QUADRATIC CYCLES ON $GL(2n)$ CUSP FORMS

Yuval Z. Flicker

Let E/F be a quadratic separable extension of global fields, with completions E_v/F_v, adeles $\mathbb{A}_E/\mathbb{A}$, ideles $\mathbb{A}_E^\times/\mathbb{A}^\times$. A central simple algebra over F is a matrix $m \times m$ algebra $\mathbf{M}_m(\mathbf{D}_d)$ with entries in a division algebra $\mathbf{D} = \mathbf{D}_d$ central of degree ($\deg = \sqrt{\dim}$) d over F. We shall consider only $\mathbf{D}$ *unsplit* by E, by which we mean that at v where E_v is a field, the exponent (order in the Brauer group $\mathrm{Br}\,(F_v) = Q/\mathbb{Z}$) of $D_v = \mathbf{D}(F_v)$ is odd (when D_v is split, namely is a matrix algebra, its exponent (= order in $\mathrm{Br}\,(F_v)$) is one). In this case $\exp D_v$ is equal to $\exp D_v^E$ (= order in $\mathrm{Br}\,(E_v)$), where $D_v^E = D_v \otimes_{F_v} E_v$ (= $\mathbf{D}^E(F_v)$, where $\mathbf{D}^E = \mathbf{D} \times_F E = \mathbf{D} \times_{\mathrm{Spec}\ F} \mathrm{Spec}\ E$). If $\mathbf{D}$ is not unsplit by E then $\exp D_v^E$ may be half of $\exp D_v$, for some v where E_v is a field. Under our assumption that $\mathbf{D}$ is unsplit by E, the algebra $\mathbf{D}^E$ is division, central over E of degree $d = \deg \mathbf{D}$.

Let $\mathbf{H}$ be a simple algebra of degree 2 central over F. The multiplicative group $\mathbf{G}$ of $\mathbf{M}_m(\mathbf{D}^{\mathbf{H}})$, where $\mathbf{D}^{\mathbf{H}} = \mathbf{D} \times_F \mathbf{H}$, is an algebraic F-group, which is an inner form of the split group $\mathbf{G}^{sp} = GL(2n)/F$, $n = md$. Put $G = \mathbf{G}(F)$, $\mathbb{G} = \mathbf{G}(\mathbb{A})$; $Z = \mathbf{Z}(F)$, $\mathbb{Z} = \mathbf{Z}(\mathbb{A})$, where $\mathbf{Z}$ is the center of $\mathbf{G}$; and $C = \mathbf{C}(F)$, $\mathbb{C} = \mathbf{C}(\mathbb{A})$, where $\mathbf{C}$ is the multiplicative group of $\mathbf{M}_m(\mathbf{D}^E)$. Then $\mathbf{C}$ is an inner form of the split group $\mathbf{C}^{sp} = GL(n)/E$. The group $\mathbb{G}$ can be realized as consisting of the invertible matrices of the form $\begin{pmatrix} A & B \\ \varepsilon\overline{B} & \overline{A} \end{pmatrix}$, A, B in $\mathbb{C}$, bar indicating the $\mathrm{Gal}\,(E/F)$-action on the second factor in $\mathbb{D}^E = \mathbb{D} \otimes_{\mathbb{A}} \mathbb{A}_E$. Here ε is a fixed element in $F^\times$, outside the norm subgroup $NE^\times = N_{E/F}E^\times$ from E, unless $\mathbf{H} = GL(2)/F$, in which case we take $\varepsilon = 1$. Indeed, $\mathbf{H}$ can be realized by such matrices with entries A, B in $\mathbb{A}_E$. The group $\mathbb{C}$ embeds in $\mathbb{G}$ via $A \mapsto \mathrm{diag}\,(A, \overline{A})$.

This note concerns the *periods* $P(\phi) = \int_{\mathbb{Z}C\backslash\mathbb{C}} \phi(h)dh$ of cusp forms ϕ in $L_0^2(\mathbb{Z}G\backslash\mathbb{G})$ over the *cycle* $\mathbb{Z}C\backslash\mathbb{C}$. This cycle has finite volume and the convergence of the integral follows at once from the rapid decay of the cusp form ϕ on $\mathbb{Z}G\backslash\mathbb{G}$. Cuspidal (automorphic) representations π (= irreducible submodules of the $\mathbb{G}$-module $L_0^2(\mathbb{Z}G\backslash\mathbb{G})$ of cusp forms in the space $L^2(\mathbb{Z}G\backslash\mathbb{G})$ of automorphic forms) which contain a form ϕ with a non-zero period are called here *cyclic*.

The interest in such cyclic π originates from studies of arithmetic cohomology, and liftings of automorphic forms. Such studies were initiated by Waldspurger [Wa] using the theory of the Weil representation, in the case of $m = d = n = 1$.

Jacquet [J1] introduced a new technique for the study of such cusp forms, which he named the "relative trace formula". It is based on integrating the kernel of the convolution operator $K_f(x, y)$ over x and y in two cycles $\mathbb{Z}C_1\backslash\mathbb{C}_1$ and $\mathbb{Z}C_2\backslash\mathbb{C}_2$.

The case of the group $\mathbf{C} \times \mathbf{C}$ and the subgroups $\mathbf{C}_1 = \mathbf{C}_2 = \mathbf{C}$ embedded diagonally, coincides with the standard trace formula. In general Jacquet's relative trace formula involves no traces; it is a summation formula, equating a geometric with a spectral sums. The case $\mathbf{C}_1 = \mathbf{C}_2$ considered in this note is called here the "bi-period summation formula".

Another notable case is introduced in Jacquet [J2] (see also [F3]); there $\mathbf{C}_2$ is a unipotent subgroup, and Fourier coefficients of the cusp forms (in addition to cycles) are obtained. We

Department of Mathematics, 231 W. 18th Ave., The Ohio State University, Columbus, OH 43210-1174; email: flicker@math.ohio-state.edu

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

then refer to this special case of Jacquet's relative trace formula as the "Fourier summation formula".

In this note we study a general case of Jacquet's bi-period summation formula, with arbitrary m, d, n. The general case – introduced here – poses several interesting questions, some of which are answered below, some are stated as "working hypothesis", and some are conjectured.

The simple algebra $\mathbf{M}_{m'}(\mathbf{D}_{d'})$ over F ($m'd' = md = n$) is *more split* than $\mathbf{M}_m(\mathbf{D}_d)$ if $\exp D'_v$ divides $\exp D_v$ for all v. In particular, ord D'_v is odd when E_v is a field. The multiplicative group of $\mathbf{M}_{m'}(\mathbf{D}^{\mathbf{H}}_{d'})$ is denoted by $\mathbf{G}'$. Similar terminology applies to the group, and locally: $\mathbf{G}'$ splits more than $\mathbf{G}$ precisely when G'_v splits more than G_v for all v. In particular, the split group $\mathbf{G}^{sp}$ is more split than any of its inner forms. Denote by V the set of places where $G_v = \mathbf{G}(F_v)$ is not isomorphic to $G'_v = \mathbf{G}'(F_v)$.

Theorem A. *Suppose that $\mathbf{G}'$ is more split than $\mathbf{G}$, and that $\mathbf{G}$ (hence also $\mathbf{G}'$) is unsplit by E. Let π be a cuspidal $\mathbb{G}$-module whose component π_u at some place u of F where $G_u \simeq G'_u$, is supercuspidal. Denote by π' the cuspidal $\mathbb{G}'$-module which corresponds to π (see [FK2]). Fix a place $u' \neq u$, and put $V' = V \cup \{u'\}$.*

Assume that (WH1) and (WH2) hold for each component π_v of π ($v \in V'$), and that $\pi_{u'}$ is bi-elliptic (definitions below). If π is cyclic then so is π' (thus $P(\phi') = \int_{\mathbb{Z}C'\backslash\mathbb{C}'} \phi'(h)dh \neq 0$ for some $\phi' \in \pi'$, where $\mathbf{C}' = \mathbf{M}_{m'}(\mathbf{D}^{E}_{d'})$). Moreover, the bi-character of π'_v is not identically zero on the set of bi-regular elements of G'_v which come from G_v, for all v.

Accept (WH1) and (WH2) as valid for π'_v ($v \in V'$). If π' is cyclic, $\pi'_{u'}$ is bi-elliptic, and the bi-character of π'_v is not identically zero on the set of bi-regular elements of G'_v which come from G_v ($v \in V$), then π is cyclic.

Two extreme cases where the Theorem applies are (1) when $\mathbf{G}' = GL(2n)/F$ is split; (2) $m' = m$, $d' = d$, and $\exp D'_v = \exp D_v$ for all v (in this case the invariants of D_v and D'_v can be different in $\mathrm{Br}(F_v) \simeq Q/\mathbb{Z}$, hence $D'_v \not\simeq D_v$ for some v).

The cuspidal $\mathbb{G}$-module $\pi = \otimes\pi_v$ and the cuspidal $\mathbb{G}'$-module $\pi' = \otimes\pi'_v$ *correspond* if $\pi_v \simeq \pi'_v$ for almost all v (where $G_v \simeq G'_v$). It is shown in [FK2] that the cuspidal $\mathbb{G}$-modules π with a supercuspidal component π_u at some place $u \notin V$ occur with multiplicity one in $L^2_0(\mathbb{Z}G\backslash\mathbb{G})$; that they satisfy the rigidity theorem: if $\pi_1 = \otimes\pi_{1v}$ and $\pi_2 = \otimes\pi_{2v}$ have supercuspidal components $\pi_{1u} \simeq \pi_{2u}$, and $\pi_{1v} \simeq \pi_{2v}$ for almost all v, then $\pi_1 \simeq \pi_2$; and that the correspondence defines an embedding of the set of the cuspidal π with a supercuspidal π_u into the set of the cuspidal π' with a supercuspidal π'_u. The image consists of the π' whose local components π'_v are obtained by the local correspondence of relevant representations of G_v to relevant representations of G'_v, for all v. In particular, if π corresponds to π' then $\pi_v \simeq \pi'_v$ for all $v \notin V$.

In fact [FK2] sharpens the work of Bernstein-Deligne-Kazhdan-Vigneras [BDKV] and [F1; III] where the case of π' with a supercuspidal and in addition another square-integrable component, is dealt with. The global theorem requires in particular establishing the local correspondence not only for tempered local representations, but also for relevant local representations (since the generalized Ramanujan conjecture – asserting that all components of a cuspidal π' are tempered – is merely a conjecture).

The notion of relevant representations (the representations which may be components of a cuspidal $\mathbb{G}$-module) is introduced in [FK1] in a similar context (of an r-fold covering of $GL(n)$), where they are shown to be irreducible and unitarizable. This notion was later used e.g. by Patterson and Piatetski-Shapiro [PPS]. Of course all the main ideas in the proof of the correspondence are due to Deligne and Kazhdan. Their proof in the case of $m = 1$ ($d = n$; i.e. $\mathbf{G}$ is anisotropic) – which is remarkably simple – is explained in [F2].

The proofs of [BDKV], [F2], [F1; III] and [FK1] are based on the "Deligne-Kazhdan" simple trace formula, and that of [FK2] on a sharper form of the simple trace formula, where regular, Iwahori-invariant functions, are used. The proof here does not involve any trace formula, yet we do use some of the ideas which play key roles in the development of the simple trace formula. Our main global tool is a new "bi-period summation formula", obtained on integrating over two copies of $\mathbb{Z}C\backslash\mathbb{C}$ the spectral and geometric expressions for the kernel of the convolution operator $r(f)$ on $L^2(\mathbb{Z}G\backslash\mathbb{G})$ for a test function f with a supercuspidal component f_u. An observation of Kazhdan implies that $r(f)$ factorizes through the natural projection to the space $L^2_0(\mathbb{Z}G\backslash\mathbb{G})$ of cusp forms. On the spectral side of our formula we obtain the periods of the cyclic cusp forms. On the geometric side we obtain a new type of bi-orbital integrals. As in [BDKV], [F2], [F1; III], [FK1], we choose another component – say $f_{u'}$ – of the test function f, and restrict its support to a certain set of "bi-elliptic bi-regular" elements in our bi-periodic sense. This choice of $f_{u'}$ greatly simplifies our study of the geometric side, indeed it makes our study possible. Yet the choice of $f_{u'}$ restricts the applicability of our technique to π and π' with a "bi-elliptic" (a notion presently to be defined) components at u'.

Our proof is based on two statements, (WH1) and (WH2), which we accept here as "working hypotheses". In Proposition 0 we prove (WH1) in a special case. The (WH1) and (WH2) are analogues of similar statements for characters, whose proofs – we hope – are applicable (after some work) in our case too. As noted above, the present note can be viewed also as a motivation to study these hypotheses. Both hypotheses are local. They concern an irreducible admissible G_v-module π_v (see [BZ]), where $G_v = \mathbf{G}(F_v)$.

Working hypothesis (WH1). *Let π_v be an admissible irreducible G_v-module. Then there exists at most one (up to a scalar multiple) C_v-invariant linear form on π_v (thus there is a single form $P_{\pi_v} : \pi_v \to \mathbb{C}$ with $P_{\pi_v}(\pi_v(h)\xi) = P_{\pi_v}(\xi)$ for all $h \in C_v$ and $\xi \in \pi_v$).*

Alternatively put, $\dim \operatorname{Hom}_{C_v}(\pi_v, 1) \leq 1$, or: the restriction of π_v to C_v has the trivial quotient with multiplicity at most one. A G_v-module π_v with $P_{\pi_v} \neq 0$ is called *cyclic*. Each local component of a cyclic cuspidal π is cyclic, but a cuspidal π whose local components are all cyclic is not necessarily cyclic. Statements similar to (WH1) were established using techniques of Gelfand-Kazhdan [GK] (cf. [BZ], (5.16)-(5.17), (7.6)-(7.10), [R], [NPS]) to prove (existence in the case of $GL(n)$ and) uniqueness of Whittaker models, the uniqueness of a $GL(n, F_v)$-invariant linear form on an irreducible $GL(n, E_v)$-module where E_v/F_v is a quadratic field extension ([F3], p. 163), the uniqueness of a $GL(2, F_v)$-invariant form on a $GL(2, K_v)$-module where K_v is a cubic extension of F_v (Prasad [P], p. 1327), as well as in the cases of such pairs as $(GL(n-1), GL(n))$, $(O(n-1), O(n))$, $(U(n-1), U(n))$ by Bernstein, Piatetski-Shapiro, Rallis. The case where E/F and $\mathbf{D}$ are split has recently been treated by Jacquet and Rallis (in fact, after this note was written). These techniques would eventually lead to a proof of (WH1). Let us verify (WH1) in a special case.

0. Proposition. *Let D_v be a division algebra of odd degree n central over F_v, E_v a quadratic field extension of F_v, and H_v a central simple algebra of degree 2 over F_v. Note that $C_v = (D_v \otimes_{F_v} E_v)^\times$ embeds in $G_v = (D_v \otimes_{F_v} H_v)^\times$. For any admissible irreducible G_v-module π_v there exists at most one (up to a scalar multiple) C_v-invariant linear form on π_v; namely,* $\mathrm{Hom}_{C_v}(\pi_v, 1)$ *has dimension* ≤ 1.

Proof. By a well-known criterion of Gelfand-Kazhdan [GK] (recorded also in [P], p. 1327; [F3], p. 163), it suffices to find a non-trivial involution $g \mapsto g^\#$ on G_v which preserves C_v, and fixes every C_v-double coset in G_v. We shall check that the involution $g \mapsto g^{-1}$ has this property. For that, realize G_v as the group of invertible matrices $\begin{pmatrix} A & B \\ \varepsilon\overline{B} & \overline{A} \end{pmatrix}$, A, B in $D_v \otimes_{F_v} E_v$, and embed C_v in G_v via $A \mapsto \mathrm{diag}\,(A, \overline{A})$. Here $\varepsilon = 1$ if $H_v = GL(2, F_v)$, and $\varepsilon = \varepsilon(H_v) \in F_v - N_{E_v/F_v} E_v$ if H_v is a quaternion algebra. Since $\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}^{-1} = \varepsilon^{-1} \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}$, and $\begin{pmatrix} 1 & B \\ \varepsilon\overline{B} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -B \\ -\varepsilon\overline{B} & 1 \end{pmatrix} \begin{pmatrix} 1 - \varepsilon B\overline{B} & 0 \\ 0 & 1 - \varepsilon\overline{B}B \end{pmatrix}^{-1}$, and there is some x in E_v with $\overline{x} = -x \neq 0$ so that $\begin{pmatrix} x & 0 \\ 0 & \overline{x} \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ \varepsilon\overline{B} & \overline{A} \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & \overline{x} \end{pmatrix} = \begin{pmatrix} A & -B \\ -\varepsilon\overline{B} & \overline{A} \end{pmatrix}$, the proof is complete. □

Let $H_v = C_c^\infty(Z_v \backslash G_v)$ denote the convolution algebra (a choice of a Haar measure is implicit) of compactly supported (modulo Z_v) smooth (= locally constant when v is non-archimedean) complex-valued functions on G_v which transform trivially under Z_v. Fix an orthonormal basis $\{\xi_v\}$ in the space of the irreducible admissible G_v-module π_v. Introduce a *bi-period* distribution on H_v by

$$\mathbb{P}_{\pi_v}(f_v) = \sum_{\xi_v} P_{\pi_v}(\pi_v(f_v)\xi_v)\overline{P}_{\pi_v}(\xi_v).$$

The linear form P_{π_v} lies in the dual π_v^* of π_v. It also defines an element – denoted $P_{\pi_v}^\vee$ – in the dual $\widetilde{\pi}_v^*$ of the contragredient $\widetilde{\pi}_v$ of π_v. Put $\langle P_{\pi_v}, \xi_v \rangle = P_{\pi_v}(\xi_v)$, $\langle P_{\pi_v}^\vee, \xi_v^\vee \rangle = P_{\pi_v}^\vee(\xi_v^\vee)$. Then $P_{\pi_v}^\vee$ decomposes as $P_{\pi_v}^\vee = \sum_{\xi_v} \langle P_{\pi_v}^\vee, \xi_v^\vee \rangle \xi_v$, and

$$\langle P_{\pi_v}, \pi_v(f_v) P_{\pi_v}^\vee \rangle = \sum_{\xi_v} \langle P_{\pi_v}^\vee, \xi_v^\vee \rangle \langle P_{\pi_v}, \pi_v(f_v)\xi_v \rangle$$

is an alternative expression for $\mathbb{P}_{\pi_v}(f_v)$.

This $\mathbb{P}_{\pi_v}(f_v)$ is clearly independent of the choice of the basis $\{\xi_v\}$ of π_v. If $\pi_{1v}, \ldots, \pi_{kv}$ are pairwise inequivalent, then $\mathbb{P}_{\pi_{1v}}, \ldots, \mathbb{P}_{\pi_{kv}}$ are linearly independent. Since $\mathbb{P}_{\pi_v}$ is independent of the choice of basis for π_v, it is bi-C_v-invariant, namely its value at ${}^a f_v^b(g) = f_v(a^{-1}gb)$, $(a, b \in C_v)$ is equal to its value at f_v. In particular the distribution $\mathbb{P}_{\pi_v}$ depends on f_v only via the bi-period integral

$$\Xi(\gamma, f_v) = \int_{C_v/C_v \cap \gamma C_v \gamma^{-1}} \int_{C_v/Z_v} f_v(h\gamma h')\,dh\,dh'.$$

The convergence of this bi-orbital integral is obvious when γ is bi-regular (see below). Note that without assuming (WH1), the bi-period distribution $\mathbb{P}_{\pi_v}$ of π_v is not uniquely defined.

Working Hypothesis (WH2). *Let π_v be a cyclic admissible irreducible G_v-module. Then there exists a bi-C_v-invariant complex valued function $p(g,\pi_v)$, which is smooth (= locally constant if v is non-archimedean) and not identically zero on a Zariski open (hence dense) subset of G_v (named bi-regular below), such that*

$$\mathbb{P}_{\pi_v}(f_v) = \int_{Z_v\backslash G_v} f_v(g)p(g,\pi_v)dg.$$

In the archimedean case, this has been shown by Sekiguchi [S]. The function $p(g,\pi_v)$ is named here the *bi-character* of π_v. It is analogous to the character $\chi(g,\pi_v)$ or $\chi_{\pi_v}(g)$ of the trace distribution $\operatorname{tr}\,\pi_v(f_v) = \int f_v(g)\chi(g,\pi_v)dg$, shown by Howe [H] and Harish-Chandra [HC2] to be locally constant on the regular set (which is Zariski open), and moreover (see Harish-Chandra [HC3]) locally integrable on G_v. The proof of [HC2] shows that the restriction of $\mathbb{P}_{\pi_v}$ to the space of functions $f_v^{K_v}(g) = \int_{K_v} f_v(kgk^{-1})dk$ (K_v = good maximal compact subgroup of G_v) is represented by a smooth function on the regular set. Since $\operatorname{tr}\,\pi_v(f_v) = \operatorname{tr}\,\pi_v(f_v^{K_v})$, this establishes the result for the trace distribution. It would be interesting to extend this simple proof of [HC2] to apply in our case too.

A similar question is dealt with in [FH], where it is shown – using Howe's orbit method as in [HC3] – that the bi-character exists as a locally constant function on the relatively(=bi)-regular set (introduced there), in the case of $GL(n,D_v)$-invariant distributions on $GL(n,D'_v)$-modules, where D_v is a division algebra central over F_v, while $D'_v = D_v\otimes_{F_v}E_v$, where E_v/F_v is a quadratic field extension. A very recent work by Rader and Rallis extends this method to show that the bi-character is locally constant on the bi-regular set in the present case too. The case of a supercuspidal π_v is discussed in the Remark below.

The local integrability ([HC3]) implies that the character is not identically zero on the regular set, in the case of the trace. The bi-character of [FH] is also locally integrable, hence not identically zero on the bi-regular set. This quadratic case is very close to that of Harish-Chandra's group case. But in general, $p(g,\pi_v)$ often fails to be locally integrable on G_v. It may be supported on the closed proper subset of "bi-singular" elements. It will be interesting to determine which π_v satisfy (WH2). We expect all cyclic admissible G_v-modules to satisfy (WH2) (in our case), in analogy with the archimedean case; see Sekiguchi [S]. This problem seems to be accessible to available techniques, but its solution would require a separate paper.

The relation $\begin{pmatrix} A & 0 \\ 0 & \overline{A}\end{pmatrix}\begin{pmatrix} I & B \\ \varepsilon\overline{B} & I\end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & \overline{A}\end{pmatrix}^{-1} = \begin{pmatrix} I & AB\overline{A}^{-1} \\ \varepsilon\overline{AB}A^{-1} & I\end{pmatrix}$ (I is the unit in C_v) shows that on the Zariski open (dense) subset $X_v = \left\{\begin{pmatrix} A & B \\ \varepsilon\overline{B} & \overline{A}\end{pmatrix}; |A| \neq 0 \neq |B|\right\}$ of G_v, a set of representatives for $C_v\backslash X_v/C_v$ is given by the matrices $\gamma = \gamma(\beta) = \begin{pmatrix} I & \beta \\ \varepsilon\overline{\beta} & I\end{pmatrix}$, where β ranges over a set of representatives for the σ-conjugacy classes of σ-regular elements in C_v, with $|\varepsilon\beta\overline{\beta} - I| \neq 0$. Here $|A| = \det A$. Following Shintani [Sh] we define the *σ-conjugacy class* of β in C_v to consist of $\{a\beta\overline{a}^{-1};\ a \text{ in } C_v\}$. It is well-known and easy to see that the conjugacy class in C_v of $\beta\overline{\beta}$ intersects $C_v^0 = GL(m,D_v)$. The map $\beta\mapsto\beta\overline{\beta}$ yields an injection of the set of σ-conjugacy classes in C_v into the set of conjugacy classes in C_v^0;

the image is easily described by means of the norm map from E_v to F_v, and the eigenvalues of $\beta\overline{\beta}$.

We shall say that $g \in G_v$ is *bi-regular* if it is *bi-conjugate* (agb, for some a, b in C_v, is equal) to $\gamma = \gamma(\beta)$ with σ-regular β (namely $\beta\overline{\beta}$ is regular: it has distinct eigenvalues, not equal to 0 or ε^{-1}). A $g \in G_v$ is called *bi-elliptic* if it is bi-conjugate to $\gamma = \gamma(\beta)$ with a σ-elliptic β (thus $\beta\overline{\beta}$ is elliptic, namely its conjugate lies in an elliptic torus of C_v^0). The Zariski open dense set in (WH2) is the bi-regular set.

A cyclic G_v-module π_v is called *bi-elliptic* if its bi-character is not identically zero on the bi-elliptic bi-regular set of G_v. Theorem A concerns π with a bi-elliptic component $\pi_{u'}$. Note that if g is bi-elliptic then $\beta\overline{\beta}$ lies in an elliptic torus of C_v^0, which is the multiplicative group $K_v^\times$ of a separable field extension K_v of degree n of F_v. The image of the map $\beta \mapsto \beta\overline{\beta}$ consists of the subgroup $N_{K_vE_v/K_v}(E_vK_v)^\times$ of $K_v^\times$. A general bi-regular bi-conjugacy class can easily be described in terms of these σ-elliptic β.

Denote by $p_\beta(z) = \det(z - \beta\overline{\beta})$ (the reduced norm is defined by the determinant on the group of points over a splitting field, such as a separable closure of F_v) the characteristic polynomial of the conjugacy class in C_v^0 of $\beta\overline{\beta}$. In the case where C_v^0 is the multiplicative group of a division algebra, the map $\beta \mapsto p_\beta$ is a bijection from the set of σ-regular (necessarily σ-elliptic) σ-conjugacy classes in C_v, to the set of irreducible separable polynomials of degree d over F_v whose eigenvalues lie in $N_{K_vE_v/K_v}(K_vE_v)^\times$, where K_v is the separable extension of F_v of degree d generated by the eigenvalue. The analogous statement holds globally with $(F, D, \dots)$ replacing $(F_v, D_v, \dots)$.

In general, the map $\beta \mapsto p_\beta$ is an injection from the set of σ-regular (resp. σ-elliptic σ-regular) σ-conjugacy classes in C_v, to the set of separable (resp. irreducible separable) polynomials of degree $\deg C_v^0$ over F_v whose irreducible factors have degrees which are multiples of $\deg D_v$. The image is easily described on expressing $\beta\overline{\beta}$ as a product of elliptic regular factors (i.e. conjugating $\beta\overline{\beta}$ into the standard Levi factor of a minimal parabolic containing a conjugate of $\beta\overline{\beta}$).

In particular, the set of bi-regular bi-conjugacy classes in G_v embeds as a subset of the set of bi-regular bi-conjugacy classes in G'_v. A bi-regular bi-conjugacy class in G'_v so obtained is said here to *come from* G_v. The set of bi-regular bi-elliptic bi-conjugacy classes in G_v bijects with the set of bi-regular bi-elliptic bi-conjugacy classes in G'_v. With this definition, the statement of Theorem A is now complete.

Remark. If π_v is cyclic and supercuspidal, then its bi-character is smooth on the bi-regular bi-elliptic set. Indeed, the linear form $\mathbb{P}_{\pi_v}$ is the unique (up to a scalar multiple) non-zero bi-C_v-invariant linear form on H_v which vanishes on the orthogonal complement of the span of the space of matrix coefficients of π_v. Hence $\mathbb{P}_{\pi_v}(f_v)$ is a constant multiple of

$$\int_{C_v/Z_v}\int_{C_v/Z_v} \langle \pi_v(f_v)\pi_v(h)\xi, \widetilde{\pi}_v(h')\xi^\vee\rangle dh\, dh'$$

$$= \int_{C_v/Z_v}\int_{C_v/Z_v}\int_{G_v/Z_v} f_v(g)\langle \pi_v(h'gh)\xi, \xi^\vee\rangle dg\, dh\, dh',$$

for any vector $\xi \neq 0$ in π_v.

If g is bi-regular bi-elliptic then it is of the form $g = c'\gamma(\beta)c$. Its bi-centralizer

$$Z_v(g) = \{(h', h) \in C_v \times C_v; h'gh = g\}$$

is equal to

$$\left\{\left(h' = c'\begin{pmatrix} t & 0 \\ 0 & \bar{t}\end{pmatrix}c'^{-1},\ h = c^{-1}\begin{pmatrix} t & 0 \\ 0 & \bar{t}\end{pmatrix}^{-1}c\right); t \in Z_v(\beta\sigma)\right\} \simeq Z_v(\beta\sigma),$$

where $Z_v(\beta\sigma) = \{t \in C_v; t\beta\bar{t}^{-1} = \beta\}$ is the σ-centralizer of β in C_v. Since β is assumed to be σ-regular σ-elliptic, the σ-centralizer $Z_v(\beta\sigma)$ is an elliptic torus in C_v^0, isomorphic to the multiplicative group of the separable extension of F_v of degree $n(= \deg C_v^0)$ generated by the elliptic regular elements $\beta\overline{\beta}$. In particular the volume $|Z_v(g)/Z_v|$ is finite, for such g.

Now suppose that f_v is supported on the bi-regular bi-elliptic set in G_v. Then we may change the order of integration, obtaining (equality up to a scalar multiple depending on the choice of ξ):

$$\mathbb{P}_{\pi_v}(f_v) = \int\limits_{G_v/Z_v} f_v(g)|Z_v(g)/Z_v|\Xi(g, c_{\pi_v})dg,$$

where $c_{\pi_v}(g) = \langle \pi_v(g)\xi, \xi^\vee\rangle$ is a matrix coefficient of π_v. In particular the bi-character $p(g, \pi_v)$ of a supercuspidal cyclic π_v is given on the bi-regular bi-elliptic set by $p(g, \pi_v) = |Z_v(g)/Z_v|\Xi(g, c_{\pi_v})$. It is therefore smooth on the bi-regular bi-elliptic set.

However, we have not verified that $p(g, \pi_v)$ is not identically zero on the bi-regular bi-elliptic set. In the classical case of characters, it is verified in [HC1] that the characters of the supercuspidal representations are locally integrable functions, and that their restrictions to the elliptic regular subset satisfy orthonormality relations. In particular the character of a supercuspidal representation is not identically zero on the elliptic regular set. It will be interesting to establish an analogue in our case. □

We obtain also purely local results. The following is a bi-analogue of Kazhdan's density theorem for characters (see [K; Appendix]). It does not rely on (WH2). Let E_w be a commutative separable semi-simple algebra of dimension two over F_w (thus E_w is a separable quadratic field extension of F_w, or $E_w = F_w \oplus F_w$). Assume that there exists a supercusp form f_u on G_u with $\Xi(g, f_u) \not\equiv 0$, or alternatively that there exists a supercuspidal cyclic π_u, in which case f_u can be taken to be its matrix coefficient. Of course, when n is odd and E_u is a field, we may take $f_u = 1$ on $G_u = (D_u \otimes_{F_u} H_u)^\times$, where D_u is a division algebra of degree n central over F_u.

Theorem B. *Assume that (WH1) holds for every irreducible admissible cyclic representation π_w of $G_w = GL(m_w, D_w^H)$, where D_w is a division algebra central over F_w of odd degree if F_w is a field, and H_w is a simple algebra of degree two central over F_w ($D_w^H = D_w \otimes_{F_w} H_w$). Then $\mathbb{P}_{\pi_w}$ is defined. If $f_w \in H_w$ is a test function such that $\mathbb{P}_{\pi_w}(f_w) = 0$ for all cyclic π_w, then the bi-orbital integral $\Xi(\gamma, f_w)$ is zero on the bi-regular set of γ in G_w.*

A local analogue of Theorem A is stated next. Suppose that the bi-elliptic part of (WH2) holds for every admissible irreducible representation $\pi_{u'}$ of $G_{u'} = (D_{u'} \otimes_{F_{u'}} H_{u'})^\times$, where

$D_{u'}$ is a division algebra central of degree n over $F_{u'}$, and $H_{u'}$ is a quaternion algebra central over $F_{u'}$ when $E_{u'}$ is a field, $GL(2, F_{u'})$ if $E_{u'} = F_{u'} \oplus F_{u'}$ (namely no assumption when n is odd). In other words, we assume that the bi-character $p(g, \pi_{u'})$ of any such $\pi_{u'}$ (not only supercuspidal as in the Remark following the statement of (WH2)), is locally constant on the bi-regular (necessarily bi-elliptic) set of $G_{u'}$. When $D_{u'} \otimes_{F_{u'}} H_{u'}$ is a division algebra, each representation of its multiplicative group is supercuspidal, and its bi-character is clearly locally constant.

Theorem C. *Let π_u be a cyclic supercuspidal G_u-module satisfying (WH1) and (WH2), where $G_u = GL(m_u, D_u^H)$, D_u an F_u-central division algebra unsplit by E_u. Then the corresponding square-integrable $G'_u = GL(m_u, {D'_u}^H)$-module π'_u is cyclic; here D'_u is an F_u-central simple algebra of the same degree as D_u, with $\exp D'_u$ dividing $\exp D_u$ (D'_u is more split than D_u).*

Recall that the local correspondence is defined by means of character relations (see [F1; III]). The corresponding π'_u is square-integrable, but not necessarily supercuspidal.

A "split" analogue – where $E = F \oplus F$ globally – of our work, is the subject matter of [F4]. Our local Theorems B and C specialize to the corresponding local theorems of [F4] when $E_u = F_u \oplus F_u$. At least when n is odd the statements of our local theorems are preferable to those of [F4], since then the requirement at u' (or w) is not present here. When n is odd our proofs of the local results can be considered better than the analogous proof in [F4] as we can work here with a global anisotropic group, for which the analysis simplifies. Although the present note overlaps with [F4], we decided to separate the two in an attempt to make each of them readable independently of the other. We hope to compare our results here, in the quadratic case, with the results of [F4], in the split case, on another occasion.

The global tool in our proofs is the following *bi-period summation formula.*

1. Proposition. *Let $f = \otimes f_v$ be a test function on $\mathbb{G}$ ($f_v \in H_v$ for all $v, f_v = f_v^0$ for almost all v) which has a supercuspidal component f_u and a component $f_{u'}$ supported on the bi-regular bi-elliptic set. Then*

$$\sum_{\pi} \sum_{\Phi} P(\pi(f)\Phi) P(\overline{\Phi}) = \sum_{\{\beta\}} |\mathbb{Z}(\beta\sigma)/\mathbb{Z}Z(\beta\sigma)| \prod_v \Xi(\gamma(\beta), f_v).$$

Here π ranges over the set of cuspidal cyclic $\mathbb{G}$-modules with a supercuspidal component at u, Φ ranges over an orthonormal basis of smooth vectors in the space of π, and $\{\beta\}$ ranges over a set of representatives for the σ-elliptic σ-regular σ-conjugacy classes in G.

Proof. Let $K_f(x,y)$ be the kernel of the convolution operator $(r(f)\phi)(x) = \int_{\mathbb{Z}\backslash\mathbb{G}} f(g)\phi(xg)dg$ on $L^2(\mathbb{Z}G\backslash\mathbb{G})$. Here $f = \otimes f_v$ is a product over all places v of F of $f_v \in H_v$, such that f_v is the unit element f_v^0 in the convolution algebra $\mathbb{H}_v$ of spherical (bi-K_v-invariant, K_v being the standard maximal compact subgroup of G_v) function in H_v, for almost all v. It is easy to see that $(r(f)\phi)(x) = \int_{\mathbb{Z}\backslash\mathbb{G}} K_f(x,y)\phi(y)dy$, where $K_f(x,y) = \sum_{\gamma \in Z\backslash G} f(x^{-1}\gamma y)$. This is the geometric expansion of the kernel.

We take the component f_u of f to be a supercusp form. A well-known observation of Kazhdan (see [F1; III]) asserts that $r(f)$ then factorizes through the natural projection into the subspace $L_0^2(\mathbb{Z}G\backslash\mathbb{G})$ of cusp forms in $L^2(\mathbb{Z}G\backslash\mathbb{G})$. Then the kernel has the spectral expansion

$$K_f(x,y) = \sum_\pi \sum_\Phi (\pi(f)\Phi)(x)\overline{\Phi}(y).$$

The first sum ranges over the set of cuspidal $\mathbb{G}$-modules π (in fact with a supercuspidal component at u), and Φ ranges over an orthonormal basis of smooth vectors in the space of π. Note that it is π – and not its equivalence class – which occurs here, by virtue of the multiplicity one theorem for such π of [F1; III] and [FK2].

Our formula is obtained on integrating these two expressions for the kernel over x, y in $\mathbb{Z}C\backslash\mathbb{C}$. The integral of the spectral expression is

$$\sum_\pi \sum_\Phi P(\pi(f)\Phi)P(\overline{\Phi}), \quad P(\Phi) = \int_{\mathbb{Z}C\backslash\mathbb{C}} \Phi(h)dh.$$

The integral over x, y in $\mathbb{Z}C\backslash\mathbb{C}$ of the geometric expression for the kernel is

$$\int\limits_{\mathbb{C}/C\mathbb{Z}} dx \int\limits_{\mathbb{Z}C\backslash\mathbb{C}} \sum_{\gamma\in G/Z} f(x\gamma y)\,dy = \sum_{\gamma\in C\backslash G/C} \int\limits_{\mathbb{C}/\mathbb{Z}\cdot C\cap\gamma C\gamma^{-1}} dx \int\limits_{\mathbb{Z}\backslash\mathbb{C}} f(x\gamma y)\,dy.$$

We take the component $f_{u'}$ of f at u' to be supported on the bi-regular bi-elliptic set. Consequently the rational bi-$\mathbb{C}$-orbits (the set of $x\gamma y$, with x, y in $\mathbb{C}$, and γ in G) on which f is non-zero, are those of the bi-regular bi-elliptic γ, represented by $\gamma = \gamma(\beta) = \begin{pmatrix} I & \beta \\ \epsilon\overline{\beta} & I \end{pmatrix}$, where β is a σ-elliptic σ-regular element of C. A complete set of representatives of these rational bi-orbits is given by $\gamma(\beta)$, as β ranges over a set of representatives $\{\beta\}$ for the σ-conjugacy classes of σ-elliptic σ-regular elements in C, $|I - \epsilon\beta\overline{\beta}| \neq 0$.

Note that $a\gamma(\beta)a^{-1} = \gamma(A\beta\overline{A}^{-1})$, where $a = \operatorname{diag}(A, A)$. Since $C \cap \gamma C\gamma^{-1} = Z(\beta\sigma)$, where $Z(\beta\sigma)$ is the group of a in C such that $A\beta\overline{A}^{-1} = \beta$, our double integral is equal to

$$= \sum_{\{\beta\}} |\mathbb{Z}(\beta\sigma)/\mathbb{Z}Z(\beta\sigma)| \int\limits_{\mathbb{C}/\mathbb{Z}(\beta\sigma)} dx \int\limits_{\mathbb{Z}\backslash\mathbb{C}} f(x\gamma(\beta)y)dy.$$

The double integral here can be expressed as a product, for $f = \otimes f_v$, of local bi-orbital integrals. Thus we obtain

$$= \sum_{\{\beta\}} |\mathbb{Z}(\beta\sigma)/\mathbb{Z}Z(\beta\sigma)| \prod_v \Xi(\gamma(\beta), f_v),$$

where the sum is finite and the product is absolutely convergent, as required. □

The following is clear.

2. Lemma. *Let $f_v \in H_v$ be a function on G_v supported on the bi-regular set. If T_v is a torus in $C_v^0 = GL(m_v, D_v)$, and T_v' is its centralizer in $C_v = GL(m, D_v^E)$, denote by $T_v'/N^\sigma(T_v')$ the quotient of T_v' by the equivalence relation $t' \sim t$ if $t' = \nu t \overline{\nu}^{-1}$ ($\nu \in C_v$), and by $\gamma(T_v'/N^\sigma(T_v'))$ the set of $\gamma(\beta)$, $\beta \in T_v'/N^\sigma(T_v')$. Then $\Xi(\gamma, f_v)$ is a smooth function with compact support on the union of $\gamma(T_v'/N^\sigma(T_v'))$ over a set of representatives $\{T_v\}$ of the conjugacy classes of F_v-tori in C_v^0.*

Conversely, given a smooth compactly supported function $\Xi(\gamma)$ on the bi-regular subset of $\cup_{\{T_v\}}\gamma(T_v'/N^\sigma(T_v'))$, there exists an $f_v \in H_v$ supported on the bi-regular set of G_v, with $\Xi(\gamma) = \Xi(\gamma, f_v)$. Both statements hold with "bi-regular" replaced by "bi-regular and bi-elliptic" throughout, except that T_v ranges then only over the elliptic conjugacy classes of F_v-tori. □

Of course the discussion above holds not only for $\mathbf{G}$ but for any inner form of it, in particular for $\mathbf{G}'$. To establish the comparison of the Theorem, we compare the geometric sides of the bi-periodic summation formula for $f = \otimes f_v$ on $\mathbb{G}$ and for $f' = \otimes f_v'$ on $\mathbb{G}'$. For this comparison fix a non degenerate differential form of highest degree on $\mathbf{G}$ over F. It defines a Haar measure on G_v and G_v', hence on $\mathbb{G}$ and $\mathbb{G}'$, in a compatible way. These measures, dg_v, dg, $d'g_v$ and $d'g$, are used to define the bi-period orbital integrals $\Xi(\gamma, f_v)$ and $\Xi(\gamma, f_v')$, as well as the distributions $P(\Phi)$ and $P(\Phi')$.

Definition. The functions $f_v \in H_v$ and $f_v' \in H_v'$ are called *matching* if $\Xi(\gamma', f_v')$ is zero on the bi-regular γ' which do not come from G_v, while if γ is a bi-regular element of G_v' which comes from γ in G_v, then $\Xi(\gamma', f_v') = \Xi(\gamma, f_v)$.

For all $v \notin V$, where V is the finite set of places where G_v is not isomorphic to G_v', we take f_v and f_v' to correspond to each other under this isomorphism. At the remaining finite number of places v in V, Lemma 2 guarantees the existence of f_v' matching any f_v which is supported on the bi-regular set of G_v. This f_v' can be taken to be supported on the bi-regular set of G_v', in fact on the (open) set of such elements which come from G_v.

Conversely, given any f_v' whose bi-period orbital integrals are supported on the set of bi-regular elements of G_v' which come from G_v, Lemma 2 guarantees the existence of an f_v, supported on the bi-regular set of G_v, matching f_v'.

3. Lemma. *For any test functions $f = \otimes f_v$ on $\mathbb{G}$ and $f' = \otimes f_v'$ on $\mathbb{G}'$ such that $f_v = f_v'$ for all $v \notin V$, $f_v = f_v^0$ for almost all v, f_u is a supercusp form and $f_{u'}$ supported on the bi-regular bi-elliptic set of $G_{u'}$, u outside V), and f_v, f_v' matching for all $v (\in V)$, we have*

$$\sum_{\pi'}\sum_{\Phi'} P(\pi'(f')\Phi')P(\overline{\Phi}') = \sum_{\pi}\sum_{\Phi} P(\pi(f)\Phi)P(\overline{\Phi}).$$

The sums range over the cuspidal $\mathbb{G}'$-modules π' and cuspidal $\mathbb{G}$-modules π, whose components at u are supercuspidal, and over orthonormal bases of smooth vectors Φ' in π' and Φ in π.

Proof. Our choice of matching f and f', as well as matching measures, guarantees the equality of the geometric sides of the bi-period summation formulae for f on $\mathbb{G}$ and f' on $\mathbb{G}'$ of Proposition 1. Hence the spectral sides are equal. □

4. Lemma. *Let π be a cuspidal $\mathbb{G}$-module with a supercuspidal component $\pi_u(u \notin V)$, and π' the corresponding cuspidal $\mathbb{G}'$-module. Let S be a finite set of places of F containing V, u, u', and all archimedean places and those where π_v is not unramified. If $f_v \in H_v$ and $f'_v \in H'_v$ are matching ($v \in V$), $f_u = f'_u$ is a supercusp form, and $f_{u'}, f'_{u'}$ are supported on the bi-regular bi-elliptic sets of $G_{u'}$, $G'_{u'}$, and $f_v = f'_v$ ($v \in S - V$), then*

$$\sum_{\Phi' \in \pi'^{\mathbb{K}'(S)}} P(\pi'_S(f'_S)\Phi')P(\overline{\Phi}') = \sum_{\Phi \in \pi^{\mathbb{K}(S)}} P(\pi_S(f_S)\Phi)P(\overline{\Phi}).$$

Here $\mathbb{K}(S) = \prod_{v \notin S} K_v$ $(\simeq \mathbb{K}'(S))$, where K_v is the standard maximal compact $GL(2n, R_v^E)$ of $G_v \simeq G'_v \simeq GL(2n, E_v)$; $\pi^{\mathbb{K}(S)}$ is the space of $\mathbb{K}(S)$-fixed vectors in π; Φ ranges over an orthonormal basis of smooth vectors in $\pi^{\mathbb{K}(S)}$. Finally $\pi_S(f_S)$ is $\prod_{v \in S} \pi_v(f_v)$.

Proof. We work with f and f' whose components are spherical (K_v-biinvariant) at each $v \notin S$. Note that $\pi_v(f_v)$ acts as 0 on Φ unless Φ is K_v-invariant, in which case $\pi_v(f_v)$ acts as multiplication by a scalar, denoted again by $\pi_v(f_v)$. Putting $\pi^S(f^S) = \prod_{v \notin S} \pi_v(f_v)$, the identity of Lemma 3 can be written as

$$\sum_{\pi'} \sum_{\Phi' \in \pi'^{\mathbb{K}'(S)}} \pi'^S(f'^S)P(\pi'_S(f'_S)\Phi')P(\overline{\Phi}') = \sum_{\pi} \sum_{\Phi \in \pi^{\mathbb{K}(S)}} \pi^S(f^S)P(\pi_S(f_S)\Phi)P(\overline{\Phi}).$$

A standard argument – originally due to Langlands (in the case of $GL(2)$) – used in [F2], [F1], [FK1], [FK2], ..., of "linear independence of characters", based on varying the spherical components of f at the $v \notin S$, using standard unitarity estimates, the Stone-Weierstrass theorem and the absolute convergence of the sums in Lemma 3, implies our claim. Of course, we use in the statement of the Lemma multiplicity one theorem for $\mathbb{G}'$ and $\mathbb{G}$ ([F1;III], [FK2]), as well as rigidity theorem for $\mathbb{G}'$ and $\mathbb{G}$ ([F1;III], [FK2]). $\square$

5. Proposition. *Suppose that π is a cuspidal cyclic $\mathbb{G}$-module with a supercuspidal component π_u $(u \notin V)$ and a bi-elliptic component $\pi_{u'}(u' \neq u)$. Suppose that (WH1) and (WH2) hold for π_v for all $v \in V$ and $v = u'$. Then the corresponding cuspidal $\mathbb{G}'$-module π' is cyclic, its component at u' is bi-elliptic, and the bi-character of π'_v $(v \in V)$ is not identically zero on the set of bi-regular elements of G'_v which come form G_v.*

Proof. It suffices to show that the side of π in the identity displayed in Lemma 4 is non zero. Consider a smooth Φ_1 in $\pi^{\mathbb{K}(S)}$ such that $P(\Phi_1) \neq 0$. In this proof we regard $\pi^{\mathbb{K}(S)}$ as an abstract representation, rather than in its automorphic realization. Denote by $\xi_0 = \xi_0^S$ the preferred $\mathbb{K}(S)$-fixed vector in $\pi^S = \bigotimes_{v \notin S} \pi_v$, and fix an orthonormal basis $\{\xi_v\}$ of smooth vectors in π_v. Then $\{\xi_0 \otimes (\bigotimes_{v \in S} \xi_v); \xi_v \in \{\xi_v\}, v \in S\}$ is an orthonormal basis of $\pi^{\mathbb{K}(S)}$. Any smooth vector in $\pi^{\mathbb{K}(S)}$ is a finite linear combination of such factorizable vectors.

Expressing Φ_1 as a linear combination of vectors including $\xi_1 = \xi_0 \otimes (\bigotimes_{v \in S} \xi_{1v})$ etc., since $P(\Phi_1) \neq 0$ we may assume that the restriction of P to ξ_1 is non zero. At each $v \in S - V$,

$v \neq u$, u', we choose $f_{1v} \in H_v$ such that $\pi_v(f_{1v})\xi_v = 0$ for all $\xi_v \in \{\xi_v\}$, $\xi_v \neq \xi_{1v}$, and $\pi_v(f_{1v})\xi_{1v} = \xi_{1v}$. Such a choice is possible since H_v spans the algebra of endomorphisms of π_v.

In fact this choice can be made also at the place u, where π_u is supercuspidal. Indeed, by the Schur orthogonality relations the matrix coefficient $f_{1u}(x) = (\pi_u(x)\xi_{1u}, \xi_{1u}^\vee)$ acts as zero on any ξ_u orthogonal to ξ_{1u}, and as a scalar multiple (we assume it is 1 on multiplying f_{1u} by a scalar) on ξ_{1u}. Moreover, such a matrix coefficient is a supercusp form (since π_u is supercuspidal), as required to apply Lemma 4. With this choice of $f_v = f_{1v}$ ($v \in S - V, v \neq u'$), our sum $\sum P(\pi_S(f_S)\Phi)P(\overline{\Phi})$ ranges over the vectors Φ whose component outside $V' = V \cup \{u'\}$ is $\xi^{V'} = \xi_0 \otimes (\bigotimes_{v \in S-V'} \xi_{1v})$. Put also $f^{V'} = (\bigotimes_{v \in S-V'} f_{1v}) \otimes (\bigotimes_{v \notin S} f_v^0)$.

The side of π in the identity of Lemma 4 can now be expressed as

$$\langle P_{V'}, \pi_{V'}(f_{V'})P_{V'}^\vee \rangle,$$

where $P_{V'}$ is the restriction of the linear form $\langle P, \Phi\rangle = P(\Phi) = \int_{\mathbb{Z}C\backslash\mathbb{C}} \Phi(h)dh$ to $\xi^{V'} \otimes \pi_{V'}$; $P_{V'}$ lies in the dual $\pi_{V'}^*$ of $\pi_{V'}$. The integral analogously defines a linear form $P^\vee$ in the dual $\widetilde{\pi}^*$ of the contragredient $\widetilde{\pi}$ of π, which consists of the $\overline{\Phi}$, $\Phi \in \pi$. Namely $\langle P^\vee, \overline{\Phi}\rangle = \int_{\mathbb{Z}C\backslash\mathbb{C}} \overline{\Phi}(h)dh$. Denote by $P_{V'}^\vee$ the restriction of $P^\vee$ to $(\xi^{V'})^\vee \otimes \widetilde{\pi}_{V'}$. Here $\{\xi_v^\vee\}$ signifies the dual basis of $\{\xi_v\}$, and $\xi_v^{0\vee} = \xi_v^0(\widetilde{\pi}_v)$. Note that $\pi_{V'}(f_{V'})P_{V'}^\vee \in \pi_{V'}$. Hence $\langle P_{V'}, \pi_{V'}(f_{V'})P_{V'}^\vee\rangle$ is defined. It is equal to the side of π in the identity of Lemma 4 as explained when $\mathbb{P}_{\pi_v}(f_v)$ was introduced, before (WH2) was stated. Note that $\pi^{V'}(f^{V'})\Phi$ is a cusp form for each cusp form Φ.

We shall now use (WH1) for π_v ($v \in V'$). It asserts the uniqueness of the C_v-invariant form P_{π_v} on π_v, up to a scalar multiple. The existence of P_{π_v} follows from the cyclicity of π. Since the components of Φ outside $V \cup \{u'\}$ are fixed, there is a constant $c(\pi)$, depending on these components, such that

$$P_V = c(\pi) \bigotimes_{v \in V'} P_{\pi_v}.$$

Our sum then takes the form

$$c(\pi)^2 \prod_{v \in V \cup \{u'\}} \mathbb{P}_{\pi_v}(f_v), \quad \mathbb{P}_{\pi_v}(f_v) = \langle P_{\pi_v}, \pi_v(f_v)P_{\pi_v}^\vee\rangle.$$

At the place u' we use (WH2). We take $f_{u'}$ which is supported on the bi-elliptic bi-regular set, such that

$$\mathbb{P}_{\pi_{u'}}(f_{u'}) = \int_{Z_{u'}\backslash G_{u'}} f_{u'}(g)p(g, \pi_{u'})dg$$

is non-zero. The choice of such $f_{u'}$ is clearly possible, since the bi-character $p(g, \pi_{u'})$ of $\pi_{u'}$ is locally constant on the bi-regular set, and is assumed to be non-zero on the bi-regular bi-elliptic set.

Similarly, at each $v \in V$ other than u', we can choose f_v which is supported on the bi-regular set of G_v, with $\mathbb{P}_{\pi_v}(f_v) \neq 0$, again using (WH2): the bi-character is smooth on the bi-regular set, and is not identically zero there. As noted following Lemma 2, there are functions f'_v $(v \in V)$ matching the f_v. The matching f'_v will be supported on the set of bi-regular (also bi-elliptic when $v = u'$) elements of G'_v which come from G_v.

With this choice of f_v $(v \in S)$, since π is cyclic, the right side of the identity displayed in Lemma 4 is non-zero. Hence the left side is non-zero. This means that π' is cyclic, and $\mathbb{P}_{\pi'_v}(f'_v) \neq 0$ $(v \in S)$ for the matching function f'_v. Since the matching function f'_v is supported on the bi-regular (also bi-elliptic when $v = u'$) elements of G'_v which come from G_v, and $\int_{Z_v \backslash G'_v} f'_v(g)p(g, \pi'_v)dg \neq 0$, the bi-character $p(g, \pi'_v)$ is not identically zero on this set, as asserted. □

6. Proposition. *Let π' be a cuspidal cyclic $\mathbb{G}'$-module which corresponds to a cuspidal $\mathbb{G}$-module π. Suppose that π'_u is supercuspidal $(u \notin V)$, that $\pi'_{u'}$ is bi-elliptic, and that for each $v \in V$, the bi-character of π'_v is not identically zero on the set of bi-regular elements which come from G_v. Suppose also that (WH1) and (WH2) hold for π'_v $(v \in V \cup \{u'\})$. Then π is cyclic.*

Proof. The discussion at the places $v \in S - V \cup \{u'\}$, including the case of the supercuspidal component at u, is as in Proposition 5. The assumptions at u' and $v \in V$ permit producing matching functions $f_{u'}$ and f_v for functions $f'_{u'}$ and f'_v for which the left side of the identity displayed in Lemma 4 is non-zero. The proof then proceeds as that of Proposition 5. □

This completes our proof of Theorem A. □

Proof of Theorem B. Choose global fields E/F with completions E_u/F_u, E_w/F_w, as well as a division algebra $\mathbf{D}$ over F unsplit by E, and a quaternion algebra H, such that the group of points over F_u, F_w of $\mathbf{G} = GL(m, \mathbf{D}^H)$ is G_u, G_w. Assume that $\Xi(g, f_w)$ is not identically zero on the bi-regular set of G_w. We shall show that this leads to a contradiction.

Since $\Xi(\gamma, f_u)$, $\Xi(\gamma, f_w)$ are locally constant on the bi-regular sets of G_u, G_w (Lemma 2), we can fix a third place u', a bi-elliptic bi-regular global element γ_0 in G, which is bi-elliptic in $G_{u'}$, and $f_{u'} \in H_{u'}$ which is supported on the bi-elliptic bi-regular set in $G_{u'}$, such that $\Xi(\gamma_0, f_v) \neq 0$ $(v = u, w, u')$.

Since $\gamma_0 \in K_v^E$ for almost all v, and $f_v^0 \geq 0$, the integral $\Xi(\gamma_0, f_v^0)$ is non zero for all v outside some finite set S of places of F. At the remaining finite set of places we choose f_v to be the characteristic function of a small neighborhood of γ_0 in G_v; then $\Xi(\gamma_0, f_v) \neq 0$. It follows that $\Xi(\gamma_0, f) \neq 0$, where $f = \otimes f_v$, and that if γ is rational (in G) with $\Xi(\gamma, f) \neq 0$, then γ is bi-regular bi-elliptic (since it is such in $G_{u'}$).

Since f is compactly supported, such $\gamma = \gamma(\beta)$ lies in a finite set of bi-orbits; indeed, the set of characteristic polynomials of the associated $\beta\overline{\beta}$ is both compact – depending on the support of f – and discrete (since β is rational) in the set of polynomials of degree n over $\mathbb{A}(\simeq \mathbb{A}^{n+1})$.

The totally disconnected topology on $G_{u'}$ permits choosing an open closed neighborhood of the orbit of γ_0 which does not intersect the orbits of the other rational γ with $\Xi(\gamma, f) \neq 0$. Replacing $f_{u'}$ by its product with the characteristic function of this neighborhood, we obtain f such that $\Xi(\gamma, f) \neq 0$ for a rational γ implies that γ is in the bi-orbit of γ_0.

We now apply the bi-period summation formula of Proposition 1, to our function f on $\mathbb{G}$. The requirements of this Proposition 1 are satisfied. Indeed, f_u is supercuspidal, and $f_{u'}$ is supported on the bi-elliptic bi-regular set. Our assumption that $\mathbb{P}_{\pi_w}(f_w)$ vanishes for all π_w implies the vanishing of the spectral (left) side of the summation formula. Hence the geometric side is zero. But it contains a single term, indexed by γ_0. So $\Xi(\gamma_0, f) = 0$, a contradiction to the assumption that $\Xi(g, f_w)$ is not identically zero on the bi-regular set of G_w, as required. □

Remark. Theorem B and its proof remain valid if we do not assume (WH1), but instead we assume for all π_w that $\mathbb{P}_{\pi_w}(f_w) = 0$, where $\mathbb{P}_{\pi_w}$ is defined by means of *any* C_w-invariant linear form P_{π_w} on the space of π_w.

Proof of Theorem C. Suppose that $\exp G_u = d_u$. We shall work with global fields E/F whose completions at the places $u_1 = u, \dots, u_{2d_u}$ are isomorphic to E_u/F_u, and at the places $u'_1 = u', \dots, u'_n$ they are $E_{u'}/F_{u'}$, and with a group $\mathbf{G} = GL(m, \mathbf{D}^H)$ over F with $G_{u_i} \simeq G_u$ $(1 \le i \le 2d_u)$ and $G_{u'_i} \simeq G_{u'}$ $(1 \le i \le n)$. We shall carry out a comparison with the inner form $\mathbf{G}'$ of $\mathbf{G}$ which is split at the places $u_{d_u+1}, \dots, u_{2d_u}$, but with $G_v \simeq G'_v$ for all other v.

We compare the bi-period summation formulae for $\mathbb{G}$ and $\mathbb{G}'$ of Proposition 1. At the places $u_{d_u+1}, \dots, u_{2d_u}$ we use matrix coefficients of π_u, while at the places $u'_1, \dots, u'_u$ we take the test functions to be supported on the bi-elliptic bi-regular set. At the places $u_1, \dots, u_{d_u}$ we take the f_{u_i} and f'_{u_i} to be matching and supported on the bi-regular set (of elements which come from G_{u_i} in the case of f'_{u_i}), as in Lemma 2. At all other places, $f_v = f'_v$ under $G_v \simeq G'_v$. Since both $f = \otimes f_v$ and $f' = \otimes f'_v$ have supercuspidal components and components supported on the bi-elliptic bi-regular sets, Proposition 1 applies. Since f and f' are matching the geometric parts of these formulae are equal.

Note that f can be chosen so that the geometric side of the bi-period summation formula is non-zero. Indeed, since $\Xi(g, f_v)$ is locally constant on the bi-regular set (Lemma 2), and is not identically zero there for $v = u_i$ or u'_i by our assumption on π_u and $f_{u'_i}$, there is some rational bi-regular bi-elliptic element γ_0 with $\Xi(\gamma_0, f_v) \ne 0$ for such v. This relation clearly holds with $f_v = f_v^0$ for almost all v. At the remaining finite set of places we choose f_v supported on a small neighborhood of γ_0, and argue as in the proof of Theorem B that f can be chosen so that $\Xi(\gamma, f) \ne 0$ for a rational γ implies that γ is in the bi-orbit of γ_0. Applying Proposition 1 with such an f we conclude that there exists a cuspidal cyclic $\mathbb{G}$-module π, in fact with the component π_v at $v = u_1, \dots, u_{2d_u}$, and a bi-elliptic component at $u'_1, \dots, u'_n$.

Since we have (WH1) and (WH2) for π_v $(v = u_i)$ by assumption, and also at $v = u'_i$ ((WH1) by Proposition 0, (WH2) on the bi-elliptic set by assumption), the proof of Proposition 5 implies that the corresponding cuspidal $\mathbb{G}'$-module π' is cyclic. In particular its components, including π'_u, are cyclic, as required. □

References

[BZ] J. Bernstein, A. Zelevinskii, Representations of the group $GL(n, F)$ where F is a nonarchimedean local field, *Russian Math. Surveys* 31 (1976), 1-68.

[BDKV] J. Bernstein, P. Deligne, D. Kazhdan, M.-F. Vigneras, *Reprśentations des groupes réductifs sur un corps local*, Hermann, Paris (1984).

[BJ] A. Borel, H. Jacquet, Automorphic forms and automorphic representations, *Proc. Sympos. Pure Math.* 33 (1979), I 189-202.

[F1] Y. Flicker, Rigidity for Automorphic forms, *J. Analyse Math.* 49 (1987), 135-202.

[F2] Y. Flicker, Transfer of orbital integrals and division algebras, *J. Ramanujan math. soc.* 5 (1990), 107-122.

[F3] Y. Flicker, On distinguished representations, *J. reine angew. Math.* 418 (1991), 139-172.

[F4] Y. Flicker, Cusp forms on $GL(2n)$ with $GL(n) \times GL(n)$ periods, and simple algebras, preprint (1993).

[FH] Y. Flicker, J. Hakim, Quaternionic distinguished representations, *Amer. J. Math.* (1994).

[FK1] Y. Flicker, D. Kazhdan, Metaplectic correspondence, *Publ. Math. IHES* 64 (1987), 53-110.

[FK2] Y. Flicker, D. Kazhdan, A simple trace formula, *J. Analyse Math.* 50 (1988), 189-200.

[GK] I. Gelfand, D. Kazhdan, Representations of $GL(n, K)$ where K is a local field, in *Lie groups and their representations*, John Wiley and Sons (1975), 95-118.

[HC1] Harish-Chandra, notes by G. van Dijk, Harmonic analysis on reductive p-adic groups, Springer Lecture Notes 162 (1970).

[HC2] Harish-Chandra, Admissible invariant distributions on reductive p-adic groups, *Queen's Papers in Pure and Applied Math.* 48 (1978), 281-346.

[HC3] Harish-Chandra, A submersion principle and its applications, in *Collected Papers*, vol. IV, Springer-Verlag, New-York, 1984, 439-446.

[H] R. Howe, Some qualitative results on the representation theory of GL_n over a p-adic field, *Pacific J. Math.* 73 (1977), 479-538.

[J1] H. Jacquet, Sur un résultat de Waldspurger, *Ann. Sci. ENS.* 19 (1986), 185-229.

[J2] H. Jacquet, On the nonvanishing of some L-functions, *Proc. Indian Acad. Sci.* 97 (1987), 117-155.

[K] D. Kazhdan, Cuspidal geometry on p-adic groups, *J. Analyse Math.* 47 (1986), 1-36.

[Ke] T. Kengmana, Characters of the discrete series for pseudo Riemannian symmetric spaces, in *Representation theory of reductive groups*, Proc. Univ. Utah conf. 1982 (P. Trombi, ed), Birkhauser, Boston-Basel (1983), 177-183.

[NPS] M. Novodvorski, I. Piatetski-Shapiro, Generalized Bessel models for a symplectic group of rank 2, *Math. USSR Sbornik* 19 (1973), 243-255, 275-286, and 21 (1973), 499-509.

[PPS] S. J. Patterson, I. Piatetski-Shapiro, The symmetric-Square L-function attached to a cuspidal automorphic representation of GL_3, *Math. Ann.* 283 (1989), 551-572.

[P] D. Prasad, Invariant forms for representations of GL_2 over a local field, *Amer. J. Math.* 114 (1992), 1317-1363.

[R] F. Rodier, Whittaker models for admissible representations of reductive p-adic split groups, in *Proc. Sympos. Pure Math.* 26, AMS, Providence, (1974), 425-430.

[S] J. Sekiguchi, Invariant spherical hyperfunctions on the tangent space of a symmetric space, in *Algebraic Groups and Related Topics*, Advanced Studies in Pure Mathematics 6 (1985), 83-126.

[Sh] T. Shintani, On liftings of holomorphic cusp forms, *Proc. Symp. Pure Math.* 33 II (1979), 97-110.

[Wa1] J.-L. Waldspurger, Correspondence de Shimura, *J. Math. Pure Appl.* 59 (1980), 1-113.

[Wa2] J.-L. Waldspurger, Sur les coefficients de Fourier des formes modulaires de poids demi-entier, *J. Math. Pure Appl.* 60 (1981), 375-484.

[We] A. Weil, *Basic Number Theory*, Springer-Verlag, Berlin 1974.