = \operatorname{ord}_F, \operatorname{ord}_3 = \operatorname{ord}_{E_3}$.*

Proof. Note that $n = \operatorname{ord}_3 b = \operatorname{ord}_3(\beta_1 + \beta_2\sqrt{A}) = \frac{1}{e}\min(2\operatorname{ord}_F\beta_1, 2\operatorname{ord}_F\beta_2 + \operatorname{ord}_F A)$, since $\operatorname{ord}_3(\boldsymbol{\pi}_F) = \operatorname{ord}_3(\boldsymbol{\pi}_3^{2/e}) = 2/e$ so that $\operatorname{ord}_3(x) = \frac{2}{e}\operatorname{ord}_F(x)$ for $x \in F^\times$. Further we have $\chi = \operatorname{ord}_3(a - \tau a) = \operatorname{ord}_3(\alpha_2\sqrt{A}) = \frac{1}{e}(\operatorname{ord}_F A + 2\operatorname{ord}_F\alpha_2)$, and noting that $A_1 + A_2 = 2a\tau a$ is a unit, also

$$\begin{aligned} X &= \operatorname{ord}_F(A_1 - A_2) = \operatorname{ord}_F(A_1^2 - A_2^2) = \operatorname{ord}_F(B_1^2 D - B_2^2 AD) \\ &= \operatorname{ord}_F D + \min(2N_1, \operatorname{ord}_F A + 2N_2) = \operatorname{ord}_F(Db\tau b) = \operatorname{ord}_F D + e\operatorname{ord}_3(b) \\ &= \operatorname{ord}_F D + \min(2\operatorname{ord}_F\beta_1, \operatorname{ord}_F A + 2\operatorname{ord}_F\beta_2) = \operatorname{ord}_F D + en. \end{aligned}$$

Hence $\{\operatorname{ord}_F\beta_1, \operatorname{ord}_F\beta_2\} = \{N_1, N_2\}$, with $\operatorname{ord}_F\beta_i = N_i (i = 1,2)$ if $\operatorname{ord}_F A = 1$. In fact, if $|\beta_2| = |B_1| = |\alpha_1\beta_1 - \alpha_2\beta_2 A|$, since $|A\alpha_2| < 1 = |\alpha_i|$, we must have $|\beta_1| = |\beta_2|$, hence $|B_i| = |\beta_i|$ also for $A \in R^\times$.

Now $\left(\begin{smallmatrix} a & bD \\ b & a \end{smallmatrix}\right)$ lies in $SL(2, E_3)$, hence

$$1 = a^2 - b^2 D = \alpha_1^2 + \alpha_2^2 A + 2\alpha_1\alpha_2\sqrt{A} - (\beta_1^2 + \beta_2^2 A + 2\beta_1\beta_2\sqrt{A})D$$

$$= \alpha_1^2 + \alpha_2^2 A - (\beta_1^2 + \beta_2^2 A)D + 2(\alpha_1\alpha_2 - \beta_1\beta_2 D)\sqrt{A}$$

implies that $\alpha_2 = \beta_1\beta_2 D/\alpha_1$. Since α_1 is a unit, the expression for χ follows. □

The computation of the stable θ-orbital integral for an element of type (III) follows closely the computation of the stable orbital integral of the norm of an element of type (II). In both cases the integral is a sum $\Phi_{1_K}\left(\begin{smallmatrix}\mathbf{a} & \mathbf{b}D\\ \mathbf{b} & \mathbf{a}\end{smallmatrix}\right) + \Phi_{1_K}\left(\begin{smallmatrix}\mathbf{a} & \mathbf{b}D\rho\\ \mathbf{b}/\rho & \mathbf{a}\end{smallmatrix}\right)$ over $\rho \in E_3^\times/N_{E/E_3}E^\times$, the difference being that in case (II) the integration is performed over $GSp(2,F)$, while in case (III) the integration is over $Sp(2,F)$. However, the result in case (III) is exactly the same as in case (II), since $T_S\backslash Sp(2,F) = T\backslash T\cdot Sp(2,F)$ is $T\backslash GSp(2,F)$ where $T_S = T\cap Sp(2,F)$. Indeed, $\det T = \{a\sigma a\tau a\sigma\tau a;\ a\in E^\times\} = N_{E/F}E^\times = F^{\times 2}$.

2. Lemma. *The stable θ-orbital integral of a strongly θ-regular topologically unipotent θ-fixed element of $GL(4,F)\times R^\times$ is equal to*

$$2q_0^{2n+1}\cdot\frac{1-q_0^{-n-1}}{1-q_0^{-1}}\left(\frac{q_0^\chi}{q_0-1} - \frac{q_0^{n+1}}{q_0^3-1} - \frac{1+q_0^{-n-1}}{q_0^3-1}\right)$$

when E/E_3 is ramified, while when E/E_3 is unramified the integral is equal to

$$\frac{q+1}{(q-1)^2}q^n\Big[\delta(n\text{ is even})\Big(q^{\frac{\chi+1}{2}} - \frac{2}{q+1}q^{\frac{\chi+1-n}{2}} + \frac{q-1}{q+1}q^{\frac{n}{2}}\Big)$$
$$+\Big(q^{\frac{\chi+1}{2}} - q^{\frac{\chi-n}{2}}\Big)\delta(n\text{ is odd}) - \frac{2}{q-1}\,\frac{q^{3([n/2]+1)}-1}{q^3-1}.$$

Proof. When E/E_3 is ramified, the computation is immediately adapted from the case of the norm of an element of type (II), and we obtain the expression $(**)$ of Section I, except that in our notations (N,X) have to be replaced by (n,χ). Similarly, when E/E_3 is unramified, the expression of the lemma appears in the part dealing with the computation of the stable orbital integral of the norm of an element of type (II), in Section J, except that our current notations are (n,χ) instead of (N,X). □

Similarly, the computation of the stable orbital integral of the norm of an element of type (III) is immediately reduced to the computation of the stable θ-orbital integral of an element of type (II). Of course, the θ-integral ranges over $Sp(2,F)$, and the stable θ-integral is a sum over 4 θ-conjugacy classes.

The orbital integral of the norm of an element of type (III) is already stable, and the integration ranges over $GSp(2,F)$. In fact the orbital integral over $GSp(2,F)$ is a stable orbital integral over $Sp(2,F)$, and each of the conjugacy classes in the stable orbit in $Sp(2,F)$ is represented by conjugation within $GSp(2,F)$. Moreover, $T_S\backslash Sp(2,F) = T\backslash T\cdot Sp(2,F)$, and $[GSp(2,F) : T\cdot Sp(2,F)] = [F^\times : F^{\times 2}] = 4$, since the factors of similitude of $t\in T$ with eigenvalues $(a,b,\sigma b,\sigma a), a\in E_1^\times, b\in E_2^\times, a\sigma a = b\sigma b$, are in $N_{E_1/F}E_1^\times \cap N_{E_2/F}E_2^\times = N_{E/F}E^\times = F^{\times 2}$, while those of $GSp(2,F)$ are in $F^\times$. Consequently, we obtain

3. Lemma. *The orbital integral of the norm of a strongly θ-regular topologically unipotent element of type (III) is equal to*

$$4q_0^{N+N'+2}(q_0-1)^{-2}(1-q_0^{-N-1})\big(q_0^X - q_0^{-N'-1}\frac{1+q_0^{1+n}+q_0^{2+2N}}{1+q_0+q_0^2}\big)$$

where $N=\min(N_1,N_2)$, $N'=\max(N_1,N_2)$, in the case where E/E_3 is ramified, while when E/E_3 is unramified, the integral is

$$2(q-1)^{-2}[(q+1)q^{2+N_1+3N_2}-(q+1)q^{1+N_1+2N_2}-2\frac{q-1}{q^3-1}(q^{3+3N_2}-1)]$$

if $N_2<N_1$, while if $N_1\le N_2$ it is

$$2(q-1)^{-2}[(q+1)q^{2+3N_1+N_2}-2q^{1+2N_1+N_2}+\frac{q-1}{q^3-1}\big(2-(q^3+1)q^{3N_1}\big)].$$

Proof. When E/E_3 is ramified, our expression is obtained from $(*)$ of Section I on replacing $n_1\le n_2$ there by $N\le N'$ here, and χ by X. When E/E_3 is unramified, our expressions are obtained from $(*)$ and $(**)$ of Section J, on replacing n_1,n_2 there by N_1,N_2 here. □

To compare the stable θ-orbital integral and the stable orbital integral when $e=2$, note that $\operatorname{ord}_F A=0$ and so $\chi=1+N_1+N_2$ and $n=\min(N_1,N_2)=N$, and $X=1+2n$. Put $n'=\max(N_1,N_2)$. The θ-expression is then

$$2q_0^{2n+2}(q_0-1)^{-1}(1-q_0^{-n-1})(q_0^3-1)^{-1}(\frac{q_0^3-1}{q_0-1}\cdot q_0^{1+n+n'}-q_0^{n+1}-1-q_0^{-n-1}),$$

and the integral of the norm is twice that.

When $e=1$, $\operatorname{ord}_F A=1$, we have $X=n=\min(2N_1,1+2N_2)$ and $\chi=1+2N_1+2N_2$. When $N_2<N_1$ we have that $X=n=1+2N_2$ is odd, and the θ-expression is

$$\frac{q+1}{(q-1)^2}q^{1+2N_2}(q^{1+N_1+N_2}-q^{N_1})-\frac{2}{q-1}\frac{q^{3(N_2+1)}-1}{q^3-1},$$

while the integral at the norm is twice that. When $N_1\le N_2$, $X=n=2N_1$, the θ-integral is

$$\frac{q+1}{(q-1)^2}q^{2N_1}\big(q^{1+N_1+N_2}-\frac{2}{q+1}q^{1+N_2}+\frac{q-1}{q+1}q^{N_1}\big)-\frac{2}{q-1}\frac{q^{3N_1+3}-1}{q^3-1},$$

while the integral of the norm is twice this expression.

We are then done once we show that in the case of type (III), the measure factor is $\frac{1}{2}$.

4. Lemma. *For tori T of type (III), the measure factor $[T^{*\theta}(R):(1+\theta)T^*(R)]/[T^*_H(R):N\big(T^*(R)\big)]$ is $\frac{1}{2}$.*

Proof. We first compute the index in $T^*_H(R)=\{(x,y,\sigma y,\sigma x);x\in R_1^\times,y\in R_2^\times,x\sigma x=y\sigma y\}$ of the image $\{N(a,\tau a,\sigma\tau a,\sigma a)=(a\tau a,a\sigma\tau a,\tau a\sigma a,\sigma a\sigma\tau a);a\in R_E^\times\}$ of $T^*(R)$ under N. Thus we need to solve in $a\in R_E^\times$ the equation $x/\sigma y=a/\sigma a$. Since $(x/\sigma y)\sigma(x/\sigma y)=1$, there is a solution a in $E^\times$, and as usual we note that the index in $\{a/\sigma a;a\in E^\times\}$ of $\{a/\sigma a;a\in R_E^\times\}$ is the ramification index $e(E/E_3)$, where $E_3=E^\sigma$.

Given a solution $a\in R_E^\times$, put $x'=x/a\tau a,y'=y/a\sigma\tau a$. Then $x'=\sigma y'\in R_1^\times\cap R_2^\times=R^\times$, and it remains to find $b\in R_E^\times$ such that $x'(\in R^\times)$ is equal to $N(b,\tau b,\sigma\tau b,\sigma b)$, thus $x'=b\tau b=b\sigma\tau b=\tau b\sigma b=\sigma b\sigma\tau b$, or $\tau b=\sigma\tau b=\sigma b=b$. Hence only the x' in $R^{\times 2}$ are obtained by the norm, and we pick the factor $[R^\times:R^{\times 2}]$ in the index of the image of the norm in $T^*_H(R)$. Thus $[T^*_H(R):N\big(T^*(R)\big)]=2e(E/E_3)$.

The index in $T^{*\theta}(R)=\{(x,\tau x,\sigma\tau x,\sigma x);x\in R_E^\times,x\sigma x=1\}$ of the image $(1+\theta)T^*(R)=\{(1+\theta)(a,\tau a,\sigma\tau a,\sigma a)=(a/\sigma a,\tau a/\sigma\tau a,\sigma\tau a/\tau a,\sigma a/a);a\in R_E^\times\}$ of $T^*(R)$ under $(1+\theta)$ is computed next. Since $x\sigma x=1$, there is a in $E^\times$ with $x=a/\sigma a$. We can solve in $a\in R_E^\times$ only up to the index $e(E/E_3)$. Then the quotient $e(E/E_3)/2e(E/E_3)$ is 1/2, and the lemma follows. $\square$

Unstable twisted case. Twisted endoscopic group of type I.F.2.

The explicit computation of the θ-orbital integrals will now be used to compute the unstable, κ-θ-orbital integrals, at a strongly θ-regular topologically θ-unipotent element $t^*=(t,\tau t,\sigma\tau t,\sigma t)$ of type (III). The character κ is the $\neq 1$ character on the group $E_3^\times/N_{E/E_3}E^\times$ of θ-conjugacy classes within the stable θ-conjugacy class of t^*. The associated endoscopic group is $\mathbf{C}=\big(GL(2)\times GL(2)\big)'$. The norm N_Ct^* is $\big(\left(\begin{smallmatrix}t\tau t&0\\0&\sigma(t\tau t)\end{smallmatrix}\right),\left(\begin{smallmatrix}t\sigma\tau t&0\\0&\tau t\sigma t\end{smallmatrix}\right)\big)$. Recall that $x_1=t\tau t=A_1+B_1\sqrt{D}$ lies in $E_1^\times$, and $x_2=t\sigma\tau t=A_2+B_2\sqrt{AD}$ lies in $E_2^\times$. The Jacobian is

$$\Delta_{G,C}(t^*)=|(t-\sigma t)\tau(t-\sigma t)|_F/|t\tau t|_F=|b\tau bD|_F=|b|_3|D|_F=q^{-n}|D|_F$$

as $t=a+b\sqrt{D},\sigma t=a-b\sqrt{D},n=\operatorname{ord}_3(b)$, thus it is q_0^{-n} when $|D|_F=1$ (as then $q=q_0$), and q_0^{-2n-1} when $|D|_F=q_0^{-1}$ (as then $q=q_0^2$; recall that $E_3=F(\sqrt{A})=E^\sigma$ and $E_1=F(\sqrt{D})=E^\tau$).

The orbital integral of 1_{K_C} on C at N_Ct^* is the product of two integrals. If $N_i=\operatorname{ord}B_i$, Lemma I.I.2 asserts that one of the factors, the orbital integral of 1_K on $GL(2,F)$, at the class with eigenvalues x_2 and σx_2, as E_2/F is ramified, is $(q_0^{N_2+1}-1)/(q_0-1)$. The other factor is such an integral at the class with eigenvalues x_1 and σx_1 in E_1. Then it is $(q_0^{N_1+1}-1)/(q_0-1)$ if E_1/F is ramified, and $\big((q_0+1)q_0^{N_1}-2\big)/(q_0-1)$ if E_1/F is unramified.

Theorem. *Let t^* be a topologically θ-unipotent strongly θ-regular element of type (III). Then*

$$\kappa\big((t-\sigma t)/2\sqrt{D}\big)\Delta_{G,C}(t^*)\Phi^\kappa_{1_K}(t^*\theta)=\Phi^C_{1_{K_C}}(N_Ct^*).$$

Proof. Consider first the case where E/E_3 is ramified. Then E_1/F is ramified, $\operatorname{ord}(D)=1$, and since E/E_3 is ramified, $N_{E/E_3}E^\times=R_3^{\times 2}\boldsymbol{\pi}^{\mathbb{Z}}$, thus ρ ranges over $R_3^\times/R_3^{\times 2}$, and the unstable

θ-orbital integral of type (III), which is described also as the unstable orbital integral of type (II), is a sum over ρ of $\kappa(\rho)$, as well as sums over $\nu(0 \leq \nu \leq n)$ and $m(0 \leq m \leq \chi)$ as in the stable case. The sum over $m(0 \leq m \leq \nu)$ is zero since the only dependence on ρ is via $\kappa(\rho)$, and $\sum_\rho \kappa(\rho) = 0$. On the range $m(\nu < m \leq \chi - \nu)$, we have the requirement $u \in BR_3^{\times 2}$, thus $\kappa(\rho) = \kappa(B) = \kappa(b) = \kappa\big((t - \sigma t)/2\sqrt{D}\big)$ there (as $\kappa(\boldsymbol{\pi}_3) = 1$). The κ-θ-integral is then

$$\begin{aligned}
&\kappa(B) \sum_{0 \leq \nu \leq n} q^{n-\nu} \sum_{\nu < m \leq \chi - \nu} 2 q_0^m q^\nu \\
&= 2\kappa(B) q^n q_0 (q_0 - 1)^{-1} \sum_{0 \leq \nu \leq n} (q_0^{\chi - \nu} - q_0^\nu) \\
&= 2\kappa(B) q^n q_0 (q_0 - 1)^{-1} \big[q_0^{\chi+1}(1 - q_0^{-n-1})/(1 - q_0^{-1}) - (q_0^{n+1} - 1)/(q_0 - 1)\big] \\
&= 2\kappa(B) q_0^{2n+1} (q_0 - 1)^{-2} (q_0^{\chi - n} - 1)(q_0^{n+1} - 1).
\end{aligned}$$

Since $n = \min(N_1, N_2)$, and $\chi = 1 + N_1 + N_2$, the set $\{n+1, \chi - n\}$ is $\{N_1 + 1, N_2 + 1\}$, and the theorem follows when E/E_3 is ramified (the factor 2 is due to choice of transported measure).

Next we consider the case where E/E_3 is unramified, in which case $q = q_0$ and ρ ranges over a set $\{1, \boldsymbol{\pi}_3\}$ of representatives for $E_3^\times / N_{E/E_3} E^\times = R_3^\times \boldsymbol{\pi}_3^{\mathbb{Z}} / R_3^\times \boldsymbol{\pi}_3^{2\mathbb{Z}}$. The unstable, or κ-θ-integral, contains a factor $(-1)^{\overline{\rho}} = (-1)^j = (-1)^{n-\nu}$. Otherwise it is the same as described in the proof of Lemma J.2, namely $\sum_{0 \leq \nu \leq n} \sum_{0 \leq m' = 2m \leq \chi - 1} (-q)^{n-\nu} *$. As there, we write this as a sum of two terms. The first is

$$\begin{aligned}
&\sum_{0 \leq \nu \leq n} \sum_{0 \leq m' \leq \nu} = \sum_{0 \leq m' \leq n} q^{3m'/2} \Big(\sum_{\nu = n} 1 + (1 + q^{-1}) \sum_{m' \leq \nu < n} (-q)^{n-\nu}\Big) \\
&= \sum_{0 \leq m \leq n/2} q^{3m} \Big(1 + (1 + q^{-1}) \sum_{0 < \nu \leq n - m'} (-q)^\nu\Big) = \sum_{0 \leq m \leq n/2} q^{3m} (-q)^{n - m'} \\
&= (-q)^n \sum_{0 \leq m \leq n/2} q^m = (-q)^n (q^{[n/2]+1} - 1)/(q - 1).
\end{aligned}$$

The second is the product of $(-1)^j = (-1)^{n - \nu'} = (-1)^n$, as ν' is even, and the second term in the proof of Lemma J.2, namely it is

$$\begin{aligned}
&\frac{(-q)^n}{q - 1} \Big\{\delta(2|n)\Big[\frac{q+1}{q-1}(q^{(\chi+1-n)/2} - 1)(q^{n/2} - 1) + q^{(\chi+1-n)/2} - q^{n/2+1}\Big] \\
&\qquad + \big(1 - \delta(2|n)\big)\Big[\frac{q+1}{q-1}(q^{(\chi - n)/2} - 1)(q^{(n+1)/2} - 1)\Big]\Big\}.
\end{aligned}$$

Recall that $\chi = 1 + 2N_1 + 2N_2$, and $n = \min(2N_1, 2N_2 + 1)$, as $e = e(E/E_3) = 1$, and $\operatorname{ord} A = 1$. Then n is even if $n = 2N_1$, and the sum is $q^n/(q-1)$ times

$$\begin{aligned}
&q^{N_1 + 1} - 1 + \frac{q+1}{q-1}(q^{1+N_2} - 1)(q^{N_1} - 1) + q^{1+N_2} - q^{1+N_1} \\
&\qquad = \big((q+1)q^{N_1} - 2\big)(q^{N_2+1} - 1)/(q - 1),
\end{aligned}$$

as required. When n is odd, then $n = 2N_2 + 1$, and we get the product of $(-q)^n/(q-1)$ and

$$q^{N_2+1} - 1 + \frac{q+1}{q-1}(q^{N_1} - 1)(q^{N_2+1} - 1).$$

This is the same expression as for even n, so that we are done. □

M. Comparison in case (IV).

Strongly θ-regular elements of type (IV) lie in the stable θ-orbits of elements $t^* = (t, \sigma t, \sigma^3 t, \sigma^2 t; e)$ in the diagonal F-torus T^*. This torus is isomorphic to $E^\times$, where E is an extension of F of degree 4, which is not the compositum of the quadratic extensions of F. To study the orbital integrals of 1_K we may and do as usual assume that $e = 1$, and omit e from the notations. Recall that E is a quadratic extension $E_3(\sqrt{D}) = F(\sqrt{D})$ of a quadratic extension $E_3 = F(\sqrt{A})$ of F, which can be described as follows.

The element A is either a uniformizer $\boldsymbol{\pi}$ in $R \subset F$ or a unit $\varepsilon \in R^\times - R^{\times 2}$, taken to be -1 if $-1 \notin R^{\times 2}$. The element $D \in R_3 - R_3^2$ can be described as $D = \alpha + \beta\sqrt{A}$ with $\alpha = 0, \beta = 1$ if $A = \boldsymbol{\pi}$; $\alpha = 0$, $\beta = 1$ or $\boldsymbol{\pi}$, if $-1 \in R^{\times 2}$ and $A \in R^\times - R^{\times 2}$; $(\alpha, \beta) \in R^{\times 2}$ or $\in (\boldsymbol{\pi} R^\times)^2$ if $A = -1 \in R^\times - R^{\times 2}$.

The Galois closure $\tilde{E}$ of E/F is E unless $A = \boldsymbol{\pi}$ and $-1 \notin R^{\times 2}$, in which case $\tilde{E}/E$ is quadratic and $\mathrm{Gal}(\tilde{E}/F) = D_4$. The field embeddings $E \hookrightarrow \tilde{E}$ which fix F are generated by $\sigma, \sigma(\sqrt{D}) = \sqrt{\sigma D}, \sigma^2(\sqrt{D}) = -\sqrt{D}, \sigma\sqrt{A} = -\sqrt{A}$. Writing $t = a + b\sqrt{D}$, with $a = a_1 + a_2\sqrt{A}$ and $b = b_1 + b_2\sqrt{A}$, we have $\sigma a = a_1 - a_2\sqrt{A}$ and $\sigma b = b_1 - b_2\sqrt{A}$.

1. Lemma. *The parameters $\chi = \mathrm{ord}_3(a - \sigma a)$ and $n = \mathrm{ord}_3(b)$ associated with the strongly θ-regular topologically unipotent elements of type (IV) are equal to the corresponding parameters X and N associated with the norm Nt of t. Further, $\chi \geq 2n + \mathrm{ord}_3 D$.*

Proof. The parameter $\chi = \mathrm{ord}_3(a - \sigma a) = \mathrm{ord}_3(a_2\sqrt{A})$ is $\mathrm{ord}_3(a_2)$ if $A \in R_3^\times$, and $1 + \mathrm{ord}_3(a_2) = 1 + 2\,\mathrm{ord}_F(a_2)$ if $A = \boldsymbol{\pi}_F = \boldsymbol{\pi}_3^2$ (then $\mathrm{ord}_3 = 2\,\mathrm{ord}_F$, we usually omit the subscript F). The parameter $n = \mathrm{ord}_3(b) = \mathrm{ord}_3(b_1 + b_2\sqrt{A}) = \min\big(\mathrm{ord}_3(b_1), \mathrm{ord}_3(b_2\sqrt{A})\big)$ is $\min\big(\mathrm{ord}_F(b_1), \mathrm{ord}_F(b_2)\big)$ if $A \in R^\times$, and $\min\big(2\,\mathrm{ord}_F(b_1), 1 + 2\,\mathrm{ord}_F(b_2)\big)$ if $A = \boldsymbol{\pi}_F$.

The norm Nt^* of t^* is (we put $e = 1$ and omit it from the notations) equal to $(x = t\sigma t, t\sigma^3 t, \sigma t \sigma^2 t, \sigma^2 t \sigma^3 t)$. We claim that the element Nt^* is of type (IV), associated with an extension E' of degree 4 of F. This E' is Galois and it coincides with E, unless $A = \boldsymbol{\pi}$ and $-1 \notin R^{\times 2}$. In this last case, $E' = E_3'(\sqrt{D'}) = F(\sqrt{D'})$ and $E_3' = F(\sqrt{A'})$, where $A' = -4\boldsymbol{\pi}$ and $D' = \sqrt{A'}$, and E'/F is not Galois.

To verify this, put $\zeta = \sqrt{\sigma D}/\sqrt{D}$, and note that

$$x = t\sigma t = (a + b\sqrt{D})(\sigma a + \sigma b\sqrt{\sigma D}) = (a\sigma a + \zeta b \sigma b D) + (b\sigma a + \zeta a \sigma b)\sqrt{D} = A_* + B_*(1+\zeta)\sqrt{D}$$

defines elements A_* and B_*. These A_* and B_* lie in E_3' when $A = \boldsymbol{\pi}$ and $-1 \notin R^{\times 2}$, since then $\zeta = \sqrt{-1}, 2\zeta D = \sqrt{-4\boldsymbol{\pi}}, (1-\zeta)/(1+\zeta) = -\zeta = -\sqrt{-1}$, and

$$B_* = (b\sigma a + \zeta a \sigma b)/(1+\zeta) = (a_1 b_1 - a_2 b_2 A) + (a_1 b_2 - a_2 b_1)\frac{1-\zeta}{1+\zeta}\sqrt{A}.$$

Further $x \in E'$, since $(1+\zeta)^2 D = 2\zeta D = \sqrt{-4\boldsymbol{\pi}}$.

In all other cases we define E_3' to be E_3 and E' to be E. In fact, if $A = \boldsymbol{\pi}$ and $-1 \in R^{\times 2}$, or $A \in R^\times$ and $-1 \in R^{\times 2}$, we have that $\zeta = \sqrt{-1} \in R^\times$.

In the remaining case $A = -1 \in R^\times - R^{\times 2}$, and D (or $D/\boldsymbol{\pi}$) lies in $R_3^\times - R_3^{\times 2}$, hence so does σD (or $\sigma D/\boldsymbol{\pi}$), and so $\sigma D/D$ lies in $R_3^{\times 2}$, and ζ lies in $R_3^\times$. Then $E_3 = F(\zeta D)$ and $E = E_3\big((1+\zeta)\sqrt{D}\big)$.

When computing the parameters X, N associated with A_*, B_*, the index 3 refers to E_3'.

Let us now show that $n = N$, namely that

$$|b|_3 = |b_1 + b_2\sqrt{A}|_3 \text{ is } |(a_1b_1 - a_2b_2A) + (a_1b_2 - a_2b_1)\frac{1-\zeta}{1+\zeta}\sqrt{A}|_3 = |B_*|_3.$$

Since $t = a + b\sqrt{D}$ is topologically unipotent, we have that $a = a_1 + a_2\sqrt{A}$ is topologically unipotent, and $|b| < 1$ if $|D| = 1$. Hence a_1 is topologically unipotent, and $|a_2| < 1$ if $|A| = 1$. Suppose that $|A| = 1$. If $|b_1| \le |b_2|$ then $|b| = |b_2|$ and $|B_*| = |b_2|$ (of course, $|1-\zeta| = |1+\zeta|$), and if $|b_1| > |b_2|$ then $|b| = |b_1| = |B_*|$. Suppose that $|A| = |\boldsymbol{\pi}|$. If $|b_1| \ge |b_2|$ then $|b| = |b_1| = |B_*|$, and if $|b_1| < |b_2|$ then $|b| = |b_2\sqrt{A}| = |B_*|$.

Finally, let us show that $\chi = X$, namely that $|a - \sigma a|_3 = |a_2\sqrt{A}|_3$ is equal to $|A_* - \sigma A_*|_3 = |b\sigma b\zeta D|_3$. For that note that the element $t = a + b\sqrt{D}$ of type (IV) is represented by a matrix $\left(\begin{smallmatrix} \mathbf{a} & \mathbf{bD} \\ \mathbf{b} & \mathbf{a} \end{smallmatrix}\right)$ in $GL(2, E_3)'$, whose determinant lies in $F^\times$. Thus $t\sigma^2 t$ lies in $F^\times$. Since

$$t\sigma^2 t = a^2 - b^2 D = a_1^2 + a_2^2 A + 2a_1a_2\sqrt{A} - (b_1^2 + b_2^2 A + 2b_1b_2\sqrt{A})D,$$

and $D = \Pi(\alpha + \beta\sqrt{A})$ with $\Pi = 1$ or $\boldsymbol{\pi}$, it follows that the coefficient of $\sqrt{A}$ is zero, hence $|a_2| = |2a_1a_2|$ equals$|\Pi| \cdot |(b_1^2 + b_2^2 A)\beta + 2b_1b_2\alpha|$.

There are three cases to be considered. If $A = \boldsymbol{\pi}$ then $\alpha = 0$ and $\beta = 1, \Pi = 1$, so $|b_1^2 + b_2^2\boldsymbol{\pi}| = |b_1^2 - b_2^2\boldsymbol{\pi}| = |b\sigma b|$ implies that $|a_2\sqrt{A}| = |b\sigma b\zeta D|$. If $A \in R^\times$ and $-1 \in R^{\times 2}$, then $D = \Pi\sqrt{A}$, and $|\Pi(b_1^2 + b_2^2 A)| = |\Pi(b_1^2 - b_2^2 A)| = |\zeta D b\sigma b|$. If $A = -1 \in R^\times - R^{\times 2}$, then $D = \Pi(\alpha + \beta\sqrt{-1})$ with $\alpha, \beta \in R^{\times 2}$, and we claim that $|(b_1^2 - b_2^2)\beta + 2b_1b_2\alpha|$ is equal to $|b\sigma b| = |b_1^2 + b_2^2| = \max(|b_1^2|, |b_2^2|)$. This is obvious when $|b_1| \ne |b_2|$ or when $|b_1| = |b_2|$ and $|b_1^2 - b_2^2| < |b_1|^2$.

Suppose that $|b_1| = |b_2| = 1$, put $x = b_1/b_2$ and $\gamma = \alpha/\beta$. To show: $|x^2 + 2\gamma x - 1|$ is 1. This quantity can be expressed as $|(x+\gamma)^2 - \gamma^2 - 1|$, or $|(\beta x + \alpha)^2 - \alpha^2 - \beta^2|$, or $|D_1\sigma D_1 - (\alpha + \beta x)^2|$, where $D_1 = \alpha + \beta\sqrt{-1}$. Now $D_1 \in R_3^\times - R_3^{\times 2}$, and $N_{E_3/F}D_1 \notin R^{\times 2}$ (otherwise $N_{E_3/F}R_3^\times = R^{\times 2}$, but E_3/F is unramified so $N_{E_3/F}R_3^\times = R^\times$). Hence $|D_1\sigma D_1 - y^2| = 1$ for any $y \in R$, and we are done.

The final claim of the lemma follows from the fact that

$$a^2 - b^2 D = t\sigma^2 t = \sigma(t\sigma^2 t) = \sigma a^2 - \sigma b^2 \sigma D.$$

This implies that $a^2 - \sigma a^2 = \sigma b^2\sigma D - b^2 D = D(\zeta^2\sigma b^2 - b^2)$. Since $t = a + b\sqrt{D}$ is topologically unipotent, $|a + \sigma a| = 1$. Hence $|a - \sigma a| \le |D||b|^2$, namely $\chi \ge 2n + \operatorname{ord}_3 D$. $\square$

We proceed to compute the orbital integral of the function 1_K at a regular absolutely unipotent element u of $GSp(2,F)$. [This element is the norm of an absolutely unipotent strongly θ-regular element t of $GL(4,F)\times F^\times$, the computation of whose stable θ-orbital integral – which is the analogous case of u_ρ, $\rho\in E_3^\times/N_{E/E_3}E^\times$, in $Sp(2,F)$ – will be reduced to that of u later below, but we also deal with it parenthetically now].

Note that the stable orbit of u reduces to a single orbit. The element u can be presented as $\left(\begin{smallmatrix}\mathbf{a} & \mathbf{bD}\\ \mathbf{b} & \mathbf{a}\end{smallmatrix}\right)=h^{-1}(t,\sigma t,\sigma^3 t,\sigma^2 t)h, t\in F(\sqrt{D})$ with $t\sigma^2 t=\sigma t\sigma^3 t, t=a+b\sqrt{D}$, and $\mathbf{a}=\left(\begin{smallmatrix}a_1 & a_2/A\\ a_2 & a_1\end{smallmatrix}\right)$ if $a=a_1+a_2/\sqrt{A}$ lies in $E_3=F(\sqrt{A})$ (similarly for $b=b_1+b_2/\sqrt{A}$; $a_i,b_i\in F$). If $D=\alpha+\beta/\sqrt{A}$, we put $\mathbf{D}=\left(\begin{smallmatrix}\alpha & \beta/A\\ \beta & \alpha\end{smallmatrix}\right)$. [$u_\rho=\left(\begin{smallmatrix}\mathbf{a} & \mathbf{bD}\boldsymbol{\rho}\\ \mathbf{b}\boldsymbol{\rho}^{-1} & \mathbf{a}\end{smallmatrix}\right)$ with $t\sigma^2 t=1$]. As in the study of the case (II), the centralizer T' of u in $GSp(2,F)$ lies in $C_A=\{\left(\begin{smallmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{c} & \mathbf{d}\end{smallmatrix}\right);\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in GL(2,E_3)'\}$, where the prime indicates determinant in $F^\times$. [For u_ρ, replace T' by T^1_ρ, C_A by C^1_A, GL' by SL, K by K^1 below, and R'_E by R^1_E].

2. Lemma. *The integral* $\Phi^{GSp(2,F)}_{1_K}(u)$ *is equal to* $\sum_{m=0}^{\infty}[K_0:K_m]\int_{T\backslash C_A}1_{K_m}(h^{-1}uh)dh$. *Here* $K_m=GL\big(2,R_3(m)\big)'$, *where* $R_3(m)=R+\boldsymbol{\pi}^m\sqrt{A}R=R+\boldsymbol{\pi}^m R_3$.

Proof. The decomposition $G=GSp(2,F)=\bigcup_{m\geq 0}C_A u_m K, K=GSp(2,R)$, implies that

$$\int_{T\backslash G}1_K(g^{-1}ug)dg=\sum_{m=0}^{\infty}|K|_G\int_{T\backslash C_A/C_A\cap u_m K u_m^{-1}}1_K(u_m^{-1}h^{-1}uhu_m)dh.$$

Put $K^A_m=C_A\cap u_m K u_m^{-1}$. The integrand on the right is non zero precisely when $h^{-1}uh\in u_m K u_m^{-1}\cap C_A$, so we obtain

$$=\sum_{m\geq 0}|K|_G|K^A_m|^{-1}_{C_A}\int_{T\backslash C_A}1_{K^A_m}(h^{-1}uh)dh=\sum_{m\geq 0}[K_0:K_m]\int_{T\backslash C_A}1_{K_m}(h^{-1}uh)dh.$$

□

The decomposition $C_A=\cup_r T'rK'$ can be used to rewrite our integral as

$$=\sum_{m\geq 0}\sum_r[T'_0:T'\cap rK'r^{-1}][K_0:K_m]\int_{K_0}1_{K_m}(k^{-1}r^{-1}urk)dk,$$

where $T'_0=T'\cap K'=T'(R)\simeq R'^{\times}_E$. Here $R'_E=\{x\in R^\times_E; N_{E/E_3}x\in F^\times\}$. As usual, $q=q_3=q_{E_3}$ denotes the residual cardinality of E_3. Put $e=e(E/E_3)$ for the ramification index of E/E_3. Denote by $\boldsymbol{\pi}_3$ a uniformizer of R_3. It is taken to be $D=\boldsymbol{\pi}\varepsilon_3$, $\varepsilon_3\in R^\times_3-R^{\times 2}_3$, if E_3/F is unramified and further E/E_3 is ramified; then $\boldsymbol{\pi}_E=\sqrt{-D}$ has norm $N_{E/E_3}\boldsymbol{\pi}_E=D=\boldsymbol{\pi}\varepsilon_3$.

3. Lemma. *When E/E_3 is ramified, we have $GL(2,E_3)'=\cup_{j\geq 0}T'r_jK', K'=GL(2,R_3)'$, where $r_j\in T\left(\begin{smallmatrix}1 & 0\\ 0 & \boldsymbol{\pi}_3^j\end{smallmatrix}\right)$ has determinant 1, and T is the centralizer of T' in $GL(2,E_3)$; here $\boldsymbol{\pi}_3$ is $D\in\det T$ (if $\boldsymbol{\pi}_E=\sqrt{-D}$, then $N_{E/E_3}\boldsymbol{\pi}_E=D$). If E/E_3 is unramified then (E_3/F is* *unramified and)* $GL(2,E_3)' = \cup T' r_{j,\varepsilon} K'$, *union over* $j \geq 0$ *and over* $\varepsilon \in R_3^\times / R_3^{\times 2}$ *if* $j \geq 1$, *where* $r_{j,\varepsilon} = t_\varepsilon \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varepsilon \boldsymbol{\pi}_3^j \end{smallmatrix}\right)$, $\det(t_\varepsilon) = \varepsilon^{-1}$. *Further, the index* $[T'_0 : r_j K' r_j^{-1} \cap T']$ *is* q^j *if* $j = 0$ *or* E/E_3 *is ramified, and it is* $\frac{q+1}{2q} q^j$ *if* E/E_3 *is unramified and* $j \geq 1$.

[Remark. The case of $SL(2,E_3)$ is dealt with in Lemma I.I.3. If E/E_3 is ramified and E_3/F is unramified, then $\boldsymbol{\pi}_3 = D = \boldsymbol{\pi}\varepsilon_3 \in \det T_\rho$. If E/E_3 is unramified then $SL(2,E_3) = \cup T_\rho^1 r_{j\,\varepsilon} K^1$, $j \geq 0$, 2 divides $j - \overline{\rho}$, where $\overline{\rho} = \mathrm{ord}_3\, \rho$, and $r_{j\,\varepsilon} = t_\varepsilon \operatorname{diag}(\boldsymbol{\pi}_3^{-(j-\overline{\rho})/2}, \varepsilon \boldsymbol{\pi}_3^{(j-\overline{\rho})/2})$].

Proof. We use the disjoint decomposition $GL(2,E_3) = \cup_{j \geq 0} T r'_j K, r'_j = \operatorname{diag}(1, \boldsymbol{\pi}_3^j)$. Here $T = \{\left(\begin{smallmatrix} a & bD \\ b & a \end{smallmatrix}\right)\}$, $a, b \in E_3$. When $E = E_3(\sqrt{D})/E_3$ is ramified, we can take $\boldsymbol{\pi}_E = \sqrt{-D}$, then $N_{E/E_3}(\sqrt{-D}) = D = \boldsymbol{\pi}_3$ is a uniformizer of E_3. As $\det T = N_{E/E_3} E^\times$ contains $\boldsymbol{\pi}_3^{\mathbb{Z}}$, if $h = trk$ lies in $GL(2,E_3)'$ then we may assume that $\det h = \|h\|$ lies in $R^\times$, and there is some $t_0 \in T$ with $\|t_0 r\| = 1$. Then $\|t\| \in R_3^\times \cap N_{E/E_3} E^\times = R_3^{\times 2}$, so $\|t\| = \varepsilon^2$ for some $\varepsilon \in R^\times$, and $h = \varepsilon^{-1} t \cdot t_0 r \cdot \varepsilon k$, $\|\varepsilon^{-1} t\| = 1$ and $\|\varepsilon k\| \in R^\times$.

When E/E_3 is unramified then so is E_3/F, and $\boldsymbol{\pi}_3 = \boldsymbol{\pi} (= \boldsymbol{\pi}_F)$. Since $N_{E/E_3} E^\times = \boldsymbol{\pi}^{2\mathbb{Z}} R_3^\times$, if $h = trk \in GL(2,E_3)'$ then by changing t we may assume that $\|h\| = \|r\| = \boldsymbol{\pi}^j$. Now k can be changed by $r^{-1} t r \in K$, so $\|k\| \in R_3^\times$ can be changed by $\|r^{-1} t r\| = N_{E/E_3}(R_3 + \boldsymbol{\pi}_3^j \sqrt{D} R_3)^\times$, which is $R_3^\times$ if $j = 0$, and $R_3^{\times 2}$ if $j \geq 1$.

The intersection $T' \cap r_j K' r_j^{-1} = \{t \in T'; r^{-1} t r \in K'\}$ consists of the $a + b\sqrt{D} \in E^\times$ with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varepsilon^{-1} \boldsymbol{\pi}_3^{-j} \end{smallmatrix}\right) \left(\begin{smallmatrix} a & bD \\ b & a \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varepsilon \boldsymbol{\pi}_3^j \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & bD\varepsilon\boldsymbol{\pi}_3^j \\ \varepsilon^{-1} b \boldsymbol{\pi}_3^{-j} & a \end{smallmatrix}\right)$ in K, thus $b \in \boldsymbol{\pi}_3^j R_3$, namely it is $R_E(j)^\times \cap R'_E, R_E(j) = R_3 + \boldsymbol{\pi}_3^j R_E = R_3 + \boldsymbol{\pi}_3^j \sqrt{D} R_3$. Note that $R'_E = \{x \in R_E^\times; N_{E/E_3} x \in R^\times\}$ contains $\ker N_{E/E_3}$. Put $R_E(j)' = R_E(j)^\times \cap R'_E$. When $e = 2$ or $j \geq 1, N R_E^\times = N R_E(j)^\times = R_3^{\times 2}$, where $N = N_{E/E_3}$; when $e = 1$ and $j = 0$, we have $N R_E^\times = R_3^\times$. The index of the lemma is the kernel in the following exact sequence:

$$1 \to R'_E / R_E(j)' \to R_E^\times / R_E(j)^\times \to R_E^\times / R'_E R_E(j)^\times \to 1.$$

The term on the right is isomorphic via the norm N to $N R_E^\times / N R_E^\times \cap R^\times \cdot N R_E(j)^\times$, which is trivial if $e = 2$ or $j = 0$, since then $N R_E(j)^\times = N R_E^\times$, while if $e = 1$ and $j \geq 1$, it is the group $R_3^\times / R^\times R_3^{\times 2} \simeq \mathbb{Z}/2$. Consequently it remains to compute

$$[R_E^\times : R_E(j)^\times] = [R_E^\times : 1 + \boldsymbol{\pi}_3^j R_E] / [R_E(j)^\times : 1 + \boldsymbol{\pi}_3^j R_E].$$

The denominator here is $[R_3^\times : R_3^\times \cap (1 + \boldsymbol{\pi}_3^j R_E)] = [R_3^\times : 1 + \boldsymbol{\pi}_3^j R_3] = (q-1) q^{j-1}$, when $j \geq 1$. To compute the numerator, note that when $e = 2, \boldsymbol{\pi}_3 = \boldsymbol{\pi}_E^2$ and $q_E = q_3 = q$, so the numerator is $(q-1) q^{2j-1}$; when $e = 1, \boldsymbol{\pi}_E = \boldsymbol{\pi}_3$ and $q_E = q_3^2 = q^2$, so the numerator is $(q^2 - 1) q^{2(j-1)}$. The lemma follows. $\square$

Our orbital integral then takes the form

$$\int_{T\backslash G} 1_K(g^{-1} u g)\, dg = \sum_{m \geq 0} \sum_{j,\varepsilon} [R'_E : R_E(j)'] [K_0 : K_m] \int_{K_0} 1_{K_m}(k^{-1} r_{j,\varepsilon}^{-1} u r_{j,\varepsilon} k)\, dk.$$

If the integrand on the right is non zero then $u \in T' \cap r_{j\varepsilon} K r_{j\varepsilon}^{-1} = R_E(j)'$. Further, $[K_0 : K_m] \int_{K_0} dk$ can be written as $\int_{K_0/K_m} dk$, and this last integral is in fact a sum. To describe this sum, put $S_m = R_3/\boldsymbol{\pi}^m R_3 \supset R_m = R/\boldsymbol{\pi}^m R = R_3(m)/\boldsymbol{\pi}^m R_3$, where $R_3(m) = R + \boldsymbol{\pi}^m R_3$. Recall that $K_m = GL\big(2, R_3(m)\big)'$. Put $K(\boldsymbol{\pi}^m) = \{k \in GL(2, R_3)'; k \equiv I(\operatorname{mod} \boldsymbol{\pi}^m)\}$. Then $K_m/K(\boldsymbol{\pi}^m) = GL(2, R_m)(m \geq 1)$ and $K_0/K(\boldsymbol{\pi}^m) = GL(2, S_m)'$, where the last prime indicates determinant in R_m. In these notations, we have

4. Lemma. *The integral $\int_{K_0/K_m} 1_{K_m}(k^{-1} r_{j,\varepsilon}^{-1} u r_{j,\varepsilon} k) dk$ is equal to the cardinality of the set*

$$L'_m = \{y \in GL(2, S_m)'/GL(2, R_m); y^{-1} r_{j,\varepsilon}^{-1} u r_{j,\varepsilon} y \in GL(2, R_m)\}.$$

$\square$

[Remark. In the case of Sp, replace u by u_ρ, and note that $GL(2, S_m)'/GL(2, R_m)$ is $SL(2, S_m)/SL(2, R_m)$, so the same answer is obtained].

5. Lemma. *$\# L_m = e_3 \cdot \# L'_m$ where $e_3 = e(E_3/F)$ is the ramification index of E_3/F, and $L_m = \{x \in SL(2, S_m); \sigma x = x^{-1}, x \overline{u}_{j,\varepsilon} x^{-1} = \sigma(\overline{u}_{j,\varepsilon})\}$, where $\overline{u}_{j,\varepsilon}$ is the image of $r_{j,\varepsilon}^{-1} u r_{j,\varepsilon}$ in $GL(2, S_m)'$.*

Proof. The map $y \mapsto x = \sigma(y) y^{-1}$ is an injection of L'_m in L_m. Indeed, if $\sigma(y_1) y_1^{-1} = \sigma(y_2) y_2^{-1}$ then $\sigma(y_1^{-1} y_2) = y_1^{-1} y_2$ lies in $GL(2, R_m)$. The map is surjective if $e_3 = 1$. Indeed, in this case the map $GL(2, R_3)' \to \{x \in SL(2, R_3); \sigma x = x^{-1}\}$, by $y \mapsto \sigma(y) y^{-1}$, is onto by Hensel's Lemma. When $e_3 = 2$, we claim that L_m is the disjoint union of the sets $\operatorname{Im}(L'_m)$ and $-\operatorname{Im}(L'_m)$. Indeed, when E_3/F is ramified, we have $\sigma x \equiv x (\operatorname{mod} \boldsymbol{\pi}_3)$. If $\sigma x = x^{-1}$, then $x^2 \equiv I (\operatorname{mod} \boldsymbol{\pi}_3)$. Since $\|x\| = 1$, this implies that $x \equiv \pm I (\operatorname{mod} \boldsymbol{\pi}_3)$. Clearly, $x \in L_m$ if and only if $-x \in L_m$. Now $x \equiv I (\operatorname{mod} \boldsymbol{\pi}_3)$ if and only if $x = \sigma(y) y^{-1}$, for some y in $GL(2, S_m)'$, again by Hensel's Lemma. $\square$

Our aim is then to determine when is L_m non-empty, and to compute its cardinality. Recall that $b = B \boldsymbol{\pi}_3^N$, and $r_{j\varepsilon} = \operatorname{diag}(1, \varepsilon \boldsymbol{\pi}_3^j)$, so we put $b' = B' \boldsymbol{\pi}_3^\nu, \nu = N - j$ and $B' = B/\varepsilon$, and $D' = D \varepsilon^2 \boldsymbol{\pi}_3^{2j}$. [In the Sp case: Recall that $b = B \boldsymbol{\pi}_3^N$, $\overline{\rho} = u \boldsymbol{\pi}_3^{\overline{\rho}}$, and $r_{j\,\varepsilon} = \operatorname{diag}(1, \varepsilon (\varepsilon_3 \boldsymbol{\pi}_3)^{j - \overline{\rho}})$, where $\varepsilon_3 = 1$ unless E/E_3 is ramified, E_3/F is unramified, and then $\boldsymbol{\pi}_3$ is $\boldsymbol{\pi}$. So we put $b' = B' \boldsymbol{\pi}_3^\nu$, $\nu = N - j$, and $B' = B/\varepsilon \varepsilon_3^j u$ ($\varepsilon_3 = 1 = \varepsilon$ if E/E_3 is ramified), and $D' = D \varepsilon^2 u^2 (\varepsilon_3 \boldsymbol{\pi}_3)^{2j}$]. Note that $\overline{b}' \neq 0$ in $S_m = R_3/\boldsymbol{\pi}^m R_3 = R_3/\boldsymbol{\pi}_3^{e_3 m} R_3$ precisely when $\nu < m' = m e_3$.

6. Lemma. *The set L'_m is non empty precisely when $0 \leq \nu \leq N, 0 \leq m' = e_3 m \leq X = \operatorname{ord}_3(a - \sigma a)$. In this case, if $m' > \nu$ then we have that $\nu + m' \leq X$ as well as: ν is even when E_3/F is ramified; $\varepsilon \in B R_3^{\times 2}$ and $j \geq 1$ (namely $\nu < N$) when E/F is unramified [$\varepsilon \in u B \varepsilon_3^j R_3^{\times 2}$ and $\nu < N$ when E/F is unramified, while if E_3/F is unramified and E/E_3 is ramified, then $u \in B \varepsilon_3^j R_3^{\times 2}$].*

If L'_m is non empty, its cardinality is as follows. $\# L'_0 = 1$; if $1 \leq m' \leq \nu$ then L'_m has cardinality $q^{3m'/2}$ if $e_3 = 2$, and $q^{3m'/2}(1 + q^{-1})$ if $e_3 = 1$; if $\nu < m' \leq X - \nu$ then $\# L'_m = (2/e_3) q_0^m q^\nu$ if E/F is ramified or $\nu < N$.

Proof. The element $\overline{u}_{j,\varepsilon}$ is $\left(\begin{smallmatrix} \overline{a} & \overline{b}' \overline{D} \\ \overline{b}' & \overline{a} \end{smallmatrix}\right)$, hence $0 \leq \nu \leq N$. If $x \in L_m$ then $x \overline{u}_{j,\varepsilon} x^{-1} = \sigma(\overline{u}_{j\varepsilon})$. Taking traces we conclude that $\overline{a} = \sigma \overline{a}$ lies in R_m, and since $\boldsymbol{\pi}$ is $\boldsymbol{\pi}_3^{e_3}$, we have $0 \leq m' \leq X$.

Clearly $\#L'_0=1$, while when $1\le m'\le\nu$ we have $\overline{b}'=0$, and $L'_m=GL(2,S_m)'/GL(2,R_m)=SL(2,S_m)/SL(2,R_m)$. Recall that $\#SL(2,R_m)=(q_0^2-1)q_0^{3m-2}=(q_0^2-1)q_0^{3(m'/e_3)-2}$, as $R_m=R/\boldsymbol{\pi}^mR, m'=me_3$. Also $S_m=R_3/\boldsymbol{\pi}_3^{m'}R_3$, hence $\#SL(2,S_m)=(q^2-1)q^{3m'-2}$. When $e_3=1, q=q_0^2$, and $\#L'_m=(q+1)q^{3m'/2-1}$. When $e_3=2, q=q_0$, and $\#L'_m=q^{3m'/2}$. Consider then from now on the case $m'>\nu$, namely $\overline{b}\neq 0$ in S_m.

Suppose that x lies in L_m. From $\|x\|=1$ and $\sigma x=x^{-1}$ we deduce that

$$\begin{pmatrix}\sigma x_1 & \sigma x_2\\ \sigma x_3 & \sigma x_4\end{pmatrix}=\begin{pmatrix}x_4 & -x_2\\ -x_3 & x_1\end{pmatrix},\qquad \text{thus } x=\begin{pmatrix}x_1 & r_2\sqrt{A}\\ r_3\sqrt{A} & \sigma x_1\end{pmatrix}$$

with $x_1\in S_m$ and $r_2,r_3\in R_m$. The relation $x(\overline{u}_{j\varepsilon}-\overline{a})=\sigma(\overline{u}_{j\varepsilon}-\overline{a})x$ implies

$$\begin{aligned}\begin{pmatrix}\overline{b}'r_2\sqrt{A} & x_1\overline{b}'\overline{D}'\\ \overline{b}\sigma(x_1) & \overline{b}'\overline{D}'r_3\sqrt{A}\end{pmatrix}&=\begin{pmatrix}x_1 & r_2\sqrt{A}\\ r_3\sqrt{A} & \sigma(x_1)\end{pmatrix}\begin{pmatrix}0 & \overline{b}'\overline{D}'\\ \overline{b}' & 0\end{pmatrix}\\ &=\begin{pmatrix}0 & \sigma\overline{b}'\cdot\sigma\overline{D}'\\ \sigma\overline{b}' & 0\end{pmatrix}\begin{pmatrix}x_1 & r_2\sqrt{A}\\ r_3\sqrt{A} & \sigma(x_1)\end{pmatrix}=\begin{pmatrix}\sigma(\overline{b}')\sigma(\overline{D}')r_3\sqrt{A} & \sigma(x_1)\sigma(\overline{b}')\sigma(\overline{D}')\\ x_1\sigma(\overline{b}') & \sigma(\overline{b}')r_2\sqrt{A}\end{pmatrix}.\end{aligned}$$

This relation consists of four relations, which we denote by $(u,v)=(\text{row, column}), 1\le u,v\le 2$.

We claim that there is $\eta'=\sigma(\eta)/\eta$, with $\eta\in S_m^\times$, and even $\eta\in S_m^{\times 2}$ unless E/F is unramified and $j=0$, such that x lies in $\left(\begin{smallmatrix}1&0\\0&\eta'\end{smallmatrix}\right)Z_{GL(2,S_m)}(\overline{u}_{j\varepsilon})$, namely $\sigma\overline{u}_{j\varepsilon}=\left(\begin{smallmatrix}1&0\\0&\eta'\end{smallmatrix}\right)\overline{u}_{j\varepsilon}\left(\begin{smallmatrix}1&0\\0&\eta'\end{smallmatrix}\right)^{-1}$, or $\left(\begin{smallmatrix}1&0\\0&\eta\end{smallmatrix}\right)^{-1}\overline{u}_{j\varepsilon}\left(\begin{smallmatrix}1&0\\0&\eta\end{smallmatrix}\right)\in GL(2,R_m)$. For this purpose, note that the relation $(2,1)$ implies that $\sigma x_1\equiv x_1\sigma(\overline{b}')/\overline{b}'(\mathrm{mod}\,\boldsymbol{\pi}_3^{m'-\nu})$, while $(2,2)$ implies that $r_2\sqrt{A}\equiv r_3\sqrt{A}\cdot\overline{D}'\overline{b}'/\sigma(\overline{b}')(\mathrm{mod}\,\boldsymbol{\pi}_3^{m'-\nu})$. Put $\eta'=\sigma(\overline{b}')/\overline{b}'\in S_m^\times$. In other words, for some f,g,g' in R_3 we have

$$\begin{aligned}x&=\begin{pmatrix}x_1 & r_3\sqrt{A}\overline{D}'\overline{b}'/\sigma(\overline{b}')+\boldsymbol{\pi}_3^{m'-\nu}f\\ r_3\sqrt{A} & x_1\sigma(\overline{b}')/\overline{b}'+\boldsymbol{\pi}_3^{m'-\nu}g'\end{pmatrix}\\ &=\begin{pmatrix}1 & 0\\ 0 & \eta'\end{pmatrix}\begin{pmatrix}x_1 & (\overline{b}'/\sigma\overline{b}')r_3\sqrt{A}\cdot\overline{D}'+\boldsymbol{\pi}_3^{m'-\nu}f\\ (\overline{b}'/\sigma\overline{b}')r_3\sqrt{A} & x_1+\boldsymbol{\pi}_3^{m'-\nu}g\end{pmatrix}.\end{aligned}$$

If E_3/F is unramified then $\boldsymbol{\pi}_3=\boldsymbol{\pi}, \overline{b}'=\overline{B}'\boldsymbol{\pi}^\nu$, and so $\eta'=\sigma(\overline{B}')/\overline{B}'$. Using $\|x\|=1$ we note that

$$\overline{b}'/\sigma\overline{b}'=x_1\cdot(\overline{b}'/\sigma\overline{b}')\sigma x_1-(\overline{b}'/\sigma\overline{b}')r_2\sqrt{A}\cdot r_3\sqrt{A}.$$

Again $(2,1)$: $(\overline{b}'/\sigma\overline{b}')\sigma x_1\equiv x_1$, and $(1,1)$: $(\overline{b}'/\sigma\overline{b}')r_2\sqrt{A}\equiv\sigma(\overline{D}')r_3\sqrt{A}$, imply that $\overline{b}'/\sigma\overline{b}'$ lies in $x_1^2-r_3^2A\sigma(\overline{D}')+\boldsymbol{\pi}_3^{m'-\nu}S_m$, which is $x_1^2=\boldsymbol{\pi}_3S_m$ unless E/F is unramified and $j=0$. In this case (E/F unramified and $j=0$) x_1 lies in $S_m^\times$, and $(2,1)$ implies that $\sigma\overline{b}'/\overline{b}'\equiv\sigma x_1/x_1(\mathrm{mod}\,\boldsymbol{\pi}_3^{\nu'-m})$. Together with $\overline{b}'/\sigma\overline{b}'\equiv x_1^2$, we obtain $x_1\sigma x_1\equiv 1(\mathrm{mod}\,\boldsymbol{\pi}_3)$. If $x_1=\alpha_1+\beta_1\sqrt{A}, A=\boldsymbol{\pi}$, then $\alpha_1^2\equiv 1(\mathrm{mod}\,\boldsymbol{\pi})$, and $\eta'=\sigma x_1/x_1=\sigma\eta/\eta, \eta=1+(\beta_1/\alpha_1)\sqrt{A}\in S_m^{\times 2}$.

If E_3/F is unramified, the norm map $N=N_{E_3/F}$ induces a surjection from $\mathbb{F}_q^\times$, where $\mathbb{F}_q$ is the residue field of R_3, to $\mathbb{F}_{q_0}^\times$, where $\mathbb{F}_{q_0}$ is the residue field of R. Note that $q=q_3$ is q_0^2 here. Hence $\ker(N|R_3^\times)$ has index q_0+1 in $R_3^\times$, so it is contained in the subgroup $R_3^{\times 2}$ of index 2 in $R_3^\times$, hence $x_1\in S_m^{\times 2}$ (unless E/F is unramified and $j=0$).

In particular, if E_3/F is ramified, since $\overline{b}' = \overline{B}'\boldsymbol{\pi}_3^{\nu}$, $(2,1)$ implies that $x_1/\sigma x_1 \equiv (\overline{B}'/\sigma\overline{B}')$ $(-1)^{\nu}$. As $S_m^{\times} \cap R_m\sqrt{A}$ is empty, $\mathrm{Re}(x_1/\overline{B}') = x_1/\overline{B}' + \sigma(x_1/\overline{B}')$ is non zero, and equal to itself times $(-1)^{\nu}$. Then ν is even when $e_3 = 2$.

Let us show that if $m' > 2N - \nu + \mathrm{ord}_3 D (\geq \nu)$, then $m' \leq X - \nu$. We shall use the auxiliary result, that there is $\eta \in S_m^{\times}$ such that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \eta \end{smallmatrix}\right)\left(\begin{smallmatrix} \overline{a} & \overline{b}'\overline{D}' \\ \overline{b}' & \overline{a} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & \eta \end{smallmatrix}\right)^{-1}$ is in $GL(2, R_m)$. Namely $\eta\overline{b}' = \sigma(\eta\overline{b}')$ and $\overline{b}'\overline{D}'/\eta = \sigma(\overline{b}'\overline{D}'/\eta)$. Recall that $b' = (B/\varepsilon)\boldsymbol{\pi}_3^{\nu}$ and $D' = D\varepsilon^2\boldsymbol{\pi}_3^{2j}$, so $b'D' = DB\varepsilon\boldsymbol{\pi}_3^{2N-\nu}$. $[b' = (B/\varepsilon u\varepsilon_3^j)\boldsymbol{\pi}_3^{\nu}$, $D' = D\varepsilon^2 u^2(\varepsilon_3\boldsymbol{\pi}_3)^{2j}$, $b'D' = BD\varepsilon u\varepsilon_3^j\boldsymbol{\pi}_3^{2N-\nu}]$. If $m' > \nu$, then $(B/\varepsilon)/\sigma(B/\varepsilon) \equiv \sigma\eta/\eta(\mathrm{mod}\,\boldsymbol{\pi}_3^{m'-\nu})$. If $m' > 2N - \nu + \mathrm{ord}_3 D$, then $BD\varepsilon/\tau(BD\varepsilon) \equiv \eta/\sigma\eta(\mathrm{mod}\,\boldsymbol{\pi}_3^{m'-(2N-\nu+\mathrm{ord}_3 D)})$. [Replace ε by $\varepsilon u\varepsilon_3^j$]. Since $m' - \nu > m' - (2N - \nu + \mathrm{ord}_3 D)$, together we have $DB^2/\sigma(DB^2) \equiv 1(\mathrm{mod}\,\boldsymbol{\pi}_3^{m'-(2N-\nu+\mathrm{ord}_3 D)})$, namely $(a - \sigma a)(a + \sigma a) = a^2 - \sigma a^2 \equiv Db^2 - \sigma(Db^2) \equiv 0(\mathrm{mod}\,\boldsymbol{\pi}_3^{m'+\nu})$. Since a is topologically unipotent, $|a - \sigma a| \leq |a + \sigma a| = 1$, and so $X \geq m' + \nu$ as asserted. In particular, $X \geq 2N + \mathrm{ord}_3 D$.

In fact, unless E/F is unramified and $j = 0$, we have that η lies in $S_m^{\times 2}$. Then $m' > \nu$ implies that $\eta\overline{B}/\varepsilon \in R_m^{\times}$, or $\varepsilon \in \overline{B}\eta R_m^{\times} \subset \overline{B}S_m^{\times 2}R_m^{\times}$. [Replace ε by $\varepsilon u\varepsilon_3^j$]. If $e_3 = 2$ then $R_m^{\times}S_m^{\times 2}$ is $S_m^{\times}$ and no new information on ε is obtained. But when $e_3 = 1$ we have $R_m^{\times}S_m^{\times 2} = S_m^{\times 2}$. Hence when $j \geq 1$ and E/F is unramified, there are two choices for ε, but only one contributes to our orbital integral, namely $\varepsilon \in \overline{B}S_m^{\times 2}$, or $\varepsilon \in BR_3^{\times 2}$ (the two possibilities for ε were in $R_3^{\times}/R_3^{\times 2}$). [Replace ε by $\varepsilon u\varepsilon_3^j$]. Note that in case (IV), if $e(E/E_3) = 1$ then $e(E_3/F) = 1$ ($e(E/E_3) = 1$ and $e(E_3/F) = 2$ is case (II)). If $e(E/E_3) = 2$ or $j = 0$, then ε can be taken to be any representative of $R_3^{\times}/R_3^{\times}$, so we obtain no constraint on ε. [If $e(E/E_3) = 2$ then $\varepsilon = 1$ and we get $u \in B\varepsilon_3^j R_3^{\times 2}$].

It remains to compute the cardinality of L_m when $\nu < m' \leq \chi - \nu$. First, L_m consists of $x = \left(\begin{smallmatrix} x_1 & (\overline{B}'/\sigma\overline{B}')\overline{D}' r_3\sqrt{A}+a \\ r_3\sqrt{A} & \sigma x_1 \end{smallmatrix}\right)$ such that $\|x\| = 1, a \in \boldsymbol{\pi}_3^{m'-\nu}S_m \cap R_m\sqrt{A}; r_1, r_3 \in R_m, x_1 = \overline{B}' r_1(1+\delta), \delta \in \boldsymbol{\pi}_3^{m'-\nu}S_m$. Here we used the relation $r_2\sqrt{A} \equiv r_3\sqrt{A}\cdot\overline{D}'\overline{b}'/\sigma\overline{b}'(\mathrm{mod}\,\boldsymbol{\pi}_3^{m'-\nu})$, and $x_1/\overline{B}' = r_1 + \boldsymbol{\pi}_3^{m'-\nu}S_m$ for some $r_1 \in R_m$. Consequently, L_m is the set of $(r_1, r_3, a, \delta) \in R_m^2 \times (\boldsymbol{\pi}_3^{m'-\nu}S_m)^2$, such that $\sigma a = -a$, and $r_1^2\overline{B}'\sigma\overline{B}'(1+\delta)(1+\sigma\delta) - r_3^2 A(\overline{B}'/\sigma\overline{B}')\overline{D}' - ar_3\sqrt{A} = 1$, taken under the quotient by the equivalence relation $(r_1, \delta) \sim (r_1', \delta')$ if $r_1(1+\delta) = r_1'(1+\delta')$, in other words we take the quotient by $1 + R_m \cap \boldsymbol{\pi}_3^{m'-\nu}S_m$.

To count the number of elements in L_m, we need to solve the defining equation. Thus we take any $r_3 \in R_m, \delta \in \boldsymbol{\pi}_3^{m'-\nu}R_3/\boldsymbol{\pi}_3^{m'}R_3 = R_3/\boldsymbol{\pi}_3^{\nu}R_3$, and $a = \alpha\sqrt{A}, \alpha$ in $R_m \cap \boldsymbol{\pi}_3^{m'-\nu-\mathrm{ord}_3(\sqrt{A})}S_m \simeq R_m \cap \boldsymbol{\pi}_3^{m'-\nu}S_m$ (when $e_3 = 1, A \in R^{\times}$; when $e_3 = 2, \nu$ is even and $A = \boldsymbol{\pi} = \boldsymbol{\pi}_3^2$). If $j \geq 1$ or E/F is ramified, we saw above that $\overline{b}'/\sigma\overline{b}' = x_1^2 + \boldsymbol{\pi}_3 S_m$. Since $x_1 = \overline{B}' r_1 + \boldsymbol{\pi}_3^{m'-\nu}S_m$, we have $\overline{B}'/\sigma\overline{B}' \equiv \overline{B}'^2 r_1^2$, namely $1 \equiv \overline{B}'\sigma\overline{B}' r_1^2$, so that there are two solutions in r_1 to the equation which defines L_m (as L_m is non empty; note that $N_{E_3/F}R_3^{\times} = R^{\times e_3}$, $R^{\times}$ is contained in $R_3^{\times\, 2/e_3}$). We conclude that L_m consists of $2q_0^m q^{\nu}$ elements (2 for r_1, q^{ν} for δ, q_0^m for r_3, a cancels the relation $\sim$). This completes the proof of the lemma when $j \geq 1$ or E/F is ramified.

Suppose now that $j = 0$ and E/F is unramified (and $m' > \nu$). We claim that L_m is empty. If not, let x be in L_m. The relations $(1,1)$ and $(2,2)$ imply that $r_2\sqrt{A} \equiv r_3\sqrt{A}\cdot Db/\sigma b \equiv r_3\sqrt{A}\cdot\sigma D\sigma b/b(\mathrm{mod}\,\boldsymbol{\pi}_3^{\nu'-m})$. Note that $D' = D$ and $b' = b, B' = B$, when $j = 0$. If $r_3 \neq 0$

in R_m, then $(b/\sigma b)^2 D \equiv \sigma D = \zeta^2 D$, where $\zeta = \sqrt{\sigma D/D} \in R_3^\times$. Hence $\sigma b/b = \pm\zeta$, and so $b = r\sqrt{D}, r \in R$ (or $b = r/\sqrt{D}$). This is impossible, since $b \in R_3$ (and $\sqrt{D} \notin R_3$). If $r_3 = 0$ in R_m, then $r_2 = 0$ in R_m, and $\|x\| = 1$ implies that $1 \equiv x_1\sigma x_1$, hence $x_1 \in S_m^\times$. Then $(2,1)$ implies that $b/\sigma b \equiv x_1/\sigma x_1$, and $(1,2)$ that $b/\sigma b \equiv (\sigma x_1/x_1)(\sigma D/D)$. Together $(b/\sigma b)^2 \equiv \sigma D/D$. But $\sigma D/D = \zeta^2$, so $\sigma b/b = \pm\zeta = \pm\sigma\sqrt{D}/\sqrt{D}$, so $b = r\sqrt{D}$ or $b = r/\sqrt{D}$, with $r \in R$, and $b \notin R_3$, a contradiction. The lemma follows. □

This completes our discussion of the orbital integral in case (IV). The twisted θ-orbital integral of a strongly θ-regular topologically unipotent element of type (IV) is a sum of two integrals, which can be reduced as usual to orbital integrals $\Phi_{1_K}^{Sp(2,F)}(t_\rho)$ in $Sp(2,F)$. The sum of these integrals is the stable orbital integral of 1_K at t_1 (or $t_\rho, \rho \in E_3^\times/N_{E/E_3}E^\times$), on $Sp(2,F)$. It coincides with the orbital integral of 1_K at any element in the stable orbit, on $GSp(2,F)$ (the stable orbit on $Sp(2,F)$ is the intersection with $Sp(2,F)$ of the orbit in $GSp(2,F)$). Consequently, to show that $\Phi_{1_K}^{GL(4,F)\times GL(1,F),st}(t\theta)$ is equal to $\Phi_{1_K}^{GSp(2,F)}(Nt)$, we simply need to observe that both $\Phi_{1_K}^{GSp(2,F)}(t)$ and $\Phi_{1_K}^{GSp(2,F)}(Nt)$ depend only on the parameters N and X attached to Nt (and n, χ attached to t). Since $n = N$ and $\chi = X$, the comparison is complete, once we show that the measure factor in the case of type (IV) is equal to one. This we do next.

7. Lemma. *For tori of type (IV), the measure factor* $[T^{*\theta}(R) : (1+\theta)T^*(R)]/[T_H^*(R) : NT^*(R)]$ *is equal to* 1.

Proof. First we compute the index in $T_H^*(R) = \{(x, \sigma^3 x, \sigma x, \sigma^2 x); x \in R_{E'}^\times; x\sigma^2 x = \sigma(x\sigma^2 x)\}$ of the image $NT^*(R) = \{N(a, \sigma a, \sigma^3 a, \sigma^2 a) = (a\sigma a, a\sigma^3 a, \sigma a\sigma^2 a, \sigma^3 a\sigma^2 a); a \in R_E^\times\}$ of $T^*(R)$.

Note that the extension E/F of degree 4 is Galois, in which case $E' = E$, except in the totally ramified case, where $E = F(\sqrt{\sqrt{\boldsymbol{\pi}}})$, and $-1 \notin R^{\times 2}$. In this case $\sigma\sqrt{\boldsymbol{\pi}} = -\sqrt{\boldsymbol{\pi}}$ and $\sigma\sqrt{\sqrt{\boldsymbol{\pi}}} = \zeta\sqrt{\sqrt{\boldsymbol{\pi}}}$, where $\zeta = \sqrt{-1} \notin R_E^\times$, and $E' = F(\sqrt{\sqrt{-\boldsymbol{\pi}}})$. Let us verify this, namely that if $a \in E$, then $a\sigma a \in E'$. Write $a = a_1 + a_2\sqrt{\sqrt{\boldsymbol{\pi}}}$, with $a_i = b_i + c_i\sqrt{\boldsymbol{\pi}}$. Then $a\sigma a = \big(a_1 + a_2\sqrt{\sqrt{\boldsymbol{\pi}}}\big)\big(\overline{a}_1 + \overline{a}_2\sqrt{\sqrt{-\boldsymbol{\pi}}}\big) = a_1\overline{a}_1 + a_2\overline{a}_2\sqrt{-\boldsymbol{\pi}} + (\overline{a}_1 a_2 + \zeta a_1\overline{a}_2)\sqrt{\sqrt{\boldsymbol{\pi}}}$, where $\overline{a}_i = b_i - c_i\sqrt{\boldsymbol{\pi}}$. So $a_i\overline{a}_i \in F^\times$, we write $\sqrt{\sqrt{\boldsymbol{\pi}}}$ as the product of $\sqrt{\sqrt{-\boldsymbol{\pi}}}$ and $a\sqrt{\zeta}$, and it remains to show that $(\overline{a}_1 a_2 + \zeta a_1\overline{a}_2)/\sqrt{\zeta}$ lies in $F(\sqrt{-\boldsymbol{\pi}})$. Note that since $-1 \notin R^{\times 2}$, one of 2 and -2 is in $R^{\times 2}$, and $\zeta = \big((1\pm\zeta)/\sqrt{\pm 2}\big)^2$. To simplify the notations, suppose that $2 \in R^{\times 2}$. Then $\sqrt{\zeta} = (1+\zeta)/\sqrt{2}$, and $1/\sqrt{\zeta} = (1-\zeta)/\sqrt{2}$. Then the sum of

$$\overline{a}_1 a_2(1-\zeta) = (b_1 - c_1\sqrt{\boldsymbol{\pi}})(b_2 + c_2\sqrt{\boldsymbol{\pi}})(1-\zeta) = \big(b_1 b_2 - c_1 c_2\boldsymbol{\pi} + (b_1 c_2 - b_2 c_1)\sqrt{\boldsymbol{\pi}}\big)(1-\zeta)$$

and

$$a_1\overline{a}_2(1+\zeta) = (b_1 + c_1\sqrt{\boldsymbol{\pi}})(b_2 - c_2\sqrt{\boldsymbol{\pi}})(1+\zeta) = \big(b_1 b_2 - c_1 c_2\boldsymbol{\pi} - (b_1 c_2 - b_2 c_1)\sqrt{\boldsymbol{\pi}}\big)(1+\zeta)$$

is $2b_1 b_2 - 2c_1 c_2\boldsymbol{\pi} - 2(b_1 c_2 - b_2 c_1)\sqrt{-\boldsymbol{\pi}}$. It lies in $F(\sqrt{-\boldsymbol{\pi}})$ as required.

Thus we need to solve in $a \in R_E^\times$ the equation $x = a\sigma a$, where $x \in R_{E'}^\times$ satisfies $x\sigma^2 x = \sigma(x\sigma^2 x)$. For this, note that the product $a\sigma^2 a\sigma(a\sigma^2 a), (a \in R_E^\times)$, ranges over $R^{\times 4}$ when

$D=\sqrt{\boldsymbol{\pi}}$, over $R^{\times 2}$ when E/E_3 is ramified and E_3/F is unramified ($D\in R^\times$), over $R^\times$ if E/F is unramified (we simply use the fact that in a quadratic extension K/k, $N_{K/k}R_K^\times$ is $R_k^\times$ if K/k is unramified, and $R_k^{\times 2}$ if K/k is ramified, or $R_k^{\times e(K/k)}$ in general). For the same reason, $x\sigma^2 x, (x\in R_E^\times)$, ranges over $R_3^{\times 2}$ if E/E_3 is ramified, and over $R_3^\times$ if E/E_3 is unramified. If further $x\sigma^2 x$ is fixed under σ, then the σ-fixed $x\sigma^2 x$ ranges over: $R^{\times 2}$ if E/E_3 and E_3/F are both ramified, $R^\times$ if E_3/F is unramified. Indeed, $(\alpha+\beta\sqrt{D})^2=\alpha^2+\beta^2 D+2\alpha\beta\sqrt{D}$ is σ-fixed precisely when $\alpha\beta=0$. When $D=\boldsymbol{\pi}$, $\alpha+\beta\sqrt{D}\in R_3^\times$ only when $\alpha\in R^\times,\beta\in R$, thus we have $\beta=0$, and the σ-fixed elements of $R_3^{\times 2}$ are $R^{\times 2}$. If $D\in R^\times$ then the σ-fixed elements of $R_3^{\times 2}$ are $R^{\times 2}\cup DR^{\times 2}=R^\times$. In conclusion, the index $[T_H^*(R):NT^*(R)]$ is equal to $1=[R^\times:R^\times]$ when E/F is unramified, to $[R^\times:R^{\times 2}]=2$ when $D\in R^\times$ and E/E_3 is ramified, and to $[R^{\times 2}:R^{\times 4}]=2$ when both E_3/F and E/E_3 are ramified, namely to the ramification index $e(E/E_3)$ in all cases.

We also need to compute the index in $T^{*\theta}(R)=\{(x,\sigma x,\sigma^3 x,\sigma^2 x); x\in R_E^\times, x\sigma^2 x=1\}$ of

$$(1+\theta)T^*(R)=\{(1+\theta)(a,\sigma a,\sigma^3 a,\sigma^2 a)=(a/\sigma^2 a,\sigma a/\sigma^3 a,\sigma^3 a/\sigma a,\sigma^2 a/a); a\in R_3^\times\}.$$

Since $x\sigma^2 x=1$, there is a solution $a\in E^\times$ to $x=a/\sigma^2 a$, and as usual, the index of $\{a/\sigma^2 a; a\in R_E^\times\}$ in $\{a/\sigma^2 a; a\in E^\times\}$ is $e(E/E_3)$. The lemma follows. $\square$

With this, the comparison in the case of type (IV) is complete. But for completeness, and possible future applications, we now write out this integral. I am grateful to J.G.M. Mars for pointing out errors in an earlier version of the formulae below, and suggesting corrections. It is best to deal separately with three cases: When $e(E/E_3)=e(E_3/F)=2$, when $e(E/E_3)=2, e(E_3/F)=1$, and when $e(E/F)=1$.

The stable θ-orbital integral of 1_K at a strongly θ-regular topologically unipotent element $u=\theta(u)$ of $GL(4,F)\times GL(1,F)$ of type (IV), with invariants n,χ, is given by the following expressions.

If $e(E_3/F)=2$, then $e(E/E_3)=2$, and we get a sum over $m'=2m$ and $0\le\nu\le n$, of $q^{n-\nu}$ times: $q^{3m'/2}=q^{3m}$ if $1\le m'\le\nu$, and $q_0^m q^{\nu'}$ if $\nu'=2\nu$ and $\nu'<m'\le\chi-\nu'$. Since $q=q_0$ (also note that $\chi=2n+1$ in this case), our sum is

$$\begin{aligned}
&q^n\sum_{0\le\nu\le n}q^{-\nu}\sum_{0\le m<\nu/2}q^{3m}+q^n\sum_{0\le\nu\le n/2}\Big(\sum_{\nu\le m\le\chi/2-\nu}q^m\Big)\\
&=\sum_{0\le m\le(n-1)/2}q^{3m}\sum_{0\le j\le n-2m-1}q^j+q^n\sum_{0\le\nu\le n/2}\frac{q^{[\chi/2]+1-\nu}-q^\nu}{q-1}\\
&=\frac{q^n(q^{[(n+1)/2]}-1)}{(q-1)^2}-\frac{q^{3[(n+1)/2]}-1}{(q-1)(q^3-1)}+q^n\frac{(q^{1+[\chi/2]-[n/2]}-1)(q^{[n/2]+1}-1)}{(q-1)^2}.
\end{aligned}$$

If $e(E_3/F)=1$ and $e(E/E_3)=2$, we have $q=q_0^2$, and $m'=m$. Our sum is then over $m(0\le m\le\chi)$ and $\nu(0\le\nu\le n)$ of the product of $q^{n-\nu}$, and of: 1 if $m=0$, $(1+q^{-1})q^{3m/2}$ if $1\le m\le\nu, 2q^{\nu+m/2}$ if $\nu<m\le\chi-\nu$. (Note that $\chi=2n+1$ in this case). Namely we have

$$\sum_{0\le\nu\le n}q^{n-\nu}\left[1+(1+q^{-1})\sum_{1\le m\le\nu}q^{3m/2}+2q^\nu\sum_{\nu<m\le\chi-\nu}q^{m/2}\right],$$

which is

$$\frac{q^{n+1}-1}{q-1}+\frac{q^{1/2}(q+1)}{q^{3/2}-1}\Big[q^n\frac{q^{(n+1)/2}-1}{q^{1/2}-1}-\frac{q^{n+1}-1}{q-1}\Big]+\frac{q^{n+\frac12}(q^{(n+1)/2}-1)(q^{(\chi-n)/2}-1)}{(q^{1/2}-1)^2}.$$

Finally, when E/F is unramified, $q=q_0^2$, $m'=m$ and the sum ranges over $0\le\nu\le n$, and $0\le m\le\chi-\nu$. (In this case $\chi=2n$). It takes the form

$$\begin{aligned}\frac{q+1}{q}\sum_{0\le\nu<n}q^{n-\nu}\big(1+&\sum_{1\le m\le\nu}(1+q^{-1})q^{3m/2}+q^{\nu}\sum_{\nu<m\le\chi-\nu}q_0^m\big)+1+\sum_{1\le m\le n}(1+q^{-1})q^{3m/2}\\ =&\frac{(q+1)q^n-2}{q-1}+\frac{q^{n-\frac12}(q+1)(q^{n/2}-1)(q^{(\chi-n+1)/2}-1)}{(q^{1/2}-1)^2}\\ &+\frac{q^{1/2}(q+1)^2}{(q^{3/2}-1)}\Big[q^{n-1}\frac{q^{n/2}-1}{q^{1/2}-1}-\frac{q^n-1}{q-1}\Big]+q^{1/2}(q+1)\frac{q^{3n/2}-1}{q^{3/2}-1}.\end{aligned}$$

To repeat, we have no use for these explicit expressions, except the observation that the final expression depends only on the parameters n and χ attached to u, since the parameters N and X attached to the norm Nu of u are equal to n,χ.

This completes our discussion of the comparison of the stable θ-orbital integral of 1_K at the strongly θ-regular element $us\theta=s\theta u$ of $GL(4,R)\times GL(1,R)$, with the stable orbital integral of 1_K at the norm $N(us)$ of us, in the case where $s=I$. It remains to compare these integrals when s is not (stably) θ-conjugate to the identity.

Unstable twisted case. Twisted endoscopic group of type I.F.2.

The computations of the θ-orbital integrals of a strongly θ-regular topologically unipotent element $t'=h^{-1}t^*\theta(h), t^*=(t,\sigma t,\sigma^3 t,\sigma^2 t)$, of type (IV), can be used to compute the κ-θ-orbital integral too. In this case κ is the non trivial character of the group $E_3^\times/N_{E/E_3}E^\times$, and it defines the endoscopic group $\mathbf{C}_3=\mathbf{C}_{E_3}$. The Jacobian factor is

$$\Delta_{G,C_3}(t,\sigma t,\sigma^3 t,\sigma^2 t)=|\frac{(t-\sigma^2t)^2\sigma(t-\sigma^2 t)^2}{t\sigma t\sigma^2 t\sigma^3 t}|_F^{1/2}=|b\sqrt{D}|_3=q^{-n}|D|_3^{1/2}.$$

Note that $|b|_3=q^{-n}$, as $n=\mathrm{ord}_3(b)$, while $|D|=1$ when E/E_3 is unramified, or $|D|_3=q^{-1}$ when E/E_3 is ramified.

Theorem. *If $t'=h^{-1}t^*\theta(h)$ is a strongly θ-regular topologically unipotent element of type (IV), then $\Phi^\kappa_{1_K}(t'\theta)$ is 0 if E_3/F is ramified, while if E_3/F is unramified then*

$$\Delta_{G,C_3}(t^*)\kappa\big((t-\sigma^2 t)/2\sqrt{D}\big)\Phi^\kappa_{1_K}(t\theta)=\Phi^{C_3}_{1_{K_3}}(N_{C_3}t^*).$$

Proof. Note that $\kappa\big((t-\sigma^2t)/2\sqrt{D}\big)=\chi_{E/E_3}(b)=\chi_{E/E_3}(B\boldsymbol{\pi}_3^n)$. When E/E_3 is ramified and E_3/F is unramified, we take $D=-\boldsymbol{\pi}\varepsilon_3(\varepsilon_3\in R_3^\times-R_3^{\times 2})$, then $N_{E/E_3}(\sqrt{D})=\boldsymbol{\pi}\varepsilon_3$, and so

$N_{E/E_3}E^\times = (\boldsymbol{\pi}\varepsilon_3)^{\mathbb{Z}} R_3^{\times 2}$, and $\chi_{E/E_3} : E_3^\times / N_{E/E_3}E^\times \tilde{\to} \{\pm 1\}$ has $\chi_{E/E_3}(\varepsilon_3) = \chi_{E/E_3}(\boldsymbol{\pi}) = -1$. Then $\chi_{E/E_3}(B\boldsymbol{\pi}_3^n) = \chi_{E/E_3}(b)(-1)^n$. When E/F is unramified, $D = \varepsilon_3, N_{E/E_3}R_E^\times = R_3^\times, \chi_{E/E_3}(B) = 1, \chi_{E/E_3}(\boldsymbol{\pi}_3) = -1$, and $\chi_{E/E_3}(B\boldsymbol{\pi}_3^n) = (-1)^n$. Moreover, the norm $N_{C_3}t^*$ is the elliptic element in $C_3 = GL(2, E_3)$ with eigenvalues $x = t\sigma t$ and $\sigma^2 x$. As $x = A_* + B_*(1+\zeta)\sqrt{D}$ and $|B_*|_3 = q^{-n}$ by Lemma 1, Lemma I.I.2 implies that the right hand side is $(q^{n+1}-1)/(q-1)$ when E/E_3 is ramified, and it is $\big((q+1)q^n - 2\big)/(q-1)$ when E/E_3 is unramified. We then turn now to the computation of $\Phi_{1_K}^\kappa(t\theta)$.

When E/E_3 and E_3/F are both ramified, as $N_{E/E_3}E^\times = \boldsymbol{\pi}_3^{\mathbb{Z}} R_3^{\times 2}$, the unstable θ-integral includes a sum over $\rho \in E_3^\times / N_{E/E_3}E^\times = R_3^\times / R_3^{\times 2}$ of $\kappa(\rho)$, while no other term depends on ρ. Hence $\kappa(1) + \kappa(\varepsilon_3) = 0$, and $\Phi_{1_K}^\kappa(t\theta)$ is zero in this case.

When E/E_3 is ramified and E_3/F is unramified, in addition to the sums over ν and m which appear in the stable θ-orbital integral, we have an additional sum over $\rho = u \in E_3^\times / N_{E/E_3}E^\times = R_3^\times / R_3^{\times 2}$, of $\kappa(u)$ times the terms indexed by ν, m (and we need to divide at the end by 2, a measure factor). If $0 \le m \le \nu$, the term indexed by ν, m is independent of u, and $\kappa(1)+\kappa(\varepsilon_3) = 0 (\varepsilon_3 \in R_3^\times - R_3^{\times 2})$. If $\nu < m \le \chi - \nu$, then we have the relation $u \in B\varepsilon_3^j R_3^{\times 2}$, and $\kappa(\varepsilon_3) = -1$, hence $\kappa(u) = \kappa(B)\kappa(\varepsilon_3)^j = \kappa(B)(-1)^j$. The κ-θ-orbital integral is then

$$\begin{aligned}
\kappa(B) \sum_{0 \le \nu \le n} \sum_{\nu < m \le \chi - \nu} (-q)^{n-\nu} 2 q^\nu q_0^m &= 2\kappa(B)(-q)^n \sum_{0 \le \nu \le n} (-1)^\nu (q_0^{\chi - \nu + 1} - q_0^{\nu+1})/(q_0 - 1) \\
&= \big(2\kappa(B)(-q)^n q_0/(q_0-1)\big)\Big(q_0^\chi \frac{1-(-q_0)^{-n-1}}{1-(-q_0)^{-1}} - \frac{(-q_0)^{n+1}-1}{-q_0-1}\Big) \\
&= 2\kappa(B) q_0 (-q)^n \big(1 - q_0^{\chi - n}(-1)^n\big)\big(1 + q_0^{n+1}(-1)^n\big)/(1-q_0^2) \\
&= 2\kappa(B) q_0 (-q)^n (q^{n+1}-1)/(q-1),
\end{aligned}$$

since $q = q_0^2$ and $\chi = 2n+1$ in our case. The theorem follows in this case too.

It remains to deal with the case where E/F is unramified. Since $N_{E/E_3}E^\times = R_3^\times \boldsymbol{\pi}_3^{2\mathbb{Z}}$ we have that $\rho = \boldsymbol{\pi}^{\overline{\rho}}, \overline{\rho}$ ranges over $\{0, 1\}$. The decomposition of $SL(2, E_3)$ was such that $j \ge 0$ and 2 divides $j - \overline{\rho}$, and when $j \ge 1$ we have the additional sum over $\varepsilon \in R_3^\times / R_3^{\times 2}$. In summary we have a sum over $\nu (0 \le \nu \le n)$, of 1 if $\nu = n$, and of $\big((q+1)/2q\big)(-q)^{n-\nu}$ if $0 \le \nu < n$, and a sum over m, of 1 if $m = 0$, of $q_0^{3m}(1+q^{-1})$ if $1 \le m \le \nu$, both terms are multiplied by 2 (two ε's) if $\nu < N$, and of $2q_0^m q^\nu$ if $\nu < m \le \chi - \nu$, in which case $\varepsilon \in BR_3^{\times 2}$ (so we have only one ε). In other words we have the sum of

$$\Big(\sum_{\nu = n} 1 + \sum_{0 \le \nu < n} (q+1)q^{-1}(-q)^{n-\nu}\Big)\Big(\sum_{m=0} 1 + q^{-1}(q+1) \sum_{1 \le m \le \nu} q_0^{3m}\Big)$$

and

$$\sum_{0 \le \nu < n} \frac{q+1}{2q}(-q)^{n-\nu} \sum_{\nu < m \le \chi - \nu} 2 q_0^m q^\nu$$

(since $\chi = 2n$, the sum in the last row can extend to $0 \le \nu \le n$). The first sum adds up to

$$\sum_{0 \le m \le n} \sum_{m \le \nu \le n} = \Big(\sum_{m=0} 1 + \frac{q+1}{q} \sum_{1 \le m \le n} q_0^{3m}\Big)\Big(\sum_{\nu = n} 1 + \frac{q+1}{q} \sum_{m \le \nu < n} (-q)^{n-\nu}\Big).$$

The inner sum is $(-q)^{n-m}$, so we get

$$(-q)^n + \frac{q+1}{q}(-q)^n \sum_{1\le m\le n} (-q_0)^m = (-q)^n\big(1+\frac{(q+1)}{q_0}\frac{\big((-q_0)^n-1\big)}{q_0+1}\big)$$
$$= \frac{(-q)^n}{q_0+1}\big(\frac{q+1}{q_0}(-q_0)^n + 1 - q_0^{-1}\big).$$

The second sum adds up to

$$(q+1)q^{-1}(-q)^n \sum_{0\le\nu<n}(-1)^\nu \sum_{\nu<m\le\chi-\nu} q_0^m = \frac{(q+1)(-q)^n}{q_0(q_0-1)} \sum_{0\le\nu\le n}\big((-q_0)^{\chi-\nu}-(-q_0)^\nu\big).$$

As $\chi = 2n$, the inner sum is $\big((-q_0)^n-1\big)\sum_{0\le\nu\le n}(-q_0)^\nu$, and we finally get

$$\frac{(q+1)(-q)^n}{(q_0+1)q_0(q_0-1)}\big(q_0^{2n+1}+(-q_0)^{n+1}+(-q_0)^n-1\big).$$

The sum of our two sums is

$$\frac{(-q)^n}{q_0+1}\left[\frac{q+1}{q_0-1}q_0^{2n} + \frac{(q+1)(1-q_0)}{q_0(q_0-1)}(-q_0)^n - \frac{q+1}{q_0(q_0-1)} + 1 - q_0^{-1} + \frac{q+1}{q_0}(-q_0)^n\right]$$
$$= (-q)^n\frac{(q+1)q^n-2}{q-1},$$

since $q = q_0^2$, as required. □

PART III. Semi simple reduction.

A. Review.

To compute the stable θ-orbital integral of 1_K at a strongly θ-regular element t in $G = GL(4,R) \times GL(1,R)$, we may assume that the centralizer $T = Z_G(t)$ of t in G is a θ-invariant torus in G, and so the centralizer $Z_G(t\theta)$ of $t\theta$ in G is the centralizer T^θ of θ in T. Decomposing $t\theta$ as $t\theta = u \cdot s\theta = s\theta \cdot u$, a product of an absolutely semi simple element $s\theta$ and a topologically unipotent element u, which commute with each other, we deduce that $u \in T^\theta$. We have $\Phi^G_{1_K}(t\theta) = \Phi^G_{1_K}(us\theta) = \Phi^{Z_G(s\theta)}_{1_{Z_K(s\theta)}}(u)$; moreover, when t, t' are stably θ-conjugate, so are s, s', and if $s = s'$, then u, u' are stably θ-conjugate. Here $t'\theta = u's'\theta = s'\theta \cdot u'$.

The decomposition of the norm NT of t is $Ns \cdot Nu = Nu \cdot Ns$, where Ns is absolutely semi simple and Nu is topologically unipotent. Indeed, we expressed the tori T in the form $h^{-1}T^*h$, where T^* is the diagonal torus and $h = \theta(h) \in \mathbf{G}$. Correspondingly $t = h^{-1}t^*h, s = h^{-1}s^*h, u = h^{-1}u^*h$, and the norm is defined by $N(a,b,c,d;e) = (abe, ace, bde, cde; abcde^2)$, namely it is defined purely in terms of the (absolutely semi simple in the case of s^*, topologically unipotent in the case of u^*) entries of s^*, u^*. Hence $\Phi^H_{1_K}(Nt) = \Phi^{Z_H(Ns)}_{1_{Z_K(Ns)}}(Nu)$, and we are then reduced to the study of $\Phi^{Z_G(s\theta)}_{1_K}(u)$ and $\Phi^{Z_H(Ns)}_{1_K}(u)$.

We shall then distinguish the cases according to the values taken by Ns^*. The main case is that of $Ns = I$, dealt with above; here s is θ-conjugate to I and $Z_G(\theta) = Sp(2,F)$. We shall proceed now to deal with each of the θ-elliptic tori, of types (I) – (IV), and list the various possibilities for Ns^* other than I. Then we compute $Z_G(s\theta)$ $\big(\subset Z_G\big(s\theta(s)\big)\big)$ and the integral $\Phi^{Z_G(s\theta)}_{1_K}(u)$, as well as the centralizer $Z_H(Ns)$ and the integral $\Phi^{Z_H(Ns)}_{1_K}(Nu)$. Fortunately the centralizer $Z_G(s\theta)$ and $Z_H(Ns)$ are just various forms of groups closely related with $GL(2,F)$, whose orbital integrals are well known. To simplify the notations we note that the entry e in $GL(1,F)$ must be in $GL(1,R)$ for our integrals to be non zero, and then our integrals are independent of this e. Hence we take $e = 1$, and omit it from the notations.

B. Case of torus of type (I).

In our usual notations, $s = h^{-1}(s_1, s_2, \sigma s_2, \sigma s_1)h$, and so $s\theta(s) = h^{-1}(s_1/\sigma s_1, s_2/\sigma s_2, \sigma s_2/s_2, \sigma s_1/s_1)h$, where $\sigma h \cdot h^{-1} = \left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right) = (14)(23)$.

(1) Suppose that $s_1/\sigma s_1 = s_2/\sigma s_2 \neq \pm 1$. Then $Z_{\mathbf{G}(E)}\big(s\theta(s)\big)$ consists of $h^{-1}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)h; A, B \in GL(2,E)$. The subgroup $Z_{\mathbf{G}(E)}(s\theta)$ consists of $g = h^{-1}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)h$ with $gs\theta(g)^{-1} = s(= h^{-1}s^*h)$. Putting $f = -s_2/s_1 = -\sigma s_2/\sigma s_1 \in R^\times$, this relation amounts to $B = \|A\|^{-1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & f \end{smallmatrix}\right)^{-1}A\left(\begin{smallmatrix} 1 & 0 \\ 0 & f \end{smallmatrix}\right)$. The group $Z_G(s\theta)$ of F-rational points is determined by the relation $h^{-1}\left(\begin{smallmatrix} A & 0 \\ 0 & \|A\|^{-1}\tilde f^{-1}A\tilde f \end{smallmatrix}\right)h = \sigma h^{-1}\left(\begin{smallmatrix} \sigma A & 0 \\ 0 & \|\sigma A\|^{-1}\tilde f \cdot \sigma A \cdot \tilde f \end{smallmatrix}\right)\sigma h$, which amounts to $\|A\| \cdot \|\sigma A\| = 1$, and $\|A\|\sigma A = \tilde f w A w \tilde f^{-1}$, $\tilde f = \left(\begin{smallmatrix} 1 & 0 \\ 0 & f \end{smallmatrix}\right)$. Consequently $\|A\| = \alpha/\sigma\alpha$ for some $\alpha \in E^\times$, and $A' = \alpha^{-1}A$ satisfies $\sigma A' = \tilde f w A' w \tilde f^{-1}$, so $A' = \left(\begin{smallmatrix} \sigma a & \sigma b \\ f\sigma b & \sigma a \end{smallmatrix}\right)$ ranges over a group which is F-isomorphic to $GL(2,F)'$ if $f \in N_{E/F}E^\times$, or over an anisotropic inner form $D'^\times$ thereof if $f \in F - N_{E/F}E$. Here the prime indicates: determinant in $N_{E/F}E^\times$. Indeed, the determinant of $\alpha^{-1}A$ is $1/\alpha\sigma\alpha$.

The topologically unipotent element $u = h^{-1}(u_1, u_2, \sigma u_2, \sigma u_1)h$ lies in $Z_G(s\theta)$. It commutes with $s\theta$, and with $s \in T$, hence with θ. Also $h = \theta(h)$. Hence $u_1\sigma u_1 = 1, u_2\sigma u_2 = 1$, and so $u_i = \alpha_i/\sigma\alpha_i$, and $A = \left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right)$ has $\|A\| = \frac{\alpha_1\alpha_2}{\sigma(\alpha_1\alpha_2)}$. Then $\alpha^{-1}A = \left(\begin{smallmatrix} 1/\alpha_2\sigma\alpha_1 & 0 \\ 0 & 1/\alpha_1\sigma\alpha_2 \end{smallmatrix}\right)$.

The stable θ-orbital integral is the sum of θ-orbital integrals, parametrized by $(F^\times/NE^\times)^2$. Let us show that precisely two such θ-orbits intersect K. Recall that if $t\theta = us\theta = s\theta u$ and $t'\theta = u's'\theta = s'\theta u'$ are stably θ-conjugate then so are $s\theta, s'\theta$, and if $s = s'$ then u, u' are.

Lemma. *Only one θ-conjugacy class in the stable θ-conjugacy class of s intersects K.*

Proof. The element $s_1 = x + y\sqrt{D} \in R_E^\times(x, y \in R; D \in R^\times$ or $\boldsymbol{\pi}R^\times)$ is absolutely semi simple. Hence $|x| = |y| = |D| = 1$. Indeed, if $|yD| < 1$ then $x \in R^\times$ (since $s_1 \in R_E^\times$) and $s_1 = x(1 + y\sqrt{D}/x)$ has a non trivial topologically unipotent part $1 + y\sqrt{D}/x$, contrary to the uniqueness of the decomposition into absolutely semi simple and topologically unipotent parts. Moreover, $|x| = 1$. Indeed, if $|x| < 1$, then $s_1 = y\sqrt{D}(1 + x/y\sqrt{D})$ again has a non trivial topologically unipotent part. Now the group $F^\times/NE^\times$ is represented by $R = 1, \boldsymbol{\pi}$, and the θ-conjugacy classes within the stable θ-conjugacy class of s are represented by $[\left(\begin{smallmatrix} x & yDR_1 \\ y/R_1 & x \end{smallmatrix}\right), \left(\begin{smallmatrix} x & yDR_2 \\ y/R_2 & x \end{smallmatrix}\right)], R_1, R_2 \in \{1, \boldsymbol{\pi}\}$. By Part I, Proposition H.3, the θ-conjugacy class does not intersect K unless it is represented by $s(R_1 = R_2 = 1)$. $\square$

Consequently the θ-conjugacy classes of $t\theta = us\theta$ which contribute to the stable θ-orbital integral of 1_K have absolutely semi simple part represented by $s\theta$. They are represented by $t\theta = us\theta = s\theta u$ and $t'\theta = u's\theta = s\theta u'$, when u and u' are topologically unipotent stably conjugate elements of $Z_G(s\theta)$. As noted above, $Z_G(s\theta)$ is F-isomorphic to (an inner form of) $GL(2, F)'$. A regular elliptic element of this group has two conjugacy classes within its stable conjugacy class, parametrized by $F^\times/NE^\times$. In fact the stable class is the intersection with $GL(2, F)'$ of the orbit in $GL(2, F)$. We conclude that (when the stable integral is non zero, E/F is unramified and)

$$\Phi_{1_K}^{G,st}(us\theta) = \Phi_{1_{Z_K(s\theta)}}^{Z_G(s\theta),st}(u) = \Phi_{1_K}^{GL(2,F)}(u),$$

where the last u is the conjugacy class in $GL(2, F)$ determined by the eigenvalues $1/\alpha_2\sigma\alpha_1$, $1/\alpha_1\sigma\alpha_2$, where $u_i = \alpha_i/\sigma\alpha_i$.

We now turn to the norm of $s = h^{-1}s^*h$. It is determined by $Ns^* = \big(s_1s_2, s_1\sigma s_2, s_2\sigma s_1, \sigma(s_1s_2)\big)$, which is $s_1\sigma s_2(s_2/\sigma s_2, 1, 1, \sigma s_2/s_2)$, since $s_2\sigma s_1 = s_1\sigma s_2 \in R^\times$. Note that $s_1/\sigma s_2 = x_1 + y_1\sqrt{D}$ is absolutely semi simple ($\neq \pm 1$), hence x_1, y_1 lie in $R^\times$. The stable conjugacy class of Ns consists of a single conjugacy class, represented (in $GSp(2, F)$) by (the product of $s_2\sigma s_1 \in R^\times$ with) $[\left(\begin{smallmatrix} x_1 & y_1D \\ y_1 & x_1 \end{smallmatrix}\right), I]$. The centralizer of Ns is

$$Z_{GSp(2,F)}(Ns) = \{[t, A]; t \in T, A \in GL(2, F)', \|t\| = \|A\|\},$$

$$T = \{\left(\begin{smallmatrix} x' & y'D \\ y' & x' \end{smallmatrix}\right) \in GL(2, F)'\}.$$

The norm Nu of u lies in $Z_{GSp(2,F)}(Ns)$; it is determined by $Nu^* = \big(u_1u_2, u_1\sigma u_2, u_2\sigma u_1, \sigma(u_1u_2)\big)$, and we have $u_i\sigma u_i = 1$. The "t" part of Nu^* is determined by $\big(u_1u_2, \sigma(u_1u_2)\big)$.

The "A" part is determined by the eigenvalues $(u_1\sigma u_2, u_2\sigma u_1) = (u_1/u_2, u_2/u_1)$. The two conjugacy classes of Nt within its stable conjugacy class are represented by $NsNu$ and $NsNu'$, where the "A" parts of Nu, Nu', denoted A, A', are stably conjugate, but not conjugate, in $GL(2,F)'$. It follows that

$$\Phi_{1_K}^{GSp(2,F)}(NsNu)^{st} = \Phi_{1_{Z_K(Ns)}}^{Z_{Gsp(2,F)}(Ns)}(Nu)^{st} = \Phi_{1_K}^{GL(2,F)}(u_1/u_2, u_2/u_1).$$

The last term is the orbital integral of 1_K on $GL(2,F)$ at the elliptic regular orbit with eigenvalues $u_1/u_2, u_2/u_1$.

To compare our orbital integrals on $GL(2,F)$ of 1_K, at the class determined by the eigenvalues $(u_1/u_2, u_2/u_1)$, and (u_1, u_2) in the θ-case, note that we have seen above that the integral is given by an explicit expression, depending only on $|u_1/u_2 - u_2/u_1|$, respectively $|u_1 - u_2|$. Since $|u_1| = |u_2| = 1$ (u is topologically unipotent), these two terms are equal, and so are our stable θ- and stable orbital integrals, when $s_1/\sigma s_1 = s_2/\sigma s_2 \neq \pm 1$.

(2) The second case to be considered is when $s_1/\sigma s_1 = \sigma s_2/s_2 \neq \pm 1$. In this case $Z_{\mathbf{G}(E)}\big(s\theta(s)\big)$ consists of $g = h^{-1}(23)\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)(23)h, (23) = \operatorname{diag}(1, w, 1)$. Then $Z_{\mathbf{G}(E)}(s\theta)$ consists of g with $gs\theta(g)^{-1} = s = h^{-1}s^*h$, thus

$$(23)\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)(23)s^*\left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right)(23)\left(\begin{smallmatrix} {}^tA & 0 \\ 0 & {}^tB \end{smallmatrix}\right)(23)\left(\begin{smallmatrix} 0 & w \\ -w & 0 \end{smallmatrix}\right) = s^*,$$

namely if $\tilde{f} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & f \end{smallmatrix}\right)$ with $f = \sigma s_1/s_2 = s_1/\sigma s_2 \in R^\times$, then $B = \left(\begin{smallmatrix} s_2 & 0 \\ 0 & \sigma s_1 \end{smallmatrix}\right)w\varepsilon {}^tA^{-1}\varepsilon w\left(\begin{smallmatrix} s_2 & 0 \\ 0 & \sigma s_1 \end{smallmatrix}\right)^{-1}$ $= \|A\|^{-1}\tilde{f}A\tilde{f}^{-1}$.

Now $Z_G(s\theta)$ consists of $h^{-1}(23)\left(\begin{smallmatrix} A & 0 \\ 0 & \|A\|^{-1}\tilde{f}A\tilde{f}^{-1} \end{smallmatrix}\right)(23)h$ which are equal to

$$\sigma h^{-1}(23)\operatorname{diag}(\sigma A, \|\sigma A\|^{-1}\tilde{f}\cdot\sigma A\cdot\tilde{f}^{-1})(23)\sigma h.$$

Since $\sigma h\cdot h^{-1} = \left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right)$, and so $(23)\sigma h\cdot h^{-1}(23) = \left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right)$, this relation amounts to $\|A\|\cdot\|\sigma A\| = 1$ and $A = \|A\|\tilde{f}^{-1}w\sigma Aw\tilde{f}$. Then $\|A\| = \alpha/\sigma\alpha, \alpha \in E^\times$, and $A' = \alpha^{-1}A = \left(\begin{smallmatrix} a & b \\ f^{-1}\sigma b & \sigma a \end{smallmatrix}\right)$. As in the previous case we see that there is only one θ-orbit of s in its stable θ-orbit which intersects K. It is represented by $(23)\operatorname{diag}\big(\left(\begin{smallmatrix} x & yD \\ y & x \end{smallmatrix}\right), f\left(\begin{smallmatrix} x & -yD \\ -y & x \end{smallmatrix}\right)\big)(23)$, if $s_1 = x + y\sqrt{D}$, and $|x| = |y| = |D| = 1$. The θ-orbits within the stable θ-orbit of $t\theta = s\theta\cdot u$ are two, the other is $t'\theta = s\theta\cdot u'$, where u, u' are stably conjugate in the group whose F-points are $\left(\begin{smallmatrix} a & b \\ f^{-1}\sigma b & \sigma a \end{smallmatrix}\right) \in GL(2,E)$ (thus this group is $GL(2,F)'$ or an anisotropic inner form $D'^\times$, depending on whether $f \in NE^\times$ or $f \notin NE^\times$). Note that $u = h^{-1}(23)(u_1, \sigma u_2, u_2, \sigma u_1)(23)h$, and $u_i\sigma u_i = 1$. If $u_i = \alpha_i/\sigma\alpha_i, A = \left(\begin{smallmatrix} u_1 & 0 \\ 0 & \sigma u_2 \end{smallmatrix}\right), A' = \frac{1}{\alpha}A = \left(\begin{smallmatrix} 1/\sigma(\alpha_1\alpha_2) & 0 \\ 0 & 1/\alpha_1\alpha_2 \end{smallmatrix}\right)$, then

$$\Phi_{1_K}^{G,st}(us\theta) = \Phi_{1_{Z_K(s\theta)}}^{Z_G(s\theta),st}(u) = \Phi_{1_K}^{GL(2,F)}(A').$$

Here we noted that the stable orbital integral in $GL(2,F)'$ of the elliptic regular element with the same eigenvalues as A', is equal to the orbital integral in $GL(2,F)$ of the orbit determined by A'.

The norm of $t\theta = us\theta$ is determined by Ns^*Nu^*. Here $Ns^* = \big(s_1s_2, s_1\sigma s_2, s_2\sigma s_1, \sigma(s_1s_2)\big)$ is the product of $s_1s_2 = \sigma(s_1s_2) \in R^\times$ with $(1, \sigma s_2/s_2, s_2/\sigma s_2, 1)$. Since $\sigma s_1/s_1 = (\sigma s_2/s_2)^{-1} \neq \pm 1$ lies in $E^1 - F \cap E^1$, we have that $Z_{GSp}(Ns) = \{[A,t]; A \in GL(2,F)', t \in T, \|A\| = \|t\|\}, T = \{\left(\begin{smallmatrix} x & yD \\ y & x \end{smallmatrix}\right) \in GL(2,F)\}$. The stable θ-orbit of Ns consists of a single orbit, which intersects K as can be shown by the arguments of the previous case. There are two orbits in the stable orbit of $NsNu$. They are represented by the two orbits in the stable orbit of A_u in $GL(2,F)'$, where Nu is $[A_u, t]$. In other words,

$$\Phi_{1_K}^{GSp(2,F),st}(NsNu) = \Phi_{1_{Z_K(Ns)}}^{Z_{GSp(2,F)}(Ns),st}(Nu) = \Phi_{1_K}^{GL(2,F)}(A_u).$$

Now A_u is the elliptic regular orbit in $GL(2,F)$ with eigenvalues $u_1u_2, \sigma(u_1u_2)$, so the last orbital integral is given by a closed formula depending on $|u_1u_2 - \sigma(u_1u_2)|$. In the θ-case, the final orbital integral on $GL(2,F)$ is given by the same formula, depending on $|u_1 - \sigma u_2|$. But $\sigma u_i = u_i^{-1}$ and u_i are topologically unipotent. Hence $|u_1u_2 - 1/u_1u_2| = |(u_1u_2)^2 - 1| = |u_1u_2 - 1| = |u_1 - u_2^{-1}|$, and the equality of the stable integral at $N(us)$ with the stable θ-integral at us follows.

(3) The third case to be considered is that when $s_1/\sigma s_1 = \sigma s_2/s_2 = -1$. In this case $\sigma s_1 = -s_1 = -x\sqrt{D}$ and $\sigma s_2 = -s_2 = -y\sqrt{D}$, and s can be represented by $s = \left(\begin{smallmatrix} 0 & & & xD \\ & 0 & yD & \\ & y & 0 & \\ x & & & 0 \end{smallmatrix}\right)$. Only one θ-conjugacy orbit in the stable θ-conjugacy class of s intersects K. It is the one represented by s, thus $|x| = |y| = |D| = 1$; all other θ-orbits are represented by $\left(\begin{smallmatrix} 0 & & & xDR_1 \\ & 0 & yDR_1 & \\ & y/R_1 & 0 & \\ x/R_2 & & & 0 \end{smallmatrix}\right), R_i \in \{1, \boldsymbol{\pi}\}$. The centralizer $Z_{\mathbf{G}}(s\theta)$ of $s\theta$ in $\mathbf{G}$ consists of $g = h^{-1}g_1h$ with $gs\theta(g)^{-1} = s$. It is then isomorphic to $SO\left(\begin{smallmatrix} 0 & & & x \\ & 0 & y & \\ & y & 0 & \\ x & & & 0 \end{smallmatrix}\right)$ (recall that $\theta(g,e) = \big(\theta(g), e\|g\|\big)$. This group is isomorphic to $\big(GL(2) \times GL(2)\big)'$, where the prime denotes the group of pairs (x_1, x_2) with equal determinants. An isomorphism is given by mapping $\big(x_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), x_2\big)$ to $\|x_2\|^{-1}\left(\begin{smallmatrix} a_1x_2 & b_1x_2 \\ c_1x_2 & d_1x_2 \end{smallmatrix}\right)$. In particular, an elliptic conjugacy class with eigenvalues $\big(\left(\begin{smallmatrix} \alpha_1 & 0 \\ 0 & \sigma\alpha_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \beta_1 & 0 \\ 0 & \sigma\beta_1 \end{smallmatrix}\right)\big)$ will be mapped to the class of $(\alpha_1/\sigma\beta_1, \alpha_1/\beta_1, \sigma\alpha_1/\sigma\beta_1, \sigma\alpha_1/\beta_1)$. There are two conjugacy classes within the stable conjugacy class of an elliptic regular element in $\big(GL(2,F) \times GL(2,F)\big)'$. Indeed, if T is the centralizer of this element, we need to compute $H^1(F,T) = H^{-1}\big(\mathrm{Gal}(E/F), X_*(T)\big)$, where $X_*(T) = \{X = (x_1, x_2, y_1, y_2) \in \mathbb{Z}^4; x_1 + x_2 = y_1 + y_2\}$, and $\sigma X = (x_2, x_1, y_2, y_1)$. Thus we need to compute the quotient of the group of $X \in X_*(T)$, with the property $NX = 0$, where $NX = X + \sigma X$, by the span of $X - \sigma X = (x_1 - x_2, x_2 - x_1, y_1 - y_2, y_2 - y_1)$. Note that $y_1 + y_2 = x_1 + x_2$ implies that $y_1 - y_2 = x_1 + x_2 - 2y_2 = (x_1 - x_2) - 2x_2 - 2y_2$. Hence our quotient is $\mathbb{Z}/2\mathbb{Z}$, as asserted. Now if $t\theta = s\theta u = us\theta$, then $u = h^{-1}u^*h$ lies in the centralizer $Z_G(s\theta) = SO(2,2) = \big(GL(2,F) \times GL(2,F)\big)'$. Since u commutes with s, it commutes with θ. Also $\theta(h) = h$, hence $\theta(u^*) = u^*$, and as $u^* = (u_1, u_2, \sigma u_2, \sigma u_1)$, we have $u_1\sigma u_1 = 1 = u_2\sigma u_2$. Consequently the θ-conjugacy classes within the stable θ-conjugacy class of $t\theta = us\theta$, which intersect K, are given by $us\theta$ and $u's\theta$, where u, u' represent the two conjugacy classes within the stable conjugacy class of u in $\big(GL(2,F) \times GL(2,F)\big)'$. This last stable class is the intersection

with $\big(GL(2,F)\times GL(2,F)\big)'$ of the orbit in $GL(2,F)\times GL(2,F)$ which is determined by the eigenvalues $\big((\alpha_1,\sigma\alpha_1);(\beta_1,\sigma\beta_1)\big)$ with $\alpha_1\sigma\alpha_1=\beta_1\sigma\beta_1$, and $u_1=\alpha_1/\sigma\beta_1, u_2=\alpha_1/\beta_1$ (thus $u_1u_2=\alpha_1/\sigma\alpha_1, u_1/u_2=\beta_1/\sigma\beta_1$). We conclude that

$$\Phi_{1_K}^{G,st}(us\theta)=\Phi_{1_{Z_k(s\theta)}}^{Z_G(s\theta),st}(u)=\Phi_{1_K}^{GL(2,F)\times GL(2,F)}\big(\left(\begin{smallmatrix}\alpha_1&0\\0&\sigma\alpha_1\end{smallmatrix}\right),\left(\begin{smallmatrix}\beta_1&0\\0&\sigma\beta_1\end{smallmatrix}\right)\big).$$

Consider next the norm $N(us)$, which is determined by $Nu^*\cdot Ns^*$. For Ns^* we have $N\Big(\operatorname{diag}\big(\sqrt{D}(x,y,-y,-x)\big)\Big)=xyD\operatorname{diag}(1,-1,-1,1)$. Its centralizer $Z_{GSp(Ns)}$ in $GSp(2,F)$ is $\big(GL(2,F)\times GL(2,F)\big)'$, consisting of the matrices $[g_1,g_2]$. The norm

$$Nu^*=N(u_1,u_2,\sigma u_2,\sigma u_1)\qquad\text{is}\qquad\big(u_1u_2,u_1\sigma u_2,\sigma u_1\cdot u_2,\sigma(u_1u_2)\big).$$

There are two conjugacy classes in the stable class of $Ns\cdot Nu$ in $GSp(2,F)$, given by $Ns\cdot U$, where U is the intersection of the orbit of Nu in $GL(2,F)\times GL(2,F)$, with $\big(GL(2,F)\times GL(2,F)\big)'$. Hence our stable orbital integral is

$$\Phi_{1_K}^{GSp(2,F),st}(Ns\cdot Nu)=\Phi_{1_{Z_K(Ns)}}^{Z_{GSp(2,F)}(Ns),st}(Nu)$$

$$=\Phi_K^{GL(2,F)\times GL(2,F)}\big(\left(\begin{smallmatrix}u_1u_2&0\\0&\sigma(u_1u_2)\end{smallmatrix}\right),\left(\begin{smallmatrix}u_1\cdot\sigma u_2&0\\0&u_2\cdot\sigma u_1\end{smallmatrix}\right)\big).$$

On the right we wrote the eigenvalues which determine the orbit, not a representative in $GL(2,F)$.

We can now compare the stable with the θ-stable orbital integral. Both are given by explicit closed formulae, which depend only on the Δ-factor, which in the θ-case is the product of

$$|(\frac{\alpha_1}{\sigma\alpha_1}-1)(\frac{\sigma\alpha_1}{\alpha_1}-1)|^{1/2}=|(u_1u_2-1)\sigma(u_1u_2-1)|^{1/2}=|(u_1u_2)^2-1|=|u_1u_2-\sigma(u_1u_2)|$$

and

$$|(\frac{\beta_1}{\sigma\beta_1}-1)(\frac{\sigma\beta_1}{\beta_1}-1)|^{1/2}=|(\frac{u_1}{u_2}-1)(\frac{u_2}{u_1}-1)|^{1/2}=|(\frac{u_1}{u_2})^2-1|=|\frac{u_1}{u_2}-\frac{u_2}{u_1}|=|u_1\sigma u_2-u_2\sigma u_1|,$$

since u_i are topologically unipotent and $u_i\sigma u_i=1$. But the product of the right hand sides is the factor which appears in the non twisted case, and our comparison is then complete.

This completes our discussion of the proof of the Theorem for elements of type (I). We dealt with $t\theta=us\theta$ according to the values taken by $s\theta(s)$. The main case is that where the orbit of $s\theta(s)$ contains the identity. Above we dealt with the cases where $s\theta(s)$ is $-I$, or its eigenvalues take precisely two values $\big((t_1,t_1,t_1^{-1},t_1^{-1})$ or $(t_1,t_1^{-1},t_1,t_1^{-1})\big)$. The remaining cases are where the eigenvalues of $s\theta(s)$ take the form $(1,-1,-1,1),(1,t,t^{-1},1)$, $(-1,t,t^{-1},-1),(t_1,t_2,t_2^{-1},t_1^{-1})$, with $t,t_i\neq\pm1$. They can similarly be handled. The centralizer will even be of smaller rank. We leave these cases to the reader.

C. Case of torus of type (II).

In this case E is the composition of the quadratic extensions of F, which are $E_1 = F(\sqrt{D}) = E^\tau, E_2 = F(\sqrt{AD}) = E^{\sigma\tau}, E_3 = F(\sqrt{A}) = E^\sigma$, and the θ-conjugacy classes within the stable θ-conjugacy class of $s = h^{-1}s^*h = h^{-1}(s_1, s_2, \tau s_2, \sigma s_1)h$, a θ-elliptic strongly θ-regular element of type (II), are represented by $[(\begin{smallmatrix} a_1 & a_2DR_1 \\ a_2/R_1 & a_1 \end{smallmatrix}), (\begin{smallmatrix} b_1 & b_2ADR_2 \\ b_2/R_2 & b_1 \end{smallmatrix})], R_1 \in F^\times/N_{E_1/F}E_1^\times, R_2 \in F^\times/N_{E_2/F}E_2^\times$. Further $h = \theta(h) = [h'_D, h'_{AD}]$, where $h'_D = (\begin{smallmatrix} -1/2\sqrt{D} & -1/2 \\ 1 & -\sqrt{D} \end{smallmatrix})$, and $s_1 = a_1 + a_2\sqrt{D} \in E_1^\times, s_2 = b_1 + b_2\sqrt{AD} \in E_2^\times$. Now we consider such $t\theta = us\theta = s\theta u$, where $s\theta$ is absolutely semi simple. So is $(s\theta)^2 = s\theta(s) = h^{-1}(s_1/\sigma s_1, s_2/\tau s_2, \tau s_2/s_2, \sigma s_1/s_1)h$. Since $\mathrm{val}(AD) = 1, s_2/\tau s_2$ must be 1. Indeed, had it been -1, we would have had that $s_2 = \alpha\sqrt{AD}, \alpha \in F^\times$, but then s_2 cannot be a unit. Note that $\alpha + \beta\sqrt{\boldsymbol{\pi}}$ can be absolutely semi-simple only when $\beta = 0, \alpha \in R^\times$. Then, multiplying s by the scalar s_2^{-1} in $R^\times$, we may assume that $s_2 = 1$. The case where $s_1/\sigma s_1 = 1$ is the main case, considered in Part II above. Suppose that $s_1/\sigma s_1 \neq 1$. As we just noted, D must then be a unit. Put $s_1 = \alpha + \beta\sqrt{D}$. If $\alpha = 0$ (and $|\beta| = 1$), then $s_1/\sigma s_1 = -1$. Otherwise, since $s_1/\sigma s_1$ is absolutely semi simple, we have that both α, β lie in $R^\times$.

In the first case, where $s_1/\sigma s_1 = -1$, we have $s^*\theta(s^*) = (-1, 1, 1, -1)$, and $Z_{\mathbf{G}}\big(s\theta(s)\big) = \{h^{-1}[A, B]h\}$. Then $Z_{\mathbf{G}}(s\theta)$ is the set of $g = h^{-1}g_1h$, such that $\|g\| = 1$ and $gs\theta(g)^{-1} = s = h^{-1}s^*h$. The last relation is $g_1\begin{pmatrix} 0 & & & \beta\sqrt{D} \\ & & 1 & \\ & -1 & & \\ \beta\sqrt{D} & & & 0 \end{pmatrix}{}^tg_1 = \begin{pmatrix} 0 & & & \beta\sqrt{D} \\ & & 1 & \\ & -1 & & \\ \beta\sqrt{D} & & & 0 \end{pmatrix}$, or, since $g_1 = [A, B], B\varepsilon w^tB = \varepsilon w$, namely $B \in SL(2)$, and $Aw^tA = w$. Since $\|A\| = \|g_1\|/\|B\| = 1$, we have $A = \varepsilon A\varepsilon = \mathrm{diag}(a, a^{-1})$. In summary, $Z_{\mathbf{G}}(s\theta) = \{h^{-1}\,\mathrm{diag}(a, B, a^{-1})h; a \in GL(1), B \in SL(2)\}$. In the second case the same is true, since $s_1/\sigma s_1 \neq \pm 1$ implies that $Z_{\mathbf{G}}\big(s\theta(s)\big) = \{h^{-1}[(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}), B]h\}$, hence $Z_{\mathbf{G}}(s\theta) = \{h^{-1}[(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}), B]h; a \in GL(1), B \in SL(2)\}$ (using $\|g\| = 1$). To find the rational points, note that $\tau h \cdot h^{-1} = [I, (\begin{smallmatrix} 0 & 1/2\sqrt{AD} \\ -2\sqrt{AD} & 0 \end{smallmatrix})]$, and that $\sigma\tau h \cdot h^{-1} = [(\begin{smallmatrix} 0 & 1/2\sqrt{D} \\ -2\sqrt{D} & 0 \end{smallmatrix}), I]$. The relation $g = \sigma\tau g$ then translates into $B = \sigma\tau B \in SL(2, E_2)$, and $a\sigma\tau a = 1$. The relation $g = \tau g$ implies $a = \tau a \in E_1^\times$, and $\tau B = dwBwd^{-1}$, where $d = \mathrm{diag}(1/2\sqrt{AD}, -2\sqrt{AD})$. Hence $B = (\begin{smallmatrix} x & y \\ -4AD\tau y & \tau x \end{smallmatrix})$. Since $-AD = N_{E_2/F}(\sqrt{AD})$, B ranges over the group $SL(2, F)$. In conclusion, the stable θ-orbital integral is

$$\Phi_{1_K}^{G,st}(us\theta) = \Phi_{1_{Z_K(s\theta)}}^{Z_G(s\theta),st}(u) = \Phi_{1_K}^{SL(2,F),st}\begin{pmatrix} u_2 & 0 \\ 0 & \tau u_2 \end{pmatrix} = \Phi_{1_K}^{GL(2,F)}\begin{pmatrix} u_2 & 0 \\ 0 & \tau u_2 \end{pmatrix}.$$

Indeed, $u = h^{-1}(u_1, u_2, \tau u_2, \sigma u_1)h$ has "B" part $(u_2, \tau u_2)$, which in the last integral above is interpreted as the conjugacy class in $GL(2, F)$ with eigenvalues $u_2, \tau u_2$. This integral is given by an explicit expression, depending on $|u_2 - \tau u_2|$.

The norm of t lies in a torus of type (II) in $GSp(2, F)$, whose elements are of the form $(\begin{smallmatrix} \mathbf{a} & \mathbf{b}D\mathbf{R} \\ \mathbf{b}\mathbf{R}^{-1} & \mathbf{a} \end{smallmatrix})$, where R ranges over a set of representatives for $E_3^\times/N_{E/E_3}E^\times$. We have that $D \in R^\times$, hence $A = \boldsymbol{\pi}$, hence $E_3 = F(\sqrt{A})$ is ramified over F and E/E_3 is unramified, and so R ranges over $\{1, \boldsymbol{\pi}_3 = \sqrt{A}\}$. Now the norm Ns of $s = \tilde{h}^{-1}s^*\tilde{h}, s^* = (s_1, s_2, \tau s_2, \sigma s_1)$, is $h^{-1}(s_1s_2, s_1\tau s_2, \sigma s_1 \cdot s_2, \sigma s_1 \cdot \tau s_2)h$; but $s_2 = 1$, so this is $h^{-1}(\begin{smallmatrix} s_1 & 0 \\ 0 & \sigma s_1 \end{smallmatrix})h = (\begin{smallmatrix} \alpha & \beta D \\ \beta & \alpha \end{smallmatrix})$, whose stably conjugate but not conjugate is represented by $\left(\begin{smallmatrix}\alpha & \beta D\mathbf{R}\\ \beta\mathbf{R}^{-1} & \alpha\end{smallmatrix}\right)$. Here we denoted by $\tilde{h}$ the h used above in the description of representatives for the θ-conjugacy classes, and use h to denote $h=\left(\begin{smallmatrix}-1/2\sqrt{A} & 0\\ 0 & I\end{smallmatrix}\right)\left(\begin{smallmatrix}h_A & 0\\ 0 & \varepsilon h_A\varepsilon\end{smallmatrix}\right)\left(\begin{smallmatrix}-1/2\sqrt{D} & -1/2\\ 1 & -\sqrt{D}\end{smallmatrix}\right)$, which realizes the torus of type (II) in $GSp(2,F)$. Since $\beta\in R^\times, D\in R^\times$ and α is 0 or in $R^\times$, $\left(\begin{smallmatrix}\alpha & \beta D\\ \beta & \alpha\end{smallmatrix}\right)$ lies in K. Its non-conjugate but stably conjugate orbit, represented by $\left(\begin{smallmatrix}\alpha & \beta D\mathbf{R}\\ \beta\mathbf{R}^{-1} & \alpha\end{smallmatrix}\right)$, $\mathbf{R}=\left(\begin{smallmatrix}0 & \boldsymbol{\pi}\\ 1 & 0\end{smallmatrix}\right)$, does not intersect K (use Proposition I.H.3, and the fact that Ns is absolutely semi simple). Hence the stable orbital integral of 1_K at $N(su)$ will be reduced to a single orbital integral.

The centralizer $Z_{\boldsymbol{GSp}}(Ns)$ of $Ns=h^{-1}(\alpha+\beta\sqrt{D},\alpha-\beta\sqrt{D})h$, consists of

$$g=h^{-1}\begin{pmatrix}X & 0\\ 0 & Y\end{pmatrix}h=\lambda h^{-1}\begin{pmatrix}0 & -w\\ w & 0\end{pmatrix}\begin{pmatrix}{}^tX^{-1} & 0\\ 0 & {}^tY^{-1}\end{pmatrix}\begin{pmatrix}0 & w\\ -w & 0\end{pmatrix}h$$
$$=h^{-1}\begin{pmatrix}X & 0\\ 0 & \lambda w{}^tX^{-1}w\end{pmatrix}h=h^{-1}\begin{pmatrix}X & 0\\ 0 & \lambda\|X\|^{-1}\varepsilon X\varepsilon\end{pmatrix}h.$$

To find the rational points, note that as $h_A=\left(\begin{smallmatrix}1 & \sqrt{A}\\ 1 & -\sqrt{A}\end{smallmatrix}\right)$, $\tau h_A\cdot h_A^{-1}=w$, and $\tau h\cdot h^{-1}=\left(\begin{smallmatrix}-w & 0\\ 0 & -w\end{smallmatrix}\right)$. The relation $g=\tau g$ then reads $\tau X=wXw$, hence also $\|X\|=\|\tau X\|$, and $\lambda=\tau\lambda\in E_1^\times$. Further we have $\sigma h\cdot h^{-1}=\left(\begin{smallmatrix}0 & -w\varepsilon/4\sqrt{AD}\\ 4\sqrt{AD}\varepsilon w & 0\end{smallmatrix}\right)$. Hence $g=\sigma g$ implies $\sigma X=\lambda\|X\|^{-1}wXw$ and that $\|X\|\cdot\|\sigma X\|=\lambda\sigma\lambda$. We then write $\lambda/\|X\|=v/\sigma v$ with $v\in E_1^\times$. Since $E_1=F(\sqrt{D})/F$ is unramified, we may and do take v in $R_1^\times$. Then $\sigma(vX)=w\cdot vX\cdot w$, and so $\sigma\tau(vX)=vX$ lies in $GL(2,E_2)$, and $\tau(vX)=w\cdot vX\cdot w$ further implies that vX ranges over a group F-isomorphic to $GL(2,F)$ (namely $\left(\begin{smallmatrix}a & b\\ \tau b & \tau a\end{smallmatrix}\right)$, a,b in E_2). In particular, for our $u^*=(u_1,u_2,\tau u_2,\sigma u_1)$, we have $Nu=h^{-1}(u_1u_2,u_1\tau u_2,u_2\sigma u_1,\tau u_2\sigma u_1)h$, whose "X" is $u_1\left(\begin{smallmatrix}u_2 & 0\\ 0 & \tau u_2\end{smallmatrix}\right)$, and $\|X\|=u_1^2, v=u_1^{-1},\lambda=1$ (thus $Nu=h^{-1}\left(\begin{smallmatrix}X & 0\\ 0 & u_1^{-2}X\end{smallmatrix}\right)h$). The stable orbital integral of 1_K at $NsNu$ is then

$$\Phi_{1_K}^{GSp,st}(NsNu)=\Phi_{1_{Z_K(Ns)}}^{Z_{GSp}(Ns),st}(Nu)=\Phi_{1_K}^{GL(2,F)}\left(\left(\begin{smallmatrix}u_2 & 0\\ 0 & \tau u_2\end{smallmatrix}\right)\right).$$

Again this is given by an explicit formula, depending only on $|u_2-\tau u_2|$, and the equality of the stable θ-integral with the stable integral follows.

D. Case of torus of type (III).

In this case E again is the compositum of the quadratic extensions $E_1=F(\sqrt{D})=E^\tau$, $E_2=F(\sqrt{AD})=E^{\sigma\tau}$, $E_3=F(\sqrt{A})=E^\sigma$, of F, and the θ-conjugacy classes within a strongly θ-regular stable θ-orbit are represented by $t_1=h^{-1}t^*h=\left(\begin{smallmatrix}\mathbf{a} & \mathbf{b}D\mathbf{R}\\ \mathbf{b}\mathbf{R}^{-1} & \mathbf{a}\end{smallmatrix}\right)$ with $R=1$ or $R\in E_3-N_{E/E_3}E$. Here $t^*=(t,\tau t,\sigma\tau t,\sigma t)$, $t=a+b\sqrt{D}$, $a=a_1+a_2\sqrt{A}$, $b=b_1+b_2\sqrt{A}$, $a_i,b_i\in F$. Further, h is such that $\theta(h)=h$, $\tau(h)h^{-1}=\left(\begin{smallmatrix}-w & 0\\ 0 & -w\end{smallmatrix}\right)$, and $\sigma(h)h^{-1}=\left(\begin{smallmatrix}-1/4\sqrt{AD} & 0\\ 0 & 4\sqrt{AD}\end{smallmatrix}\right)\left(\begin{smallmatrix} & w\varepsilon\\ \varepsilon w & \end{smallmatrix}\right)$. As usual we distinguish the cases according to the values of

$$s_1\theta(s_1)=h^{-1}\big(s/\sigma s,\tau(s/\sigma s),\sigma\tau(s/\tau s),\sigma(s/\tau s)\big)h,$$

where $t_1\theta = u_1 s_1 \theta = s_1 \theta u_1$ is the decomposition of t_1 into a product of commuting absolutely semi simple $s_1\theta$, and topologically unipotent u_1, elements. Also $s_1 = h^{-1}(s, \tau s, \sigma\tau s, \sigma s)h$. Now $s_1\theta(s_1)$ is absolutely semi simple, hence so is $s/\sigma s = a' + b'\sqrt{D}(a', b' \in R_3)$. If $D = \boldsymbol{\pi}(A \in R^\times)$, then $b' = 0$, and so $a' = \pm 1$. If $\sigma s = -s$, then $s = b\sqrt{D}$ and $s_1 \notin K$. Hence $\sigma s = s$ is the case where $s_1\theta(s_1) = I$, which is handled above. Hence $D \in R^\times$, and $A = \boldsymbol{\pi}$, so E/E_3 is unramified, and F ranges over $\{1, \boldsymbol{\pi}_3\}$. Then we write $\sigma s/s = x + y\sqrt{A}(x, y \in R_1)$, and conclude again that $y = 0$ (since $\sigma s/s$ is absolutely semi simple and $A = \boldsymbol{\pi}$), hence $\tau(\sigma s/s) = \sigma s/s \in R_1^\times$. The cases to be considered are $s/\sigma s = 1$ – but this is the main case considered above – or $\sigma s = -s$, or $\sigma s/s \neq \pm 1$.

If $\sigma s = -s$, then $s = b\sqrt{D}, \sigma b = b = b_1 + b_2\sqrt{A} \in R_3^\times$, from which it follows that the stable θ-conjugacy class of s_1 intersects K in a single θ-conjugacy class, represented by s_1 with $R = 1$. The centralizer $Z_{\mathbf{G}}(s\theta)$ consists of $g = h^{-1}g_1 h$, such that $\|g\| = 1$ and $gs_1\theta(g)^{-1} = s_1 = h^{-1}s^* h$. The last relation can be read as $g_1 s^* \left(\begin{smallmatrix} 0 & -w \\ w & 0 \end{smallmatrix}\right) {}^t g_1 = s^* \left(\begin{smallmatrix} 0 & -w \\ w & 0 \end{smallmatrix}\right)$. Hence $Z_{\mathbf{G}}(s\theta)$ is the group of $g = h^{-1}xg_2x^{-1}h$, where $x = (b, \tau b, 1, 1)$, and $g_2 = (B, B')$ in $SO\left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right)$. The relation $\|g\| = 1$ implies that g_2 indeed lies in the special orthogonal group. Note that $\|B\| = \|B'\|$.

Next we determine the rational points $Z_G(s\theta)$. The relation $\tau(h)h^{-1} = \left(\begin{smallmatrix} -w & 0 \\ 0 & -w \end{smallmatrix}\right)$ implies that if $g = \tau g$, and $g_2 = (B, B')$, then $\tau g_2 = \tau(B, B') = (B, wBw)$, since $\left(\begin{smallmatrix} -w & 0 \\ 0 & -w \end{smallmatrix}\right)$ is (I, w) under the isomorphism $SO\left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right) \simeq \big(GL(2) \times GL(2)\big)'/Z$. Thus $B \in GL(2, E_1)$, and B' lies in a group isomorphic to $GL(2, E_1)$. Further, the relation $g = \sigma g$ can be expressed as $\sigma(B, B') = \mathrm{Int}(x)(B, B')$, where $x = \big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & b\tau b \end{smallmatrix}\right), \left(\begin{smallmatrix} \tau b & 0 \\ 0 & b \end{smallmatrix}\right)\big)\big(\left(\begin{smallmatrix} -1/4\sqrt{AD} & 0 \\ 0 & 4\sqrt{AD} \end{smallmatrix}\right), I\big)(w\varepsilon, \varepsilon w)$, since $\mathrm{diag}(b^{-1}, \tau b^{-1}, \tau b, b) = \big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & b\tau b \end{smallmatrix}\right), \left(\begin{smallmatrix} \tau b & 0 \\ 0 & b \end{smallmatrix}\right)\big)$. It follows that B takes the form $\left(\begin{smallmatrix} \alpha & \beta \\ 16b\tau bAD\sigma\beta & \sigma\alpha \end{smallmatrix}\right)$, and B' is $\left(\begin{smallmatrix} x & y \\ -\sigma(y)b/\tau b & \sigma x \end{smallmatrix}\right)$, namely B, B' range over groups isomorphic to $GL(2, F)$, and they satisfy $\|B\| = \|B'\|$. The element $u = h^{-1}u^* h, u^* = (u, \tau u, \sigma\tau u, \sigma u)$, commutes with θ, hence $u\sigma u = 1$. Then there is $v \in R_E^\times$ with $u = v/\sigma v$, and as an element of $SO\left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right), u^*$ can be expressed as $v^* = \big(\left(\begin{smallmatrix} v\tau v & 0 \\ 0 & \sigma(v\tau v) \end{smallmatrix}\right), \left(\begin{smallmatrix} v\sigma\tau v & 0 \\ 0 & \tau v\sigma v \end{smallmatrix}\right)\big)$. As noted above, there is only one θ-conjugacy class in the stable θ-class of $t_1\theta = u_1 s_1\theta$, which intersects K. Moreover, there is only one conjugacy class in the stable conjugacy class of v^* in $\big(GL(2, F) \times GL(2, F)\big)'$. Indeed, if T is the centralizer of v^* in this group, $H^1(F, T)$ is the quotient of $X = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$ with $x_1 + x_2 = 0 = x_3 + x_4\big(\tau X = (x_1, x_2, x_4, x_3), \sigma X = (x_2, x_1, x_4, x_3)$, and $NX = X + \sigma X + \tau X + \sigma\tau X$ is $0\big)$, by the span of $X - \tau X = (0, 0, y, -y), \sigma X - \tau X = (x, -x, 0, 0)$, namely it is zero. Hence

$$\Phi_{1_K}^{G,st}(us\theta) = \Phi_{1_K}^{GL(2,F)\times GL(2,F)}\left(\begin{pmatrix} v\tau v & 0 \\ 0 & \sigma(v\tau v) \end{pmatrix}, \begin{pmatrix} v\sigma\tau v & 0 \\ 0 & \tau v\sigma v \end{pmatrix} \right)$$

is a product of two orbital integrals on $GL(2, F)$, which depend on the factors

$$|v\tau v - \sigma(v\tau v)| = |u\tau u - 1|, \text{ and } |v\sigma\tau v - \tau v\sigma v| = |u/\tau u - 1|.$$

The norm Ns_1 is determined by $Ns_1^* = (s\tau s, s\sigma\tau s, \tau s\sigma s, \sigma\tau s\sigma s) = b\tau bD(1, -1, -1, 1)$. Hence the centralizer $Z_{\boldsymbol{GSp}}(Ns_1) = Z_{\boldsymbol{GSp}}\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{smallmatrix}\right)$ consists of $[B, B']$ with $\|B\| = \|B'\|$. The element $u^* = (u, \tau u, \sigma\tau u, \sigma u)$ has norm $Nu^* = (u\tau u, u\sigma\tau u, \tau u\sigma u, \sigma u\sigma\tau u) = [\left(\begin{smallmatrix} u\tau u & 0 \\ 0 & \sigma u\sigma\tau u \end{smallmatrix}\right), \left(\begin{smallmatrix} u\sigma\tau u & 0 \\ 0 & \tau u\sigma u \end{smallmatrix}\right)]$. The stable conjugacy class of an element of type (III) in $GSp(2,F)$ consists of a single conjugacy class. We conclude that

$$\Phi_{1_K}^{GSp(2,F)}(Ns_1Nu_1) = \Phi_{1_K}^{GL(2,F)\times GL(2,F)}\left(\begin{pmatrix} u\tau u & 0 \\ 0 & \sigma(u\tau u) \end{pmatrix}, \begin{pmatrix} u\sigma\tau u & 0 \\ 0 & \tau u\sigma u \end{pmatrix}\right)$$

is a product of two orbital integrals on $GL(2,F)$, which depend on the factors $|u\tau u - \sigma(u\tau u)| = |(u\tau u)^2 - 1| = |u\tau u - 1|$, and $|u\sigma\tau u - \sigma(u\sigma\tau u)| = |(u\sigma\tau u)^2 - 1| = |u\sigma\tau u - 1| = |u/\tau u - 1|$. Here we used the fact that $u\sigma u = 1$, and that u is topologically unipotent, so that $|u\tau u + 1| = 1$. This completes the comparison when $\sigma s = -s$.

The remaining case is when $\sigma s/s \neq \pm 1$. Since $\tau(\sigma s/s) = \sigma s/s$, we have that $s_1\theta(s_1) = h^{-1}(s/\sigma s, s/\sigma s, \sigma s/s, \sigma s/s)h$, and so $Z_{\mathbf{G}}(s_1\theta(s_1)) = \{h^{-1}\left(\begin{smallmatrix} B & 0 \\ 0 & B' \end{smallmatrix}\right)h\}$. Further $Z_{\mathbf{G}}(s_1\theta)$ is the set of $g = h^{-1}g_1h, g_1 = \left(\begin{smallmatrix} B & 0 \\ 0 & B' \end{smallmatrix}\right)$, with $\|g\| = 1$ and $gs_1\theta(g)^{-1} = s_1$. This translates to ($\|B\| \cdot \|B'\| = 1$ and) $B\left(\begin{smallmatrix} 0 & s \\ \tau s & 0 \end{smallmatrix}\right)^t B' = \left(\begin{smallmatrix} 0 & s \\ \tau s & 0 \end{smallmatrix}\right)$, thus $B' = \left(\begin{smallmatrix} \tau s & 0 \\ 0 & s \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^t B^{-1}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \tau s & 0 \\ 0 & s \end{smallmatrix}\right)^{-1} = \|B\|^{-1}\left(\begin{smallmatrix} \tau s & 0 \\ 0 & -s \end{smallmatrix}\right)B\left(\begin{smallmatrix} \tau s & 0 \\ 0 & -s \end{smallmatrix}\right)^{-1}$. Note that $s/\tau s = \sigma(s/\tau s)$. Thus $Z_{\mathbf{G}}(s_1\theta)$ consists of $g = h^{-1}\operatorname{diag}\left(B, \|B\|^{-1}\left(\begin{smallmatrix} \tau s & 0 \\ 0 & -s \end{smallmatrix}\right)B\left(\begin{smallmatrix} \tau s & 0 \\ 0 & -s \end{smallmatrix}\right)^{-1}\right)h$.

The rational points on this group, $Z_G(s_1\theta)$, are obtained on solving $g = \tau g$ and $g = \sigma g$. Since $\tau(h)h^{-1} = \left(\begin{smallmatrix} -w & 0 \\ 0 & -w \end{smallmatrix}\right)$, we have $\tau B = wBw$, thus B lies in a group isomorphic to $GL(2,E_1)$. The equation $g = \sigma g$ leads to $(\sigma B, \sigma B') = (w\varepsilon B'\varepsilon w, \varepsilon wBw\varepsilon)$, or to $\sigma\tau B = \|B\|^{-1}\left(\begin{smallmatrix} \tau s & 0 \\ 0 & s \end{smallmatrix}\right)B\left(\begin{smallmatrix} \tau s & 0 \\ 0 & s \end{smallmatrix}\right)^{-1}$, and $\|B\| = \|\tau B\| = \|\sigma\tau B\|^{-1}$. Hence $\|B\| = v/\sigma v, v = \tau v$, v can be taken to be a unit since E_1/F is unramified. So $\sigma\tau(v^{-1}B) = \left(\begin{smallmatrix} \tau s & 0 \\ 0 & s \end{smallmatrix}\right)v^{-1}B\left(\begin{smallmatrix} \tau s & 0 \\ 0 & s \end{smallmatrix}\right)^{-1}$, and $v^{-1}B$ lies in a group isomorphic to $GL(2,F)$. Now $u_1 = h^{-1}(u, \tau u, \sigma\tau u, \sigma u)h$, so $B = \left(\begin{smallmatrix} u & 0 \\ 0 & \tau u \end{smallmatrix}\right)$, and

$$\Phi_{1_K}^{G,st}(u_1s_1\theta) = \Phi_{1_K}^{GL(2,F)}(B)$$

depends only on $|u - \tau u|$.

The norm of s_1 is obtained from $Ns_1^* = (s\tau s, s\sigma\tau s, \tau s\sigma s, \sigma\tau s \cdot \sigma s)$. The two middle entries are equal, hence $Z_{\boldsymbol{GSp}}(Ns_1) = \{g = h^{-1}\operatorname{diag}(a, B, b)h; ab = \|B\|\}$. Here $h = [h'_D, h'_{AD}]$, since Ns_1Nu_1 is an element of type (III) in $GSp(2,F)$. Since $\sigma\tau(h)h^{-1} = \left(\begin{smallmatrix} & & & 1/2\sqrt{D} \\ & & I & \\ & & & \\ -2\sqrt{D} & & & \end{smallmatrix}\right)$,
$Z_{GSp}(Ns)$ consists of g with $b = \sigma\tau a$ and $B \in GL(2,E_2)$. The relation

$$\tau(h)h^{-1} = \begin{pmatrix} 1 & & & \\ & 0 & 1/2\sqrt{A} & \\ & -2\sqrt{A} & 0 & \\ & & & 1 \end{pmatrix}$$

further implies that $a = \tau a$ and B ranges over the matrices $B = \left(\begin{smallmatrix} \alpha & \beta \\ -4A\tau\beta & \tau\alpha \end{smallmatrix}\right)$ with $\|B\| = a\sigma a$. The stable conjugacy class of Nu in $Z_{GSp}(Ns)$ consists of a single conjugacy class (the corresponding $H^1(T)$ is $\{X = (x_1, x_2, x_3, x_4); NX = 0\}/\langle X - \tau X, X - \sigma X\rangle$, where $\tau X =$

$(x_1, x_3, x_2, x_4), \sigma X = (x_4, x_3, x_2, x_1)$, thus it is zero). Now Nu is $h^{-1}(u\tau u, u\sigma\tau u, \tau u\sigma u, \sigma\tau u\sigma u)h$, with $B = \left(\begin{smallmatrix} u\sigma\tau u & 0 \\ 0 & \tau u\sigma u \end{smallmatrix}\right)$. Then

$$\Phi_{1_K}^{GSp(2,F)}(Ns_1Nu_1) = \Phi_{1_K}^{GL(2,F)}\left(\begin{pmatrix} u\sigma\tau u & 0 \\ 0 & \tau u\sigma u \end{pmatrix}\right),$$

and this integral on $GL(2,F)$ is determined by the factor $|u\sigma\tau u - \tau u\sigma u|$, which is equal to $|u/\tau u - \tau u/u| = |u^2 - (\tau u)^2| = |u - \tau u|$, since $u\sigma u = 1$ and u is topologically unipotent. This is the factor obtained in the twisted case, and so the comparison is complete for strongly θ-regular elements of type (III).

E. Case of torus of type (IV).

In this case $E = F(\sqrt{D})$ is a quadratic extension of $E_3 = F(\sqrt{A})$, which is a quadratic extension of F. There are three cases: $A = \boldsymbol{\pi}$ and $D = \sqrt{\boldsymbol{\pi}}$; $-1 \in R^{\times 2}, A \in R^\times$ and $D = \sqrt{A}$ or $\boldsymbol{\pi}\sqrt{A}$; $-1 \notin R^{\times 2}, A = -1$, and $D = \alpha + \beta\sqrt{A} \in R_3 - R_3^2$, with $\alpha, \beta \in R^\times$ or $\boldsymbol{\pi}R^\times$. In all cases $\sigma\sqrt{D} = \sqrt{\sigma D}, \sigma^2\sqrt{D} = -\sqrt{D}, \sigma^3\sqrt{D} = -\sqrt{\sigma D}, \sigma\sqrt{A} = -\sqrt{A}$, so E_3 is the fixed field of σ^2 in E. The strongly θ-regular θ-orbits are represented by $t_1 = h^{-1}t^*h = \left(\begin{smallmatrix} \mathbf{a} & \mathbf{bDR}^{-1} \\ \mathbf{bR} & \mathbf{a} \end{smallmatrix}\right), t^* = (t, \sigma t, \sigma^3 t, \sigma^2 t), t = a + b\sqrt{D}, \mathbf{a} = \left(\begin{smallmatrix} a_1 & a_2 A \\ a_2 & a_1 \end{smallmatrix}\right)$ if $a = a_1 + a_2\sqrt{A}, \mathbf{D} = \left(\begin{smallmatrix} \alpha & \beta A \\ \beta & \alpha \end{smallmatrix}\right)$ if $D = \alpha + \beta\sqrt{A}, \mathbf{R} = \left(\begin{smallmatrix} R_1 & R_2 A \\ R_2 & R_1 \end{smallmatrix}\right)$ for $R = R_1 + R_2\sqrt{A}$, taken over a set of representatives for $E_3^\times/N_{E/E_3}E^\times$. Note that $\theta(h) = h$, and $\sigma(h)h^{-1}$ is $(1, 1/4\sqrt{AD}, -4\sqrt{AD}, 1)(2431)$, where (2431) denotes the matrix with rows $(0,1,0,0), (0,0,0,1), (1,0,0,0), (0,0,1,0)$. As usual, we consider the decomposition $t_1\theta = s_1\theta u_1 = u_1 s_1\theta$, and $s_1\theta(s_1) = h^{-1}\big(s/\sigma^2 s, \sigma(s/\sigma^2 s), \sigma(\sigma^2 s/s), \sigma^2 s/s\big)h$. If $\sigma^2(s/\sigma^2 s) = s/\sigma^2 s$ then it is ± 1.

Consider first the case where $s/\sigma^2 s \neq \pm 1$. Then $\sigma(s/\sigma^2 s) \neq s/\sigma^2 s, \sigma^2 s/s$, hence the eigenvalues of $s_1\theta(s_1)$ are distinct. Moreover, $s/\sigma^2 s = a' + b'\sqrt{D}$ with $b' \in R_3^\times$ and $D \in R_3^\times$, since $s/\sigma^2 s$ is absolutely semi simple and it is not in $R_3^\times$. Then $s = a + b\sqrt{D}, a, b, D \in R_3^\times, s_1 = \left(\begin{smallmatrix} \mathbf{a} & \mathbf{bDR} \\ \mathbf{bR}^{-1} & \mathbf{a} \end{smallmatrix}\right)$ lies in K for $R = 1 (\mathbf{R} = I)$, but when $R \neq 1$ in $E_3^\times/N_{E/E_3}E^\times$ (which is represented by $\{1, \boldsymbol{\pi}_3\}$, since E/E_3 is unramified), the θ-conjugacy class does not intersect K. Thus the stable θ-orbital integral reduces to a single θ-orbital integral.

Now the eigenvalues of $s_1\theta(s_1)$ are distinct, hence $Z_{\mathbf{G}}\big(s_1\theta(s_1)\big) = \{h^{-1}dh; d = \text{diagonal in } \mathbf{G}\}$, and $Z_{\mathbf{G}}(s_1\theta)$ consists of $h^{-1}g_1 h, g_1$ is diagonal matrix with $g_1 = \theta(g_1)$, namely $g_1 = (x, y, y^{-1}, x^{-1})$. The rational points are given by $\sigma g = g$, thus $\sigma(x, y, y^{-1}, x^{-1}) = (2431)(x, y, y^{-1}, x^{-1}) = (y, 1/x, x, 1/y)$, and so $g = h^{-1}(x, \sigma x, 1/\sigma x, 1/x)h, x \in E^\times$ with $x\sigma^2 x = 1$. The absolutely unipotent element u_1 has the form $u_1 = h^{-1}(u, \sigma u, \sigma^3 u, \sigma^2 u)h$, where $u\sigma^2 u = 1$. Then

$$\Phi_{1_K}^{G,st}(u_1 s_1\theta) = \Phi_{1_{GL(1,R_E)'}}^{GL(1,E)'}(u) = 1,$$

where the prime indicates here the property $x\sigma^2 x = 1$.

The norm Ns_1 of $s_1 = h^{-1}s^*h$ is obtained from $Ns^* = (s\sigma s, s\sigma^3 s, \sigma s\sigma^2 s, \sigma^3 s\sigma^2 s)$, which has distinct eigenvalues $\big(s \neq \sigma^2 s$, hence $\sigma s \neq \sigma^3 s$, and $s/\sigma^2 s \neq \sigma(s/\sigma^2 s), \sigma(\sigma^2 s/s)\big)$. The centralizer in $\boldsymbol{GSp}(2)$ is then $h^{-1}(\text{diagonal})h$, and Nu^* is $\big(u\sigma u, \sigma^3(u\sigma u), \sigma(u\sigma u), \sigma^2(u\sigma u)\big)$.

Then $\Phi_{1_K}^{GSp(2,F)}(Ns_1Nu_1) = \Phi_{1_{GL(1,R_E)'}}^{GL(1,E)'}(u\sigma u) = 1$, since $Z_{GSp(2,F)}(Ns) = \{h^{-1}(x,\sigma^3 x,\sigma x,\sigma^2 x)h; x \in E^\times, x\sigma^2 x = 1\}$. This establishes the matching in the case where $s/\sigma^2 s \neq \pm 1$.

The case where $\sigma^2 s/s = 1$ is the main case ($s_1 = 1$) considered first. So it remains to deal with the case where $\sigma^2 s = -s$. Here $s = b\sqrt{D}$, and b, D lie in $R_3^\times$. Hence E/E_3 is unramified, $\{1, \boldsymbol{\pi}_3\}$ represents the θ-conjugacy class within the stable θ-class, $s_1 = \left(\begin{smallmatrix} 0 & \mathbf{bD} \\ \mathbf{b} & 0 \end{smallmatrix}\right)$ lies in K, but the θ-orbit of $\left(\begin{smallmatrix} 0 & \mathbf{bDR} \\ \mathbf{bR}^{-1} & 0 \end{smallmatrix}\right)$ does not intersect K, and our stable θ-orbital integral reduces to the θ-orbital integral of 1_K at s_1. Now $Z_{\mathbf{G}}(s\theta)$ is the set of $g = h^{-1}g_1 h$ with $\|g\| = 1$ and $gs\theta(g)^{-1} = s = h^{-1}s^*h$, thus $g_1 s^* \left(\begin{smallmatrix} 0 & w \\ -w & 0 \end{smallmatrix}\right)^t g_1 = s^*\left(\begin{smallmatrix} 0 & w \\ -w & 0 \end{smallmatrix}\right)$. Since $\sigma^2 s = -s$, we have that $s^* = (s, \sigma s, \sigma^3 s, \sigma^2 s)$ is $(s, \sigma s, -\sigma s, -s)$, hence $Z_{\mathbf{G}}(s\theta)$ consists of $g = h^{-1}Sg_1S^{-1}h$, where $S = (s, \sigma s, 1, 1)$ and $g_1\left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right)^t g_1 = \left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right)$, $\|g_1\| = 1$. Thus g_1 lies in $SO\left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right)$, and under the usual isomorphism with $\big(GL(2)\times GL(2)\big)'/Z$, we write $g_1 = (B, B')$. The group $Z_G(s\theta)$ of rational points is obtained on solving $g = \sigma g$. Thus $\sigma g_1 = Xg_1X^{-1}$, where

$$X = \begin{pmatrix} \sigma s & & & 0 \\ & s & 0 & \\ & 0 & 1 & \\ 0 & & & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/4\sqrt{AD} \\ -4\sqrt{AD} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s & & & 0 \\ & \sigma s & 0 & \\ & 0 & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 0 \\ & 0 & 1 & \\ & 1 & 0 & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & 0 \\ & -4s\sqrt{AD} & 0 & \\ & 0 & -1/4s\sqrt{AD} & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 0 \\ & 0 & 1 & \\ & 1 & 0 & \\ 0 & & & 1 \end{pmatrix} \left(\begin{pmatrix} -4s\sqrt{AD} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -4s\sqrt{AD} \end{pmatrix} \right) (-I, w).$$

Consequently $\sigma(B, B') = \big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -4s\sqrt{AD} \end{smallmatrix}\right) wB'w \left(\begin{smallmatrix} 1 & 0 \\ 0 & -4s\sqrt{AD} \end{smallmatrix}\right)^{-1}, \left(\begin{smallmatrix} -4s\sqrt{AD} & 0 \\ 0 & 1 \end{smallmatrix}\right) B \left(\begin{smallmatrix} -4s\sqrt{AD} & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\big)$, and $\sigma^2 B = dwBwd^{-1}$, where $d = \operatorname{diag}(1, -16s\sigma sA\sqrt{D\sigma D})$. In conclusion, B lies in a group isomorphic to $GL(2, E_3)'$, where the prime means determinant in $F^\times$. If g lies in $Z_G(s\theta)$, then in $g_1 = (B, B')$, B' is determined by B. Hence $Z_G(s\theta)$ is isomorphic to $GL(2, E_3)'$. Moreover, $u_1 = h^{-1}u^*h$, and $u^* = (u, \sigma u, \sigma^3 u, \sigma^2 u), u\sigma^2 u = 1$. Choose $v \in R_E^\times$ with $v/\sigma^2 v = u$; it exists since E/E_3 is unramified. Under the isomorphism of $SO\left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right)$ with $\big(GL(2)\times GL(2)\big)'/Z$, u^* is $\big(\left(\begin{smallmatrix} u\sigma u & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} u & 0 \\ 0 & \sigma u \end{smallmatrix}\right)\big)$, and σu^* is $\big(\left(\begin{smallmatrix} \sigma u/u & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \sigma u & 0 \\ 0 & 1/u \end{smallmatrix}\right)\big) = \big(w\left(\begin{smallmatrix} u & 0 \\ 0 & \sigma u \end{smallmatrix}\right)w, \left(\begin{smallmatrix} u\sigma u & 0 \\ 0 & 1 \end{smallmatrix}\right)\big)$ (thus indeed $u_1 \in Z_G(s_1\theta)\big)$. Then

$$\Phi_{1_K}^{G,st}(u_1 s_1\theta) = \Phi_{1_{GL(2,R_3)'}}^{GL(2,E_3)'} \left(\begin{pmatrix} v\sigma v & 0 \\ 0 & \sigma^2 v\sigma^3 v \end{pmatrix} \right).$$

This is an orbital integral of 1_K on $GL(2, E_3)$ (K is the maximal compact $GL(2, R_3)$ of $GL(2, E_3)$ on the right), and it is given by a closed formula, depending only on

$$|v\sigma v - \sigma^2 v\sigma^3 v|_{E_3} = |u\sigma u - 1|_{E_3} = |u\sigma u - 1|_F |u/\sigma u - 1|_F,$$

since $u\sigma^2 u = 1$ and $|x|_{E_3} = |N_{E_3/F}x|_F = |x\sigma x|_F$.

The norm Ns is $h^{-1}\big(s\sigma s, \sigma^3(s\sigma s), \sigma(s\sigma s), \sigma^2(s\sigma s)\big)h$, which is equal to $h^{-1}s\sigma s(1,-1,-1,1)h$ since $\sigma s^2 = -s$. Hence $Z_{\boldsymbol{GSp}}(Ns)$ consists of $g = h^{-1}[B,B']h$, $\|B\| = \|B'\|$, and is isomorphic to $\big(GL(2)\times GL(2)\big)'$. To determine its rational points, $Z_{GSp}(Ns)$, we consider the g fixed by σ. The relation $\sigma^2 g = g$ (on considering $\sigma^2(h)h^{-1}$) leads to the statement that B, B' lie in a group isomorphic to $GL(2, E_3)$.

Since $\sigma h \cdot h^{-1}$, up to a multiple by a diagonal matrix, is $\left(\begin{smallmatrix} 0 & w \\ w & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & & 0 \\ & w & \\ 0 & & 1\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 1 & & 0 \\ & w & \\ 0 & & 1\end{smallmatrix}\right)$ acts on (B,B') by mapping it to (B',B), we conclude that the relation $\sigma g = g$ has the solutions $h^{-1}[B,B']h$, B' determined by B, and B ranging over a group isomorphic to $GL(2,E_3)'$ (prime = determinant in $F^\times$). The norm of u_1 is $Nu = h^{-1}(u\sigma u, u\sigma^3 u, \sigma u\sigma^2 u, \sigma^2 u\sigma^3 u)h$, thus the B here has the eigenvalues $u/\sigma u, \sigma u/u$. We conclude that

$$\Phi_{1_K}^{GSp(2,F)}(Ns_1 Nu_1) = \Phi_{1_K}^{GL(2,E_3)}\left(\begin{pmatrix} u/\sigma u & 0 \\ 0 & \sigma u/u\end{pmatrix}\right),$$

and this integral over $GL(2,E_3)$ is given by the usual formula, which depends explicitly on the factor

$$|u/\sigma u - \sigma u/u|_{E_3} = |u/\sigma u - 1|_{E_3} = |u/\sigma u - 1|_F |u\sigma u - 1|_F$$

(since u is topologically unipotent and $u\sigma^2 u = 1$). This factor is the same as in the θ-case, and the matching of the stable orbital integrals follow in all cases.

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