10 Geometric Ramanujan Conjecture and Drinfeld Reciprocity Law*

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The purpose of this article is to describe and explain some of our recent work, which concerns, in particular, the following themes:

- The Ramanujan or purity conjecture for cuspidal automorphic forms π with a super cuspidal component of $\mathrm{GL}(r)$ over a global field F of characteristic $p > 0$
- The reciprocity law relating the above π with irreducible continuous r-dimensional $\ell(\neq p)$-adic representations ρ of the galois group $\mathrm{Gal}(\bar{F}/F)$ whose restriction to some decomposition group $\mathrm{Gal}(\bar{F}_v/F_v)$ is irreducible
- Drinfeld's explicit reciprocity law, which realizes the above conjectured correspondence $\pi \leftrightarrow \rho$ as the irreducible factors $\pi \otimes \rho$ in the composition series as a $G(\mathbb{A}) \times \mathrm{Gal}(\bar{F}/F)$-module of ℓ-adic cohomology with compact support and coefficients in a smooth sheaf of the geometric generic fiber of the Drinfeld Moduli scheme

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- Deligne's conjecture on the Lefschetz fixed point formula in étale topology for a finite flat correspondence, multiplied by a sufficiently high power of the Frobenius, on a separated scheme of finite type over a finite field
- Higher congruence relations

Our proof of the purity conjecture and our deduction of the Drinfeld explicit reciprocity law from Deligne's conjecture are based, in particular, on a new form of the Selberg trace formula for a test function with at least one supercusp component. This new trace formula is a representation theoretic analogue of Deligne's conjecture; some of its applications to lifting problems are studied in [FK], [Fi] ($i = 2, 3, 4$). It is our pleasure to dedicate this paper to Atle Selberg in appreciation of his work in general and his trace formula in particular.

Let F be a geometric global field, namely the field of rational functions on a smooth projective absolutely irreducible curve over a finite field of characteristic p. Let $\mathbb{A}$ be the ring of F-adèles, $G = \mathrm{GL}(r)$, and π an irreducible admissible $G(\mathbb{A})$-module over the field of complex numbers. Then π is the restricted direct product $\otimes_v \pi_v$ over all places v of F of irreducible admissible $G_v = G(F_v)$-modules π_v. Let R_v be the ring of integers in the completion F_v of F at v, and put $K_v = G(R_v)$. For almost all v the component π_v is unramified, namely, has a K_v-fixed nonzero vector, and consequently there are nonzero complex numbers $z_{1,v}, \dots, z_{r,v}$, uniquely determined up to permutation by π_v, with the following property: π_v is the unique irreducible unramified constituent $\pi((z_{i,v}))$ of the unramified G_v-module $I(\mathbf{z}_v) = \mathrm{Ind}(\delta^{1/2}\mathbf{z}_v; B_v, G_v)$, which is normalizedly induced from the unramified character $\mathbf{z}_v\colon (b_{ij}) \to \prod_i z_{i,v}^{\mathrm{val}_v(b_{ii})}$ of the upper triangular subgroup B_v of G_v; val_v is the order valuation of F_v, normalized by $\mathrm{val}_v(\boldsymbol{\pi}_v) = 1$ where $\boldsymbol{\pi}_v$ is any generator of the maximal ideal in R_v.

Definition. The elements $z_{1,v}, \dots, z_{r,v}$ are called the *Hecke eigenvalues* of π at v.

In this work we are concerned with cuspidal $G(\mathbb{A})$-modules π. The space $L_0(G, C)$ of complex-valued cusp forms on $G(\mathbb{A})$ consists of all functions ϕ on $G(F)\backslash G(\mathbb{A})$ which are compactly supported modulo the center $Z(\mathbb{A})$ of $G(\mathbb{A})$ and transform under $Z(\mathbb{A})$ according to a unitary character, with the property that for every proper F-parabolic subgroup P of G, whose unipotent radical is denoted by N, we have

$\int_{N(F)\backslash N(A)} \phi(nx)\,dn = 0$ for all x in $G(\mathbf{A})$. A *cuspidal* $G(\mathbf{A})$-module is an admissible irreducible $G(\mathbf{A})$-module π which occurs as a direct summand in the representation of $G(\mathbf{A})$ on $L_0(G, C)$ by right translation. These cuspidal π are representation theoretic analogues, for function fields, of the (spaces spanned by the translates of) holomorphic cusp forms on the upper-half complex plane.

Our proofs involve ℓ-adic techniques. We can obtain results concerning modules over the complex numbers, due to a rationality property of the cuspidal π, analogous to that which holds for holomorphic cusp forms. Let $\overline{Q}$ be the field of algebraic numbers.

Proposition. *Let π be a cuspidal $G(\mathbf{A})$-module with algebraically valued central character $\omega\colon Z(\mathbf{A})/Z(F) = \mathbf{A}^\times/F^\times \to \overline{Q}^\times$. Then* (1) *$\pi$ can be realized in the space $L_0(G, \overline{Q})$ of $\overline{Q}$-valued cusp forms, and* (2) *all Hecke eigenvalues $z_{i,v}$ of π are algebraic.*

In fact we obtain more precise results. Namely, for every cuspidal π with central character of finite order, the galois closure $Q(\pi)$ of the field generated by all Hecke eigenvalues of π at the unramified places is a finite extension of Q. We call $Q(\pi)$ the *field of definition* of π. Moreover, this π can be realized in the space $L_0(G, Q(\pi))$ of $Q(\pi)$-valued cusp forms on G.

Let $L^2(G, \mathbf{C})$ be the space of complex valued functions ϕ on $G(F)\backslash G(\mathbf{A})$ which transform under $Z(\mathbf{A})$ by a unitary character, such that $|\phi|^2$ is integrable on $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$. An admissible irreducible $G(\mathbf{A})$-module π is called *discrete-series* if it occurs as a direct summand of the representation of $G(\mathbf{A})$ on $L^2(G, \mathbf{C})$ by right translation. Every cuspidal $G(\mathbf{A})$-module is discrete series. Let q_v be the cardinality of the residue field $R_v/(\pi_v)$, and $|\,.\,|_{\mathbf{C}}$ is the absolute value on $\mathbf{C}$, normalized by $|a|_{\mathbf{C}} = a$ for every positive real number a. Our first theme is the following

Integrality Conjecture. *The absolute value of each Hecke eigenvalue of any unramified component of a discrete series $G(\mathbf{A})$-module π is equal to an integral power $q_v^{i/2}$ of $q_v^{1/2}$ with $|i| < n$.*

This conjecture has an obvious extension to all components of π in terms of their central exponents. It can be made also for number fields, and other reductive groups.

Recall that an irreducible G_v-module π_v is *supercuspidal* if it has a nonzero coefficient ϕ_v which satisfies $\int_{N_v} \phi_v(xny)\,dn = 0$ for all x, y in

G_v, and every proper parabolic subgroup P_v of G_v, whose unipotent radical is denoted by N_v. Let ∞ be a place of F. We prove the Integrality Conjecture for almost all components of any cuspidal π which has a supercuspidal component π_∞. Now each component of a cuspidal π is unitary and nondegenerate. A well-known estimate asserts that if π_v is an irreducible nondegenerate unramified unitary G_v-module over $\mathbf{C}$, then $q_v^{-1/2} < |z_{i,v}|_{\mathbf{C}} < q_v^{1/2}$ for every Hecke eigenvalue $z_{i,v}$ of π_v. As noted by Laumon, combining these two estimates we obtain the following

Purity Theorem. *Let π be a cuspidal $G(\mathbf{A})$-module with a supercuspidal component π_∞. Then each conjugate of each Hecke eigenvalue $z_i(\pi_v)$, for almost all unramified components π_v of π, lies on the unit circle in* $\mathbf{C}$.

The proof of this result is algebro-geometric. It is patterned along lines suggested by the work of Langlands [Ls] and Drinfeld [D2]. It relies on a comparison of the Grothendieck form [G] of the Lefschetz fixed-point formula for powers of the Frobenius, with a new form of the Selberg trace formula. The Integrality Conjecture is analogous to Deligne's theorem [De 2] on the eigenvalues of the action of the Frobenius on ℓ-adic cohomology groups, which plays a key role in our proof. The Purity Theorem is a representation theoretic analogue for $\mathrm{GL}(r)$ over a function field of Ramanujan's well-known conjecture concerning the Hecke eigenvalues (or rather Fourier coefficients) of the cusp form $\Delta(z) = e^{2\pi i z}\prod_1^\infty(1 - e^{2\pi i z n})$ of weight 12 on the upper half-plane $\mathrm{Im}(z) > 0$ for the group $\mathrm{SL}(2, \mathbf{Z})$. Our methods are likely to extend and prove the Integrality Conjecture for any discrete-series π with an elliptic component π_∞ on using a stronger form of the trace formula, but this has not been done as yet.

The Purity Theorem is one of our two main absolute results. The other concerns a higher rank analogue of the Eichler-Shimura and Ihara congruence relations, and not the trace formula. These are relations between certain Frobenius eigenvalues and the Hecke eigenvalues. These relations imply strong purity results for these Frobenius eigenvalues. We delay stating this result until the proof of the Purity Theorem is explained, as it concerns objects which we anyway introduce below for the proof of the Purity Theorem. We next explain our main relative result, which concerns the reduction of the reciprocity law, and Drinfeld's explicit form of it, to Deligne's conjecture. The proof is similar to that of the Purity Theorem and relies on the trace formula.

The reciprocity law concerns continuous r-dimensional irreducible ℓ-adic ($\ell \neq p$) representations ρ: $W(\bar{F}/F) \to \mathrm{GL}(r, \bar{Q}_\ell)$ of the Weil group of F. We say that ρ is *constructible* if it is unramified for almost all v. Here $\bar{F}$ denotes a separable closure of F. Note that it is not yet known whether every continuous irreducible finite-dimensional representation of $W(\bar{F}/F)$ is necessarily constructible. From now on we deal only with constructible ρ. For constructible ρ, for almost all v the restriction ρ_v of ρ to the decomposition subgroup $W(\bar{F}_v/F_v)$ at v factorizes through the Weil group $W(\bar{\mathbf{F}}_v/\mathbf{F}_v) \simeq \mathbf{Z}$ of the field $\mathbf{F}_v$ of constants of F_v, and the isomorphism class of ρ_v is determined uniquely by the unordered r-tuple $\{u_{i,v} = u_i(\rho_v); 1 \leq i \leq r\}$ of eigenvalues of the Frobenius automorphism $\rho_v(\mathrm{Fr}_v)$. Here Fr_v: $x \to x^{q_v}$, where $q_v = |\mathbf{F}_v|$, is a generator of the subgroup $W(\bar{F}_v/F_v)$ of $\mathrm{Gal}(\bar{\mathbf{F}}_v/\mathbf{F}_v) \simeq \hat{\mathbf{Z}} = \lim \mathbf{Z}/n\mathbf{Z}$. It is useful to note that in our case of function fields we have the following

Proposition. *Let ρ be a continuous irreducible finite dimensional ℓ-adic representation of $W(\bar{F}/F)$ whose determinant* $\det \rho$ *has finite order. Then ρ extends by continuity to a representation of the galois group* $\mathrm{Gal}(\bar{F}/F)$.

Definition. We say that a constructible ρ and the $G(\mathbf{A})$-module $\pi = \otimes \pi_v$ *correspond* if for almost all v the unordered r-tuples $\{u_i(\rho_v)\}$ and $\{z_i(\pi_v)\}$ are equal.

We state below Deligne's conjecture and indicate by a superscript * any statement which depends on it. The first is the following higher

Reciprocity Law*. *Fix a place ∞ of F and a rational prime $\ell \neq p$. The correspondence defines a bijection between the sets of equivalence classes of* (1) *cuspidal G-modules π whose component π_∞ at ∞ is supercuspidal, and* (2) *irreducible r-dimensional continuous ℓ-adic constructible representations ρ of $W(\bar{F}/F)$ whose restriction ρ_∞ to $W(\bar{F}_\infty/F_\infty)$ is irreducible. The determinant* $\det \rho$ *of ρ corresponds by class-field theory to the central character of π. In particular, the π of* (1) *whose central characters are of finite order correspond to the ρ of* (2) *whose determinants* $\det \rho$ *are of finite order.*

In particular, for each constructible ρ whose determinant has finite order there is a finite galois extension $Q(\rho)$ of Q which contains the Frobenius eigenvalues $u_i(\rho_v)(1 \leq i \leq r)$ for almost all v. Moreover, each constructible ρ belongs to a compatible system of ℓ-adic representations, in the following sense. For each prime $\ell' \neq \ell$, p there exists a continuous irreducible constructible r-dimensional ℓ'-adic representation ρ' of $W(\bar{F}/F)$ which is irreducible at ∞ such that the unordered r-tuples $\{u_i(\rho_v)\}$ and $\{u_i(\rho'_v)\}$ are equal for all v where ρ_v and ρ'_v are unramified.

When $r = 1$ the Reciprocity Law above reduces to global class-field theory for function fields. For $r = 2$ this is a theorem of Drinfeld [D1], [D2], and our work is merely a higher rank extension of Drinfeld's amazingly original work.

It is clear that at most one cuspidal π can correspond to a given ρ, by virtue of the rigidity theorem for cusp forms of GL(r), and at most one ℓ-adic ρ can correspond to a given π. Indeed, by the Chebotarev Density Theorem if K is a galois extension of F which is unramified outside a finite set, then the Frobenius elements of the unramified places of K are dense in Gal(K/F). The difficulty is in proving the *existence* of a π corresponding to ρ, and ρ to π.

The first existence assertion (given π there is ρ) is reduced to Deligne's conjecture using the trace formula by the same proof which establishes the Purity Theorem. It is the following

Existence Theorem 1*. *For every cuspidal G-module π whose component π_∞ is supercuspidal and its central character is of finite order, and for any rational prime $\ell \neq p$, there exists a corresponding irreducible r-dimensional ℓ-adic continuous representation ρ of* Gal($\bar{F}/F$) (*which is necessarily constructible*).

Before explaining Deligne's conjecture and the proof* of the Existence Theorem, we proceed to discuss the reduction of the Reciprocity Law* to the Existence Theorem. This reduction is done by induction on r, using the theory of L-functions. Let $\rho\colon \mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}_{\bar{Q}_\ell} V$ be an r-dimensional constructible ℓ-adic representation, and $L(t, \rho) = \prod_v L(t, \rho_v)$ the Euler product attached to ρ by Grothendieck. Here

$$L(t, \rho_v) = \det[(1 - \rho(\mathrm{Fr}_v) t^{\log_p q_v}) | V]^{-1} \in Q_\ell[[t]],$$

for almost all v. Then $L(t, \rho)$ is a rational function in t which satisfies the functional equation

$$L(p/t, \check{\rho}) = \varepsilon(t, \rho) L(t, \rho),$$

where $\check{\rho}$ is the dual of ρ, and $\varepsilon(t, \rho)$ is a monomial in t depending on ρ. We use a result of Laumon [L], conjectured by Deligne, that when ρ is constructible and has virtual degree zero, then $\varepsilon(t, \rho)$ is equal to the product $\Pi_v\ \varepsilon(t,\ \rho_v,\ \psi_v)$ over all v of the local constants $\varepsilon(t, \rho_v, \psi_v)$ of [Del].

Using this product formula for $\varepsilon(t, \rho)$ we apply a variant of the Piatetski–Shapiro converse theorem [PS]; assuming the validity of the Reciprocity Law by induction for $r-1$, we prove* the following converse direction of the Existence Theorem 1* (given ρ there is π):

Existence Theorem 2*. *Given an irreducible continuous ℓ-adic r-dimensional constructible representation ρ of $W(\bar{F}/F)$, there exists an automorphic G-module π which corresponds to ρ.*

The variant of the Converse Theorem which we use is the following. Put $G'' = \mathrm{GL}(r-1)$.

Converse Theorem. *Fix a place ∞ of F and a supercuspidal G''_∞-module π^0_∞. Suppose that $\pi = \otimes\pi_v$ is an admissible nondegenerate (i.e., a constituent of the space of Whittaker functions) $G(\mathbf{A})$-module such that the standard Euler product $L(s, \pi, \tau)$ attached to the pair (π, τ) is entire (namely, it is a polynomial in p^s and p^{-s}) and satisfies the usual functional equation for all cuspidal G''-modules τ whose component τ_∞ at ∞ is the fixed supercuspidal τ^0_∞. Then there is a constituent of τ which is automorphic.*

The proof of this form of the Converse Theorem, which assumes the functional equation of $L(s, \pi, \tau)$ only for τ with the component τ^0_∞, is similar to that of Piatetski–Shapiro [PS]. We need this form since our induction assumption can be made only for the τ provided by the Existence Theorem 1*, namely, for those with a supercuspidal component τ^0_∞.

Arguing along lines suggested by [Del], we use the Existence Theorems*, the functional equations of the L-functions, and properties of variation of local L and ε factors under twists by highly ramified characters to establish* the existence of a local correspondence $\pi_v \leftrightarrow \rho_v$ with the property that the global π and ρ correspond if and only if π_v and ρ_v correspond for all places v of F. More precisely, we prove the

Local Reciprocity Law*. *For every local field F_v of positive characteristic and $r \geq 1$ there is a unique bijection $\pi_v \leftrightarrow \rho_v$ between the sets of* *equivalence classes of* (1) *irreducible supercuspidal G_v-modules π_v, and* (2) *continuous ℓ-adic r-dimensional irreducible representations ρ_v of $W(\bar{F}_v/F_v)$, which reduces to local class-field theory for $r = 1$, with the following properties*: (*A*) *If π_v corresponds to ρ_v then* (1) $\pi_v \otimes \chi_v$ *corresponds to $\rho_v \otimes \chi_v$ for every character χ_v of $F_v^\times \simeq \overline{W}(F_v/F_v)^{ab}$*; (2) *the central character of π_v corresponds to* det ρ_v *by local class-field theory*; (3) *the contragredient of π_v corresponds to the contragredient of ρ_v*. (*B*) *If the* GL(n, F_v)*-module π_n corresponds to ρ_n, and the* GL(m, F_v)*-module π_m corresponds to ρ_m, then*

$$L(s; \pi_m, \pi_n) = L(p^{-s}; \rho_m \otimes \rho_n), \qquad \varepsilon(s; \pi_m, \pi_n) = \varepsilon(p^{-s}; \rho_m \otimes \rho_n).$$

where $L(s; \pi_m, \pi_n)$ is the L-function of [JPS].

By [GK] or the converse theorem, the supercuspidal π_n is uniquely determined by the family $L(s; \pi_n, \pi_{n-1})$ for all π_{n-1}. By virtue of the work of Bernstein and Zelevinsky [Z], there is a unique natural extension of this correspondence to relate the sets of equivalence classes of all (1) irreducible G_v-modules π_v, and (2) continuous ℓ-adic r-dimensional representations ρ_v of $W(\bar{F}_v/F_v)$, which satisfies (A), commutes with induction, and bijects square integrable π_v with indecomposable ρ_v. Moreover, π corresponds to ρ if and only if π_v corresponds to ρ_v for almost all v.

The Local Reciprocity Law* is stated for every $\ell \neq p$, and for every local field F_v of characteristic p. However, it is possible that the translation principle of [K2] and [De3] can be used to deduce the validity of the Local Reciprocity Law* also for F_v of characteristic zero and residual characteristic p, for any rational prime $\ell \neq p$ (cf. Henniart (in preparation)).

Our study of the Purity Theorem and the Existence Theorem 1* is based on Drinfeld's ([D1], [D2]) explicit construction of a Gal($\bar{F}/F$) × $G(\mathbb{A}_f)$-module H. Here $\mathbb{A}_f$ is the ring of finite adèles, namely the adèles without component at ∞. Drinfeld's Explicit Reciprocity Law would conjecture that the irreducible constituents $\tilde{\rho} \times \tilde{\pi}_f$ of H realize the Reciprocity Law*. In [D1], [D2] Drinfeld introduces the notion of elliptic modules and their level structures, and constructs a moduli scheme M of isomorphism classes of such pairs (of rank r). When $r = 2$ the moduli scheme is a curve, and Drinfeld proves the Purity Theorem and Reciprocity Law, in particular, for cuspidal GL(2)-modules π with a supercuspidal component π_∞ on studying $H = H_c^1(\bar{M})$. Here $H_c^1(\bar{M})$ is the first ℓ-adic cohomology group with compact support and coefficients in a smooth sheaf defined by π_∞, of the geometric generic fiber $\overline{M} = M \times_A \overline{F}$ of the moduli curve M. In the higher dimensional case we work instead with the virtual representation $H^+ - H^-$, where $H^+ = \oplus_i H_c^i(\overline{M})(r-1-i$ is even) and $H^- = \oplus_i H_c^i(\overline{M})(r-1-i$ is odd), with coefficients in a sheaf determined by the supercuspidal π_∞. We conjecture (see below) that $H_c^i(M) = 0$ for $i \neq r-1$. Had this been proven we would take $H = H_c^{r-1}(\overline{M})$.

We use below the following definitions and notations. Let A be the ring of functions in F which are regular outside ∞, namely, the elements of F which are integral outside ∞. Let I be a nonzero ideal in A. Let $\hat{A}$ be the profinite completion of A, and $U_I = \mathrm{GL}(r, \hat{A}) \cap [1 + M(r, I\hat{A})]$ the congruence subgroup mod I of $\mathrm{GL}(r, \hat{A})$. Fix the Haar measure on $G(\mathbf{A}_f)$ which assigns U_I the volume one. Let $\mathbf{H}_I$ be the convolution algebra of compactly supported U_I-biinvariant functions on $G(\mathbf{A}_f)$. It is naturally isomorphic to the algebra (under product in $G(\mathbf{A}_f)$) spanned by the double cosets $U_I g U_I$, g in $G(\mathbf{A}_f)$. If π_f is an admissible $G(\mathbf{A}_f)$-module, let π_f^I denote the space of U_I-fixed vectors in π_f. Then π_f^I is an $\mathbf{H}_I$-module, and $\pi_f \to \pi_f^I$ is a bijection from the set of equivalence class of irreducible $G(\mathbf{A}_f)$-modules generated by their U_I-fixed vectors to the set of equivalence classes of irreducible $\mathbf{H}_I$-modules.

In [D1] it is shown that there exists an affine scheme $M_{r,I}$ of finite type over A, parametrizing the set of isomorphism classes of elliptic modules of rank r with structure of level I. It is affine, smooth but not proper. The adèle group $G(\mathbf{A}_f)$ acts on the scheme $M_r = \varprojlim M_{r,I}$. We have that $U_I \backslash M_r$ is equal to $M_{r,I}$.

The scheme which plays a key role in the work is a covering scheme of $M_{r,I}$, introduced in [D2]. Let D_∞ be a division algebra of dimension r^2 central over F_∞, with invariant $1/r$. Let π be a uniformizer in F_∞. Let U_∞ be a congruence subgroup in the multiplicative group $D_\infty^\times$. Then U_∞ is normal, compact, and open in $D_\infty^\times$, and has finite index in $D_\infty^\times/\langle\pi\rangle$. There exists a finite étale galois covering $\tilde{M}_{r,U_\infty}$ of M_r, and $\tilde{M}_{r,I,U_\infty}$ of $M_{r,I}$, with galois group $D_\infty^\times/U_\infty\langle\pi\rangle$, and the direct product $[(U_\infty \backslash D_\infty^\times) \times \mathrm{GL}(r, \mathbf{A}_f)]/F^\times$ acts on $\tilde{M}_{r,U_\infty}$.

Let $\bar{\rho}$ be an irreducible representation of $D_\infty^\times/U_\infty\langle\pi\rangle$, and $\ell \neq p$ a rational prime. Then there is a smooth $\overline{Q}_\ell$-adic sheaf $\mathbf{L} = \mathbf{L}(\bar{\rho})$ on $X = M_r$ associated with $\bar{\rho}$, and one defines the ℓ-adic cohomology groups $H_c^i(\overline{X}, \mathbf{L})$ of the geometric generic fiber $\overline{X} = X \times_A \overline{F}$ with compact support and coefficients in the $\overline{Q}_\ell$-sheaf $\mathbf{L}(\bar{\rho})$. Given a finite correspondence $(f,\ h: X' \to X)$ with flat f and a sheaf morphism

$\alpha: h^*\mathbf{L} \to f^!\mathbf{L}$, one defines an endomorphism $H_c^i(f, \alpha, h)$ of the $\overline{Q}_\ell$-module $H_c^i(X, \mathbf{L})$ as the composition of the following natural maps:

$$H_c^i(\overline{X}, \mathbf{L}) \xrightarrow{h^*} H_c^i(\overline{X}', h^*\mathbf{L}) \xrightarrow{\alpha} H_c^i(\overline{X}', f^!\mathbf{L}) = H_c^i(\overline{X}, f_* f^!\mathbf{L}) \xrightarrow{t} H_c^i(\overline{X}, \mathbf{L}).$$

Here $t: f_* f^!\mathbf{L} \to \mathbf{L}$ is the morphism adjoint to the identity morphism $f^!\mathbf{L} \to f^!\mathbf{L}$. In our case, for any double coset $U_I g U_I$ in $U_I \backslash G(\mathbf{A}_f)/U_I$ the action of $G(\mathbf{A}_f)$ on M_r defines a finite flat correspondence

$$(f_g, h_g: X_g \to X)$$

on X such that the natural map $j: f_g^*\mathbf{L}(\bar{\rho}) \to f_g^!\mathbf{L}(\bar{\rho})$ is an isomorphism, and a sheaf morphism $\alpha = \alpha(g): h_g^*\mathbf{L}(\rho) \to f_g^!\mathbf{L}(\rho)$. Then $g \mapsto H_c^i(g) = H_c^i(f_g, \alpha(g), h_g)$ defines an action of the algebra $\mathbf{H}_I$ on $H_c^i(\overline{X}, \mathbf{L}(\bar{\rho}))$. Taking the direct limit over I one obtains an action of $G(\mathbf{A}_f)$ on $H_c^i(M_r \times_A \overline{F}, \mathbf{L}(\bar{\rho}))$.

Now the galois group $\mathrm{Gal}(\overline{F}/F)$ acts on $H^i = H_c^i(\overline{X}, \mathbf{L}(\bar{\rho}))$; so does the Hecke algebra $\mathbf{H}_I$. Denote the irreducible composition factors of $H = \sum_i (-1)^i H^i (0 \le i \le 2(r-1))$ as a virtual $\mathbf{H}_I \times \mathrm{Gal}(\overline{F}/F)$-module by $\tilde{\pi}_f \times \tilde{\rho}$. Let $\pi_\infty = \pi_\infty(\bar{\rho})$ be the square-integrable G_∞-module which corresponds to the $D_\infty^\times$-module $\bar{\rho}$ (thus π_∞ and $\bar{\rho}$ satisfy the character relation $\chi_{\pi_\infty}(\gamma) = (-1)^{r-1}\chi_{\bar{\rho}}(\gamma')$ for all elliptic regular γ in G_∞ and γ' in $D_\infty^\times$ with equal characteristic polynomials). Suppose that π_∞ is supercuspidal and its central character has finite order. Then we prove* the following

Drinfeld Reciprocity Law*. (1) *For every constituent $\tilde{\pi}_f \times \tilde{\rho}$ of H there is a cuspidal G-module $\pi_f \otimes \pi_\infty$ such that $\tilde{\pi}_f \simeq \pi_f^I$ as $\mathbf{H}_I$-modules.* (2) *For every cuspidal G-module $\pi_f \otimes \pi_\infty$ with $\pi_v^I \ne 0$ there is a unique factor $\tilde{\pi}_f \otimes \tilde{\rho}$ in H such that $\pi_f^I \simeq \tilde{\pi}_f$.* (3) *The dimension of every irreducible $\tilde{\rho}$ in H is r. The restriction $\tilde{\rho}_\infty$ of $\tilde{\rho}$ is irreducible and corresponds to π_∞ by the Local Reciprocity Law.* (4) *For every constructible irreducible ℓ-adic representation ρ of dimension r such that ρ_∞ is irreducible, there exists $I \ne 0$ and a supercuspidal π_∞, such that $\rho \simeq \tilde{\rho}$ for some constituent $\tilde{\pi}_f \times \tilde{\rho}$ of H.* (5) *If $\tilde{\pi}_f \times \tilde{\rho}$ is a constituent of H then $\tilde{\rho}$ and $(\tilde{\pi}_f \times \pi_\infty)\nu^{-(r-1)/2}$ correspond (also in the strong sense of all places), where ν is the volume character on $\mathbf{A}^\times/F^\times$.* (6) *The multiplicity of $\tilde{\pi}_f \otimes \pi_\infty$ in H is one.* (7) *H is the direct sum of the $\tilde{\pi}_f \times \tilde{\rho}$ if $H^i = 0$ for all i such that $r - 1 - i$ is odd.*

This we reduce to Deligne's conjecture. We conjecture that

$$H_c^i(\overline{X}, \mathbf{L}(\bar{\rho})) = H^i(\overline{X}, \mathbf{L}(\bar{\rho}))$$

for all i, and, in particular, $H^i = 0$ for $i \neq r-1$. Using congruence relations we derive below some evidence for the conjecture that H^i vanishes for $i \neq r-1$. In our case of positive characteristic the $\tilde{\pi}_f$ are *a priori* only $G(\mathbf{A}_f)$-modules; the $\tilde{\pi}_f \otimes \pi_\infty$ are shown* to be cuspidal, and the sum is shown* to be direct, using a comparison of Deligne's conjecture and the Selberg Formula. This is in contrast with the analogous theory in characteristic zero, where one uses the de Rham cohomology to have the *a priori* statement that the sum is direct and the $\tilde{\pi}_f$ that occur in H are automorphic. The Explicit Reciprocity Law is likely to follow* from our methods also when π_∞ is square-integrable but not supercuspidal. However, then the $\tilde{\pi}_f$ are conjecturally (parts of) discrete series automorphic representations which are not necessarily cuspidal (for example, $\tilde{\pi}_f$ may be one-dimensional if π_∞ is the Steinberg representation, namely, $\dim \bar{\rho} = 1$). As the reduction of this statement to Deligne's conjecture requires a stronger form of the Trace Formula then we use, we do not discuss this here. Of course, in this case we do not expect H^i to vanish for $i \neq r-1$.

The proof of the Purity Theorem and the Existence Theorem 1* depends on certain fixed-point formulae. These formulae apply to schemes over finite fields and in our case to the special fiber $X_v = X \times_A \mathbf{F}_v$ of $X = M_{r,I}$ at the place v of F; here $\mathbf{F}_v$ is the residue field A/v, and $\overline{\mathbf{F}}_v (= \overline{F}/\bar{v}$, where $\bar{v}$ is an extension of v to $\overline{F})$ is its algebraic closure. Put $\overline{X}_v = X_v \times_{F_v} \overline{\mathbf{F}}_v$. By constructibility of our smooth sheaf $\mathbf{L} = \mathbf{L}(\bar{\rho})$ on X, for almost all primes v of A not in I we have that $H^i = H^i_c(\overline{X}, \mathbf{L}(\bar{\rho}))$ is isomorphic to $H^i_v = H^i_c(\overline{X}_v, \mathbf{L}(\bar{\rho}))$ as $\mathbf{H}_I$-modules. Moreover, H^i and H^i_v are isomorphic as $\mathbf{H}_I \times \mathrm{Gal}(\overline{F}_v/F_v)$-modules, for almost all v.

The fixed point formula used in the proof of the Purity Theorem is the following. Let X be a separated scheme of finite type over the finite field $\mathbf{F}_q$ of $q = p^d$ elements. Let $\mathbf{L}$ be a smooth $\overline{Q}_\ell$-adic sheaf. Put $\overline{X} = X \times_{\mathbf{F}_q} \overline{\mathbf{F}}_q$. The geometric $\mathrm{Fr}_q \times 1$ and arithmetic $1 \times \mathrm{Fr}_q$ Frobenii act on X, and on the sheaf $\mathbf{L}$, and their product as an endomorphism of $H^i_c(\overline{X}, \mathbf{L})$ acts trivially. Let m be an integer. At each point x in the set $X(\mathbf{F}_{q^{|m|}})$, $(\mathrm{Fr}_q \times 1)^m$ fixes x, and it acts on the stalk $\mathbf{L}_x \simeq \overline{Q}_\ell^t$. Put $\mathrm{tr}((\mathrm{Fr}_q \times 1)^m | \mathbf{L}_x)$ for the trace of $(\mathrm{Fr}_q \times 1)^m$ on the stalk $\mathbf{L}_x$.

Grothendieck Fixed Point Formula. *For every separated scheme X of finite type over $\mathbf{F}_q$, and an $\overline{Q}_\ell$-adic sheaf $\mathbf{L}$ on X, for every $m \neq 0$ we have*

$$\sum_{x \in X(\mathbf{F}_{q^{|m|}})} \mathrm{tr}((\mathrm{Fr}_q \times 1)^m | \mathbf{L}_x) = \sum_i (-1)^i \, \mathrm{tr}((\mathrm{Fr}_q \times 1)^m | H^i_c(\overline{X}, \mathbf{L})).$$

This is due to Grothendieck [G]; see also [SGA5], Exp. III, (6.13.3), p. 134 and [SGA4½], p. 86. Here X is not required to be (smooth and) proper. In particular, it applies with our noncompact scheme $M_{r,I,v}$. Underlying the proof is the observation that in characteristic $p > 0$ one has $\frac{d}{dx}(x^p) = 0$, hence the graph of the Frobenius is transverse to the diagonal. In particular the fixed points of the Frobenius are isolated.

If $\overline{X}$ is proper and smooth over an algebraically closed field k, and $\mathbf{L}$ is a smooth $\overline{Q}_\ell$- sheaf on $\overline{X}$, then a stronger variant of the Fixed-Point Formula (which we do not use) is known; see [SGA4½], p. 151, for the case of a constant $\mathbf{L}$, and [SGA5; Exp. III], Theorem 4.4 (p. 102) and (4.12) (p. 111) for the case of any smooth $\overline{Q}_\ell$-sheaf $\mathbf{L}$. To state this, let $i: \overline{X}' \hookrightarrow \overline{X} \times_k \overline{X}$ be a closed subscheme which is transverse to the diagonal morphism $\overline{\Delta}: \overline{X} \hookrightarrow \overline{X} \times_k \overline{X}$. Suppose that $f = \mathrm{pr}_1 \circ i$ is finite and flat. Put $h = pr_2 \circ i$. Let $\alpha: h^*\mathbf{L} \to f^!\mathbf{L}$ be a sheaf morphism. Suppose that α factorizes through the morphism $j: f^*\mathbf{L} \to f^!\mathbf{L}$, which is obtained by adjunction from the trace map $\mathrm{tr}: f_! f^*\mathbf{L} \to \mathbf{L}$($f_! = f_*$ for our proper f) of [SGA4; Exp. XVIII], Theorem 2.9 (p. 553)(SLN 305 (1973)). Then $\alpha = j \circ \beta$ for some $\beta: h^*\mathbf{L} \to f^*\mathbf{L}$.

For each point x' of $\overline{X}'$ we have $(h^*\mathbf{L})_{x'} = \mathbf{L}_{h(x')}$ by definition. If $h(x') = x$ and $f(x') = x$, then the sheaf morphism $\beta: h^*\mathbf{L} \to f^*\mathbf{L}$ induces a morphism $\beta_{x'}: (h^*\mathbf{L})_{x'} \to (f^*\mathbf{L})_{x'}$ on the stalks, namely $\beta_{x'}$ is an endomorphism of the finite dimensional vector space $\mathbf{L}_x$ over $\overline{Q}_\ell$. Then we have

Lefschetz Fixed-Point Formula. *If $\overline{X}$ is proper and smooth over an algebraically closed field, and $\mathbf{L}$ is a smooth $\overline{Q}_\ell$-sheaf on $\overline{X}$, then*

$$\sum_{\substack{x' \in \overline{X}' \\ h(x') = f(x') = x}} \mathrm{tr}[\beta_{x'} | \mathbf{L}_x] = \sum_i (-1)^i \, \mathrm{tr}[H_c^i(f, \alpha, h) | H_c^i(\overline{X}, \mathbf{L})].$$

However, the Drinfeld moduli scheme $M_{r,I,v}$ is not proper. In the late 1970's Deligne suggested that the following variant could be used to imply Drinfeld's reciprocity Law.

Deligne's Conjecture. *Suppose that X is a separated scheme of finite type over $\mathbf{F}_q$; $(f, h: X' \to X)$ is a correspondence where f is finite and flat; $\mathbf{L}$ a smooth $\overline{Q}_\ell$-adic sheaf on X, and $\alpha: h^*\mathbf{L} \to f^!\mathbf{L}$ a sheaf morphism which factorizes as the composition of the natural morphism $j: f^*\mathbf{L} \to f^!\mathbf{L}$*

and a morphism $\beta\colon h^*\mathbf{L} \to f^*\mathbf{L}$. Then there exists an integer m_0 such that for every integer m with $|m| \geq m_0$ we have

$$\sum_{x'} \operatorname{tr}[(\beta \circ (\mathrm{Fr}_q \times 1)^m)_{x'} | \mathbf{L}_x]$$
$$= \sum_i (-1)^i \operatorname{tr}[H_c^i(f, \alpha \circ (\mathrm{Fr}_q \times 1)^m, h \circ (\mathrm{Fr}_q \times 1)^m) | H_c^i(\bar{X}, \mathbf{L})].$$

On the left the sum ranges over all x' in $\bar{X}'$ with $(h \circ (\mathrm{Fr}_q \times 1)^m)(x') = f(x')$, and we put $x = f(x')$.

Underlying this conjecture is the hope that multiplying the correspondence $(f, h\colon X' \to X)$ by a sufficiently high power of the Frobenius one obtains a correspondence transverse to the diagonal $\Delta\colon X \hookrightarrow X \times_{\mathbf{F}_q} X$. In [D2] Drinfeld already worked only with high powers of the Frobenius. In [SGA 5; Exp. III], Theorem 4.4, Illusie expresses the alternating sum on the right in terms of local data for a quasifinite, flat correspondence (see (4.12), p. 111), and a complex $\mathbf{L}$ of sheaves in $D_c^b(\bar{X}, \bar{Q}_\ell)$. When X is one-dimensional, for a correspondence multiplied by a high power of the Frobenius, this local data is known to be the trace on the stalk $\mathbf{L}_x$ as in the equality above, and so Deligne's conjecture follows. In addition Deligne's conjecture holds in the cases when X is proper and smooth, and when $f = h = id$, as mentioned above. Deligne-Lusztig [DL], p. 119, noted that Deligne's conjecture holds for an automorphism of finite order of the scheme X; they multiplied the automorphism by a Frobenius and considered the result as a Frobenius (with respect to another structure on the scheme) for which the Grothendieck Fixed-Point Formula is valid. Our form [FK] of the Simple Trace Formula is a representation theoretic analogue of Deligne's conjecture.

A brief sketch of our study of the Purity Theorem (and Drinfeld's Law*) by means of the Grothendieck Formula (and Deligne's conjecture, respectively) will now follow. As noted above, the Fixed-Point Formula expresses the alternating sum of traces of the action of the Frobenius on the modules H^i by means of the set of points in $M_{r,I,v}(\bar{\mathbf{F}}_v)$ fixed by the action of the Frobenius, and the traces of the resulting morphisms on the stalks of the sheaf $\mathbf{L}(\bar{\rho})$ at the fixed points. Drinfeld describes in [D2] the set $M_{r,I,v}(\bar{\mathbf{F}}_v)$ as a disjoint union of isogeny classes of elliptic modules over $\bar{\mathbf{F}}_v$. Their types are studied in analogy with the Honda-Tate Theory. A type is described in group theoretic terms as an elliptic torus in $G(F)$, and the cardinality of the set

$M_{r,I,v}(\mathbf{F}_{v,m})([\mathbf{F}_{v,m}:\mathbf{F}_v]=m)$ is expressed in terms of orbital integrals of conjugacy classes γ in $G(F)$ which are elliptic in $G(F_\infty)$. Drinfeld showed (unpublished; we give a simpler proof, based on [K1]) that these are the orbital integrals of a test function whose component at v is the spherical function $f_m = f_m^{(r)}$ on G_v defined by the relation $\operatorname{tr}(\pi_v(z))(f_m) = q_v^{m(r-1)/2}\sum_{i=1}^r z_i^m$, where $\mathbf{z}=(z_i)$. This explains the shift $q_v^{(r-1)/2}$ in the relation between the Frobenius and Hecke eigenvalues and the fact that when $\tilde{\rho}_v$ is unramified, so is $\tilde{\pi}_v$.

Our form of the Trace Formula is the following

Trace Formula. *Let $f=\otimes f_w$ be a test function on $G(\mathbf{A})$ whose component f_∞ is a supercusp form. Suppose that its component f_v at v is the above spherical $f_m^{(r)}$, where m is sufficiently large, depending on the support of the other components $f_w(w\neq v)$. Then*

$$\sum_\pi \operatorname{tr}\pi(f) = \sum_\gamma c(\gamma)\Phi(\gamma, f).$$

On the left the sum ranges over all cuspidal G-modules. On the right the sum is finite and ranges over the elliptic conjugacy classes γ in G. $\Phi(\gamma, f)$ is the orbital integral of f at γ, and $c(\gamma)$ are standard volume factors.

The computations mentioned prior to the statement of the Trace Formula imply that the right (geometric) side of the Selberg Trace Formula is equal to the left (stalk) side of the Fixed Point Formula, namely, we have

$$\sum_\gamma c(\gamma)\Phi(\gamma, f) = \sum_{x\in M_{r,I,v}(\mathbf{F}_{q_v^{|m|}})} \operatorname{tr}[(\mathrm{Fr}_v\times 1)^m|\mathbf{L}(\bar{\rho})_x] \qquad (*)$$

for a function $f = f_\infty f^{v,\infty} f_{m,v}^{(r)}$ specified below. By virtue of the Fixed-Point Formula the right side of $(*)$ is equal to the following *cohomological side* (of the Fixed-Point Formula):

$$\sum_{i=0}^{2(r-1)} (-1)^i \sum \operatorname{tr}\tilde{\pi}_f(f^\infty)\operatorname{tr}\tilde{\rho}_v((\mathrm{Fr}_v\times 1)^m).$$

The inner sum ranges over all irreducible constituents $\tilde{\pi}_f\times\tilde{\rho}$ in H^i, and f^∞ is the product of the characteristic function f_v^0 of K_v in G_v, with a locally constant compactly supported function $f^{v,\infty}$ on $G(\mathbf{A}^{v,\infty})$, which is specified below, where $\mathbf{A}^{v,\infty}$ is the ring of adèles without components at v and ∞. Note that

$$\operatorname{tr}\tilde{\rho}_v((\mathrm{Fr}_v\times 1)^m) = \sum_{j=1}^r u_j(\tilde{\rho}_v)^m,$$

where $u_j(\tilde{\rho}_v)$ are the Frobenius eigenvalues. By virtue of the Trace Formula, the left side of (*) is equal to the following *automorphic side* (of the Selberg Trace Formula):

$$\sum_{\pi} \operatorname{tr} \pi(f) = (-1)^{r-1} \sum_{\pi} \operatorname{tr} \pi^{v,\infty}(f^{v,\infty}) \operatorname{tr} \pi_v(f^{(r)}_{m,v}).$$

The sums range over all cuspidal G-modules $\pi = \otimes \pi_w$ whose component at ∞ is the supercuspidal π_∞ which corresponds to the $D_\infty^\times$-module $\tilde{\rho}$. The component f_∞ of $f = f^{v,\infty} f_\infty f^{(r)}_{m,v}$ satisfies $\operatorname{tr} \pi_\infty(f_\infty) = (-1)^{r-1}$ and $\operatorname{tr} \pi'_\infty(f_\infty) = 0$ for every irreducible G_∞-module π'_∞ inequivalent to π_∞. Recall that

$$\operatorname{tr} \pi_v(f^{(r)}_{m,v}) = q_v^{m(r-1)/2} \sum_{i=1}^{r} z_i(\pi_v)^m,$$

where $z_i(\pi_v)$ are the Hecke eigenvalues.

All in all, we obtain the following *fundamental identity*:

$$\sum_{i=0}^{2(r-1)} (-1)^i \sum_{\tilde{\pi}_f} \operatorname{tr} \tilde{\pi}_f(f^\infty) \sum_{j=1}^{r} u_j(\tilde{\rho})^m$$
$$= (-1)^{r-1} \sum_{\pi} \operatorname{tr} \pi^{v,\infty}(f^{v,\infty}) \sum_{j=1}^{r} (z_j(\pi_v) q_v^{(r-1)/2})^m.$$

Now the scheme $M_{r,I,v}$ is not proper, and the Grothendieck Formula is available only for powers of the Frobenius. Hence the components $f_w (w \neq v, \infty)$ of the test function f are taken to be the characteristic function of $U_I \cap G_w$. Then $\operatorname{tr} \pi^{v,\infty}(f^{v,\infty})$ is a non-negative integer, namely, the dimension of the space of U_I-fixed vectors in $\pi^\infty = \otimes_{w \neq \infty} \pi_w$. Since both sides of the fundamental identity consist of finite sums (for a fixed I and π_∞), m ranges over the infinite set of $m \geq m_0$, and the coefficients of the Hecke eigenvalues are all positive (if r is odd) or negative (if r is even), we conclude from linear independence of the characters $m \mapsto u_j^m$ and $m \mapsto (q_v^{(r-1)/2} z_i)^m$ the following.

For every cuspidal G-module π with a supercuspidal component π_∞ and a nonzero U_I-fixed vector, there is a finite set of places of F such that for every v outside this set, for every Hecke eigenvalue $z_i(\pi_v)$ there is a Frobenius eigenvalue $u_j(\tilde{\rho}_v)$ of some constituent $\tilde{\pi}_f \times \tilde{\rho}$ of H, such that $u_j(\tilde{\rho}_v) = q_v^{(r-1)/2} z_i(\pi_v)$. Deligne's purity result [De 2] asserts that the complex absolute values of the algebraic numbers $u_j(\tilde{\rho}_v)$ are integral powers of $q_v^{1/2}$, while the unitarity of the nondegenerate component π_v of the cusp form π implies the bound $q_v^{-1/2} < |z_i(\pi_v)| < q_v^{1/2}$. Consequently all conjugates of the Hecke eigenvalues lie on the unit circle in the complex plane. This completes our sketch of the proof of the Purity Theorem for cusp forms.

The proof does not imply any relation between the cuspidal π and the $G(\mathbf{A}_f)$-module $\tilde{\pi}_f$ attached to $\tilde{\rho}$. It does not show that the $\tilde{\pi}_f$ are automorphic. It does not show that every Frobenius eigenvalue is related to a Hecke eigenvalue and hence has complex absolute values equal to $q_v^{(r-1)/2}$; indeed, there might be cancellations among the coefficients in the cohomological side. However, assuming Deligne's conjecture, we may take any correspondence associated to a U_I-biinvariant function $f^\infty = f^{v,\infty}f_v^0$, in the fundamental identity. Using linear independence of characters of the Hecke algebra of U_I-biinvariant functions, we conclude that the sum in the cohomological side is taken over the same set as in the automorphic side, namely, each $\tilde{\pi}_f$ is the component outside ∞ of a cuspidal π, the multiplicity of $\tilde{\pi}_f \times \tilde{\rho}$ is one, and $\tilde{\pi} = \tilde{\pi}_f \otimes \pi_\infty$ corresponds to $\tilde{\rho}$ (multiplied by $\nu^{(r-1)/2}$). This completes our sketch of the reduction of the Existence Theorem 1* (for every cuspidal π there is an ℓ-adic ρ) to Deligne's conjecture.

Our last theme in this work is a higher rank generalization of the classical theory of congruence relations. This method is entirely different than the previous one. It relies on a study of the geometry of certain correspondences on $M_{r,I,v}$, and not on the Trace Formula. It applies to $H_c^i(M_{r,I} \times_A \overline{F}, \mathbf{L}(\tilde{\rho}))$ with any $D_\infty^\times$-module $\tilde{\rho}$, not necessarily one which corresponds to a supercuspidal G_∞-module π_∞, and also to cohomology without compact supports. The result is the following; it does not depend on Deligne's conjecture.

Congruence Relations. *For any $\tilde{\rho}$, I and i as above, and for almost all v (depending on I), for every irreducible constituent $\tilde{\pi}_f \times \tilde{\rho}$ of $H_c^i(M_{r,I} \times_A \overline{F}, \mathbf{L}(\tilde{\rho}))$ we have the following. For every eigenvalue u of the endomorphism $\tilde{\rho}(\mathrm{Fr}_v \times 1)$ there is a Hecke eigenvalue $z(\tilde{\pi}_v)$ such that $u = q_v^{(r-1)/2} z(\tilde{\pi}_v)$.*

In particular, $\tilde{\rho}(\mathrm{Fr}_v \times 1)$ has at most r distinct eigenvalues.

This is the only result which we prove which relates the Frobenius and Hecke eigenvalues of $\tilde{\rho}$ and $\tilde{\pi}_f$ which occur together as an irreducible constituent $\tilde{\pi}_f \times \tilde{\rho}$ in the composition series of H^i as an $\mathbf{H}_I \times \mathrm{Gal}(\overline{F}/F)$-module. The proof is based on the study of the correspondence T_j defined by the coset $K_v g_j K_v$ in the Hecke algebra of compact K_v-double cosets in G_v; here $g_j = \mathrm{diag}(\pi_v, \ldots, \pi_v, 1, \ldots, 1)$ with $\det g_j = \pi_v^j$. We show that Fr_v satisfies the relation

$$\sum_{j\,\mathrm{odd}} q_v^{j(j-1)/2}\,\mathrm{Fr}_v^{r-j} \circ T_j = \sum_{j\,\mathrm{even}} q_v^{j(j-1)/2}\,\mathrm{Fr}_v^{r-j} \circ T_j \qquad (0 \le j \le r).$$

In particular we have

$$p_{\tilde{\pi}_v}(q_v^{(r-1)/2}\tilde{\rho}(\mathrm{Fr}_v \times 1)) = 0, \qquad \text{where} \qquad p_{\tilde{\pi}_v}(t) = \det(tI - Z(\tilde{\pi}_v))$$

is the characteristic polynomial of $\tilde{\pi}_v$, namely $Z(\tilde{\pi}_v)$ is the matrix whose r eigenvalues are the Hecke eigenvalues $z_i(\tilde{\pi}_v)$ of the unramified component $\tilde{\pi}_v$ of $\tilde{\pi}_f$.

In the classical theory of Eichler-Shimura and Ihara which concerns $\mathrm{GL}(2, Q)$, one shows on studying the de Rham cohomology that *a priori* the $\tilde{\pi}_f$ are automorphic (after tensoring with π_∞) and $\dim \tilde{\rho} = 2$. The relations of eigenvalues for almost all v, obtained from the "congruence relations" study of the correspondence attached to $K_v\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}K_v$, suffices to show that $\tilde{\pi} \to \tilde{\rho}$ so defined realizes the Reciprocity Law. Similar study is carried out in Drinfeld [D1] for GL(2) over a function field. It will be interesting to obtain such complete results also in the higher rank case on developing Drinfeld's de Rham Theory.

When π_∞ is supercuspidal, combining the Congruence Relations with the Purity Theorem, we obtain

Corollary. *If π_∞ is supercuspidal, for almost all v, each conjugate of each Frobenius eigenvalue of a constituent $\tilde{\rho}$ of $H_c^i(\overline{M}_{r,I}, \mathbf{L}(\tilde{\rho}))$ has complex absolute value $q_v^{(r-1)/2}$.*

This is used in the reduction of the Reciprocity Law* to Deligne's Conjecture.

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