COMPOSITIO MATHEMATICA

YUVAL Z. FLICKER

On endo-lifting

Compositio Mathematica, tome 67, nº 3 (1988), p. 271-300.
<http://www.numdam.org/item?id=CM_1988__67_3_271_0>

Compositio Mathematica 67: 271–300 (1988)

On endo-lifting

YUVAL Z. FLICKER
Department of Mathematics, The Ohio State University, 231 W18th Ave, Columbus OH 43210, USA and
Department of Mathematics, Harvard University, Science Center, One Oxford St., Cambridge MA 02138, USA

Received 16 October 1987; accepted 10 February 1988

Δὸς ποῦ στῶ

Introduction

One of the more fascinating chapters in the theory of automorphic forms, and in representation theory of p-adic groups, is the usage of the Selberg trace formula to study characters of representations when adequate knowledge of orbital integrals is available. The purpose of this paper is to suggest an entirely different application of the trace formula. The roles of the two sides will be changed, and orbital integrals will be studied using the trace formula, given (elementary) knowledge of characters.

Some of these "standard" applications involve lifting theories, such as the metaplectic correspondence [FK1], endo-lifting [K1], [F1], simple algebra correspondence [F2; III], symmetric-square [F5] and base-change [AC], [F3], [F4] liftings. Here two groups G and G' are related. The trace formula (for G) is an identity, for each test function f, of the form $\Sigma_\pi \operatorname{tr} \pi(f) = \Sigma_\gamma \Phi(\gamma, f)$. One begins by proving the existence of matching functions f on G and f' on G', for which the orbital integrals $\Phi(\gamma, f)$ and $\Phi(\gamma', f')$ are equal. This is a difficult step, especially when spherical functions are involved; of course, a suitable notion relating the conjugacy classes γ and γ' has to be supplied. For a matching pair (f, f') one has $\Sigma_\gamma \Phi(\gamma, f) = \Sigma_{\gamma'} \Phi(\gamma', f')$, and by a double application of the trace formula (for G and for G'), one concludes that $\Sigma_\pi \operatorname{tr} \pi(f) = \Sigma_{\pi'} \operatorname{tr} \pi'(f)$. The techniques developed in [FK1], [F2], etc., permit isolating packets of G and G'-modules in these sums, and there results a correspondence of representations of the p-adic and adelic groups attached to G and G', stated in terms of character relations (e.g., $\operatorname{tr} \pi(f) =$

Partially supported by an NSF grant.

$\operatorname{tr}\pi'(f)$ for all matching f, f', or $\chi_\pi(\gamma) \simeq \chi_{\pi'}(\gamma')$ for all matching γ, γ', where χ_π is the character of π).

The purpose of this paper is to suggest a new technique in the study of the initial step of matching f and f', especially in the crucial case of spherical functions f_v and f_v' on the p-adic groups G_v and G_v' (in a special case described below). There is a simple parametrization of unramified G_v-modules π_v and G_v'-modules π_v', by means of which a correspondence $\pi_v \leftrightarrow \pi_v'$ is defined. The spherical f_v, f_v' are called *corresponding* if $\operatorname{tr}\pi_v(f_v) = \operatorname{tr}\pi_v'(f_v')$ for all corresponding unramified pairs (π_v, π_v'). By the theory of the Satake transform the correspondence $f_v \leftrightarrow f_v'$ of spherical functions is well defined, and the problem is to show that *corresponding spherical functions are matching.*

This is a crucial question. Our initial motivation is the desire to solve this problem in the context of the metaplectic correspondence of [FK1], where G' is an mth fold covering of $G = GL(n)$. This problem is reduced in [FK1] (§12 for spherical functions, §13 for discrete functions) to the analogous problem in the context of the endo-lifting from $G' = GL(r, E)$ to $G = GL(n, F)$, where E is a cyclic field extension of F of degree $k = n/r$. The latter problem is solved in Kazhdan [K1] in the case $r = 1$, namely when G', which we now denote by H, is an elliptic torus in G. Consequently, as explained in [FK1], §§12–13, the theorems of [FK1] are proven only for pairs (m, n) where m is prime to each composite (non-prime) integer bounded by n. In conclusion we have to match functions on $H = GL(r, E)$ and $G = GL(n, F)$ for any divisor r of n.

The purpose of this paper is to reduce this last problem to a computation of the twisted character of the unramified representation π_0 of G_v which (conjecturally) corresponds to the trivial H_v- module τ_0. This computation is beyond the scope of this paper and will await another publication. Our method is as follows. Let κ be a primitive character of $\operatorname{Gal}(E/F) \simeq \mathbb{A}^\times / F^\times N_{E/F}\mathbb{A}_E^\times$. There is a bijection (called lifting or correspondence) relating $\operatorname{Gal}(E/F)$-orbits $\{\tau \circ \sigma^i (0 \leqslant i < k)\}$ of cuspidal H-modules τ with $\tau \not\simeq \tau \circ \sigma$, with cuspidal G-modules π with $\pi \simeq \pi \otimes \kappa$. This follows from the theory [AC] of base-change for $GL(n)$, and consequently does not provide the character relations of [K1] needed in [FK1] as described above.

However, for a careful choice of special test functions $\phi = \otimes\phi_v$ on $G(\mathbb{A})$ and $f = \otimes f_v$ on $H(\mathbb{A}) = GL(r, \mathbb{A}_E)$ we do have $\operatorname{tr}\pi(\phi \times \kappa) = \operatorname{tr}\tau(f)$ if $\tau \leftrightarrow \pi$; we do not know that $\Phi_\kappa(\gamma, \phi_v) = \Phi(\gamma, f_v)$ (the definitions of the twisted character $\operatorname{tr}\pi(\phi \times \kappa)$ and twisted orbital integral $\Phi_\kappa(\gamma, \phi_v)$ are given in Section I below). Thus, for each place v of F the components f_v, ϕ_v of $f = \otimes f_v$, $\phi = \otimes\phi_v$ are chosen to be corresponding, but we do not know that they are matching. For such a pair f, ϕ, the representation theoretic sides of the trace formulae: $\Sigma_\tau \operatorname{tr}\tau(f) = \Sigma_\gamma \Phi(\gamma, f)$ on H, and $\Sigma_\pi \operatorname{tr}\pi(\phi \times \kappa) = \Sigma_\gamma \Phi_\kappa(\gamma, \phi)$ on G, are equal. Consequently $\Sigma_\gamma \Phi(\gamma, f) = \Sigma_\gamma \Phi_\kappa(\gamma, \phi)$.

Now our problem is to show that for a fixed finite unramified place u of F the corresponding spherical functions f_u', ϕ_u' are matching. We choose f, ϕ with $f_u = f_u'$ and $\phi_u = \phi_u'$. Given a (regular) rational δ in $H = GL(r, E)$, $f^u = \otimes_{v \neq u} f_v$ and $\phi^u = \otimes_{v \neq u} \phi_v$ can be chosen to have $\Phi(\gamma, f) = 0$ and $\Phi_\kappa(\gamma, \phi) = 0$ unless $\gamma = \delta$. Consequently $\Phi(\delta, f) = \Phi_\kappa(\delta, \phi)$.

Had we known that $\Phi(\delta, f^u) = \Phi_\kappa(\delta, \phi^u)(\neq 0)$ we could conclude that $\Phi(\delta, f_u') = \Phi_\kappa(\delta, \phi_u')$ and be done. But we do not, and instead we apply the same arguments with a pair $f^* = f_u^* \otimes f^u$ and $\phi^* = \phi_u^* \otimes \phi^u$. Here f_u^* is a pseudo-coefficient (see [K2]) of the Steinberg H_u-module st_{H_u}, and ϕ_u^* a twisted pseudo-coefficient (see [F2]) of the corresponding G_u-module $\pi(\mathrm{st})_u$. Thus, f_u^* and ϕ_u^* correspond, and we have $\Phi(\delta, f^*) = \Phi_\kappa(\delta, \phi^*)$. Now our Conjecture B_0, which concerns the computation of the twisted character of $\pi(\mathrm{st})_u$, or equivalently that of the representation π_0 related to the trivial module τ_0 (see Conjecture B_1), implies that f_u^* and ϕ_u^* are matching. Hence, $\Phi(\delta, f_u') = \Phi_\kappa(\delta, \phi_u')$ is obtained, and we are done.

The contents of this paper are as follows. The main Theorem 1, which asserts that corresponding spherical functions are matching provided that the twisted character of the lift π_0 of the trivial H_u-module τ_0 is computed, is stated in Section II. So is Theorem 2, which recalls the required consequences of the base-change theory, and also the trace formulae (for H and G). Section III deals with the choice of the test functions, and the proof of Theorem 1. Section I states the main Conjecture B_0 (or B_1) and its consequences to lifting and to matching orbital integrals of spherical and general functions. In the Appendix Conjecture B_0 is proven in the special case of $H = GL(1, E)$ and $G = GL(2, F)$. This proves all of our conjectures but only in the easiest case of $GL(2)$.

The same technique, of reducing (by means of the trace formula) the study of orbital integrals to that of a computation of a specific twisted character of an unramified representation, is used to establish in [FK2] the unstable transfer of orbital integrals of spherical functions which is required in our proof of the absolute form of the symmetric square lifting. This technique can also be used to establish the unstable transfer in the context of base change for the unitary group $U(3, E/F)$ in three variables (see [F4]).

I. Conjectures

Let E be a cyclic extension of degree $k > 1$ of a local non-archimedean field F. Let R and R_E denote the rings of integers of F and E. Choose an isomorphism $E \xrightarrow{\sim} F^k$ over F such that $R_E \xrightarrow{\sim} R^k$ when E/F is unramified. There results an embedding $E \hookrightarrow M(k, F) = \mathrm{End}_F F^k$ and $E^\times \hookrightarrow GL(k, F) = \mathrm{Aut}_F F^k$. For every $r \geqslant 1$ we obtain $M(r, E) \hookrightarrow M(n, F)$, where $n = rk$, and so an embedding of the multiplicative group $H = GL(r, E)$ in $G = GL(n, F)$, which maps the maximal compact subgroup $K_H = GL(r, R_E)$ of H into $K = GL(n, R) = M(n, R)^\times$. We regard H as a subgroup of G, and note that the embedding of H in G is unique up to conjugation in G.

Denote by $N_{E/F}$ the norm map from E to F. Fix a complex valued character κ of $F^\times$ whose kernel is $N_{E/F}E^\times$. Its composition with the determinant det: $G \to F^\times$ is a character of G denoted again by κ. It satisfies $\kappa(h) = 1$ for every h in $H \subset G$. Our first object of study is a relation between orbital integrals on H and κ-orbital integrals on G, as follows.

Let ω be a character of the center $Z(G) = F^\times$ of G. Let ϕ be a locally constant function on G which satisfies $\phi(zg) = \kappa(z)^{rk(k-1)/2}\omega(z)^{-1}\phi(g)$ (g in G, z in $Z(G)$) whose support is compact modulo $Z(G)$. Let dg be the Haar measure on G which assigns K the volume one. For each torus T in G let d_T be the Haar measure on T which assigns the maximal compact subgroup $T(R)$ the volume one. The quotient measure dg/d_T on the homogeneous space G/T is denoted here by $d\dot{g}$. An element g of G is called *regular* if its centralizer $Z_G(g)$ is a torus. If x lies in H and it is regular (in G), then $Z_G(x) = Z_H(x)$ lies in H and $\kappa(Z_G(x)) = 1$. Hence the *κ-orbital integral*

$$\Phi_\kappa(x, \phi) = \int_{G/Z_G(x)} \kappa(\dot{g})\phi(\dot{g}x\dot{g}^{-1})\, d\dot{g}$$

of ϕ at x is well-defined.

Analogously, let dh be the Haar measure on H which assigns K_H the volume one, and $d\dot{h}$ the quotient measure dh/d_T on the homogeneous space H/T. Let f be a locally constant function on H with $f(zh) = \omega(z)^{-1}f(h)$ (h in H, z in $Z(G) = F^\times$) whose support is compact modulo $Z(G)$. The *orbital integral* of f at x in H which is regular in G is defined to be

$$\Phi(x, f) = \int_{H/Z_G(x)} f(\dot{h}x\dot{h}^{-1})\, d\dot{h}.$$

To relate these two types of orbital integrals, let $\bar{F}$ be a separable closure of F containing E, and σ an automorphism of $\bar{F}$ over F whose restriction to E generates the galois group $\mathrm{Gal}(E/F)$. The Lie algebra $L(G) = M(n, \bar{F})$ of G is the direct sum of three subspaces: the space $L(H) \simeq M(r, \bar{F})^k$ of matrices made of k blocks of size $r \times r$ along the diagonal, the space P of upper triangular matrices with non-zero entries over $L(H)$, and the complementary space $\bar{P}$ of lower triangular matrices. Then $\dim P = \dim \bar{P} = r^2k(k-1)/2$.

For h in H, put $\tilde{h} = \operatorname{diag}(h, \sigma h, \ldots, \sigma^{k-1}h)$; then $\tilde{h}$ is an invertible matrix in $L(H)$. Let $\operatorname{Ad}(h)$ denote the adjoint action $X \to \tilde{h}X\tilde{h}^{-1}$ of $\tilde{h}$ on P. Put $\tilde{\Delta}(h) = \det(\operatorname{Ad}(h) - I)$, where I is the identity. Note that if $h_l (1 \leqslant l \leqslant r)$ denote the eigenvalues of the $r \times r$ matrix h in H, then $\tilde{\Delta}(h) = \Pi(\sigma^i h_l/\sigma^j h_{l'} - 1)$, where the product ranges over all i, j, l and l' with $1 \leqslant i < j \leqslant k$; $1 \leqslant l, l' \leqslant r$. Also we put $\tilde{\Delta}'(h) = \Pi(\sigma^i h_l - \sigma^j h_{l'})$, with product ranging over all i, j, l and l' as above, and we put $\bar{\Delta}(h) = \Pi(\sigma^i h_l - \sigma^j h_l)$, with product ranging over the i, j, l with $1 \leqslant i < j \leqslant k$, $1 \leqslant l \leqslant r$.

Let $|.|$ denote the valuation of $\bar{F}$ which is normalized by $|\pi| = q^{-1}$, where π is any generator of the maximal ideal of R, and q is the cardinality of the residue field $R/(\pi)$. Then $|\tilde{\Delta}(h)| = |\tilde{\Delta}'(h)|\,|\det_G h|^{-(k-1)r/2}$, where $\det_G h$ is the determinant $\Pi_{i,l}\sigma^i h_l$ of h as a matrix in G; note that $\det_G h = N_{E/F}\det_H h$. Now for every h in H, we have $\sigma\bar{\Delta}(h) = (-1)^{(k-1)r}\bar{\Delta}(h)$. We say that h is *regular* if $\bar{\Delta}(h) \neq 0$ (namely h is regular as an element of G). Fix a regular h_0 in H. Then $\bar{\Delta}(h)/\bar{\Delta}(h_0)$ lies in F, since $\bar{\Delta}(h)$ lies in E and F is the field of $\operatorname{Gal}(E/F)$-invariant elements in E. Moreover, if k is odd then $\bar{\Delta}(h)/\bar{\Delta}(h_0)$ lies in $N_{E/F}E^\times$, while if k is even, then $(\bar{\Delta}(h)/\bar{\Delta}(h_0))^2$ is in $N_{E/F}E^\times$.

Following [K1] we put

$$\Delta(h) \quad = \quad |\tilde{\Delta}(h)|\kappa(\bar{\Delta}(h)/\bar{\Delta}(h_0))^{k(k-1)/2}.$$

Then $\Delta(h)$ is real, and moreover non-negative if k is odd or divisible by 4. In the latter case $\Delta(h)$ is independent of h_0, while if k is even but not divisible by 4, then $\Delta(h)$ is independent of h_0 only up to a sign. When E/F is unramified we choose h_0 which satisfies $|\bar{\Delta}(h_0)| = 1$.

Note that $\Delta(\sigma h) = \kappa(-1)^{rk(k-1)/2}\Delta(h)$. Since the matrices σh and h in G have equal sets of eigenvalues, they are conjugate in G if h is regular. Namely, there is α in G with $\sigma h = \alpha h \alpha^{-1}$; it is clear that the image of $\det\alpha$ in $F^\times/N_{E/F}E^\times$ is uniquely determined. It follows from [K1] that

$$\det\alpha \quad \text{lies in} \quad (-1)^{r(k-1)k/2}N_{E/F}E^\times.$$

Hence

$$\Delta(\sigma h)\Phi_\kappa(\sigma h, \phi) \quad = \quad \Delta(h)\Phi_\kappa(h, \phi)$$

for every regular h.

Definition. The functions ϕ on G and f on H are called *matching* if $\Phi(h, f) = \Delta(h)\Phi_\kappa(h, \phi)$ for every h in H regular in G.

In particular, we have $\Phi(\sigma h, f) = \Phi(h, f)$ if f matches a ϕ in G.

CONJECTURE A. *For every ϕ there is a matching f; for every f with $\Phi(\sigma h, f) = \Phi(h, f)$ for all regular h there exists a matching ϕ.*

When $r = 1$ this is proven in Kazhdan's fundamental work [K1]. As shown in [FK1], §13, Conjecture A implies that for every ϕ on $G = GL(n, F)$ there is a matching genuine function $\tilde{\phi}$ on the m-fold covering group $\tilde{G}$ of G, and for every $\tilde{\phi}$ there is a matching ϕ. This fact plays a key role in the study [FK1] of the metaplectic correspondence. It is shown in [FK1], §13, that the case of $r = 1$ in Conjecture A, established in [K1], implies the transfer between ϕ and $\tilde{\phi}$ when m is prime to every composite integer bounded by n.

A second theme in this work concerns representations of G and H, and character relations. Our next aim is to state the relevant Conjecture B, which implies Conjecture A.

We begin by fixing notations to be used below. Let π be an admissible G-module which satisfies $\pi(zg) = \kappa(z)^{rk(k-1)/2}\omega(z)\pi(g)$ (z in $Z(G)$, g in G), and ϕ a function on G as above. The convolution operator $\pi(\phi) = \int_{G/Z(G)} \phi(g)\pi(g)dg$ has finite rank and its trace is denoted by $\operatorname{tr}\pi(\phi)$. Put $\pi \otimes \kappa$ for the G-module $(\pi \otimes \kappa)(g) = \pi(g)\kappa(g)$. Suppose that π is irreducible and $\pi \otimes \kappa \simeq \pi$. Then there exists a non-zero (hence invertible) intertwining operator A on the space of π with $A\pi(g)\kappa(g) = \pi(g)A$ for all g in G. In particular, $A^k\pi(g) = \pi(g)A^k$ (g in G). Since π is irreducible, A^k is a scalar which we normalize to be one. This fixes A up to a kth root of unity.

Suppose that π has a Whittaker model, namely there exists an additive complex valued non-trivial character ψ of F, such that π is equivalent to the representation of G by right translations on a space of functions $W: G \to \mathbb{C}$ which transform under the unipotent radical $N = \{(n_{ij})\}$ of B by $W((n_{ij})g) = \psi(\Sigma_{i=1}^{n-1} n_{i,i+1})W(g)$. Then $\pi \otimes \kappa$ has a Whittaker model, which consists of the functions $W \otimes \kappa$: $g \mapsto W(g)\kappa(g)$. We choose $A: \pi \to \pi \otimes \kappa$ to be defined by $AW = W \otimes \kappa$. Then $\pi(x)A = \kappa(x)A\pi(x)$ for all x in G, since for all g in G we have

$$(\pi(x)AW)(g) = (AW)(gx) = \kappa(gx)W(gx) = \kappa(x)\kappa(g)(\pi(x)W)(g)$$
$$= \kappa(x)(A\pi(x)W)(g).$$

It is easy to see that this normalization commutes with the functor of induction. By [BZ] a tempered G-module is equivalent to one normalizedly induced from a square-integrable module, hence it has a Whittaker model. Note that by a tempered or square-integrable module we mean an irreducible one. Every irreducible G-module π is a quotient of a G-module $I(\varrho \otimes \mu)$

induced from the product $\varrho \otimes \mu$ of a tempered module ϱ and an unramified character μ. We take $A_\pi\colon \pi \mapsto \pi \otimes \kappa$ to be the one obtained from $I(A_\varrho)\colon I(\varrho \otimes \mu) \mapsto I(\varrho \otimes \kappa \otimes \mu)$, where $A_\varrho\colon \varrho \mapsto \varrho \otimes \kappa$ is the operator $A_\varrho W_\varrho = W_\varrho \otimes \kappa$ on the Whittaker space of ϱ.

Put $\pi(\kappa) = A$ and $\pi(\phi \times \kappa) = \int_{G/Z(G)} \phi(g)\pi(g)A\,dg$ for any ϕ as above. This is an operator of finite rank; the trace is finite and denoted by $\operatorname{tr}\pi(\phi \times \kappa)$.

DEFINITION. The H-module τ *corresponds* to the G-module π if $\pi \otimes \kappa \simeq \pi$ and $\operatorname{tr}\pi(\phi \times \kappa) = \operatorname{tr}\tau(f)$ for all matching functions ϕ on G and f on H. We denote π by $\pi(\tau)$ and τ by $\tau(\pi)$ if τ and π correspond.

The definition of correspondence can be stated in terms of character relations. The character χ_τ of τ is a locally constant function on the regular set (distinct eigenvalues) of H with $\chi_\tau(zh^{-1}xh) = \omega(z)\chi_\tau(x)$ (z in $Z(G)$; x, h in H) such that $\operatorname{tr}\tau(f) = \int_{H/Z(G)} \chi_\tau(h)f(h)dh$ for all f on H which vanish on the singular set of H. A well-known theorem of Harish-Chandra [H] asserts that χ_τ extends to a locally integrable function on H, hence, that $\operatorname{tr}\tau(f) = \int \chi_\tau(h)f(h)dh$ for all f. Similarly, the twisted character χ_π of $\pi(\simeq \pi \otimes \kappa)$ is a locally constant function on the regular set of G which extends to G as a locally integrable function which satisfies $\chi_\pi(zg) = \kappa(z)^{rk(k-1)/2}\omega(z)\chi_\pi(g)$ (g in G, z in $Z(G)$) and $\operatorname{tr}\pi(\phi \times \kappa) = \int_{G/Z(G)} \chi_\pi(g)\phi(g)dg$ for all ϕ (see [C1]). It is easy to check that $\chi_\pi(g^{-1}xg) = \kappa(g)\chi_\pi(x)$ (g in G, x regular in G). We have the following:

LEMMA A1. *If g is regular in G and $\chi_\pi(g) \neq 0$ then g is conjugate to an element of H.*

Proof. Since g is regular, its centralizer $Z_G(g)$ in G is isomorphic to the product $\Pi_i L_i^\times$ of the multiplicative groups of field extensions L_i of F with $\Sigma_i [L_i : F] = n$. Let $N_i\colon L_i \to F$ be the norm maps. Since $\chi_\pi(g) \neq 0$, the restriction of κ to $Z_G(g)$ is trivial. Hence, $\kappa(\Pi_i l_i) = \kappa(\Pi_i N_i l_i) = 1$ for all (l_i) in (L_i). In particular, $N_i L_i^\times \subset \ker\kappa = N_{E/F}E^\times$ for all i, hence $L_i \supset E$ for all i by local class field theory. Consequently $\Pi_i L_i^\times$ is isomorphic to a torus in H (indeed, $\Sigma_i [L_i : E] = r = n/[E : F]$), and g is conjugate to an element of H.

Let $\{T\}$ denote a set of representatives for the conjugacy classes of tori T in H and $[W(T, H)]$ the cardinality of the Weyl group $W(T, H)$ of T in H. The Weyl integration formula on H asserts

$$\int_{H/Z(G)} f(h)\chi_\tau(h)\,dh = \sum_{\{T\}} [W(T, H)]^{-1} \int_{T/Z(G)} \Delta_H(t)^2\Phi(t, f)\chi_\tau(t)\,dt,$$

where

$$\Delta_H(t) = |\det(\mathrm{Ad}(t) - I)|_{L(H)/L(T)}|^{1/2},$$

and $L(X)$ denotes the Lie algebra of a reductive group X. Let $\tau \circ \sigma^i$ be the H-module $(\tau \circ \sigma^i)(h) = \tau(\sigma^i h)$. It is clear that $\sigma_{\tau \circ \sigma^i}(h) = \chi_\tau(\sigma^i h)$, hence we have:

LEMMA A2. *If f matches ϕ, then* $\mathrm{tr}(\tau \circ \sigma^i)(f) = \mathrm{tr}\,\tau(f)$ *for every τ and i.*

The Weyl integration formula for G asserts

$$\int_{G/Z(G)} \phi(g)\chi_\pi(g)\,\mathrm{d}g = \sum_{\{T\}} [W(T, G)]^{-1} \int_{T/Z(G)} \Delta_G(t)^2 \Phi_\kappa(t, \phi)\chi_\pi(t)\,dt,$$

where

$$\Delta_G(t) = |\det(\mathrm{Ad}(t) - I)|_{L(G)/L(T)}|^{1/2}.$$

We conclude

LEMMA A3. *There exists a constant $c = c(h_0) \neq 0$ such that π and τ correspond if and only if $\Delta(h)\chi_\pi(h) = c\,\Sigma_{i=0}^{k-1}\,\chi_\tau(\sigma^i h)$ for every h regular in G.*

This gives an alternative definition of the correspondence, which implies the following:

COROLLARY. *τ corresponds to π if and only if $\tau \circ \sigma^i$ corresponds to π for every i.*

It is easy to deduce from linear independence of characters that at most one π can correspond to τ, and at most one orbit $\{\tau \circ \sigma^i(0 \leqslant i < k)\}$ can correspond to π.

CONJECTURE B. *The correspondence $\tau \mapsto \pi(\tau)$ defines a bijection from the set of orbits $\{\tau \circ \sigma^i(0 \leqslant i < k)\}$ of equivalence classes of tempered H-modules τ to the set of equivalence classes of tempered G-modules π with $\pi \otimes \kappa \simeq \pi$. It bijects orbits of length k with supercuspidal G-modules.*

A stronger form of this conjecture is obtained on replacing "tempered" by "unitary" and even "irreducible" in its statement.

PROPOSITION [B $\Rightarrow$ A]. *Conjecture* B *implies Conjecture* A.

The proof of this Proposition consists of two lemmas.

LEMMA B1. *If conjecture* B *holds, then for every f there exists ϕ, and for every ϕ there exists f, such that* $\mathrm{tr}\,\pi(\phi \times \kappa) = \mathrm{tr}\,\tau(f)$ *for all corresponding π and τ.*

Proof. Given f, the form $\Phi(\pi) = \operatorname{tr}(\tau(\pi))(f)$ is a good form in the terminology of [BDK] on the free abelian group $F_\kappa(G)$ generated by the equivalence classes of the irreducible tempered G-modules π with $\pi \otimes \kappa \simeq \pi$. By the immediate twisted analogue [F2; I.7] of the Theorem of [BDK], the form Φ is a trace form, namely there exists ϕ with $\Phi(\pi) = \operatorname{tr}\pi(\phi \times \kappa)$ for all π. Conversely, given ϕ the form $F(\tau) = \operatorname{tr}(\pi(\tau))(\phi \times \kappa)$ is a good form, hence a trace form by the Theorem of [BDK], namely there is an f with $F(\tau) = \operatorname{tr}\tau(f)$ for all τ.

LEMMA B2. *If f and ϕ satisfy the relation $\operatorname{tr}\pi(\phi \times \kappa) = \operatorname{tr}\tau(f)$ for all corresponding π and τ then ϕ and f are matching.*

Proof. Using the Weyl integration formulae, the assumption implies that

$$\sum_{\{T\}} [W(T, H)]^{-1}\Delta_H(t)^2 \int_{T/Z(G)} (\Delta(t)\Phi_\kappa(t, \phi) - \Phi(t, f))\chi_\tau(t)\, dt = 0 \qquad (*)$$

for every H-module τ. Since τ can be taken to be any H-module induced from a character of the Borel subgroup B_H of H, we conclude that $\Phi(t, f) = \Delta(t)\Phi_\kappa(t, \phi)$ for every regular t in B_H. By induction on the dimension of the minimal parabolic subgroup of H which contains T, we may assume that $\Phi(t, f) = \Delta(t)\Phi_\kappa(t, \phi)$ for every regular t which is not elliptic in H. Now, by a well-known completeness result for characters of representations of compact groups, and the Deligne–Kazhdan correspondence [F2; III] which transfers this result from the context of the multiplicative group of a division algebra to that of the general linear group, we have the following. The characters of the square-integrable H-modules form an orthonormal basis for the space of conjugacy invariant functions χ on the elliptic set of H, which transform under $Z(G)$ according to ω, with respect to the inner product

$$\langle \chi, \chi' \rangle = \sum_{\{T\}_e} [W(T, H)]^{-1} |T/Z(G)|^{-1} \Delta_H(t)^2 \int_{T/Z(G)} \chi(t)\bar{\chi}'(t)\, dt.$$

Here $\{T\}_e$ indicates the set of elliptic tori in $\{T\}$. Since $\Phi(t, f) = \Delta(t)\Phi_\kappa(t, \phi)$ on the non-elliptic set, the lemma follows from $(*)$.

Next we describe an explicit form of Conjecture A in the special case of spherical functions; this form implies (in a non-trivial way) Conjecture B. A function f is called *spherical* if it is K_H-biinvariant, and ϕ is *spherical* if it is K-biinvariant. Of course, spherical functions exist only when ω is unramified, namely it is trivial on $R^\times$. The explicit form of Conjecture A will assert that if one of ϕ or f is spherical, then the matching function can be taken to be also spherical. In fact, the matching spherical function can be specified explicitly, as follows.

Let π be a generator of the maximal ideal $R - R^\times$ in R, and ord the order (additive) valuation of F normalized by $\operatorname{ord}(\pi^m u) = m$ (u in $R^\times$, m in $\mathbb{Z}$). Then $|x| = q^{-\operatorname{ord}(x)}$, where q is the cardinality of the residue field $R/(\pi)$. For any n-tuple $\mathbf{z} = (z_1, \ldots, z_n)$ of non-zero complex numbers, define an unramified character of the upper triangular subgroup B of G by $\mathbf{z}$: $(b_{ij}) \to \Pi_i z_i^{\operatorname{ord}(b_{i,i})}$ (here $b_{i,j} = 0$ if $i > j$). Denote by $\delta^{1/2}$ the character defined by $\mathbf{z} = (q^{(n-1)/2}, q^{(n-3)/2}, \ldots, q^{(1-n)/2})$. Let $I(\mathbf{z})$ denote the G-module normalizedly induced from the B-module $\mathbf{z}$; thus $I(\mathbf{z}) = \operatorname{Ind}(\mathbf{z}\delta^{1/2}; B, G)$, where $\operatorname{Ind}(\mathbf{z}; B, G)$ signifies unnormalized induction from B to G. The symmetric group $W = S_n$ on n letters acts on $\mathbb{C}^{\times n}$ by permutation, and $I(\mathbf{z})$, $I(\mathbf{z}')$ have equal characters (namely $I(\mathbf{z})$, $I(\mathbf{z}')$ are equal as virtual representations) if and only if the images of $\mathbf{z}$ and $\mathbf{z}'$ in $\mathbb{C}^{\times n}/S_n$ are equal. An irreducible G-module π is called *unramified* if it has a non-zero K-fixed vector. The composition series of $I(\mathbf{z})$ contains a unique unramified constituent $\pi(\mathbf{z})$, and every unramified G-module π is of the form $\pi(\mathbf{z})$ where $\mathbf{z} = \mathbf{z}(\pi)$ is uniquely defined by π in $\mathbb{C}^{\times n}/S_n$. Namely the map $\mathbf{z} \mapsto \pi(\mathbf{z})$ is an isomorphism from $\mathbb{C}^{\times n}/S_n$ to the set of equivalence classes of unramified G-modules.

Let E be the unramified extension of F of degree k. For each $\mathbf{z}_H$ in $\mathbb{C}^{\times k}$ we introduce the normalizedly induced H-module $I(\mathbf{z}_H)$ and its unramified constituent $\pi(\mathbf{z}_H)$, and note that the map $\mathbf{z}_H \mapsto \pi(\mathbf{z}_H)$ defines a parametrization of the set of equivalence classes of unramified H-modules by the manifold $\mathbb{C}^{\times k}/S_k$.

If π is admissible and π_i are its composition factors (repeated according to their multiplicities) which satisfy $\pi_i \otimes \kappa \simeq \pi_i$, we write $\operatorname{tr}\pi(\phi \times \kappa)$ for $\Sigma_i \operatorname{tr}\pi_i(\phi \times \kappa)$. It is clear that if ϕ is spherical and $\operatorname{tr}\pi(\phi \times \kappa) \neq 0$, then π has a K-fixed non- zero vector. In particular, we have $\operatorname{tr}(I(\mathbf{z}))(\phi \times \kappa) = \operatorname{tr}(\pi(\mathbf{z}))(\phi \times \kappa)$ for every $\mathbf{z}$ and spherical ϕ. Since the linear forms $\operatorname{tr}\pi_1(\phi \times \kappa), \ldots, \operatorname{tr}\pi_j(\phi \times \kappa)$ in ϕ are linearly independent if $\pi_1, \ldots, \pi_j$ are irreducible, inequivalent and satisfy $\pi_i \simeq \pi_i \otimes \kappa (1 \leqslant i \leqslant j)$, it is clear that $\phi = 0$ if ϕ is spherical and $\operatorname{tr}\pi(\phi \times \kappa) = 0$ for all unramified π (which is irreducible with $\pi \simeq \pi \otimes \kappa$). A simple, standard computation of the character of an induced representation, implies the following:

LEMMA B3. *Let ζ denote a primitive kth root of 1 in $\mathbb{C}$. For any $\mathbf{z}_H$ in $\mathbb{C}^{\times k}$, the H-module $I(\mathbf{z}_H^k)$ corresponds to the G-module $I(\mathbf{z}(\mathbf{z}_H))$, where $\mathbf{z}(\mathbf{z}_H) = (\mathbf{z}_H, \zeta\mathbf{z}_H, \zeta^2\mathbf{z}_H, \ldots, \zeta^{k-1}\mathbf{z}_H)$.*

Here, $\mathbf{z}_H^k = (z_{i,H}^k)$ if $\mathbf{z}_H = (z_{i,H})$ $(1 \leqslant i \leqslant r)$, and $\alpha\mathbf{z}_H = (\alpha z_{i,H})$ for α in $\mathbb{C}$. Note that $I(\mathbf{z}_H)$ and $I(\mathbf{z})$ are irreducible if $z_{i,H} \neq q\zeta^l z_{j,H}$ for all i, j $(1 \leqslant i, j \leqslant r;\ 0 \leqslant l < k)$ by [BZ], hence, $\pi(\mathbf{z}_H)$ corresponds to $\pi(\mathbf{z})$ in this case.

DEFINITION. The spherical functions ϕ on G and f on H are called *corresponding* if $\operatorname{tr}(I(\mathbf{z}_H))(f) = \operatorname{tr}(I(\mathbf{z}(\mathbf{z}_H)))(\phi \times \kappa)$ for all $\mathbf{z}_H$ in $\mathbb{C}^{\times k}$.

In view of the comments above, for each spherical ϕ (resp. f) there is at most one corresponding spherical f (resp. ϕ). The existence of f, and analogously ϕ is assured by the theory of the Satake transform, which asserts that each rational function on $\mathbb{C}^{\times k}/S_k$ is of the form $\mathbf{z}_H \mapsto \operatorname{tr}(\pi(\mathbf{z}_H))(f)$ for a unique spherical f.

CONJECTURE C. *If the spherical ϕ and f are corresponding, then they are matching.*

Namely, corresponding spherical ϕ and f have matching orbital integrals. Note that $\Delta(h) = |\tilde{\Delta}(h)|$ if k is odd, and $\Delta(h) = |\tilde{\Delta}(h)|(-1)^{\operatorname{ord}(\tilde{\Delta}(h))}$ if k is even.

A special case of Conjecture C is of crucial importance. Let ϕ^0 be the unit element of the convolution algebra of spherical functions on G, and f^0 that on H. If π is irreducible with $\pi \otimes \kappa \simeq \pi$, then $\operatorname{tr}\pi(\phi^0 \times \kappa)$ is equal to one if π is unramified, and to zero otherwise. If τ is irreducible then $\operatorname{tr}\tau(f^0)$ is one if τ is unramified, and zero otherwise. In particular, the unit elements ϕ^0 and f^0 are corresponding.

CONJECTURE C_0. *The unit elements ϕ^0 and f^0 are matching.*

Conjecture C_0 is a special case of Conjecture C, but a straightforward analogue of the proof of Theorem 19 in [FK1] implies the following

PROPOSITION [$\mathrm{C}_0 \Rightarrow \mathrm{C}$]. *Conjecture* C_0 *implies Conjecture* C.

Note that the proof of [FK1], Theorem 19, is non-trivial; it relies on the trace formula and the usage of regular functions.

Conjecture C_0 is of crucial importance in the study [FK1] of the metaplectic correspondence. In §12 of [FK1] it is shown that Conjecture C_0 implies the analogous transfer of ϕ^0 to the unit element $\tilde{\phi}^0$ of the Hecke algebra of genuine functions on an m fold covering $\tilde{G}$ of G, namely that ϕ^0 and $\tilde{\phi}^0$ are matching. This is used in [FK1], §19, to derive the statement analogous to Conjecture C, that corresponding spherical ϕ and $\tilde{\phi}$ are matching. In the case of $r = 1$, Conjecture C is proven in Kazhdan's fundamental work [K1], and the Theorems of [FK1] are deduced for the case specified in [FK1], Corollary 12, namely the case where m is prime to all composite integers bounded by n.

Standard techniques (see, e.g., [K1], [FK1], and [F2]), which rely on the trace formula and the fact that the rigidity theorem is known for our H and G, imply the following

PROPOSITION [**C** $\Rightarrow$ **B**]. *Conjecture* C *implies Conjecture* B.

Let τ_0 be the trivial H-module, and π_0 the (irreducible) G-module $I(\kappa, \kappa^2, \ldots, \kappa^k)$ normalizedly induced from the character $(a_{i,j}) \mapsto \Pi_i \kappa^i(a_{i,i})$ of the parabolic subgroup $P_{\mathbf{r}}$ of type $\mathbf{r} = (r, r, \ldots, r)$ of G. Here $a_{i,j}(1 \leqslant i, j \leqslant k)$ are $r \times r$ matrices with entries in F, $a_{i,i}$ are invertible $(1 \leqslant i \leqslant k)$ and $a_{i,j} = 0$ if $i > j$. Note that $\kappa(a_{i,i})$ is $\kappa(\det a_{i,i})$. Both τ_0 and π_0 are unitary, and unramified if κ (hence E/F) is unramified. The arguments which imply Proposition [C $\Rightarrow$ B] establish also that Conjecture C implies the following

Conjecture B_1. *The H-module τ_0 corresponds to the G-module π_0.*

Let st_H be the Steinberg H-module, and st the Steinberg $GL(r, F)$-module. Put $\pi(\mathrm{st})$ for the irreducible G-module $I(\mathrm{st} \otimes \kappa, \mathrm{st} \otimes \kappa^2, \ldots, \mathrm{st} \otimes \kappa^k)$ normalizedly induced from the $P_{\mathbf{r}}$-module $(a_{i,j}) \mapsto \otimes_{i=1}^k [(\mathrm{st} \otimes \kappa^i)(a_{i,i})]$. By virtue of a well-known formula, expressing the character of the Steinberg representation as an alternating sum of characters of representations induced from one dimensional representations of the parabolic subgroups, Conjecture B_1 is equivalent to the following

Conjecture B_0. *The Steinberg H-module,* st_H, *corresponds to the G-module* $\pi(\mathrm{st})$.

This is a special case of Conjecture B.

Remark. As explained in [FK2], it suffices to prove our conjectures only for F of characteristic zero. They follow for F of positive characteristic on using [K3].

II. Theorems

The aim of this paper is to prove the following

Theorem 1. *Conjecture* B_0 *implies Conjecture* C.

In particular, conjectures B_0 and C_0 are equivalent; they imply all other conjectures here. In the appendix we prove Conjecture B_0 in the case where $G = GL(2, F)$ and $H = GL(1, E)$ is a torus of G.

The proof of Theorem 1 is global. It relies on the trace formula. We work with a cyclic extension E/F of degree k of global fields. At a place v of F which stays prime in E, the tensor product $E_v = E \otimes_F F_v$ is a cyclic field extension of degree k of the completion F_v of F at v; this is the case of a local field extension considered so far. At a place v of F which splits E we have $E_v = E \otimes_F F_v = F'_v \otimes \ldots \otimes F'_v$ (r' copies), where F'_v is a cyclic extension of F_v of degree k', and $k = k'r'$. There is a generator σ of $\mathrm{Gal}(E/F)$ whose restriction to the decomposition group at v maps $(x_1, \ldots, x_{r'})$ to $(\sigma' x_{r'}, \sigma' x_1, \ldots, \sigma' x_{r'-1})$, where σ' generates the galois group $\mathrm{Gal}(F'_v/F_v)$. In particular, identifying F_v with the diagonal in E_v, we obtain

$$N_{E_v/F_v}(x_1, \ldots, x_{r'}) = N_{F'_v/F_v}(x_1 x_2 \ldots x_{r'}).$$

The image of E_v under the norm map N_{E_v/F_v} is $N_{F'_v/F_v} F'_v$, and the character κ_v is taken to be a character of $F_v^\times$ whose kernel is $N_{F'_v/F_v} F'^{\times}_v$.

The group $H_v = GL(r, E_v) = GL(r, F'_v)^{r'}$ is embedded in the diagonal subgroup of type $(rk', \ldots, rk')$ in $G_v = GL(n, F_v)$, by embedding $GL(r, F'_v)$ in $GL(rk', F_v)$ as usual. Let N_v be the unipotent radical of the upper triangular parabolic subgroup P_v of G_v of type $(rk', \ldots, rk')$. For h in the standard Levi subgroup M_v of P_v, put $\delta_{N_v}(h) = |\det \mathrm{Ad}(h)_{L(N_v)}|$. Given a function ϕ_v on G_v, define a function $\phi_{v,N}$ on M_v by

$$\phi_{v,N}(h) = \delta_{N_v}^{1/2}(h) \int_{K_v} \int_{N_v} \phi_v(k^{-1} hnk)\, dn\, dk$$

where $K_v = GL(n, R_v)$. The centralizer T_v in G_v, of a regular h in M_v, lies in M_v. Put

$$\Delta_{N_v}(h) = |\det(\mathrm{Ad}(h) - I)_{L(N_v)}|,$$

$$\Phi_{\kappa_v}(h, \phi_v) = \int_{T_v \backslash G_v} \kappa_v(\dot{x})\, \phi_v(\dot{x}^{-1} h \dot{x})\, d\dot{x},$$

and

$$\Phi_{\kappa_v}(h, \phi_{v,N}) = \int_{T_v \backslash M_v} \kappa_v(\dot{x})\, \phi_{v,N}(\dot{x}^{-1} h \dot{x})\, d\dot{x}.$$

A standard integration formula asserts that for every h in M_v regular in G_v we have

$$\Delta_{N_v}(h)\, \Phi_{\kappa_v}(h, \phi_v) = \Phi_{\kappa_v}(h, \phi_{v,N}).$$

Hence, the question of matching ϕ_v with f_v on H_v is reduced to matching $\phi_{v,N}$ on M_v with f_v on H_v, namely to matching functions on $GL(r, F'_v)$ and on $GL(rk', F_v)$. By induction on k we may assume in our global study of Theorem 1 that this transfer, for $k' < k$, is available, for ordinary (Conjecture A) and spherical (Conjecture C) functions.

REMARK. Suppose, by induction on k, that Conjecture B holds for $k' < k$, namely given an irreducible $GL(r, F_v')$-module τ_v, there exists a corresponding $GL(rk', F_v)$-module $\pi(\tau_v)$. Denote by $I(\pi(\tau_{1,v}) \otimes \ldots \otimes \pi(\tau_{k',v}))$ the G_v-module normalizedly induced from the P_v-module $\pi(\tau_{1,v}) \otimes \ldots \otimes \pi(\tau_{k',v})$ (on M_v, extended trivially across N_v). Then a standard computation of the character of an induced representation implies that

$$\operatorname{tr}[I(\pi(\tau_{1,v}) \otimes \ldots \otimes \pi(\tau_{k',v}))](\phi_v \times \kappa_v) = \operatorname{tr}(\tau_{1,v} \otimes \ldots \otimes \tau_{k',v})(f_v)$$

for all matching ϕ_v on G_v and f_v on H_v. Consequently, the H_v-module $\tau_{1,v} \otimes \ldots \otimes \tau_{k',v}$ and the G_v-module $I(\pi(\tau_{1,v}) \otimes \ldots \otimes \pi(\tau_{k',v}))$ are corresponding,. It is then clear that Conjecture A, B, C, suitably stated in the case of a place v of F which splits in E, reduce at once to the analogous conjectures with k' replacing k. In particular, these conjectures hold for a place v of F which splits completely in E, for then $k' = 1$. We shall use this fact in our global study below.

To prove Theorem 1 we shall now formulate Theorem 2, which is a global lifting theorem, whose local analogue is Conjecture B.

Let E be a cyclic extension of degree $k > 1$ of a global field F. Denote by $\mathbb{A}_E$ and $\mathbb{A}$ the rings of adeles of E and F. Put $G = GL(n, F)$ and $H = GL(r, E)$, $G_v = GL(n, F_v)$ and $H_v = GL(r, E_v)$, $G(\mathbb{A}) = GL(n, \mathbb{A})$ and $H(\mathbb{A}_E) = GL(r, \mathbb{A}_E)$. Here F_v is the completion of F at a place v, and $E_v = E \otimes_F F_v$ is the direct sum $F_v' \oplus \ldots \oplus F_v'$ of r_v' cyclic extensions F_v' of F_v of degree k_v', where $r_v' k_v' = k$. Conjectures A, B, C are stated for v which stays prime in E/F, thus $r_v' = 1$ and $E_v = F_v'$ is a field; this is the crucial case, and the other cases are reduced to it by induction. Therefore, we treat the cases of v which split in E/F only paranthetically. In any case, $H_v = GL(r, F_v')^{r'}$ will be regarded as a subgroup of G_v, and $H(\mathbb{A}_E)$ of $G(\mathbb{A})$. We also denote by $Z(\mathbb{A})$ the center of $G(\mathbb{A})$, and fix a unitary character ω of $Z(\mathbb{A}) = \mathbb{A}^\times$ which is trivial on $Z(F) = F^\times$. Fix a character κ of $\mathbb{A}^\times/F^\times$ whose kernel is the norm subgroup $F^\times N_{E/F}\mathbb{A}_E^\times$. It corresponds to E by global class field theory. The local components of ω and κ will be denoted by ω_v and κ_v; they are unramified for almost all v.

If v is a non-archimedean place of F where ω_v and κ_v are unramified, we let $\mathbb{H}(G_v)$ be the Hecke algebra of complex-valued K_v-biinvariant functions ϕ_v on G_v with $\phi_v(zg) = \kappa_v(z)^{rk(k-1)/2}\omega_v(z)\phi_v(g)$ (g in G_v, z in $Z_v(G_v)$) which are compactly supported modulo Z_v. A Hecke operator is the operator of convolution with a Hecke function. Let $L(G)$ be the span of the set of complex valued functions ψ on $G\backslash G(\mathbb{A})$ with (1) $\psi(zg) = \kappa(z)^{rk(k-1)/2}\omega(z)\psi(g)$ (z in $Z(\mathbb{A})$, g in $G(\mathbb{A})$); (2) there exists an open compact subgroup U_ψ of $G(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the ring of finite adeles, with $\psi(gu) = \psi(g)$ for all y in

U_ψ; (3) ψ is an eigenvector of all Hecke operators in $\mathbb{H}(G_v)$ for infinitely many places v. Then $L(G)$ is the space of *automorphic forms* (see [Av]). $G(\mathbb{A})$ acts by right translation, and by an *automorphic* G-module we mean any irreducible constituent of the $G(\mathbb{A})$-module $L(G)$. Let $L^2(G)$ be the space of ψ in $L(G)$ such that $|\psi|^2$ is integrable on $Z(\mathbb{A})G\backslash G(\mathbb{A})$. An automorphic G-module which occurs as a direct summand in $L^2(G)$ is called a *discrete-series* G-module. The function ψ in $L^2(G)$ is called *cuspidal* if for every proper F-parabolic subgroup P of G we have $\int_{N\backslash N(\mathbb{A})} \psi(nx)\,dn = 0$ for every x in $G(\mathbb{A})$, where N is the unipotent radical of P. The space $L_0(G)$ of cusp forms splits as a direct sum with finite multiplicities (for any reductive group G) of irreducible discrete-series G-modules, called *cuspidal*. In our case of $G = GL(n)$, these multiplicities are all equal to one.

Analogously, we introduce automorphic, discrete-series and cuspidal H-modules τ on replacing G_v by H_v, K_v by $K(H_v) = GL(r, R_{E_v})$, $G(\mathbb{A})$ by $H(\mathbb{A}_E)$ etc., except that condition (1) in the definition of $L(H)$ is still phrased with $Z(\mathbb{A})$, namely, it is $\psi(zh) = \omega(z)\psi(h)$ (z in $Z(\mathbb{A})$, h in $H(\mathbb{A}_E)$), rather than with the center $Z_H(\mathbb{A}_E)$ of $H(\mathbb{A}_E)$.

We denote $G(\mathbb{A})$-modules by π, and $H(\mathbb{A}_E)$-modules by τ. If π is irreducible, then it is the restricted tensor product $\otimes_v \pi_v$ over all places v of F of G_v-modules π_v which are unramified for almost all v. If τ is irreducible then it is the product $\otimes_v \tau_v$ over all places v of F of H_v-modules τ_v which are unramified for almost all v. The notion of local correspondence is defined above for unramified τ_v and π_v.

DEFINITION. The irreducible $G(\mathbb{A})$-module π and $H(\mathbb{A}_E)$-module τ *correspond* if the G_v-module π_v and the H_v-module τ_v correspond for almost all places v of F. Write $\pi(\tau)$ for π and $\tau(\pi)$ for τ if π and τ correspond.

It follows at once from the rigidity theorem for automorphic forms on $GL(n)$ of [JS] that at most one automorphic non-degenerate G-module π may correspond to a given τ, and in this case π is equivalent to $\pi \otimes \kappa$. If τ is automorphic and corresponds to π, then $\tau \circ \sigma^i$ corresponds to π for each $i (0 \leqslant i < k)$, and it is not hard to deduce from Proposition 3.6 in [JS] that at most one orbit $\{\tau \circ \sigma^i (0 \leqslant i < k)\}$ of cuspidal H-modules may correspond to a given π. If τ is a cuspidal H-module with $\tau \simeq \tau \circ \sigma$, then there exists a cuspidal $GL(r, \mathbb{A})$-module τ_F whose base change lift (see [AC]) is τ. It is easy to see that τ corresponds to the induced automorphic G-module $I(\tau_F \otimes \tau_F\kappa \otimes \ldots \otimes \tau_F\kappa^{k-1})$.

Our proof of Theorem 1 is based on the following

THEOREM 2. *The correspondence $\tau \to \pi(\tau)$ defines a bijection from the set of orbits $\{\tau \circ \sigma^i (0 \leqslant i < k\}$ of cuspidal H-modules τ with $\tau \not\simeq \tau \circ \sigma$, to the set* *of cuspidal G-modules π with $\pi \otimes \kappa \simeq \pi$. If the cuspidal π and τ correspond then π_v and τ_v correspond at each place v where π_v and τ_v are unramified, and at each place v which splits in E/F.*

This is proven for $n = 2$ by simple means in [F3a], and in [AC], III.6, for all $n \geqslant 2$, as a consequence of the theory of base-change for $GL(n)$ ($n = 2$ in [F3a]). Theorem 2 concerns, and is used below in the context of, π with no elliptic components. Its proof relies on Arthur's computations [A] of the contribution to the trace formula from Eisenstein series; these computations are also used in the proof of the κ-Trace Formula below. The local theory of base-change for $GL(n)$, and the global theory for cuspidal representations π with a supercuspidal component, is established in [F3b] by elementary means. However, this special case does not seem to imply any form of Theorem 2 which would suffice for us to prove Theorem 1 for $n > 2$.

To explain the way in which Theorem 2 is to be used, we recall the simple trace formula of [FK1], §18, in the context of the group $H(\mathbb{A}_E)$. For every place v of F fix a Haar measure dh_v on H_v such that the product of the volumes of $K(H_v)$ converges. Put $dh = \otimes dh_v$ for the product measures on $H(\mathbb{A}_E)$. The trace formula will be stated for a measure fdh, where f is a smooth function on $H(\mathbb{A}_E)$ which transforms under $Z(\mathbb{A})$ by ω^{-1} and is compactly supported modulo $Z(\mathbb{A})$, of the form $f = \otimes_v f_v$, where f_v are functions on H_v which are equal to the unit element f_v^0 of the Hecke algebra $\mathbb{H}(H_v)$ of H_v for almost all (non-archimedean) places v of F. The trace formula is usually stated for a function which transforms according to a character of the center of $H(\mathbb{A}_E)$ rather than $Z(\mathbb{A})$, but the adjustments required in the proof are trivial.

The simple trace formula is stated not for a general function f, but for a well-chosen f. Let u'' be a non-archimedean place of F which splits completely in E such that $\omega_{u''}$ is unramified. Put $J = GL(r, F_{u''})$. Then $H_{u''} = J^k$. Fix supercuspidal J-modules $j_i^0 (1 \leqslant i \leqslant k)$ whose central characters are $\omega_{u''}^{1/k}$ such that $j_{i'}^0 \not\simeq j_i^0 \otimes \alpha$ for every $i' \neq i$ and character α of $F_{u''}^\times$. Let f_i be a normalized matrix-coefficient of j_i^0. Thus $\operatorname{tr} j_i^0(f_i) = 1$ and $\operatorname{tr} j(f_i) = 0$ for every irreducible J-module j with central character $\omega_{u''}^{1/k}$ which is inequivalent to j_i^0. Let f_i' be the product of f_i and the characteristic function of the subset $\boldsymbol{\pi}_{u''}^i J^0$ of J, where $\boldsymbol{\pi}_{u''}$ is a local uniformizer at u'', and J^0 is the set of elements in J whose eigenvalues have valuations which are all equal to one. Put f' for the function $f_1' \otimes \ldots \otimes f_k'$ on $H_{u''}$, and $f_{u''}'$ for $f_{u''}'(h) = \int f'(zh)\omega_{u''}(z)dz$, where dz is the measure on $Z_{u''}$ which assigns the volume one to $Z_{u''} \cap K_{u''}$. The function $f_{u''}'$ on $H_{u''}$ transforms under $Z_{u''}$ according to $\omega_{u''}^{-1}$, and has the property that $\operatorname{tr} \tau_{u''}(f_{u''}')$ is zero for every irreducible $H_{u''}$-module $\tau_{u''}$ which transforms by $\omega_{u''}$ on $Z_{u''}$, unless $\tau_{u''}$ is the product of $j_{1,u''}^0 \otimes \ldots \otimes j_{k,u''}^0$ with an unramified character of the center of $H_{u''}$ whose restriction to $Z_{u''}$ is one.

The simple trace formula will be stated for a function f whose component at u'' is the above $f'_{u''}$.

Let $u' \neq u''$ be a place of F which splits completely in E such that $\omega_{u'}$ is unramified. To specify the component of f at u', let $\phi'_{u'}$ be a regular function on $G_{u'}$. Recall (e.g., [FK1]) that by that we mean that $\phi'_{u'}$ is a function with the usual properties of a function $\phi_{u'}$, which is supported on the set of g in $G_{u'}$ whose eigenvalues lie in $F_{u'}^{\times}$ and have distinct valuations, such that the normalized orbital integral $F(g, \phi'_{u'}) = \Delta_{u'}(g)\Phi(g, \phi'_{u'})$ depends only on the valuations of the eigenvalues of g. If N is the unipotent radical of the standard parabolic subgroup of type $(r, \ldots, r)$ then the function $f'_{u'} = \phi'_{u',N}$ is a regular function on $H_{u'}$, which matches $\phi'_{u'}$.

TRACE FORMULA. *For $f = \otimes f_v$ whose components at u'' and u' are as above we have*

$$\sum_{\tau} \operatorname{tr} \tau(fdh) = \sum_{\{\gamma\}} |Z(\gamma, H(\mathbb{A}))/Z(\mathbb{A})Z(\gamma, H)|\Phi(\gamma, fdh).$$

On the left the sum ranges over all cuspidal H-modules τ which transform under $Z(\mathbb{A})$ by ω and whose component $\tau_{u''}$ at u'' is a multiple of $j^0_{1,u''} \otimes \ldots \otimes j^0_{k,u''}$ by an unramified character (and $\tau_{u'}$ is unramified). In particular $\tau \not\simeq \tau \circ \sigma$. The sum on the right is finite and ranges over a set of representatives $\{\gamma\}$ for the conjugacy classes of regular elliptic elements in $H/Z(G)$ which are elliptic in $H_{u''}$. In the volume factor, $Z(\gamma, X)$ is the centralizer of γ in the group X.

It is clear that the term indexed by γ on the right is independent of the choice of measure on $Z(\gamma, H(\mathbb{A}))$. Note that the choice of $f_{u''}$ amounts to a choice of $k(\geqslant 2)$ components, at k places of E, which are supercusp forms. Then our trace formula follows, with minor modifications due to the choice of center, from [FK1], §18.

We also need the trace formula for G, twisted by κ. This formula will be stated for a product measure $dg = \otimes_v dg_v$ on $G(\mathbb{A})$, and a test function $\phi = \otimes \phi_v$ on $G(\mathbb{A})$ with the usual properties. Namely, each ϕ_v is smooth, satisfies $\phi_v(zg) = \kappa_v(z)^{rk(k-1)/2}\omega_v(z)^{-1}\phi_v(g)$ (z in Z_v, g in G_v), and is compactly supported modulo Z_v. For almost all v this ϕ_v is the unit element ϕ_v^0 of the Hecke algebra $\mathbb{H}(G_v)$. The trace formula which we need is related to the operator

$$(r(\phi\, dg \times \kappa)\psi)(x) = \int_{G(\mathbb{A})/Z(\mathbb{A})} \phi(g)\kappa(x)\psi(xg)\, dg$$

on the space $L_0(G)$ of cusp forms. Namely, it is twisted by the operator $r(\kappa)$: $\psi(x) \to \kappa(x)\psi(x)$. We need this formula only for a function ϕ whose local components ϕ_v match the local components f_v for every v. In particular, we take $\phi_{u''}$ to be a function which matches the function $f'_{u''}$ which depends on the supercuspidal J-modules $j_i^0 (1 \leqslant i \leqslant k)$, and $\phi_{u'}$ is taken to be the regular function $\phi'_{u'}$ mentioned above in the context of the trace formula for H.

DEFINITION. Let m be a positive integer. The regular function $\phi'_{u'}$ is called *m-regular* if $F(g, \phi'_{u'})$ is zero unless g has the following property. For every subset A of $\{1, 2, \ldots, n\}$ of cardinality a $(1 \leqslant a \leqslant n/2)$, we have

$$\left| \frac{1}{a} \sum_{i \in A} \operatorname{ord}_{u'}(g_i) - \frac{1}{n-a} \sum_{i \notin A} \operatorname{ord}_{u'}(g_i) \right| > m,$$

where $g_1, \ldots, g_n$ are the n eigenv· ues of g.

κ-TRACE FORMULA. *For every* $\phi^{u'} = \otimes \phi_v (v \neq u')$ *as above there is* $m = m(\phi^{u'})$ *such that for every* $\phi = \phi^{u'} \otimes \phi_{u'}$ *with m-regular* $\phi_{u'}$ *we have*

$$\sum_{\pi} \operatorname{tr} \pi(\phi\, dg \times \kappa) = \sum_{\{\gamma\}} |Z(\gamma, H(\mathbb{A}))/Z(\mathbb{A})Z(\gamma, H)| \Phi_\kappa(\gamma, \phi\, dg).$$

The sum on the left ranges over all cuspidal G-modules π *with* $\pi \otimes \kappa \simeq \pi$, *central character* $\kappa^{rk(k-1)/2}\omega$ *and component at* u' *which is induced from the product of an unramified character and the supercuspidal* $H_{u'}$*-module* $j_1^0 \otimes \ldots \otimes j_k^0$. *The sum on the right is finite; it ranges over all conjugacy classes* $\{\gamma\}$ *of regular elements* γ *in* $G/Z(G)$ *which have a representative in H and are elliptic in* $H_{u''}$.

REMARK. The integer m depends only on the support of $\phi^{u'}$ modulo conjugacy in $G(\mathbb{A}^{u'})$.

Proof. The trace formula is obtained on integrating over $G(\mathbb{A})/Z(\mathbb{A})$ the restriction to the diagonal $x = y$ of two different expressions for the kernel $K(x, y)$ of the integral operator $r(\phi\, dg \times \kappa)$ on $L(G)$. One expression is $\Sigma_\gamma \kappa(x)\phi(x\gamma y^{-1})$, where γ ranges over G/Z. If $\phi(x\gamma x^{-1}) \neq 0$, then γ has distinct eigenvalues since $\phi_{u'}$ is regular. If γ is not elliptic in G then there is a $(1 \leqslant a \leqslant n/2)$ such that a conjugate (γ', γ'') (in the obvious notations) of γ lies in the standard (diagonal) Levi subgroup of type $(a, n - a)$. Put

$$s = \sum_{v \neq u'} s_v, \quad \text{where } s_v = |\operatorname{ord}_v(\det \gamma'))/a - (\operatorname{ord}_v(\det \gamma''))/(n-a)|.$$

Then $s_v = 0$ for almost all v, and there is an integer $m > 0$, depending on the support of $\phi^{u'}$, but not on a or γ, such that $s < m$. If $f_{u''}$ is m-regular then the product formula $\Sigma_v \operatorname{ord}_v(\alpha) = 0$ on (α in) $F^\times$ implies that $f(x\gamma x^{-1})$ is zero for every x in $G(\mathbb{A})$ and γ in G which is not elliptic (regular).

A standard change of integration (over $G(\mathbb{A})/Z(\mathbb{A})$) and summation (over the elliptic regular γ in G/Z) leads to

$$\sum_{\{\delta\}} |Z(\delta, G(\mathbb{A}))/Z(\mathbb{A})Z(\delta, G)| \Phi_\kappa(\delta, \phi\, dg).$$

The sum ranges over the conjugacy classes $\{\delta\}$ of the elliptic regular δ in G/Z. The proof of Lemma A1 implies that if $\Phi_\kappa(\delta, \phi\, dg) \neq 0$ then a conjugate γ of δ lies in H. Then $Z(\delta, G) = Z(\gamma, H)$. Consequently, we obtain the right side of the κ-trace formula.

The second expression for the kernel involves an orthonormal basis of the space $L(G)$. Although the left side of the κ-trace formula is very simple – involving only cuspidal G-modules – and suggests that the operator $r(\phi\, dg \times \kappa)$ on $L(G)$ factorizes through the projection to $L_0(G)$, we do not know (at present) to prove this a-priori. Instead we use the explicit computations of the integral over $G(\mathbb{A})/Z(\mathbb{A})$ of the restriction to the diagonal of the second expression for the kernel given in Arthur [A]. Our argument above, concerning the regular component $\phi_{u'}$, implies that the truncation applied in [A] is trivial for our ϕ. The computations of [A] are carried out with κ replaced by 1, but since the operator $r(k)$: $\psi(x) \to \kappa(x)\psi(x)$ fixes all parabolic subgroups, these computations apply with trivial modifications in our twisted case as well. The result of these computations is given (for $\kappa = 1$) in [A], theorem 8.2. It is too complicated to recall here fully. All that we need is the partial description, with a general κ, which is given in [F1], §12, p. 170 (our κ is denoted there by ε, and we need the case of $\sigma = 1$ in [F1]). Our argument here is similar to that of [F1], §12, where analogous (but different) vanishing result is proven.

In the discussion of [A], (8.2) (=[F1], §12), we use the following remark. Denote by $\mathbf{j}$ any supercuspidal $H_{u''}$-module which is obtained as the product of the fixed supercuspidal $\mathbf{j}^0 = j_1^0 \otimes \ldots \otimes j_k^0$ chosen above, and an unramified character. Denote by $I(\mathbf{j})$ the $G_{u''}$-module normalizedly induced from the representation $\mathbf{j} \otimes 1$ of the upper triangular parabolic subgroup $H_{u''}N$ of $G_{u''}$. The choice of the j_i^0, with the property that $j_{i'}^0 \not\simeq j_i^0 \otimes \alpha$ for all $i' \neq i$ and characters α of J, guarantees by Theorem 4.2 of [BZ] that $I(\mathbf{j})$ is irreducible for every $\mathbf{j}$. It is clear that the function $\phi_{u''}$ with $\phi_{u'',N} = f'_{u''}$ can be chosen so that the operator $\pi_{u''}(\phi_{u''} dg_{u''})$ factorizes through the projection on the connected component of the $I(\mathbf{j})$ in the Grothendieck group $R(G_{u''})$ of $G_{u''}$ (see [BD]). We record this fact as the following:

LEMMA. *For every irreducible $G_{u''}$-module $\pi_{u''}$ the convolution operator $\pi_{u''}(\phi_{u''}dg_{u''})$ is zero unless $\pi_{u''}$ is of the form $I(\mathbf{j})$.*

We now return to the computation of the representation theoretic side of the κ-trace formula, and the computations of [A], (8.2) (or [F1], §12). The terms which appear depend on various parameters. First, we have a standard Levi subgroup M of G (containing the diagonal subgroup M_0), and an element s of the Weyl group of M in G. Then we have a unitary discrete series representation ϱ of $M(\mathbb{A})$ such that $s\varrho \simeq \varrho \otimes \kappa$. If M is of type $(n_1, \ldots, n_a)$, and correspondingly $\varrho = \varrho_1 \otimes \ldots \otimes \varrho_a$, the s acts by permuting the indices. Since κ is of order k, the set $\{\varrho_i \otimes \kappa^j (0 \leqslant j < k)\}$ is contained in the set $\{\varrho_j (1 \leqslant j \leqslant a)\}$ for every $i (1 \leqslant i \leqslant a)$. The terms themselves involve an integral and a sum, and the integrand itself is the trace of some intertwining operator $\mathcal{M}_L(P, \lambda)M(P, s)$ (in the notations of [A], p. 1324, 1.-2), acting on the convolution operator $I(\varrho \otimes e^{\lambda}; \phi\, dg \times \kappa)$ (in our notations, where e^{λ} is some unramified character on $M(\mathbb{A})/M(F)$). The component at u'' of this convolution operator is of the form $I(\varrho_{u''} \otimes e^{\lambda}; \phi_{u''}dg_{u''})$; note that $\kappa_{u''}$ is 1 since u'' splits completely from F to E. The Lemma implies that this component is zero unless $\varrho_{u''}$ is of the form $\mathbf{j}$. In this case we have that n_i is a multiple of r for every $i (1 \leqslant i \leqslant a)$, hence $a \leqslant k$. Since $j_{i'}^0 \not\simeq j_i^0 \otimes \alpha$ for all $i' \neq i$ and α, considering the component at u'' (where $\kappa_{u''} = 1$) of the global relation $\varrho_{s(1)} \otimes \ldots \otimes \varrho_{s(a)} = \varrho_1\kappa \otimes \ldots \otimes \varrho_a\kappa$, we see that $s = 1$. Hence the discrete-series $GL(rb_i, \mathbb{A})$-module ϱ_i satisfies $\varrho_i \simeq \varrho_i \otimes \kappa$, and k divides $n_i = rb_i$. Moreover, it is clear from the condition on the j_i^0 that the ϱ_i are cuspidal and the automorphic $I(\varrho \otimes e^{\lambda})$ are irreducible and non-degenerate.

To prove that the representation theoretic side of the κ-trace formula is as asserted, we need to prove that $a = 1$, namely that $\pi = I(\varrho) = \varrho = \varrho_1$ is cuspidal. Indeed, if this is the case then $M = G$ and the integral of [A], Theorem 8.2, reduces to a point. The corresponding contribution to the trace formula is of the form $\operatorname{tr}\pi(\phi\, dg \times \kappa)$, as required.

If $a \neq 1$ then $b_i < k$, hence by induction on k, since the cuspidal $GL(rb_i, \mathbb{A})$-module ϱ_i satisfies $\varrho_i \otimes \kappa \simeq \varrho_i$, there exists a unique orbit $\tau_i \circ \sigma^j (0 \leqslant j < k)$ of cuspidal $GL(rb_i/k, \mathbb{A}_E)$-modules which correspond to ϱ_i. In particular, the $H(\mathbb{A}_E)$-module $I_H = I(\tau_1 \otimes \ldots \otimes \tau_a)$ corresponds to the $G(\mathbb{A})$-module $I = I(\varrho_1 \otimes \ldots \otimes \varrho_a)$. At u'', the component $\tau_{i,u''}$ of τ_i is of the form $\otimes_j \tau_{i,u'',j} (1 \leqslant j \leqslant k)$, hence the component $I_{u''} = I(\varrho_{1,u''} \otimes \ldots \otimes \varrho_{a,u''})$ of I is of the form $I(\otimes_{i,j}\tau_{i,u'',j})$, and the $\varrho_{l,u''}$ are of the form $\otimes_i j_i^0$ for some set of indices i. Now each of the $\tau_{i,u'',j}$ is a $GL(rb_i/k, F_{u''})$-module, while the supercuspidal j_i^0 are $GL(r, F_{u''}) = J$-modules. Since $rb_i/k < r$ for all i we obtain a contradiction, implying that $a = 1$, and the proof of the κ-trace formula is complete.

REMARK. A special case of the κ-trace formula for a test function with n elliptic components at places of F which stay prime in E would suffice for our proof of Theorem 1. For the precise statement see the "Alternative proof" at the end of Section III, below.

III. Proofs

Our proof of Theorem 1 depends on a comparison of the two trace formulae. It is a new application of the trace formula, which is different than the standard way in which it is used. The standard approach, on which the proof of Proposition [$C_0 \Rightarrow B$] is based, compares the group theoretic sides of the trace formulae, which involve orbital integrals, and extract lifting consequences from the resulting identity of the representation theoretic sides of the formulae. Our approach reverses this order. For a suitable choice of test functions f and ϕ we compare the representation theoretic sides of the formulae; careful choice of the components of f and ϕ compensates for the fact that the local correspondence is not available. We then conclude the required matching properties of the orbital integrals in question from the resulting identity of the group theoretic sides of the trace formulae.

We indicate two approaches for the construction of the test functions f and ϕ, and the completion of the proof of Theorem 1. In both of these approaches we regard the unramified cyclic extension E_u/F_u of local non-archimedean fields under construction as the completion at a place u of a cyclic extension (of the same degree) of global fields E/F. There is no difficulty in choosing the components f_v and ϕ_v of f and ϕ at a place v of F which does not ramify in E. The first approach is based on using regular functions at the ramified places; it relies on the results of Kazhdan [K1] in the case of $r = 1$. Conjecture B_0 will not be used before the final lines of the proof. The second approach is based on applying Conjecture B_0 at each place v of F where there is ramification. Moreover, as alluded to in the Remark following the proof of the κ-trace formula, assuming Conjecture B_0 at sufficiently many (n or 2, depending on which form of the trace formula is used) places of F which stay prime in E, we can give a different proof for a special case of the κ-trace formula which suffices for our proof of Theorem 1.

In the first approach we use two special cases, of Conjectures B_0 and A, when $r = 1$, due to Kazhdan [K1], which we now state. Suppose that E/F is a cyclic extension of degree k of local non-archimedean fields. Then we have:

PROPOSITION 1. *The trivial character of $GL(1, E) = E^{\times}$ corresponds to the normalizedly induced irreducible $GL(k, F)$-module $I_{\kappa} = I(1, \kappa, \kappa^2, \ldots, \kappa^{k-1})$.*

COROLLARY 2. *For every character μ of $F^{\times}$, the $GL(1, E)$-module $\eta = \mu \circ N_{E/F}$ corresponds to the $GL(k, F)$-module $I_{\kappa}(\mu) = \mu \otimes I_{\kappa}$.*

COROLLARY 3. *For every r-tuple $\mu_1, \ldots, \mu_r$ of characters of $F^{\times}$, the induced H-module $I(\boldsymbol{\eta}) = I(\eta_1, \ldots, \eta_r)$, where $\eta_i = \mu_i \circ N_{E/F}$, corresponds to the induced G-module $I_{\kappa}(\boldsymbol{\mu}) = I(I_{\kappa}(\mu_1), \ldots, I_{\kappa}(\mu_r))$.*

Proposition 1 is due to [K1], Corollary 2 is an immediate consequence, and Corollary 3 follows from a standard computation of the character of an induced representation. In particular, by definition we have

COROLLARY 4. *For all matching f and ϕ we have* $\operatorname{tr}(I(\boldsymbol{\eta}))(f) = \operatorname{tr}(I_{\kappa}(\boldsymbol{\mu}))(\phi \times \kappa)$.

Of course, we also have the following easy

LEMMA 5. *For every ϕ on G such that $\Phi_{\kappa}(g, \phi)$ is supported on the set of g in G which are regular (and have a conjugate in H), there exists a matching f on H. Conversely, for every f on H such that $\Phi(h, f)$ is supported on the set of h in H which are regular in G, and satisfies $\Phi(\sigma h, f) = \Phi(h, f)$ for all h, there exists a matching ϕ on G.*

Indeed, on the regular set both $\Phi(h, f)$ and $\Delta(h)\Phi_{\kappa}(h, \phi)$ are locally constant.

Let $\mu_1', \ldots, \mu_r'$ be (unitary) characters on $F^{\times}$, put $\eta_i' = \mu_i' \circ N_{E/F}$, and suppose that η_i'/η_j' is ramified for all $i \neq j$ and $\Pi_i \eta_i' = \omega$ on $F^{\times}$. Let $f_{\boldsymbol{\eta}'}$ be a function on H with $(\Delta_H \Phi)(h, f_{\boldsymbol{\eta}'})$ equal to $\Sigma_w \Pi_i \eta_i'(h_{w(i)})^{-1}$ if h has eigenvalues $h_i (1 \leqslant i \leqslant r)$ in $E^{\times}$ with $\operatorname{ord} h_i = i$, and zero otherwise. The sum ranges over w in the symmetric group S_r on r letters. We have

PROPOSITION 6. *If τ is an irreducible H-module with* $\operatorname{tr} \tau(f_{\boldsymbol{\eta}'}) \neq 0$ *then $\tau = I(\boldsymbol{\eta})$, where $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_r)$, η_i are characters of $E^{\times}$ and η_i/η_i' are unramified.*

Proof. Apply the Weyl integration formula to $\operatorname{tr} \tau(f_{\boldsymbol{\eta}'})$. Since $f_{\boldsymbol{\eta}'}$ is regular, the Theorem of [C] (see [FK1], §14) applies; it implies that the A-module of N-coinvariants (or N-homology) τ_N of τ (A is the diagonal in H, N the unipotent upper triangular subgroup of H) contains a copy of a character $\boldsymbol{\eta}$ of A as asserted in the proposition. By Frobenius reciprocity τ is a constituent of the induced H-module $I(\boldsymbol{\eta})$. But $I(\boldsymbol{\eta})$ is irreducible by Theorem 4.2 of [BZ], since the η_i'/η_j' are chosen to be ramified, as required.

Let f^* be a pseudo-coefficient of the Steinberg H-module st_H (see [K2]). Thus, by definition $\operatorname{tr} \tau(f^*) = 0$ for every tempered irreducible H-module τ inequivalent to st_H, and $\operatorname{tr} \mathrm{st}_H(f^*) = 1$. By [Z], Theorem 9.7(b), we conclude that $\operatorname{tr} \tau(f^*) = 0$ for every non-degenerate unitary τ inequivalent to st_H. In addition to proving the existence of such f^*, it is shown in [K2] that $\Phi(h, f^*)$ is zero at each regular non-elliptic h in H, and

$$\Phi(h, f^*)|Z(h, H)/Z(H)| = (\chi(\mathrm{st}_H))(h) = (-1)^{r-1}$$

at each elliptic regular h in H, where $\chi(\mathrm{st}_H)$ is the character of st_H. Implicit is a choice of a Haar measure dh on H. It is chosen to satisfy $|Z(H)/Z(G)| = k$ (note that $E^\times/F^\times$ is compact).

Similarly, let ϕ^* be a pseudo-coefficient of the irreducible G-module $\pi(\mathrm{st}) = I(\mathrm{st} \otimes \kappa, \mathrm{st} \otimes \kappa^2, \ldots, \mathrm{st} \otimes \kappa^k)$ of Conjecture B_0. This $\pi(\mathrm{st})$ is an isolated point in the variety of κ-invariant tempered G-modules, and ϕ^* is defined to be a function with $\mathrm{tr}\,\pi(\phi^* \times \kappa) = 0$ for every tempered irreducible κ-invariant G-module π inequivalent to $\pi(\mathrm{st})$, and $\mathrm{tr}\,(\pi(\mathrm{st}))(\phi^* \times \kappa) = 1$. Further, we have $\mathrm{tr}\,\pi(\phi^* \times \kappa) = 0$ for every unitary non-degenerate κ-invariant G-module π inequivalent to $\pi(\mathrm{st})$. The methods of [K2] apply in this twisted case to show that a pseudo-coefficient ϕ^* exists, and has the property that $\Phi_\kappa(h, \phi^*)$ is zero if h in H is regular in G but not elliptic, while if h is elliptic regular in H then

$$\Phi_\kappa(h, \phi^*)|Z(h, G)/Z(G)| = (\chi(\pi(\mathrm{st})))(h).$$

In particular f^* and ϕ^* are matching, and in summary we have:

PROPOSITION 7. *There exist matching functions f^* on H and ϕ^* on G, such that* (1) *$\Phi(h, f^*)$ is zero if h is regular non-elliptic, and non-zero if h is regular elliptic;* (2) *$\mathrm{tr}\,\tau(f^*)$ is zero for every unitary non-degenerate irreducible τ inequivalent to st_H, and $\mathrm{tr}\,\mathrm{st}_H(f^*) = 1$;* (3) *$\mathrm{tr}\,\pi(\phi^* \times \kappa)$ is zero for every unitary non-degenerate irreducible π* inequivalent to $\pi(\mathrm{st})$, *and* $\mathrm{tr}\,(\pi(\mathrm{st}))(\phi^* \times \kappa) = 1$.

We shall now begin the proof of Theorem 1. We are given a cyclic unramified extension E_u/F_u of local non-archimedean fields, and corresponding spherical functions ϕ_u'' and f_u''. We have to show that ϕ_u'' and f_u'' are matching. By a standard integration formula ($F(h, f_u'') = F^M(h, f_{uN}'')$ if h is regular and lies in the Levi M of a parabolic $P = MN$ with unipotent radical N; see, e.g., [FK1], §7), it suffices to prove that $\Phi(h, f_u'') = \Delta(h)\Phi(h, \phi_u'')$ only for h in H_u, regular in G_u, which are elliptic in H_u, hence also in G_u.

For the proof we take a cyclic extension E/F of global fields with $[E:F] = [E_u:F_u] = k$, such that at some place u of F the completions of E and F are the given local E_u and F_u, and such that each archimedean place of F splits in E. Such E/F is easily constructed, but it is not clear to me if it can be chosen to be everywhere unramified. We fix a global unitary character ω whose component ω_u at u is the one which appears in the definition of ϕ'_u and f'_u. We fix a non-archimedean place u'' of F which splits completely in E such that $\omega_{u''}$ is unramified. Fix matching functions $f'_{u''}$ and $\phi'_{u''}$, related to the supercuspidal $H_{u''}$-module $j^0_{1,u''} \otimes \ldots \otimes j^0_{k,u''}$, as in the statements of the Trace Formula and the κ-Trace Formula. Let $u_i (1 \leqslant i \leqslant \mathbf{I})$ denote the places of F which ramify in E. Fix regular matching functions f'_{u_i} and ϕ'_{u_i} as in Proposition 6 for each $i (1 \leqslant i \leqslant \mathbf{I})$.

Let γ' be a regular elliptic element of H_u. Our aim is to show that

$$\Phi(\gamma', f'_u) \;=\; \Delta(\gamma')\Phi(\gamma', \phi'_u).$$

Since both sides of this equation are locally constant as functions in γ', and $H = GL(r, E)$ is dense in $H_u H_{u''} \Pi_i H_{u_i}$, we may assume that γ' is regular elliptic rational element of H, with the property that $\Phi(\gamma', f'_v)$ is non-zero for $v = u''$ and $v_i (1 \leqslant i \leqslant \mathbf{I})$. The centralizer of γ' in H is isomorphic to the multiplicative group $D^\times$ of a field extension D of F of degree r. Fix a non-archimedean place $u'(\neq u, u'')$ of F which splits completely in D, such that $\omega_{u'}$ is unramified and the eigenvalues of γ' are all units at u'. Fix a place $w(\neq u, u', u'', u_i)$ of F which splits completely in E; it is taken to be archimedean if F is a number field. At each non-archimedean place v of F other than u, u', u'', u_i, w, let f'_v and ϕ'_v be corresponding spherical functions with $\Phi(\gamma', f'_v) \neq 0$ and $\Phi_{\kappa_v}(\gamma', \phi'_v) \neq 0$. For almost all v these are taken to be the unit elements f^0_v and ϕ^0_v.

Denote by $u'_i (1 \leqslant i \leqslant n)$ the places of D over the place u' of F. For every positive integer m there exists an element $\delta = \delta_m$ in $D^\times$ such that (1) $\mathrm{ord}_v(\delta) = 0$ for every finite $v \neq u'_i$; (2) $\Sigma^n_{i=1} \mathrm{ord}_{u'_i}(\delta) = 0$; (3)

$$\left| \frac{1}{a} \sum_{i \in A} \mathrm{ord}_{u'_i}(\delta) - \frac{1}{n-a} \sum_{i \notin A} \mathrm{ord}_{u'_i}(\delta) \right| > m$$

for every non-empty proper subset A of $\{1, \ldots, n\}$ of cardinality a. If F is a function field this is a consequence of the Riemann–Roch theorem. Whatever char F is, given a finite set S of finite places of D not including the u'_i, and denoting the residual characteristic of D at v in S by d_v, on replacing δ by δ^d, where $d = \Pi_{v \in S} (d_v - 1) d_v^{m_v}$ and m_v are sufficiently large, we may assume that δ is as close to the identity as desired at the places v of S. If F is a number field, then applying the Dirichlet unit theorem we may replace δ by its product with a unit in $D^\times$ so that the following holds. There exists a compact subset $C_\infty(D)$ of $(D \otimes_Q \mathbb{R})^\times$, which depends only on D, such that δ lies in $C_\infty(D)$. Hence, we have the property (2) above.

Denote by ∞_i the archimedean places of F. We conclude that there exist matching $f'_\infty = \otimes_i f'_{\infty_i}$ and $\phi'_\infty = \otimes_i \phi'_{\infty_i}$ which depend on γ' but not on m (and matching f'_w and ϕ'_w, independent of m, if F is a function field), with the following property. Let ϕ'_u be a function on G_u, and f'_u on H_u. So far, they are not related in any particular way. Let $\phi'_{u'}$ be an m-regular function on $G_{u'}$, where m is an integer, depending on ϕ'_v for all $v \neq u'$, as asserted in the κ-Trace Formula. Then there exists $\delta = \delta_m$ as above such that $\gamma = \gamma'\delta$ satisfies

$$\Phi_{\kappa_v}(\gamma, \phi'_v) \neq 0 \quad \text{and} \quad \Phi(\gamma, f'_v) \neq 0 \quad \text{for all} \quad v \neq u, \tag{1}$$

and

$$\Phi(\gamma, f'_u) = \Phi(\gamma', f'_u) \quad \text{and} \quad \Phi_{\kappa_u}(\gamma, \phi'_u) = \Phi_{\kappa_u}(\gamma', \phi'_u). \tag{2}$$

At this stage we note that there are only finitely many conjugacy classes of h in $H/Z(G)$ with $\Phi(h, f') \neq 0$ or $\Phi_\kappa(h, \phi') \neq 0$, where $f' = \otimes f'_v$ and $\phi' = \otimes \phi'_v$. Moreover, we note that for every $v \neq u$, the choice of f'_v (as a spherical function or one which matches some ϕ'_v) guarantees that $\Phi(\sigma h, f'_v) = \Phi(h, f'_v)$ for every regular h. We work below with f'_u and ϕ'_u which also have this property. We conclude that f'_w and ϕ'_w can be replaced by matching f''_w and ϕ''_w with $\Phi(\gamma, f''_w) = \Phi(\gamma, f'_w)$, such that $\Phi(h, f''_w)$ is supported on a small neighborhood of γ which is contained in the support of $\Phi(h, f'_w)$, and $\Phi(h, f'') = 0$ and $\Phi_\kappa(h, \phi'') = 0$ for every h in $H/Z(G)$ which is not conjugate to γ in G. Here f'' is f' with f'_w replaced by f''_w, and ϕ'' is ϕ' with ϕ'_w replaced by ϕ''_w. To simplify the notations we now write f'_w for f''_w and ϕ'_w for ϕ''_w.

From now on we work with two pairs of global functions, (f'', ϕ'') and (f^*, ϕ^*). The components at $v \neq u$ of f'' and f^* are equal to f'_v. The components at $v \neq u$ of ϕ'' and ϕ^* are equal to ϕ_v'. The component at u of f'' is the spherical f''_u, and ϕ'' has as a component at u the corresponding spherical ϕ''_u. The component at u of f^* is the pseudo-coefficient f^*_u of Proposition 7, and ϕ^* has the matching function ϕ^*_u as a component at u. Note that the component $f'_{u'}$ at u' is chosen to be m-regular where m depends on (the $f'_v(v \neq u, u')$ and on) both f''_u and f_u^*. The construction of these functions guarantees the following.

PROPOSITION 8. (1) *Suppose that τ and $\pi(\tau)$ are corresponding cusp forms. Then* $\operatorname{tr} \tau(f'') = \operatorname{tr}(\pi(\tau))(\phi'' \times \kappa)$ *and* $\operatorname{tr} \tau(f^*) = \operatorname{tr}(\pi(\tau))(\phi^* \times \kappa)$. (2) *If h lies in $H/Z(G)$ but is not conjugate in $G/Z(G)$ to γ then*

$$\Phi(h, f'') = \Phi(h, f^*) = \Phi_\kappa(h, \phi'') = \Phi_\kappa(h, \phi^*) = 0.$$

(3) $\Phi(\gamma, f^*) \neq 0$ *and* $\Phi_\kappa(\gamma, \phi^*) \neq 0$, *and* $\Phi(\sigma^i\gamma, f^*) = \Phi(\gamma, f^*)$ *for all* $i (0 \leqslant i < k)$. (4) *The Trace Formula holds for* f'' and f^*. *The* κ*-Trace Formula holds for* ϕ'' *and* ϕ^*.

Denote by (f, ϕ) either of (f'', ϕ'') or (f^*, ϕ^*). By (2) we have

$$\operatorname{vol}(\gamma)\Phi_\kappa(\gamma, \phi) = \sum_{\{h\}} \operatorname{vol}(h)\Phi_\kappa(h, \phi).$$

We write $\operatorname{vol}(h)$ for the volume factor which appears in the κ-Trace Formula. By the κ-Trace Formula this is equal to

$$= \sum_\pi \operatorname{tr}\pi(\phi \times \kappa) = \frac{1}{k}\sum_\tau \operatorname{tr}\tau(f).$$

The last equality follows from Theorem 2, which asserts that precisely k cuspidal τ (namely the inequivalent $\tau \circ \sigma^i (0 \leqslant i < k)$) correspond to the π which occur in the κ-Trace Formula. By the Trace Formula we obtain

$$= \frac{1}{k}\sum_{\{h\}} \operatorname{vol}(h)\Phi(h, f) = \operatorname{vol}(\gamma)\Phi(\gamma, f).$$

Here the sum ranges over the conjugacy classes in $H/Z(G)$ (which are regular and elliptic). The only summands which are not necessarily zero are the k terms indexed by the $\sigma^i(\gamma)(0 \leqslant i < k)$, by (2). By (3), we have $\Phi(\sigma^i\gamma, f) = \Phi(\gamma, f)$ for all i, whence the last equality. We deduce the

COROLLARY 9. *We have* $\Phi_\kappa(\gamma, \phi'') = \Phi(\gamma, f'')$ *and* $\Phi_\kappa(\gamma, \phi^*) = \Phi(\gamma, f^*)$.

By (3) of Proposition 8, we have that $\Phi_\kappa(\gamma, \phi^*) = \Phi(\gamma, f^*) \neq 0$. Hence,

$$\Phi_{\kappa_u}(\gamma, \phi''_u)/\Phi_{\kappa_u}(\gamma, \phi^*_u) = \Phi_\kappa(\gamma, \phi'')/\Phi_\kappa(\gamma, \phi^*) = \Phi(\gamma, f'')/\Phi(\gamma, f^*)$$

$$= \Phi(\gamma, f''_u)/\Phi(\gamma, f^*_u).$$

Proposition 7 asserts that $\Delta(\gamma)\Phi_{\kappa_u}(\gamma, \phi^*_u) = \Phi(\gamma, f_u^*)$. Since this is non-zero, we get

$$\Delta(\gamma)\Phi_{\kappa_u}(\gamma, \phi''_u) = \Phi(\gamma, f''_u).$$

This is the assertion of Conjecture C: the corresponding spherical f''_u and ϕ''_u are matching. In the proof of Proposition 7 we used Conjecture B_0. Hence, Conjecture B_0 implies Conjecture C, and Theorem 1 follows.

Alternative proof. In the above proof of Theorem 1 we used the new, powerful technique of regular functions, in various instances. The first application of regular functions is in annihilating the orbital integrals in the trace formula which are associated with singular orbits. Our usage of the very (i.e., m-) regular functions at the place u' permitted us to annihilate the orbital integrals of all non-elliptic-regular orbits in the proof of the κ-Trace Formula. In the proof of Theorem 1 regular functions were also used at the places u_i which ramify in E/F to reduce the comparison of orbital integrals and characters to the special, easier case of split elements in H, and H-modules induced from a character of the Borel subgroup. These cases were treated by Kazhdan [K1]. Conjecture B_0, in the form of Proposition 7, was used only at the last lines of the proof.

An alternative proof, which relies more on Proposition 7 and less on regular functions, can also be given, as we shall now briefly indicate. Again, we put a regular function at u' to annihilate the orbital integrals in the κ-Trace Formula (and also in the Trace Formula) associated with non-regular orbits. However, to prove the κ-Trace Formula we choose n places of F which stay prime in E and use their pairs (f_v, ϕ_v) of functions as in Proposition 7. Since the orbital integrals ($\Phi(h, f_v)$ and) $\Phi_{\kappa_v}(h, \phi_v)$ vanish on the regular non-elliptic elements h of H_v, the multiplicative properties of the weighted orbital integrals $J_{\mathcal{O}}(\phi)$ and weighted traces $J_\chi(\phi)$ of [A] imply the vanishing of all terms in the trace formula of [A] except those mentioned in our κ-Trace Formula. Moreover, using the invariant form of the trace formula, as developed by Arthur, it is clear that it suffices to use such pairs (f_v, ϕ_v) only at two – rather than n – places v of F which stay prime in E. Of course, these places can be taken to be unramified in E.

The advantage of this proof is that it eliminates the need to use the theory of base-change for $GL(n)$ in the proof of the κ-Trace Formula given above. Its disadvantage is in using the complicated computations of the weighted orbital integrals $J_{\mathcal{O}}(\phi)$ in the κ-trace formula. Of course, one can combine the two approaches and use m-regular functions to annihilate the non-elliptic-regular orbital integrals, and n pairs (f_v, ϕ_v) to annihilate the $J_\chi(\phi)$ of [A] which are not associated with cuspidal G-modules π.

The other place where Proposition 7 can be used instead of regular functions is at the ramified places u_i. The advantage of this approach is that it eliminates the need to use the results (Proposition 1 and its Corollaries of [K1], who treated the case of $r = 1$, to deduce our results for a general r. Instead this approach can be used to deduce also the results of [K1] as the special case of $r = 1$. However, this approach assumes Conjecture B_0 also at the ramified places. The approach based on regular functions given above can be used if Conjecture B_0 is proven only for the places v of F which are unramified in E. This completes our discussion of an alternative proof of Theorem 1.

IV. Appendix

Proof of Conjecture B_0 *for* $r = 1$, $k = 2$. Here E is a quadratic extension of a local field F, and κ is the (quadratic) character of $F^\times$ whose kernel is the norm subgroup $N_{E/F}E^\times$. Let κ_1 be a character of $F^\times$ with $\kappa_1^2 = \kappa$. Denote by $|.|$ the normalized (by $|\pi| = q^{-1}$) absolute value on F. Put $\mu(x) = |x|^{1/2}\kappa_1(x)$ and $\mu_P(p) = \mu(a/b)$ if $p = \left(\begin{smallmatrix} a & c \\ 0 & b\end{smallmatrix}\right)$ lies in the upper triangular subgroup P of $G = GL(2, F)$. By definition, the space of the G-module $I(\kappa_1, \kappa_1^{-1})$, unitarily induced from the character $p \mapsto \kappa_1(a/b)$ of P, consists of the functions ψ: $G \to \mathbb{C}$ such that for some open compact subgroup U_ψ of G we have $\psi(pgu) = \mu_P(p)\psi(g)$ (p in P, u in U_ψ, g in G). G acts by right translation. Consider the space $J(\kappa_1, \kappa_1^{-1})$ of locally constant functions ϕ: $F^2 \to \mathbb{C}$ with $\phi(\lambda\mathbf{v}) = \mu(\lambda)^{-2}\phi(\mathbf{v})$ (λ in $F^\times$, $\mathbf{v} = (x, y)$ in F^2) and G-action $(\tau(g)\phi)\mathbf{v}) = \mu(\det g)\phi(\mathbf{v}g)$. Put $\mathbf{v}_0 = (0, 1)$.

LEMMA 1A. *The G-modules* $I(\kappa_1, \kappa_1^{-1})$ *and* $J(\kappa_1, \kappa_1^{-1})$ *are isomorphic.*

Proof. The isomorphism is given by $\psi(g) = \mu(\det g)\phi(\mathbf{v}_0 g)$.

Let $\mathbb{P}V$ be the projective space of lines $\{\lambda\mathbf{v};\ \lambda \text{ in } F^\times\}$, $\mathbf{v} \neq 0$, through $\mathbf{0} = (0, 0)$ in $V = F^2$. Put $\|(x, y)\| = \max\{|x|, |y|\}$ and $V_0 = \{\mathbf{v} \text{ in } V;\ \|\mathbf{v}\| = 1\}$. Denote by $R^\times$ the group of units in the ring R of integers in F. Let $\mathbb{P}V_0$ be the space of lines $\{\lambda\mathbf{v};\ \mathbf{v} \text{ in } V_0,\ \lambda \text{ in } R^\times\}$ in V_0. Then $\mathbb{P}V$ is isomorphic to $\mathbb{P}V$. Denote by $d\mathbf{v}$ the Haar measure on $\mathbb{P}V_0$ which assigns the compact space $\mathbb{P}V_0$ the volume one, and also the corresponding measure on $\mathbb{P}V$. If $\mathbf{v} = (a, b)$, $\mathbf{w} = (c, d)$, put $\langle \mathbf{v}, \mathbf{w}\rangle$ for $\det\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \mathbf{v}\, w\, {}^t\mathbf{w}$, where $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$. Put

$$(\mathbb{F}\phi)(\mathbf{v}) = \int_{\mathbb{P}V} \phi(\mathbf{w})\,\mu(\langle \mathbf{v}, \mathbf{w}\rangle)^{-2}\, d\mathbf{w}.$$

Then $(\mathbb{F}\phi)(\lambda\mathbf{v}) = \mu(\lambda)^{-2}(\mathbb{F}\phi)(\mathbf{v})$. Moreover writing $\mu(g)$ for $\mu(\det g)$, $\kappa(g)$ for $\kappa(\det g)$, and $\sigma(g)$ for $w\,{}^tg^{-1}w^{-1}$, we have

$$\begin{aligned}(\mathbb{F}(\tau(g)\phi))(\mathbf{v}) &= \int \mu(g)\,\phi(\mathbf{w}g)\,\mu(\langle \mathbf{v}, \mathbf{w}\rangle)^{-2}\, d\mathbf{w} \\ &= \mu(g) \int \phi(\mathbf{w})\,\mu(\langle \mathbf{v}, \mathbf{w}g^{-1}\rangle)^{-2}\, d(\mathbf{w}g^{-1}) \\ &= (\mu(g)/|g|) \int \phi(\mathbf{w})\,\mu(\langle \mathbf{v}\sigma(g), \mathbf{w}\rangle)^{-2}\, d\mathbf{w} \\ &= \kappa(g)\mu(g^{-1})(\mathbb{F}\phi)(\mathbf{v}g^{-1}) = \kappa(g)(\tau(g)(\mathbb{F}\phi))(\mathbf{v}),\end{aligned}$$

since $\tau(\sigma(g)) = \tau(g)$ (τ is a $PGL(2, F)$-module). We conclude the following

LEMMA 2A. *The non-scalar operator* $\mathbb{F}$ *intertwines the G-modules* $J(\kappa_1, \kappa_1^{-1})$ *and* $J(\kappa_1^{-1}, \kappa_1)$.

In particular the operator $\tau(g)\mathbb{F}$ is an integral operator with kernel $K(\mathbf{v}, \mathbf{w}) = \mu(g)\mu(\langle \mathbf{v}g, \mathbf{w}\rangle)^{-2}$ on $\mathbb{P}V_0$, namely

$$(\tau(g)\mathbb{F}\phi)(\mathbf{v}) = \int_{\mathbb{P}V_0} \phi(\mathbf{w})\,\mu(g)\,\mu(\langle \mathbf{v}g, \mathbf{w}\rangle)^{-2}\,d\mathbf{w}.$$

The character $\chi(g)$ of the operator $\tau(g)\mathbb{F}$ is given by the integral over the diagonal.

LEMMA 3A. *We have* $\chi(g) = \mu(g)\int_{\mathbb{P}V_0} \mu(\langle \mathbf{v}g, \mathbf{v}\rangle)^{-2} d\mathbf{v}$.

It is clear that $\chi(h^{-1}gh) = \kappa(h)\chi(g)$. Lemma A1 then implies that $\chi(g)$ is zero unless g lies in a torus of G isomorphic to $E^\times$. Suppose that $E = F(\theta^{1/2})$, where θ lies in $F - F^2$. We may assume that $|\theta| = 1$ if E/F is unramified, and that $|\theta| = q^{-1}$ otherwise. Up to conjugation in G, we have $g = \left(\begin{smallmatrix} a & b\theta \\ b & a\end{smallmatrix}\right)$. If $\mathbf{v} = (x, y)$, then $\langle \mathbf{v}g, \mathbf{v}\rangle = \det\left(\begin{smallmatrix} \mathbf{v}g \\ \mathbf{v}\end{smallmatrix}\right) = b(y^2 - x^2\theta)$. Hence,

$$\chi(g) = |a^2 - b^2\theta|^{1/2}\kappa_1(a^2 - b^2\theta)\kappa(b)|b|^{-1}\int_{\mathbb{P}V_0} |y^2 - x^2\theta|^{-1}\kappa(y^2 - x^2\theta)\,dv.$$

Fix $g_0 = \left(\begin{smallmatrix} 0 & \theta \\ 1 & 0\end{smallmatrix}\right)$. Then

$$\Delta(g) = \left|\frac{4\theta b^2}{a^2 - b^2\theta}\right|^{1/2}\kappa(b),$$

as defined prior to Conjecture A. Of course, $\kappa(y^2 - x^2\theta) = 1$, and $|y^2 - x^2\theta| = 1$ if $|\theta| = 1$. Note that the G-module π_0 which occurs in Conjecture B_1 is the G-module $\kappa_1^{-1} \otimes J(\kappa_1, \kappa_1^{-1})$. If χ_0 denotes the character of π_0, then $\chi_0(g) = \kappa_1(g)^{-1}\chi(g)$. We conclude the following

LEMMA 4A. *If* E/F *is unramified then* $\Delta(g)\chi_0(g) = 1$.

The same conclusion, up to a scalar, holds in the ramified case.

It follows that the trivial character of $H = GL(1, E) = E^\times$ corresponds to the G-module $I(1, \kappa) \simeq I(\kappa, 1)$. This is Conjecture B_1 for $r = 1$ and $k = 2$, hence, also Conjecture B_0 for this case, since the Steinberg $GL(1, E)$-module is the trivial character of $E^\times$.

References

[A] J. Arthur: On a family of distributions obtained from Eisenstein series. II: explicit formulas, *Amer. J. Math.* 104 (1982) 1289–1336.

[AC] J. Arthur and L. Clozel: *Base-change for GL(n)*, *Ann. of Math. Studies* (1988).

[Av] V. Averbuch: Remark on the definition of an automorphic form, *Comp. Math.* 59 (1986) 3–13.

[BD] J. Bernstein, rédigé par P. Deligne: Le "centre" de Bernstein, dans *Représentations des groupes réductifs sur un corps local*, Paris, Hermann (1984).

[BDK] J. Bernstein, P. Deligne and D. Kazhdan: Trace Paley-Wiener theorem for reductive p-adic groups, *J. Analyse Math.* 47 (1986) 180–192.

[BZ] J. Bernstein and A. Zelevinsky: Induced representations of reductive p-adic groups I, *Ann. Sci. ENS* 10 (1977) 441–472.

[C] W. Casselman: Characters and Jacquet modules, *Math. Ann.* 230 (1977) 101–105.

[Cl] L. Clozel: Characters of non-connected, reductive p-adic groups, *Canad. J. Math.* 39 (1987) 149–167.

[F1] Y. Flicker: On twisted lifting, *Trans. AMS* 290 (1985) 161–178.

[F2] Y. Flicker: Rigidity for automorphic forms, *J. Analyse Math.* 49 (1987) 135–202.

[F3a] Y. Flicker: Regular trace formula and base change lifting, *Amer. J. Math.* 110 (1988).

[F3b] Y. Flicker: *Regular trace formula and base-change for $GL(n)$*; preprint, Harvard 1987.

[F4] Y. Flicker: Unitary quasi-lifting: preparations, *Contemporary Math.* 53 (1986) 119–139; Unitary quasi-lifting: applications, *Trans. AMS* 294 (1986) 553–565; Packets and liftings for $U(3)$, *J. Analyse Math.* 50 (1988); Base change trace identity for $U(3)$, preprint, MSRI (1986).

[F5] Y. Flicker: On the symmetric-square: Total global comparison; preprint, Harvard, 1986.

[FK1] Y. Flicker and D. Kazhdan: Metaplectic correspondence, *Publ. Math. IHES* 64 (1987) 53–110.

[FK2] Y. Flicker and D. Kazhdan: On the symmetric-square: Unstable local transfer, *Invent. Math.* (1988).

[H] Harish-Chandra: Admissible invariant distributions on reductive p-adic groups, *Queen's papers in Pure and Applied Math.* 48 (1978) 281–346.

[JS] H. Jacquet and J. Shalika: On Euler products and the classification of automorphic forms. II, *Amer. J. Math.* 103 (1981) 777–815.

[K1] D. Kazhdan: On lifting, in *Lie Group Representations*. II, SLN 1041 (1984) 209–249.

[K2] D. Kazhdan: Cuspidal geometry of p-adic groups, *J. Analyse Math.* 47 (1986) 1–36.

[K3] D. Kazhdan: Representations of groups over close local fields, *J. Analyse Math.* 47 (1986) 175–179.

[Z] A. Zelevinski: Induced representations of reductive p-adic groups II. *Ann. Sci. ENS* 13 (1980) 165–210.