RIGIDITY FOR AUTOMORPHIC FORMS[†]

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[‡] This chapter can be read independently of the other chapters.

General introduction

Suppose that $\pi = \bigotimes\pi_v$ and $\pi' = \bigotimes\pi'_v$ are discrete-series representations of the adele group $\mathrm{GL}(n, \mathbf{A})$ over a number field F. The rigidity theorem for $\mathrm{GL}(n)$ of [JS] asserts that if π_v is equivalent to π'_v for almost all places v of F (that is, with at most finitely many exceptions), then π_v is equivalent to π'_v for all v. For other reductive connected F-groups G such an assertion is often false, but one hopes that the following form of the *global rigidity conjecture* is correct: Given an automorphic $\pi = \bigotimes\pi'_v$ and a finite set V of places of F, there are only finitely many automorphic $\pi' = \bar{\otimes}\, \pi'_v$ with π'_v equivalent to π_v for all v outside V. This is related to the notion of "packets", defined in Chapter IV in a special case.

The trace formula suggests a possible proof of such a global rigidity conjecture, for a group G whose automorphic representations can be compared, by means of lifting, with those of another group G', for which the rigidity theorem is known. Presently this implies that G' has to be $\mathrm{GL}(n)$, or a group which has already been compared with $\mathrm{GL}(n)$ by such a method. Our aim in this work is to establish in a general framework several tools required in such a proof, and employ these tools in two special cases. We obtain detailed information on the representation theory of (1) inner, and (2) outer, forms of $\mathrm{GL}(n)$, namely (1) multiplicative groups of simple algebras, and (2) unitary groups.

Our plan is to obtain an identity of trace formulae for matching functions on G and G', and by means of a transfer of spherical functions (established in [Sph] in the case considered in Chapter IV) to reduce the question to one in local harmonic analysis. Indeed, after a suitable reduction the identity of trace formulae yields an identity of traces of representations of the local groups G_v and G'_v. An important step towards the proof of the global rigidity conjecture becomes the *local rigidity conjecture*: Suppose we are given a certain identity relating traces of G'_v-modules with traces of G_v-modules for matching functions, see Chapter II, §3. If on the side of G' there occur only finitely many irreducible tempered local G'-modules, then on the side of G there appear only tempered G-modules, and they are finite in number.

There are four Chapters in this work, denoted by I, II, III and IV. In Chapter I we present adelic and local fundamental tools of harmonic analysis on the group. This is given in a general twisted setting required for the applications of Chapter IV. We omit the proofs of those twisted analogues which are immediate adaptations of proofs existing in the literature, and record only those proofs which are new. Some of these tools are:

(i) The simple trace formula of Deligne–Kazhdan, for a wider class of test functions $f = \bigotimes f_v$ than usual. We require f to have a discrete component (in $A(G)$ of [K]) in addition to a supercuspidal component, but we do not require f to have a component supported on the regular elliptic set. This is essential for the applications of Chapter III concerning the Deligne–Kazhdan correspondence. The case of $G = \mathrm{GL}(n)$ is given in [FK].

(ii) Kazhdan's density theorem in the twisted case. The proof of [K, Appendix] is based on the simple trace formula. In the general twisted case we cannot use the simple trace formula since σ-invariant supercuspidal representations may not exist. For example, suppose that $\sigma(g) = {}^t g^{-1}$ as in [Sym]. Then there are no σ-invariant supercuspidal $\mathrm{PGL}(3, F_v)$-modules if F_v is a local field of odd residual characteristic, hence no σ-invariant automorphic $\mathrm{PGL}(3, \mathbf{A})$-modules with a supercuspidal component if F is a function field of odd characteristic. To extend the proof to the twisted case we use (a special case of) the general trace formula of Arthur [A] (and [CLL] in the twisted case). Lemma 4, which shows that a pair (T', G'), where T' is a torus in a local group G', can be "lifted" to a pair (T, G) consisting of a torus T in a group G over a global field, was suggested to me by D. Kazhdan.

(iii) Lifting orbital integrals of a function on a Levi subgroup of G to orbital integrals of a function on G. This is a new result. Its proof is based on the trace Paley–Wiener theorem of [BDK] (the proof in the twisted case follows closely that of [BDK] in the connected case), and a suitable "representation theoretic" decomposition of the Hecke algebra, using (a twisted analogue of) the geometric lemma of [BZ; (2.12)].

(iv) The Howe [Ho], Harish-Chandra [H] theory of characters (in characteristic zero; detailed proofs of the Theorems of [H] are recorded in [Cl] also in the twisted case).

(v) Kazhdan's theory [K] relating characters and orbital integrals is recorded here in the twisted case. The only non-immediate change in the adaptation of the proof of [K] to the twisted case is that the density theorem of [K] has to be replaced by that of (ii) here.

(vi) The theory of [BZ], [C], [S] concerning exponents of modules of coinvariants.

In Chapter II, we use the tools of Chapter I and basic definitions of stable conjugacy to present a general technique which reduces the local rigidity conjecture to several Assumptions, which amount to matching orbital integrals. The inductive arguments of Chapter III, §7, show that the crucial case is that of functions in the elliptic (or discrete) class $A(G)$ of [K] (and [BDK]) (cf. (iii) above). The technique is suggested by our joint work with D. Kazhdan on the metaplectic correspondence [FK]. It replaces ad-hoc arguments which were used in the study of liftings in some low rank cases.

The general approach which is espoused in Chapters I and II (as well as [K], [K″], [F], [FK], [Sph]) is that the study of orbital integrals — which has been hard so far when buildings' combinatorics or germ computations were used — can be reduced to the more accessible study of characters. This is the approach used in

Chapters III and IV. It is used, for example, also in our joint work [Sym; V] with D. Kazhdan to carry out the unstable transfer of orbital integrals of spherical functions in the case of the symmetric square lifting.

In the second half (Chapters III and IV) of the work we use the techniques of Chapters I and II in two special cases, independent of each other, to obtain lifting theorems. In Chapter III we give a new proof of the Deligne–Kazhdan correspondence (cf. [DKV]) relating the local and adelic representation theories of the multiplicative group of a simple algebra of rank n central over F on the one hand, and $\mathrm{GL}(n, F)$ on the other. In particular we verify in Chapter III all the Assumptions of Chapter II, in our case. The assumptions of Chapter II, §5, on the elliptic set, are verified in Chapter I, §5, directly, using the relations between orbital integrals and characters. Then in Chapter III, §7, we match orbital integrals in general, proving in particular the assumptions of Chapter II, §3, by an inductive argument, involving the main local lifting theorem of Chapter III, §5, and the trace Paley–Wiener theorem of [BDK].

In Chapter IV we consider a quadratic extension E/F and study the stable base-change lifting from a unitary group $U(n)$ in n variables with respect to E/F, to $\mathrm{GL}(n, E)$. We define tempered packets, for $U(n)$ locally, and also global packets for a compact form of the unitary group. We also establish the local and global rigidity theorems in these cases. Our only assumptions are those of Chapter II, §5. They can be checked for $n = 3$ by standard techniques (see, e.g., [Sym; I]). Our usage of the "regular" functions of [Sph] eliminates the need to study those terms in the trace formula attached to singular conjugacy classes. A detailed description of the results is given in §1 of Chapter IV.

I am deeply grateful to David Kazhdan for his constant interest, constructive criticism, and instructive conversation. Much of what is new here I learned from him. This work is based on a course at Harvard University, Fall 1985, where we first explained [BZ] and some of [BZ′], then [BD] and some of [BDK], [H] and [K], terminating with the present work. This still seems to me to be a recommendable path to the heart of p-adic representation theory. Of course, on first reading it is better to assume that G is connected. The non-connected generalization is required for Saito–Shintani base-change and other lifting problems, as in Chapter IV, [Sym], [U(3)], etc.

Chapter I. Harmonic Analysis

§1. Conventions

Let F be a global field of characteristic zero with a ring $\mathbf{A}$ of adeles; the completion of F at the place v is denoted by F_v. Let G be a reductive group over F;

this is often identified with its group of $\bar{F}$-points, where $\bar{F}$ is a fixed algebraic closure of F. Put $G(K)$ for the group of K-points of G, for any extension K of F. We put G for $G(F)$, G_v for the group $G(F_v)$ of F_v-rational points on G, and $G(\mathbf{A})$ for the group of adele points; these conventions apply to any F-subgroup of $\boldsymbol{G}$. We do not assume that $\boldsymbol{G}$ (by abuse of language from now on we use the symbol G) is connected. Its connected component (of the identity) is denoted by G^0; it is a normal subgroup, and the quotient G/G^0 is finite. The other connected components are denoted by G^i ($i \geqq 0$). For example, if $\boldsymbol{Z}$ is the center of $\boldsymbol{G}^0$, we have Z, Z_v and $Z(\mathbf{A})$.

An F-subgroup P of G is called here *parabolic* if P^0 is an F-parabolic subgroup of G^0, and P/P^0 is isomorphic to G/G^0. Note the last condition, which is not standard. Denote by N the unipotent radical of P. It is equal to the unipotent radical of P^0. Fix a minimal parabolic subgroup $P_0 = M_0N_0$, and its Levi subgroup M_0. Unless otherwise specified, we consider only standard P, which contain P_0. By a Levi subgroup M of P we mean the one which contains M_0. Its connected component M^0 is a Levi subgroup of P^0, and $M/M^0 \simeq P/P^0 \simeq G/G^0$. Then $P = MN$, and N is normalized by M.

It is illuminating to consider an example. Let G be the semi-direct product of $G^0 = \mathrm{GL}(3)$, and the group $\{1, \sigma\}$, where σ is the automorphism of G^0 mapping g to $J{}^tg^{-1}J$. Here $J = (\delta_{i,3-j})$, and ${}^tg$ is the transpose of g. If P_0^0 is the upper triangular subgroup of G^0, then $P_0 = P_0^0 \times \langle\sigma\rangle$ is parabolic. But if P^0 is a parabolic subgroup of G^0 of type (2, 1), then $P = P^0 \times \langle\sigma\rangle$ is not a parabolic subgroup of G, since it is not a subgroup.

Let F be a local or global field of characteristic zero, and $L(G)$ the Lie algebra of $\boldsymbol{G}^0$. For x in $G^0 \times \sigma$, consider the polynomial $\det[(t + 1 - \mathrm{Ad}(x)) \mid L(G)]$ in t. Let d be the degree of the first non-zero power of t in this polynomial. It is called the *rank* of $G^0 \times \sigma$. Denote by $D(x)$ the coefficient of t^d. Then x is called *regular* if $D(x) \neq 0$. It is then semi-simple, and its centralizer $\boldsymbol{Z}(x)$ in $\boldsymbol{G}^0$ is a torus. A semi-simple x is called *elliptic* if the center of $Z(x)Z/Z$ (if F is local), or $Z(x, \mathbf{A})/Z(x)Z(\mathbf{A})$ (if F is global), is compact. $\boldsymbol{Z}$ is the center of $\boldsymbol{G}^0$. If x is elliptic regular, then $\boldsymbol{Z}(x)$ is an elliptic torus of $\boldsymbol{G}^0$.

Let $Z_0(\mathbf{A})$ be a closed subgroup of $Z(\mathbf{A})$ such that $Z_0(\mathbf{A})Z$ is closed and $Z(\mathbf{A})/Z_0(\mathbf{A})Z$ is compact. Suppose that $Z_0(\mathbf{A}) = \Pi_v Z_{0v}$, where the product extends over all places v of F. Put $Z_0 = Z_0(\mathbf{A}) \cap G$.

Fix a character ω of $Z_0(\mathbf{A})/Z_0$; its local components are denoted by ω_v. We now fix a place v, and omit v from the notations until the end of this §. Let $C(G)$ be the space of complex-valued functions on G which transform under Z_0 by ω^{-1}, which are compactly-supported modulo Z_0, smooth if v is archimedean and locally-constant if v is non-archimedean.

Since our main interest is in orbital integrals, and the orbit under G^0 of an element x in a connected component G^i of G is contained in G^i, we restrict our attention from now on to f in $C(G^i)$. This entails no loss of generality, as any f in $C(G)$ is the sum over the connected components G^i of G of the restriction of f to G^i. Recall that G is the semi-direct product of G^0 and the finite group G/G^0. Thus our G^i is a coset $G^0 \times \sigma$. For our study it suffices to replace G by its subgroup $G^0 \rtimes \langle \sigma \rangle$, whose quotient by G^0 is the cyclic group $\langle \sigma \rangle$ generated by σ. Thus from now on we assume that G is of this form. In particular, its parabolic subgroups are of the form $P = P^0 \rtimes \langle \sigma \rangle$, where $\sigma P^0 = P^0 \sigma$. Further, from now on the notation f in $C(G)$ will mean that f is supported on $G^0 \times \sigma$; we deal below only with such f. Assume that Z_0 contains the subgroup $Z_{00} = \{z\sigma(z^{-1});\ z \text{ in } Z\}$, and that ω attains the value one on Z_{00}.

Fix a Haar measure dy on G^0/Z. For every x in G let $Z_G(x)$ be the centralizer of x in G^0. Fix a Haar measure d_x on $Z_G(x)Z/Z$ such that if $Z_G(x)$ and $Z_G(x')$ are isomorphic then d_x and $d_{x'}$ are equal. The *orbital integral* of f at x is defined to be the integral

$$\Phi(x, f) = \int f(yxy^{-1})dy/d_x.$$

It is taken over y in $G^0/Z_G(x)Z$; it depends on the choice of dy and d_x. The orbit of a regular element x in G is closed; hence $\Phi(x, f)$ converges for a regular x for all f in $C(G)$. Moreover, $\Phi(f)$ converges for f in $C(G)$ at *any* x in G by [Rao]. At a regular x in G we shall also consider the integral $'\Phi(f)$, defined on replacing $Z_G(x)$ (in the definition of $\Phi(x, f)$) by the split component in the center of $Z_G(x)$.

Let F be local, $x = su = us$ the Jordan decomposition of x in G into semi-simple and unipotent elements s and u, $Z(s)$ the centralizer of s in G^0, $L(Z(s))$ its Lie algebra and $L(G)$ the Lie algebra of G^0. Put

$$\Delta(x) = |\det\{(1 - \mathrm{Ad}(s)) \mid L(G)/L(Z(s))\}|^{1/2},$$

and

$$F(x, f) = \Delta(x)\Phi(x, f), \qquad 'F(x, f) = \Delta(x)\,'\Phi(x, f) \qquad \text{(for regular } x).$$

For example, if $G = \mathrm{GL}(n, F)$, and $x_1, x_2, \ldots$ are the distinct eigenvalues of x in G, then

$$\Delta(x) = \left| \prod_{i<j} (x_i - x_j)^2 \right|^{1/2} \Big/ |\det x|^{(n-1)/2}.$$

If (1) E is a cyclic extension of F of degree l and σ is a generator of the galois group $\mathrm{Gal}(E/F)$; (2) G'' is the semi-direct product $G' \rtimes \langle \sigma \rangle$ of the group $G' = \mathrm{Res}_{E/F} G$ obtained from G upon restricting scalars from E to F, and $\mathrm{Gal}(E/F)$; and (3) x lies in $G'(F)$ ($\simeq G(E)$), then $\Delta(x \times \sigma)$ (with respect to G'') equals $\Delta(Nx)$ (with respect to G), where Nx is an element of G which has the same set of eigenvalues as $x\sigma(x) \cdots \sigma^{l-1}(x)$.

Let $P = MN$ be an F-parabolic subgroup of G, and K a maximal compact σ-invariant subgroup of G^0 with $G = KP$. For m in M put $\delta_P(m) = |\det \operatorname{Ad}(m) | L(N)|$, and for f in $C(G)$ put

$$f_N(m) = \delta_P(m)^{1/2} \int_N \int_K f(k^{-1}mnk)dkdn.$$

f_N depends on N, but its orbital integral at an element m of M regular in G depends only on M. Indeed, a standard computation (see, e.g., [FK], §7) shows that $F(m, f) = F^M(m, f_N)$ for such m, where F^M is the orbital integral, multiplied by Δ_M-factor, with respect to M. Note that f_N lies in $C(M)$; in particular, it is supported on $M^0 \times \sigma$, since f lies in $C(G)$. Denote by $J(G)$ the space of f in $C(G)$ such that $F(g, f) = 0$ for every regular g in G. Put $\bar{C}(G) = C(G)/J(G)$. The image of f_N in $\bar{C}(M)$ will be denoted by f_M since it depends on the Levi subgroup M but not on the unipotent radical N.

§2. Automorphic forms

Let F be a global field. At each non-archimedean place v of F denote by R_v the ring of integers in F_v, and by K_v a special maximal compact subgroup of G_v^0. Suppose that K_v is σ-invariant. At almost all v we take $K_v = G^0(R_v)$. Fix a product measure $dx = \lambda \otimes dx_v$ on $G(\mathbf{A})/Z_0(\mathbf{A})$, so that the product of the volumes $|K_v/K_v \cap Z_{0v}|$ converges. Let $f = \otimes f_v$ be the product of f_v in $C(G_v)$ over all v, where at almost all v the component f_v is the function f_v^0 which is supported on $Z_{0v}K_v \times \sigma$, and attains the value $|K_v/K_v \cap Z_{0v}|^{-1}$ on $K_v \times \sigma$. Denote by $C(G(\mathbf{A}))$ the space spanned by all such functions f.

Denote by $L(G)$ the space of functions ψ on $G \backslash G(\mathbf{A})$ which transform under $Z_0(\mathbf{A})$ by ω, and are slowly increasing (see, e.g., [BJ]) on $GZ_0(\mathbf{A}) \backslash G(\mathbf{A})$. Let r be the representation of $G(\mathbf{A})$ on $L(G)$ by right translates, The operator $r(f)$ on $L(G)$ which maps ψ to $(r(f)\psi)(x) = \int f(y)\psi(xy)dx$ (y in $G(\mathbf{A})/Z_0(\mathbf{A})$) is an integral operator on $G \backslash G(\mathbf{A})/Z_0(\mathbf{A})$ with the kernel $K(x, y) = \Sigma f(x^{-1}\gamma y)$ (γ in G/Z_0).

An irreducible subspace V of $L(G)$ is a unitary $G^0(\mathbf{A})$-module π^0, which is called *automorphic*. It is the restricted direct product $\pi^0 = \otimes \pi_v^0$ of irreducible G_v^0-modules π_v^0, which are almost all *unramified*, namely have a K_v-fixed vector (which is unique up to scalar multiples). All π_v^0 are admissible and unitary. Recall that a G_v^0-module, namely a representation $\pi_v^0 : G_v^0 \to \operatorname{Aut} V$ of G_v^0 in a complex space V, is called *smooth* (or algebraic) if for every v in V the group of x in G_v^0 with $\pi^0(x)v = v$ is open in G_v^0, and *admissible* if in addition for any open subgroup U of G_v^0 the space V^U of U-fixed vectors is finite-dimensional. Theorem 3.25 of [BZ] asserts that a smooth G_v^0-module of finite length is admissible.

When G is G_v or $G(\mathbf{A})$, a G^0-module π^0 is called *σ-invariant* if ${}^\sigma\pi^0 \simeq \pi^0$, where $\sigma(x) = \sigma x \sigma^{-1}$ and ${}^\sigma\pi^0(x) = \pi^0(\sigma(x))$. The restricion π^0 of a G-module π to G^0 is σ-invariant since ${}^\sigma\pi^0(x) = \pi(\sigma)\pi^0(x)\pi(\sigma)^{-1}$. On the other hand, an irreducible σ-invariant G^0-module π^0 extends to a G-module by putting $\pi(\sigma) = A$; A is an interwining operator with ${}^\sigma\pi^0(x) = A\pi^0(x)A^{-1}$ (x in G^0) whose order is equal to the order l or σ (by Schur's lemma A^l is a scalar which we normalize to be 1).

For an automorphic $G(\mathbf{A})$-module (π, V) we define the operator $\pi(f) = \int f(x)\pi(x)dx$ on V, where dx is the Haar measure on $G(\mathbf{A})/Z_0(\mathbf{A})$ fixed above. By definition the space V of π is spanned by vectors $\otimes\xi_v$ is a vector in the space of π_v, which is K_v-invariant for almost all v. Since π^0 is σ-invariant, $\pi_v(\sigma)\xi_v = \xi_v$ for almost all v. Hence for almost all v the operator $\pi_v(f_v^0)$ is the projection on the one-dimensional subspace of K_v-fixed vectors, and its trace $\operatorname{tr} \pi_v(f_v^0)$ is 1. Hence almost all factors in the product $\operatorname{tr} \pi(f) = \Pi_v \operatorname{tr} \pi_v(f_v)$ are equal to one. If ${}^\sigma V = r(\sigma)V$ is not equivalent to V, the operator $\pi(f) = \int f(x \times \sigma)\pi^0(x)r(\sigma)dx$ (x in $G^0(\mathbf{A})/Z_0(\mathbf{A})$) has trace equal to zero.

A function ψ in $L(G)$ is called *cuspidal* if for any proper F-parabolic subgroup of G^0 (*not* G) with unipotent radical N, the integral $\int \psi(nx)dx$ over $N \backslash N(\mathbf{A})$ is 0, for any x in $G(\mathbf{A})$. Let r_0 be the restriction of r to the space $L_0(G)$ of cuspidal functions. The space $L_0(G)$ is the direct sum of irreducible spaces π^0 which occur with finite multiplicities $m(\pi^0)$. The operator $r_0(f)$ is of trace class, and

$$\operatorname{tr} r_0(f) = \sum m(\pi) \operatorname{tr} \pi(f). \tag{2.1}$$

The sum is over the equivalence classes of the π^0 in $L_0(G)$ which are σ-invariant and extend to $G(\mathbf{A})$-modules π. Here we use the assumption that f lies in $C(G(\mathbf{A}))$, namely it is supported on $G^0(\mathbf{A}) \times \sigma$. The sum is absolutely convergent, and each π on the right is unitary.

The elements x, x' of G are called (*stably*) *conjugate* if there is y in G^0 (resp. $\boldsymbol{G}^0$) with $x' = \operatorname{Ad}(y)x$ ($= yxy^{-1}$). Here F can be local or global. The conjugacy classes within the stable conjugacy class of x in G are parametrized by the set $B(x, F) = G^0 \backslash A(x/F)/Z_{\boldsymbol{G}_0}(x)$, where $A(x/F)$ is the set of y in $\boldsymbol{G}^0$ with $[\operatorname{Ad}(y)](x)$ in G, and $Z_{\boldsymbol{G}^0}(x)$ is the centralizer of x in $\boldsymbol{G}^0$. The map

$$x \to \{\tau \to y_\tau = y^{-1}\tau(y);\ \tau \text{ in } \operatorname{Gal}(\bar{F}/F)\}$$

defines a bijection

$$B(x/F) \simeq \ker[H^1(F, Z_{\boldsymbol{G}^0}(x)) \to H^1(F, \boldsymbol{G}^0)].$$

Recall that $H^1(F, A)$ means $H^1(\operatorname{Gal}(\bar{F}/F), A(\bar{F}))$. Thus, given x, any x' stably conjugate to x determines an element in $B(x/F)$, and x' is actually conjugate to x if and only if it determines the identity in $H^1(F, Z_{\boldsymbol{G}^0}(x))$. When F is global, we also define $B(x, \mathbf{A})$ (resp. $B(x/\mathbf{A}^w)$) to be the pointed direct sum of $B(x/F_v)$ over all v (resp. $v \neq w$).

§3. Trace formula

Notations as in §2. Let u be a place of F. The function f_u in $C(G_u)$ is called *supercuspidal* if for any F_u-parabolic subgroup of G_u^0 (not G_u) with unipotent radical N_u, the integral $\int f_u(xny)dn$ over N_u is 0 for any x, y in G_u.

Lemma. *If f has a supercuspidal component at u, then $r(f)$ vanishes on the $G(\mathbf{A})$-invariant complement of $L_0(G)$ in $L(G)$.*

Proof. Put $\mathbf{N}$ for $N(\mathbf{A})$, PG for $G(\mathbf{A})/Z_0(\mathbf{A})$. Then

$$\int_{N\backslash\mathbf{N}} (r(f)\psi)(nx)dn = \int_{N\backslash\mathbf{N}} \int_{PG} f(y)\psi(nxy)dydn$$

$$= \int_{N\backslash\mathbf{N}} \int_{N\backslash PG} \left[\sum_{\gamma \text{ in } N} f(x^{-1}n^{-1}\gamma y)\psi(y)\right] dydn$$

$$= \int_{N\backslash PG} \left[\int_{N\backslash\mathbf{N}} \sum_{\gamma \text{ in } N} f(x^{-1}n^{-1}\gamma y)dn\right] \psi(y)dy$$

$$= \int_{N\backslash PG} \left[\int_{\mathbf{N}} f(x^{-1}ny)dn\right] \psi(y)dy = 0.$$

The order of integration can be changed since the second integral above is absolutely convegent: f has compact support on PG, and $N\backslash\mathbf{N}$ is compact. The lemma follows.

Remark. The Lemma implies that tr $r_0(f) =$ tr $r(f)$ for such f.

Let F be a global field of characteristic zero.

Proposition. *Let $C = \Pi\, C_v$ be a compact subset of $G(\mathbf{A})$ with $C_v = G^0(R_v)$ for almost all v. Then there are only finitely many regular conjugacy classes in $G(\mathbf{A})$ with a representative in G which intersect C non-trivially.*

Proof. We deal only with the case of a connected G. Fix a faithful representation of $\boldsymbol{G}$ in GL$(n, \bar{F})$ for some n. Thus we can define a map $G(\mathbf{A}) \to \mathbf{A}^{n-1} \times \mathbf{A}^{\times}$ by mapping x to the ordered set of coefficients in its characteristic polynomial. The image of C is compact; that of G is discrete; hence there are only finitely many semi-simple conjugacy classes in GL$(n, \mathbf{A})$ with a representative in G which intersect C non-trivially. Now two semi-simple conjugacy classes in G which are conjugate in GL$(n, \mathbf{A})$ are conjugate in GL(n, F). The Theorem of [St], p. 102, asserts that a conjugacy class of GL$(n, \bar{F})$ intersects $\boldsymbol{G}$ in only finitely many conjugacy classes of $\boldsymbol{G}$. However, by definition (see §2), a $\boldsymbol{G}$-conjugacy class with a representative in G is a stable conjugacy class. If γ_G is a stable conjugacy class in G, then there exists a finite set V of places of F such that γ_G intersects $G^0(R_v)$ at most at one conjugacy class for all v outside V. This γ_G is contained in a stable conjugacy class $\gamma_{\mathbf{A}}$, and $\gamma_{\mathbf{A}}$ is the product over all v of stable conjugacy classes γ_v in G_v. Since γ_v consists of finitely many conjugacy classes for all v, γ_G consists of only finitely many conjugacy classes in G which intersect C, as required.

The non-connected case reduces to the connected case whenever there is an injective norm map, for example in the cases of base-change and the symmetric-square.

Suppose that f is as in the Lemma, and it vanishes on the conjugacy class in $G^0(\mathbf{A})$ of any γ in G which is not elliptic regular. Then using the Lemma we have that $r(f)$ is a trace class operator, whose trace is the integral of its kernel over the diagonal, namely

$$\operatorname{tr} r(f) = \int_{G(\mathbf{A})/Z_0(\mathbf{A})G} \left[\sum_{\gamma\in G/Z_0} f(x\gamma x^{-1})\right] dx$$

$$(3.1) \qquad = \sum_{\{\gamma\}} \int_{G^0(\mathbf{A})/Z_0(\mathbf{A})\tilde{Z}(\gamma)} f(x\gamma x^{-1})dx$$

$$= \sum_{\{\gamma\}} (|Z(\gamma,\mathbf{A})Z(\mathbf{A})/Z(\gamma)Z_0(\mathbf{A})|/[\tilde{Z}(\gamma):Z(\gamma)])\Phi(\gamma,f).$$

$\boldsymbol{Z}(\gamma)$ (resp. $\tilde{\boldsymbol{Z}}(\gamma)$) is the centralizer of γ in $\boldsymbol{G}^0$ (resp. $\boldsymbol{G}^0/\boldsymbol{Z}$). $\{\gamma\}$ is the set of conjugacy classes of elliptic regular elements in $G^0 \times \sigma/Z$, due to our assumption on f. Each of the integrals in (3.1) is absolutely convergent, and the sum is finite by the Proposition. We conclude

Corollary. *Suppose that u, u', u'' are places of F with $u \neq u'$, f_u is a supercuspidal function, the orbital integral of $f_{u'}$ vanishes on the regular non-elliptic set of $G_{u'}$, and $f_{u''}$ vanishes on the singular set. Then* (2.1) *is equal to* (3.1), *where the sum of* (3.1) *is finite.*

Proof. The Proposition implies that if $f(x\gamma x^{-1}) \neq 0$ for x in $G_0(\mathbf{A})$, then γ lies in one of finitely many regular conjugacy classes (= orbits). Suppose that γ lies in such a regular non-elliptic class. Then the invariant distribution $\Phi(\gamma): h \to \Phi(\gamma, h)$ on $C(G_{u'})$ vanishes at $f_{u'}$. Let $C_0(G_{u'})$ be the span of the functions $h - h^g$, h in $C(G_{u'})$, g in $G^0_{u'}$. Denote by $C_0(G_{u'})_\gamma$ (resp. $C(G_{u'})_\gamma$) the space of restrictions of the elements of $C_0(G_{u'})$ (resp. $C(G_{u'})$) to the orbit of γ. The uniqueness of the $G^0_{u'}$-invariant measure on the orbit of γ means that any distribution on $C(G_{u'})_\gamma/C_0(G_{u'})_\gamma$ is a scalar multiple of $\Phi(\gamma)$. Thus $C(G_{u'})_\gamma/C_0(G_{u'})_\gamma$ is one-dimensional, and $C_0(G_{u'})$ is the kernel of $\Phi(\gamma)$. Hence there are h_i, g_i as above (h_i in

$C(G_{u'})$, g_i in $G_{u'}^0$), so that $f_{u'} = \Sigma_i (h_i - h_i^{g_i})$ (finite sum over i) on the orbit of γ. We may choose h_i to be zero outside a small neighborhood of the orbit of γ.

Replacing in f the component $f_{u'}$ by $f_{u'} - \Sigma_i (h_i - h_i^{g_i})$ will not change the side (2.1) of the trace formula, since $\operatorname{tr} \pi(h^g) = \operatorname{tr} \pi(h)$. On the other hand, the function f now vanishes on the orbit of γ, but its values on all other conjugacy classes with a rational representative do not change. Consequently we may assume that if $f(x\gamma x^{-1}) \neq 0$, then γ is elliptic regular. The corollary follows.

Remark (1). The fact that $f_{u'}$ is permitted to be *any* function whose orbital integrals vanish on the regular non-elliptic set of $G_{u'}$, and it is not assumed that it is supported on the elliptic regular set, is fundamental for the applications of Chapter III.

Remark (2). Supercuspidal functions are obtained as linear combinations of matrix coefficients of supercuspidal representations. In the twisted case, however, there may not exist σ-invariant supercuspidal G-modules; this is the case when the residual characteristic of F is odd, in the example of the symmetric square specified in §1. Then the condition at u cannot be made. However, for local applications such as those of the next section, we use a different form of the Corollary, based on Arthur's work. By the rank of G we mean the dimension of the quotient, by the split component of a maximal σ-invariant torus in Z, of a maximal σ-invariant split torus in G.

Corollary 1. *Let $f = \otimes f_v$ (f_v in $C(G_v)$ for all v) be a function whose components at u_i ($0 \leqq i \leqq r$), where $r \geqq \operatorname{rk} G$, are supported on the elliptic regular set of G_{u_i}, and ϕ_{u_1} is zero on the x in $G_{u_1}^0 \times \sigma$ for which there are g in $G_{u_1}^0$ and $z \neq 1$ in Z_{u_1} with $gxg^{-1} = zx$. Then*

$$\sum_{\{\gamma\}} |Z(\gamma, \mathbf{A})Z(\mathbf{A})/Z(\gamma)Z_0(\mathbf{A})| \Phi(\gamma, f) = \sum_{\pi} c_\pi \operatorname{tr} \pi(f). \tag{3.2}$$

The sum over $\{\gamma\}$ is finite. It ranges over the conjugacy classes of regular x in G which are elliptic at the u_i. The sum over π is absolutely convergent. It ranges over automorphic $G(\mathbf{A})$-modules. The c_π are complex numbers.

Proof. The assumption at u_1 alone implies that the sum $\Sigma J_o(f)$ of [A] is equal to our sum over $\{\gamma\}$. It is finite by the Proposition. The sum $\Sigma J_\chi(f)$ of [A′] consists of integrals of logarithmic derivatives of intertwining operators acting on induced representations. As the degrees of the derivatives are at most $\operatorname{rk}(G)$, our $r + 1$ assumptions imply the vanishing of all integrals. There remains a discrete sum of irreducible representations π whose components at u_i are elliptic. The c_π are integral and positive for cuspidal π.

§4. Density

Let F be a local field of characteristic zero, and $G = G^0 \rtimes \langle \sigma \rangle$ a reductive group over F, as in §1. The following is a twisted analogue of Kazhdan's [K, Appendix] density theorem.

Proposition. *Let f be a function in $C(G)$ such that $\operatorname{tr} \pi(f) = 0$ for all admissible irreducible G-modules π. Then $\Phi(x, f) = 0$ for all regular x in G.*

Remark. Consequently $J(G)$ (defined in §1) consists of all f in $C(G)$ such that $\operatorname{tr} \pi(f) = 0$ for every G-module π.

Proof. In the proof, we denote F, G^0, f by F', G'^0, f'. Due to the integration formula $F(x, f') = F^M(x, f'_N)$ (§1) we may assume that there exists an elliptic regular element $x_0 \times \sigma$ in $G'^0 \times \sigma$ with $\Phi(x_0 \times \sigma, f') \neq 0$, and that its centralizer T' in G'^0 is an elliptic torus over F' which splits over the galois extension F'' of F'.

We first prove the following

Lemma. *Let F' be a local field, G' a reductive group over F', T' a (maximal) torus of G' over F', and F'' a galois field extension of F' such that T' and G' split over F''. Then there exists a galois extension E/F of global fields such that at a set of places w of F of cardinality at least two we have $F_w \simeq F'$, $E_w = E \otimes_F F_w \simeq F''$, $\mathrm{Gal}(E_w/F_w) \simeq \Gamma$, where $\Gamma = \mathrm{Gal}(E/F)$, and a pair (T, G) consisting of a reductive group G and a torus T over F with $G(F_w) \simeq G'$, $T(F_w) \simeq T'$ (all w), such that $G(F)$ is dense in $G_w = G(F_w)$ and $T(F)$ in $T_w = T(F_w)$.*

Proof. It is clear that there exist E and F with the required properties. Once (T, G) is found, since the set of w has cardinality at least two it follows from [CF], middle of page 361, that $(T(F), G(F))$ is dense in (T_w, G_w). Now, it is well known (see [Se], p. III-1; also Sém. Grothendieck, Exp. VI, *Catégories fibrées et descente*, 1961), that if K/k is a galois field extension, A is a torus in an algebraic group H, both defined over k, then the set of K/k-forms of (A, H) is parametrized by the first cohomology group $H^1(\mathrm{Gal}(K/k), \mathrm{Aut}_K(A, H))$ of $\mathrm{Gal}(K/k)$ in the group $\mathrm{Aut}_K(A, H)$ of automorphisms of the pair (A, H) over K ($\mathrm{Aut}_K(A, H)$ consists of automorphisms of H over K which map A to A). The group $A(K)$ of K-points of A injects as a normal subgroup of $\mathrm{Aut}_K(A, H)$; denote the quotient by W_K.

Let (A, H) be a pair consisting of a reductive group H over F with $H(E_w) \simeq G'(E_w)$ and a torus A of H over F with $A(E_w) \simeq T'(E_w)$. We have

$$
\begin{tikzcd}
0 \arrow[r] & A(E) \arrow[r] \arrow[d, hook] & \mathrm{Aut}_E(A, H) \arrow[r] \arrow[d, hook] & W_E \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & A(E_w) \arrow[r] & \mathrm{Aut}_{E_w}(A, H) \arrow[r] & W_{E_w} \arrow[r] & 0
\end{tikzcd}
$$

Since $A(E)$ is normal in $\mathrm{Aut}_E(A, H)$, by [Se], Prop. 38, p. I-6, we have the associated commutative diagram

$$\begin{array}{ccccccc} W_E^\Gamma \to H^1(\Gamma, A(E)) & \to & H^1(\Gamma, \mathrm{Aut}_E(A,H)) & \xrightarrow{p} & H^1(\Gamma, W_E) \\ \downarrow\wr \qquad\quad \downarrow\psi & & \downarrow\phi & & \downarrow\wr \\ W_{E_w}^\Gamma \to H^1(\Gamma, A(E_w)) & \to & H^1(\Gamma, \mathrm{Aut}_{E_w}(A,H)) & \xrightarrow{p_w} & H^1(\Gamma, W_{E_w}). \end{array}$$

The Tate–Nakayama theory [Ta] implies that ψ is surjective.

The pair (T', G') is determined by an element α_w in $H^1(\Gamma, \mathrm{Aut}_{E_w}(A, H))$. To produce a pair (T, G) as required we have to find an element α in $H^1(\Gamma, \mathrm{Aut}_E(A, H))$ whose image under ϕ is α_w. Put $\beta = p_w(\alpha_w)$; it can be regarded as an element of $H^1(\Gamma, W_E)$. As in [Se], denote by ${}_{\alpha_w}A$ the torus determined by the cocycle α_w; since it depends only on β, we denote ${}_{\alpha_w}A$ by ${}_\beta A$. For each γ in $H^1(\Gamma, W_E)$ there exists an element $\Delta(\gamma)$ in $H^2(\Gamma, {}_\gamma A(E))$ (constucted in [Se], p. I-70), such that γ lies in the image of p if and only if $\Delta(\gamma) = 0$ (see [Se], Prop. 41, p. I-70). Also, for each γ_w in $H^1(\Gamma, W_{E_w})$ there is $\Delta_w(\gamma_w)$ in $H^2(\Gamma, {}_{\gamma_w}A(E_w))$ such that γ_w lies in the image of p_w if and only if $\Delta_w(\gamma_w) = 0$. The Tate–Nakayama theory [Ta] implies that $H^2(\Gamma, {}_\beta A(E))$ and $H^2(\Gamma, {}_\beta A(E_w))$ are isomorphic as groups. By their construction (in [Se], p. I-70), $\Delta = \Delta_w$. Since $\beta = p_w(\alpha_w)$, we have $\Delta_w(\beta) = 0$, hence $\Delta(\beta) = 0$, and β lies in the image of p. By [Se], Cor. 2, p. I-67, the inverse image by p_w of β is the quotient of $H^1(\Gamma, {}_\beta A(E_w))$ by $\mathrm{Im}\, W_{E_w}^\Gamma$, and $p^{-1}(\beta)$ is $H^1(\Gamma, {}_\beta A(E))/\,\mathrm{Im}\, W_E^\Gamma$. The Tate–Nakayama theory [Ta] implies that the map $H^1(\Gamma, {}_\beta A(E)) \to H^1(\Gamma, {}_\beta A(E_w))$ is surjective. Hence there is α in $H^1(\Gamma, \mathrm{Aut}_E(A, H))$ with $\phi(\alpha) = \alpha_w$. The pair (T, G) determined by α has the required properties, and the lemma follows.

Let E/F be a global field extension, and (T, G^0) a pair defined over F with the properties specified by the Lemma. In these notations $T_w = T'$ is the centralizer $Z_{G_w^0}(x_0 \times \sigma)$ of $x_0 \times \sigma$ in G_w^0. $T = T(F)$ is dense in T_w, and $G = G(F)$ is dense in G_w. Hence the centralizer $Z_{G_w^0 \times \sigma}(T_w)$ of T_w in $G_w^0 \times \sigma$ is equal to the centralizer $Z_{G_w^0 \times \sigma}(T)$ of T in $G_w^0 \times \sigma$, and contains the centralizer $Z_{G^0 \times \sigma}(T)$ of T in $G^0 \times \sigma$ as a dense subset. Choose $x \times \sigma$ in $Z_{G^0 \times \sigma}(T)$ sufficiently near $x_0 \times \sigma$ so that T is $Z_{G^0}(x \times \sigma)$ and $\Phi(x \times \sigma, f_w) \neq 0$. Here we denote our local function f' by f_w.

The Tate–Nakayama theory [Ta] implies that the natural homomorphism $H^1(F, T) \to H^1(\mathbf{A}^w, T)$ is an isomorphism, where $T = Z_{G^0}(x \times \sigma)$ is a torus and $H^1(\mathbf{A}^w, T)$ is the pointed direct sum of the groups $H^1(F_v, T)$ over all places $w \neq v$. If $x' \times \sigma$ is an element of G which is stably conjugate to $x \times \sigma$ in G_v for some place v, namely $x \times \sigma$ and $x' \times \sigma$ are conjugate in $\boldsymbol{G}_v^0$, then they are conjugate in $G^0(F')$ where F' is a finite extension of F, and hence in $\boldsymbol{G}^0$; consequently they are stably conjugate in G^0. If $x' \times \sigma$ is an element of $G^0 \times \sigma$ which is conjugate to $x \times \sigma$ in G_v for all $v \neq w$, then it determines the identity element in $H^1(\mathbf{A}^w, T)$, hence in $H^1(F, T)$, and hence it is conjugate to $x \times \sigma$.

Let V be a finite set of places of F where T is elliptic, of cardinality larger than the rank of G, not including the place w of the proposition. At each v in V choose f_v

in $C(G_v)$ which is supported on the elliptic regular set of $G_v^0 \times \sigma$, and with $\Phi(x \times \sigma, f_v) \neq 0$. Choose f in $C(G(\mathbf{A}))$ (see §2) whose components at v in V are those chosen above, and whose component at w is the function of the proposition. As noted in §3 there are only finitely many conjugacy classes in $G(\mathbf{A})$ with representative $x' \times \sigma$ in $G^0 \times \sigma$, necessarily elliptic regular, with $\Phi(x' \times \sigma, f) \neq 0$. We can replace finitely many of the components f_v (for $v \neq w$) of f by their product with the characteristic function of a small open closed neighbourhood of the orbit of $x \times \sigma$ in $G_v^0 \times \sigma$, to assure that if $\Phi(x' \times \sigma, f) \neq 0$ for $x' \times \sigma$ in G, then $x' \times \sigma$ is conjugate to $x \times \sigma$ in G_v^0 for all $v \neq w$. Consequently, if $\Phi(y, f) \neq 0$ for y in G then it is conjugate to $x \times \sigma$.

We can now apply the trace formula identity (3.1) of Corollary 3.1, since f is chosen to satisfy the requirements of this Corollary. The assumption of the proposition implies that the right side of (3.1) is equal to zero, since $\operatorname{tr} \pi(f) = 0$ for all π, while the left side of (3.1) is a non-zero scalar multiple of $\Phi(x \times \sigma, f)$. Since $\Phi(x \times \sigma, f_v) \neq 0$ for all $v \neq w$ by the choice of f_v, we conclude that $\Phi(x \times \sigma, f_w) = 0$, as required.

In the non-twisted case Kazhdan [K; Appendix] had proven the Proposition using the Deligne–Kazhdan trace formula, on producing a supercuspidal function f_v with $\Phi(x, f_v) \neq 0$ for the given x in $G_v^0 \times \sigma$, so that $\Phi(x', f_v) = 0$, for any x' in G_v which is stably conjugated but not conjugate to x in G_v. This construction holds also in the case of base-change, at a place which splits. However in other twisted cases it is more difficult to construct supercuspidal functions. For example, there are no σ-invariant supercuspidal G_v^0-modules if F_v has odd residual characteristic, and $G_v^0 = \mathrm{PGL}(3, F_v)$, $\sigma(x) = J{}^t x^{-1} J$ is the example of §1.

4.1. Proposition. *For f in $C(G)$ with $\operatorname{tr} \pi(f) = 0$ for all tempered G-modules π, we have $\Phi(x, f) = 0$ for all regular x in G.*

Remark 1. Theorem 10 of [H] asserts that $F(f)$ is uniquely determined by its values on the regular set. Using this, the Propositions imply that $\Phi(x, f) = 0$ for *all* x in G.

Remark 2. The Propositions are proven for a local field F of characteristic zero. They hold also for a local field F of positive characteristic by virtue of Theorem A of Kazhdan [K′]. Moreover, Theorem B of [K′] implies that if $\operatorname{tr} \pi(f) = 0$ for all G-modules π then f lies in the linear span of the commutators $[f_1, h_1] = f_1 * h_1 - h_1 * f_1$, where f_1 lies in $C(G)$ and h_1 in $C(G^0)$.

Proof. As in §1, a minimal parabolic subgroup $P_0 = M_0 N_0$ is fixed, and $P = MN$ denotes a parabolic subgroup containing P_0, such that M contains M_0. Let $A = A_M$ be the split component in the center of M, $\mathfrak{A} = \mathfrak{A}_P = \mathrm{Hom}(X(M)_Q, \mathbf{R})$ where $X(M)_Q = X(A)_Q$ is the group of characters of M defined over Q, and $H: M \to \mathfrak{A}$ the homomorphism defined by $\langle H(m), \chi \rangle = \log|\chi(m)|$ for all χ in

$X(M)_Q$. If ρ is an irreducible M-module with central character ω_ρ, define λ_ρ in $\mathfrak{A}^* = X(M)_Q \otimes \mathbf{R}$ by $\langle \lambda_\rho, H(m)\rangle = \log|\omega_\rho(m)|$ (m in M). This ρ is called *positive* if $\langle \lambda_\rho, \alpha\rangle$ is positive for every root α of A in N, and *essentially tempered* if $\rho \otimes \chi$ is tempered for some χ in $X(M)_Q$. The classification theorem of [BW], XI, (2.11), asserts that (i) if ρ is essentially tempered and positive, then the unitarily induced G-module $I_P(\rho)$ has a unique irreducible submodule $J_P(\rho)$, and (ii) any irreducible G-module is so obtained, and (iii) $J_P(\rho)$ is equivalent to $J_{P'}(\rho')$ if and only if $P = P'$ and ρ is equivalent to ρ'. The proof in the twisted case follows closely that given in [BW] for the connected case. A G-module is called *standard* if it is equivalent to $I_P(\rho)$ with a positive M-module ρ. By virtue of the relation $\mathrm{tr}(I_P(\rho))(f) = \mathrm{tr}\,\rho(f_N)$ (which follows from a standard computation of a character of an induced representation, easily adapted to the twisted case), the fact that $\mathrm{tr}\,\rho(f_N) = 0$ if and only if $\mathrm{tr}(\rho \otimes \chi)(f_N) = 0$ for any χ in $X(M)_Q$, and the relation $F(m, f) = F^M(m, f_N)$ for m in M regular in G, the proposition follows at once from Proposition 4 and the following Lemma. Let $R_{\mathbf{Z}}(G)$ be the ("Grothendieck") free abelian group generated by Irr G, where Irr G is the set of equivalence classes of (admissible) irreducible G-modules. Put also $R(G) = R_{\mathbf{Z}}(G) \otimes \mathbf{C}$. Then we have

Lemma. *The set of standard G-modules is a basis of $R(G)$ over* $\mathbf{Z}$.

Proof. Given an irreducible G-module π, it is equivalent to $J_P(\rho)$ for some pair (P, ρ). If π' is a submodule of $I_P(\rho)$ inequivalent to π, and $\pi' = J_{P'}(\rho')$, then $\lambda_{\rho'} < \lambda_\rho$ for the order $<$ on $\mathfrak{A}^*$, by [BW; XI, (2.13)]. By [BZ′] π' and π have the same cuspidal datum (L, ε), consisting of a Levi subgroup L and an irreducible L-module ε whose modules $\varepsilon_U = r_{L'U}\varepsilon$ of coinvariants (see §6) with respect to any parabolic subgroup $L'U$ of L are zero in the Grothendieck group $R(L')$ of L'. Hence π' lies in a fixed finite set, and by induction on λ_π we may assume that each such π' is a linear combination over $\mathbf{Z}$ of standard G-modules. Consequently $\pi = J_P(\rho) = I_P(\rho) - \Sigma\, \pi'$ also lies in the span of the standard G-modules.

It remains to show that standard modules are linearly independent. Fixing a cuspidal datum (M, ρ), it is shown above that all irreducible G-modules attached to (M, ρ) are linear combinations of standard G-modules attached to (M, ρ), and we obtain a (finite, square) unipotent matrix. Since irreducible G-modules are linearly independent over $\mathbf{C}$, the standard G-modules are linearly independent over $\mathbf{C}$, and the lemma follows.

This completes the proof of the Proposition.

§5. Characters

Let F be a local non-archimedean field of characteristic zero. We now recall some of the fundamental results of Howe [Ho], Harish-Chandra [H] and Kazhdan [K] about Fourier transforms of invariant distributions, characters and orbital integrals. We state this theory in the twisted setting. Detailed proofs of the theorems of [H] are recorded in [Cl] also in the twisted case. The proofs of the twisted analogues of the theorems of [K] follow closely the proofs in the connected case which are amply explained in [K] and will not be reproduced here. The proofs of [K] rely on the results recorded in §§6–7 below. The statements of [K] are independent of §§6–7. We prefer to record these fundamental statements first, as they clarify the relationship between characters and orbital integrals, and delay to §§6–7 the study of induction and restriction which is independent of the work of this §5.

Fix an F-valued symmetric non-degenerate G^0-invariant bilinear form B on the Lie algebra $L(G)$ of G^0, an additive character $\psi \neq 1$ of F, and a Haar measure dX on $L(G)$. The map $\phi \to \hat{\phi}$, where

$$\hat{\phi}(X) = \int_{L(G)} \psi(B(X, Y))\phi(Y)dY,$$

is a linear bijection of the space $C(L(G))$ of locally-constant compactly-supported functions ϕ on $L(G)$, onto itself. A *distribution* T on $L(G)$ is a linear complex-valued function on $C(L(G))$. Its Fourier transform $\hat{T}$ is defined by $\hat{T}(\phi) = T(\hat{\phi})$. For x in G, put $\phi^x(X) = \phi(\mathrm{Ad}(x)X)$ and ${}^xT(\phi) = T(\phi^x)$. T is called *invariant* if ${}^xT = T$ for all x in G. Given a set ω in $L(G)$, let $J(\omega)$ be the space of all invariant distributions T on $L(G)$ which are supported on the closure of $\mathrm{Ad}(G)\omega$. Theorem 3 of [H] asserts

5.1. Proposition. *If ω is compact and T lies in $J(\omega)$, then there exists a locally-integrable function F in $L(G)$ with $T(\phi) = \int_{L(G)} F\phi dX$ for all ϕ in $C(L(G))$.*

Let $Z(X)$ be the centralizer of X in G^0, and dx the unique (up to scalar), G^0-invariant measure on the homogeneous space $G^0/Z(X)$. By a theorem of Deligne and Rao [Rao], the integral

$$\mu_{\mathcal{O}}(\phi) = \int_{G^0/Z(X)} \phi(\mathrm{Ad}(x)X)dx \qquad (\phi \text{ in } C(L(G)))$$

is well-defined; it depends only on (dx and) the orbit $\mathcal{O} = \mathrm{Ad}(G)X$ of X. The Fourier transform $\hat{\mu}_{\mathcal{O}}$ of the measure $\mu_{\mathcal{O}}$ is a function by Proposition 5.1.

Let N be the set of all nilpotent elements in $L(G)$. It is a union of finitely many ("nilpotent") G^0-orbits. Let ω be a compact set in $L(G)$. The local behaviour of the Fourier transform of T in $J(\omega)$ is described by [H], Theorem 4:

5.2. Proposition. *There exists a G-domain D (open closed G^0-invariant subset) of $L(G)$ which contains 0, and a "nilpotent" distribution μ (a linear* *combination with complex coefficients, depending on T, of the $\mu_{\mathcal{O}}$, where $\mathcal{O}$ are the nilpotent orbits), so that $\hat{T} = \hat{\mu}$ on D.*

Fix a Haar measure dx on G. For a smooth G-module $\pi : G \to \operatorname{Aut} V$ we defined in §2 the endomorphism $\pi(f)$ of V by $\pi(f) = \int_G f(x)\pi(x)dx$; clearly, $\pi(f)$ depends on dx. If π is admissible then $\pi(f)$ has finite rank, and its trace is denoted by $\operatorname{tr} \pi(f)$. It is easy to see that if π is admissible and irreducible then there exists a complex-valued conjugacy-invariant locally-constant function χ on the regular set of G such that $\operatorname{tr} \pi(f) = \int_G f(x)\chi(x)dx$ for every f in $C(G)$ which is supported on the regular set of G. The function χ is called the *character* of π. Note that $\operatorname{tr} \pi(f)$ depends on dx, but $\chi(x)$ is independent of dx. Theorem 1 of [H] asserts

5.3. Proposition. *The character χ of an admissible irreducible G-module π is a locally integrable function on G. In particular,* $\operatorname{tr} \pi(f) = \int f(x)\chi(x)dx$ *for every f in $C(G)$.*

Theorem 5 of [H] describes the local behaviour of the character χ at a semi-simple element g in G.

5.4. Proposition. *Suppose that g is a semi-simple element in G. Let M and $L(M)$ be the centralizers of g in G^0 and $L(G)$. Then there exists a neighborhood V of 0 in $L(M)$, and an M-invariant "nilpotent" distribution μ on $L(M)$, so that $\chi(g \exp X) = \hat{\mu}(X)$ for all X in V.*

The above results of Harish-Chandra [H] are based on the technique developed by Howe [Ho] in the case of $G = \mathrm{GL}(n)$. Kazhdan [K] showed that the above local behaviour in fact characterizes the characters, and orbital integrals, at least on the elliptic set. This characterization extends to the entire (not necessarily elliptic) set G by Proposition 7 below.

To describe Kazhdan's theory [K], let S be the space of conjugation invariant functions s on G, such that for every semi-simple g in G there is a neighborhood V of 0 in the Lie algebra $L(Z(g))$ of the centralizer $Z(g)$ of g in G^0, and a $Z(g)$-invariant distribution μ on $L(Z(g))$ supported on the nilpotent set of $L(Z(g))$, so that $s(g \exp X) = \hat{\mu}(X)$ for all regular X in V. Let $\eta(X)$ be the coefficient of the smallest possible power of t in the polynomial $\det(t - \operatorname{ad}(X))$ (X in $L(G)$). η is a non-zero polynomial function on $L(G)$, and X is called *regular* if $\eta(X) \neq 0$.

Let $S_e = S_e(G)$ be the space of functions on the elliptic subset of $G^0 \times \sigma$ in G obtained by restriction of the functions of S.

Let $\operatorname{Irr} G$ be the set of equivalence classes of admissible irreducible G-modules, $R_{\mathbf{Z}}(G)$ the free abelian group generated by $\operatorname{Irr} G$, and put $R(G) = R_{\mathbf{Z}}(G) \otimes_{\mathbf{Z}} \mathbf{C}$ for the Grothendieck group of G.

5.4.1. Remark. The quotient of $R(G)$ by the equivalence relation $\pi \sim \pi \otimes \zeta$, where ζ is a character of G/G^0, is naturally isomorphic to the quotient of $R(G^0)$ by the equivalence relation $\pi \sim 0$ if π is an irreducible non-σ-invariant G^0-module.

Let M be a Levi subgroup of a parabolic subgroup $P = MN$ with unipotent radical N, and denote by $i_{GM}\rho$ the G-module unitarily induced from the M-module ρ, which is trivially extended across N. Then I_{GM} extends to a functor from the category of M-modules to the category of G-modules. Its restriction is a homomorphism from $R(M)$ to $R(G)$, and we denote by $R_I(G)$ the span in $R(G)$ of the images of i_{GM} over all $M \neq G$. Put $\bar{R}(G)$ for the quotient $R(G)/R_I(G)$. Denote by χ the character of a member π in $R(G)$; it is a finite linear combination with complex coefficients of characters of irreducible G-modules. Theorem D of Kazhdan [K] asserts

5.5. Proposition. *The map $\bar{R}(G) \to S_e$, $\pi \to \chi$, is an isomorphism.*

In particular, any function on the elliptic set of $G^0 \times \sigma$ whose local behaviour is given by the defining property of S, is the restriction to the elliptic set of a character of a virtual G-module.

Theorem C of [K] gives another characterization of S_e. Let $A(G)$ be the space of f in $C(G)$ whose orbital integrals vanish on the regular non-elliptic set. As in §1, let $J(G)$ be the space of f in $C(G)$ whose orbital integrals vanish on the regular set of G, and $\bar{A}(G)$ the quotient space $A(G)/J(G)$. Theorem C of [K] asserts

5.6. Proposition. *The map $\bar{A}(G) \to S_e, f \to {}'\Phi(f)$, is an isomorphism.*

Recall that $'\Phi(f)$ is defined in §1; $'\Phi(x, f)$ is the product of $\Phi(x, f)$ by the volume $|Z_G(x)/Z|$ at a regular elliptic x in G.

5.7. Corollary. *The spaces $\bar{A}(G)$ and $\bar{R}(G)$ are isomorphic.*

The isomorphism is defined by Propositions 5.5 and 5.6.

As an example, let G be the multiplicative group of a simple algebra A. Then A is a matrix algebra $M(m, D)$ over a division algebra D of rank d central over F, and G is an inner form of the split group $G' = \mathrm{GL}(n, F)$, where $n = md$.

5.8. Proposition. *The space $S_e(G)$ consists of the locally constant functions on the elliptic set of G.*

Proof. A stable conjugacy class in G consists of a single conjugacy class. A semi-simple conjugacy class γ in G is determined by its characteristic polynomial p_γ (which has coefficients in F); and a unipotent conjugacy class determines a conjugacy class of Levi subgroups, namely a partition $\alpha = (m_i)$ of m (here m_i are positive integers with $\Sigma_i\, m_i = m$ and $m_i \geqq m_{i+1}$ (all i)). There is a natural injection of the set of conjugacy classes in G into the set of conjugacy classes in G', denoted by $\gamma \to \gamma'$ and defined by $p_{\gamma'} = p_\gamma$ and $(m_i) \to (dm_i)$. This is an example of a norm map in the sense of Chapter II, §1. Similarly there is an injection of the nilpotent classes in the Lie algebra $M(m, D)$ of G into the set of such classes in $M(n, F)$. The nilpotent orbit $\mathcal{O}$ in $M(m, D)$ determines the partition α of m, and the corresponding standard (upper triangular) parabolic subgroup of G is denoted by P_α. Put $\mathcal{O} = \mathcal{O}_\alpha$ and $\hat{\mu}_\alpha$ for $\hat{\mu}_{\mathcal{O}}$. Let θ_α be the character of the G-module unitarily induced from the trivial P_α-module. Lemma 5 of [Ho], which is stated only for $G' = \mathrm{GL}(n, F)$ but its proof applies to any G as above, asserts that there is a small neighborhood V of zero in $M(m, D)$ such that the Fourier transform $\hat{\mu}_\alpha$ is equal to $\theta_\alpha(\exp X)$ at X for all X in V. This is zero on the set of elliptic regular $\exp X$ if α is not the trivial partition (m) of m. Moreover, the character $\theta_{(m)}$ is identically one. Since the centralizer of any elliptic element in G is of the form $\mathrm{GL}(m', D')$, where D' is a simple algebra central over a field extension F' of F, the proposition follows.

Combining this result (for G and for G') with Proposition 5.6, we obtain the following

Corollary. *For every f in $A(G)$ there exists f' in $A(G')$, and for every f' in $A(G')$ there exists f in $A(G)$, such that ${}'\Phi(\gamma, f) = {}'\Phi(\gamma', f')$ for all regular γ in G and γ' in G' with $p_\gamma = p_{\gamma'}$.*

This proves the assumptions (5.1) and (5.2) in Chapter II below in the special case of our G and G'.

§6. Coinvariants

Let F be a local non-archimedean field, $G = G^0 \rtimes \langle\sigma\rangle$ as in §1, and (π, V) an admissible G-module of finite length. If $P = MN$ is an F-parabolic subgroup with a Levi subgroup M and unipotent radical N (as in §1), then the quotient of V by $\{\pi(n)v - v;\, v \text{ in } V, n \text{ in } N\}$ is an M-module ${}'\pi_N$, since M normalizes N. Denote by ${}''\pi_N$ the image of ${}'\pi_N$ in the Grothendieck group $R(M)$. The (normalized) *M-module π_N of N-coinvariant of π* is defined to be $\delta_P^{-1/2}\,{}''\pi_N$. It is shown in [BZ] that π_N is admissible of finite length. The functor $r_{GM} : \pi \to \pi_N$, from the category $K(G)$ of G-modules to the category $K(M)$ of M-modules, is exact. Let $I_M(\rho)$ be the G-module $\mathrm{Ind}_P^G(\delta_P^{1/2}\rho)$ induced from the $P = MN$-module $\delta_P^{1/2}\rho \otimes 1$. The functor $i_{MG} : \rho \to I_M(\rho)$, from $K(M)$ to $K(G)$, is exact. Then by Frobenius reciprocity we have that $\mathrm{Hom}_G(\pi, I_M(\rho)) = \mathrm{Hom}_M(\pi_N, \rho)$ for all irreducible M-modules ρ and G-modules π. Hence $\pi_N \neq 0$ implies that π is a subquotient of $I_M(\pi_N)$. Note that since π_N is an M-module, its restriction π_N^0 to $M^0 = M \cap G^0$ is a σ-invariant M^0-module.

We shall use the following non-connected, or twisted, variant of the theorem [C] of Deligne–Casselman. Let A be a maximally split torus in G^0 which is

σ-invariant (thus $\sigma(A)=A$), B a σ-invariant minimal parabolic subgroup of G^0 containing A, Δ the set of roots of A in B. Fix a σ-invariant lattice L in A so that $|\alpha(\lambda)|=1$ if and only if $\alpha(\lambda)=1$ for all λ in L and α in Δ, and so that A/L is compact. Put L^- for the set of λ in L with $|\alpha(\lambda)| \leqq 1$ for all α in Δ. For any semi-simple $t=t_0 \times \sigma$ in $G^0 \times \sigma$, there exists a positive integer m, and y in G^0, so that $yt^{ml}y^{-1}=\lambda s$, where λ lies in L^-, and s is a compact element of G^0 (the closure of the group generated by s is compact). Let P_λ^0 be the standard (containing B) parabolic subgroup of G^0 whose Levi component M_λ^0 is the centralizer $Z_{G^0}(\lambda)$ of λ in G^0, and put $P_t^0=M_t^0N_t^0$ for $y^{-1}P_\lambda^0 y$. We shall be interested here only in a special case, where P_t^0 is σ-invariant, and then we put $P_t=P_t^0 \rtimes \langle\sigma\rangle$.

Note that our definition of the parabolic subgroup P_t is the same as in [C]. To recall the definition of [C], put $A_\theta = \bigcap \ker\alpha$ (α in θ) for any subset θ of Δ. Denote by A^- the set of x in A with $|\alpha(x)| \leqq 1$ for all α in Δ. Given a semi-simple $t=t_0 \times \sigma$ in $G^0 \times \sigma$ with $yt^{ml}y^{-1}=as$ for a in A^- and a compact element s, let θ be the set of α in Δ with $|\alpha(a)|=1$. Denote by M_θ the centralizer in G^0 of the torus A_θ. Then $M_\theta = M_\lambda^0$.

Choosing the sequence $\{K_i\}$ of open compact subgroups from [C], Lemma 2.1, to be σ-invariant, the proof of [C], Theorem 5.2, extends to the twisted case, and asserts

6.1. Proposition. *Let $t=t_0\times\sigma$ be a regular element of G so that $P=P_t=P_t^0\times\langle\sigma\rangle$ is a parabolic subgroup. Then $\chi(\pi)(t)=\chi('\pi_N)(t)(=\chi(''\pi_N)(t))$. Since $\Delta(t)=\Delta_M(t)\delta_P(t)^{-1/2}$, we have $(\Delta\chi(\pi))(t)=(\Delta_M\chi(\pi_N))(t)$ for such t.*

Here $\chi(\pi)$ denotes the character of π.

We now recall Lemma 5.1 of [C], in our non-connected, or twisted case. We consider only the case where t_0 lies in A, in which case $t^l=t_0\sigma(t_0)\cdots\sigma^{l-1}(t_0)$ (where $t=t_0\times\sigma$) is σ-invariant. We choose t_0 so that t^l lies in A^-; then the associated λ in L is σ-invariant and lies in L^-. Then $M_t^0=M_\lambda^0$ and P_t^0 are σ-invariant, and we put $P=P_{t^l}=MN$. Let C be an open compact σ-invariant congruence subgroup of G^0 with the properties of K_i in [C], Lemma 2.1, and in particular $C=(C\cap\bar N)(C\cap M)(C\cap N)$, where $\bar N$ is the unipotent radical of the parabolic $P_{t^{-l}}=\bar P$ opposite to P. Let f_t be the function in $C(G)$ supported on $ZCtC$ which attains the value $|ZCtC/Z|^{-1}$ on CtC. Let f_t^M be the function on M which is supported on $t(C\cap M)$ and attains the value $\delta_P^{1/2}(t)/|C\cap M|$ there. The proof of [C], Lemma 5.1, extends to the twisted case, and implies the following

6.2. Proposition. *We have* $\operatorname{tr}\pi(f_t)=\operatorname{tr}\pi_N(f_t^M)$ *for any G-module π.*

This Proposition is used below as follows. Let ρ be an irreducible constituent of the M-module π_N. Denote the central character of its restriction ρ^0 to M^0 by ω_ρ, and the character of ρ by χ_ρ. Note that ω_ρ is σ-invariant, since so is ρ^0. We are interested in the function f_t^M on $M^0\times\sigma$ since

$$\operatorname{tr}\rho(f_t^M) = \int \chi_\rho(tx) f_t^M(tx)dx = \omega_\rho(t_0)\delta_P^{1/2}(t)\operatorname{tr}\rho(f_1^M)$$

for $t = t_0 \times \sigma$ with t_0 in the center of M^0, and $\operatorname{tr}\rho(f_1^M)$ is the (non-negative integral) multiplicity of the trivial representation of $C \cap M$ in ρ. On the other hand, f_t is a C-biinvariant function, where C is independent of t.

Let $W(M, G) = N(M, G)/M^0$ be the quotient by M^0 on the normalizer $N(M, G)$ of M in G^0, where $P = MN$ is a parabolic subgroup of G.

6.3. Proposition. *Let $x = x_0 \times \sigma$ be a regular element in G. Then the orbital integral $F(x, f_t)$ vanishes unless x is conjugate to an element of M. For x in M we have*

$$F(x, f_t) = \sum_w F^M(wxw^{-1}, f_t^M),$$

where the sum extends over $W(M, G)$.

Remark. The proof of this Proposition relies on Corollary 7.5. It is given here since the functions f_t, f_t^M do not appear in §7, and it is clear that the work of §7 does not depend on Proposition 6.3.

Proof. Corollary 7.5 implies that given f_t^M there exists a function f on G such that $F(x, f) = 0$ unless x is conjugate in G to an element of M, and when x lies in M then

$$F(x, f) = \sum_w F^M(wxw^{-1}, f_t^M) \qquad (w \text{ in } W(M, G)).$$

The Weyl integration formula and Proposition 6.1 imply that $\operatorname{tr}\pi(f) = \operatorname{tr}\pi_N(f_t^M)$ for every G-module π, since the parabolic subgroup P_x associated with any element x in the support of $F(x, f)$ is $P_t = P$. On the other hand, Proposition 6.2 implies that $\operatorname{tr}\pi(f_t) = \operatorname{tr}\pi_N(f_t^M)$, hence $\operatorname{tr}\pi(f_t) = \operatorname{tr}\pi(f)$, for every G-module π. But then Proposition 4 implies that $F(x, f) = F(x, f_t)$ for every regular x in G, and the proposition follows.

§7. Trace functions

Let F be a local non-archimedean field, $G = G^0 \rtimes \langle\sigma\rangle$, $C(G)$ the space of functions f on G as in §1 which are supported on $G^0 \times \sigma$, $J(G)$ the space of f in $C(G)$ whose orbital integrals vanish at each regular element in G, $A(G)$ the space of f in $C(G)$ whose orbital integrals vanish on every regular non-elliptic element in G, $\bar{C}(G) = C(G)/J(G)$ and $\bar{A}(G) = A(G)/J(G)$. Our final aim in this Section is to prove the following

Proposition. *Let M be a Levi subgroup of G, and f^M an element of $C(M)$ with the property that for every m, m' in M which are regular in G and conjugate to each other by G^0 we have*

$$F^M(m, f^M) = F^M(m', f^M). \tag{7.1}$$

Then there exists f in $C(G)$ with $f_M = f^M$ (f_M is defined at the end of § 1) and $f_L = 0$ for every Levi subgroup L of G which does not contain any conjugate of M.

This Proposition, which concerns "lifting" of orbital integrals from a Levi subgroup of G to G itself, is proven below using representation theoretic techniques, in the spirit of Corollary 5.7. Thus we are to use (the twisted analogue of) the trace Paley–Wiener theorem of [BDK], and the geometric lemma of [BZ'; (2.12)], which we now proceed to state. The proof of the Proposition is new also in the connected case.

As in §5 let $R_{\mathbf{Z}}(G)$ denote the integral Grothendieck group of G-modules of finite length ($R_{\mathbf{Z}}(G)$ is a free abelian group with basis $\operatorname{Irr} G$), put $R(G) = R_{\mathbf{Z}}(G) \otimes \mathbf{C}$ and let $i_{MG}: R(M) \to R(G)$ be the induction homomorphism. As in §6, let $r_{GM}: R(G) \to R(M)$ denote the restriction homomorphism. Let $X(G)$ be the group of unramified characters ψ of G which are trivial on $\langle \sigma \rangle$; equivalently $X(G)$ is the group of σ-invariant unramified characters of G. $X(G)$ acts naturally on $\operatorname{Irr} G$ and $R(G)$ by $\psi: \pi \to \pi\psi$. $X(G)$ has a natural structure of a complex algebraic group, isomorphic to $\mathbf{C}^{\times d}$, where $d = d(G) = \dim X(G)$. As usual, fix a Haar measure dx on G (equivalently on G^0, with the convention that measures on discrete sets assign volume 1 to each point). Our convention in this section is that $C(G)$ consists of compactly supported functions; the passage to the space of functions which are compactly supported modulo the center of G^0 and transform under this center by a fixed character, is trivial, and left to the reader. Thus each function f in $C(G)$ defines a linear form $\beta_f: R(G) \to \mathbf{C}$ by $\beta_f(\pi) = \operatorname{tr} \pi(f)$. It is clear that the form $\beta = \beta_f$ satisfies the following conditions.

(PW(i)) For any Levi subgroup M and irreducible M-module ρ, the function $\psi \to \beta(i_{MG}(\rho\psi))$ is a regular function on the complex algebraic variety $X(M)$.

(PW(ii)) There exists an open compact σ-invariant subgroup K in G^0 which dominates β, in the sense that β vanishes on each G-module π which has no non-zero K-fixed vector.

Let $R^*(G) = \operatorname{Hom}_{\mathbf{C}}(R(G), \mathbf{C}) = \operatorname{Hom}(\operatorname{Irr} G, \mathbf{C})$ be the space of all linear forms on $R(G)$. A form $\beta: R(G) \to \mathbf{C}$ is called *good* if it satisfies (PW(i)) and (PW(ii)), and it is called *trace* if $\beta = \beta_f$ for some f in $C(G)$. The spaces of trace and good forms are denoted by $F_{\mathrm{tr}} = F_{\mathrm{tr}}(G)$ and $F_{\mathrm{good}} = F_{\mathrm{good}}(G)$. The trace Paley–Wiener theorem of Bernstein–Deligne–Kazhdan [BDK] is the following

PW-Theorem. *For every p-adic reductive group G we have $F_{\mathrm{tr}} = F_{\mathrm{good}}$.*

This Theorem describes the image of the natural morphism $\mathrm{tr}: C(G) \to R^*(G)$. As noted at the end of §4, Proposition 4 and Theorem B of [K′] imply that $\ker \mathrm{tr} = [C(G), G(G^0)]$ for every local non-archimedean field (of any characteristic). The PW-Theorem is proven in [BDK] when G is connected. The proof for $G = G^0 \rtimes \langle\sigma\rangle$ follows closely that of [BDK], with straightforward modifications. For example, in the proof [BDK; (5.3)] of [BDK; (3.2)] one takes a σ-invariant good K, and uses the twisted version of [C] given in §6. The proof [BDK; (5.5)] of the combinatorial lemma [BDK; (3.3)] relies on the geometric lemma of [BZ; (2.12)] which we now recall, in the twisted case, as it is used below in the proof of the Proposition.

Recall (§1) that a Levi subgroup M of G contains the fixed Levi component M_0 of the minimal parabolic subgroup P_0. Denote by W_M the quotient by (the connected component) M_0^0 (of M_0) of the normalizer $N(M_0^0, M^0)$ of M_0^0 in M^0 (W_M is the Weyl group of M_0^0 in M^0). Then σ acts on W_M. Since P_0^0 is σ-invariant we have $l(\sigma w) = l(w)$ where l denotes the length function on W_G. Let L be a Levi subgroup of G, and let $W(M^0, L^0)$ denote a set of representatives in W_G, of minimal length, for $W_M \backslash W_G / W_L$. Then σ acts on $W(M^0, L^0)$; denote by $W(M, L)$ the set of σ-invariant elements in $W(M^0, L^0)$. For every w in $W(M, L)$ put $M_w = M \cap wLw^{-1}$ and $L_w = w^{-1}Mw \cap L$. For every w in $W(M^0, L^0)$ put $M_w^0 = M^0 \cap wL^0w^{-1}$ and $L_w^0 = w^{-1}M^0w \cap L^0$.

Geometric Lemma. *For every ρ in $R(L)$ we have*

$$F(\rho) \overset{\mathrm{dfn}}{=} r_{G,M} \circ i_{L,G}(\rho) = \sum_w i_{M_w,M} \circ w \circ r_{L,L_w}(\rho) \qquad (w \text{ in } W(M, L)).$$

Proof. If G is connected ($= G^0$), this is [BZ; (2.12)], which asserts that

$$F'(\rho) \overset{\mathrm{dfn}}{=} r_{G^0,M^0} \circ i_{L^0,G^0}(\rho) \sum_w i_{M_w^0,M^0} \circ w \circ r_{L^0,L_w^0}(\rho) \qquad (w \text{ in } W(M^0, L^0))$$

for every ρ in $R(L^0)$. [BZ] choose an order $w_1, w_2, \ldots, w_k$ with $l(w_i) \geqq l(w_{i+1})$ on the elements of $W(M^0, L^0)$, and define a functorial filtration $F_1' \subset F_2' \subset \cdots \subset F_k' = F'$, with

$$F_i'(\rho)/F_{i+1}'(\rho) = i_{M_{w_i}^0,M^0} \circ w_i \circ r_{L^0,L_{w_i}^0}(\rho).$$

Put $P = MP_0$, $Q = LP_0$. If V is the space of ρ (in $R(L)$), then by definition $i_{L,G}(\rho)$ acts by right translation on the space of the locally constant functions $\psi: G^0 \to V$ such that $\psi(mng) = \delta_Q(m)^{1/2}\rho(m)\psi(g)$ (m in L^0, n in the unipotent radical of Q, g in G^0). Note that

$$([(i(\rho))(\sigma)]\psi)(g) = (\rho(\sigma)\psi)(\sigma^{-1}(g)).$$

Denote by V_i the space of ψ which are supported on $\bigcup_j Q^0 w_j P^0$ (union over $j \leqq i$). Then V_i is P^0-invariant, and $F_i'(\rho)$ is defined in [BZ] to be the image of V_i under r_{G^0,M^0}.

Denote by $A_1, \ldots, A_b$ the orbits under σ in $W(M^0, L^0)$. The elements of an orbit have equal length, denoted $l(A_i)$. We order the orbits such that $l(A_i) \geqq l(A_{i+1})$. Denote the cardinality of A_i by a_i, and order the $w_1, \ldots, w_k$ such that the first a_1 elements lie in A_1, the next a_2 elements are in A_2, and so on. Put $b_i = a_1 + \cdots + a_i$ and $F_i = F'_{b_i}$. Then F_i' is σ-invariant, and extends to an M-module, and $F_i(\rho)/F_{i-1}(\rho)$ is an M-module. The restiction of $F_i(\rho)/\mathrm{F}_{i-1}(\rho)$ to M^0 is a direct sum of a_i M^0-modules which are permuted under the action of σ. If $a_i > 1$ then $F_i(\rho)/F_{i-1}(\rho)$ lies in the linear span of the irreducible non-σ-invariant M^0-modules, hence corresponds to 0 in $R(M)$ by Remark 5.4.1. This completes the proof of the geometric lemma.

Corollary. *For each Levi subgroup M of G put $T_M = i_{M,G} \circ r_{G,M} : R(G) \to R(G)$. Then*

(i) $T_L \circ i_{M,G} = \Sigma_w\, i_{M_w,G} \circ r_{M,M_w}$,

(ii) $T_L \circ T_M = \Sigma_w\, T_{M_w}$,

where $M_w = M \cap w^{-1}Lw$ and w ranges over $W(L, M)$.

Proof. (i) $i_{L,G} \circ r_{G,L} \circ i_{M,G} = \Sigma_w\, i_{L,G} \circ i_{L_w,L} \circ w \circ r_{M,M_w} = \Sigma_w\, i_{L_w,G} \circ w \circ r_{M,M_w}$ is equal to the required expression since $i_{L_w,G} \circ w = i_{M_w,G}$ by [BDK], Lemma 5.4(iii).

(ii) $T_L \circ i_{MG} \circ r_{GM} = \Sigma_w\, i_{M_w,G} \circ r_{M,M_w} \circ r_{G,M} = \Sigma_w\, i_{M_w,G} \circ r_{G,M_w} = \Sigma_w\, T_{M_w}$.

The proof of the PW-Theorem in the twisted case can be completed now as in [BDK], and we proceed to establish the Proposition. Denote the pairing $R^*(G) \times R(G) \to \mathbf{C}$ by $(\beta, \pi) \to \langle \beta, \pi \rangle$. Let

$$i^*_{MG} : R^*(G) \to R^*(M) \quad \text{and} \quad r^*_{GM} : R^*(M) \to R^*(G)$$

be the morphisms adjoint to i_{MG} and r_{GM}. Note that $\bar{C}(G)$ is a subspace of $R^*(G)$; the function f defines the form $\beta = \beta_f : \pi \to \langle \beta, \pi \rangle = \operatorname{tr} \pi(f)$. Put $\langle f, \pi \rangle$ for $\langle \beta, \pi \rangle$ in this case.

7.2. Lemma. *For every M, i^*_{MG} maps $\bar{C}(G)$ to $\bar{C}(M)$, and r^*_{GM} maps $\bar{C}(M)$ to $\bar{C}(G)$.*

Proof. For f in $\bar{C}(G)$, f_M (defined in §1) satisfies $\langle f_M, \pi_M \rangle = \langle f, i_{MG}\pi_M \rangle$ for every π_M in $R(M)$ by virtue of a standard formula for a character of an induced representation. By virtue of Proposition 4 we have $i^*_{MG} f = f_M$, as required. For the second part of the lemma, for every f^M in $\bar{C}(M)$ define a form $\beta = r^*_{GM}(f^M)$ in $R^*(G)$ by $\langle \beta, \pi \rangle = \langle f^M, r_{GM}\pi \rangle$ (π in $R(G)$). This is clearly a good form, hence a trace form by the PW-Theorem, namely $r^*_{GM}(f^M)$ is a function in $\bar{C}(G)$, and the (second part of the) lemma follows.

Corollary. *The homomorphisms* $i_{MG}: R(M) \to R(G)$ *and* $R_{GM}: R(G) \to R(M)$ *admit adjoints* $i^*_{MG}: \bar{C}(G) \to \bar{C}(M)$ *and* $r^*_{GM}: \bar{C}(M) \to \bar{C}(G)$.

A function f in $\bar{C}(G)$ is called *discrete* if $i^*_{MG} f = 0$ for all Levi $M \neq G$. By Proposition 4 the space of discrete functions in $\bar{C}(G)$ is $\bar{A}(G)$.

Combinatorial Lemma. *For each proper Levi subgroup* M *of* G *there is a rational number* c_M *such that* $f^d = f - \Sigma_{M \neq G} c_M r^*_{GM} \circ i^*_{MG}(f)$ *is discrete for every* f *in* $\bar{C}(G)$.

Proof. This is an analogue of Lemma 3.3 of [BDK]. In [BDK] a form β in $R^*(G)$ is called discrete if $i^*_{MG}\beta = 0$ for all $M \neq G$. Lemma 3.3 of [BDK] asserts that there are c_M such that for each β in $R^*(G)$ the form

$$\beta^d = \beta - \sum_{M \neq G} c_M r^*_{GM} \circ i^*_{MG}(\beta)$$

is discrete. But Lemma 7.2 asserts that if f lies in $\bar{C}(G)$ then f^d lies in $\bar{C}(G)$, not only in $R^*(G)$, hence it is in $\bar{A}(G)$, as asserted.

7.3. Theorem. $\bar{C}(G)$ *is the direct sum over a set of representatives* N *for the conjugacy classes of Levi subgroups in* G, *of* $r^*_{GM}(\bar{A}(M))$.

Proof. To show that $\bar{C}(G)$ is the sum of $r^*_{GM}(\bar{A}(M))$, we assume by induction that this claim holds for every proper Levi subgroup M of G. Namely we assume that for each $M \neq G$, and for each $L \subset M$, there is a rational number $c_{M,L}$ with the following property. Given f^M in $\bar{C}(M)$, there are $f^{M,L}$ in $\bar{A}(L)$ for each $L \subset M$, such that $f^M = \Sigma_{L \subset M} c_{M,L} r^*_{M,L} f^{M,L}$. Hence, given f in $\bar{C}(G)$ there are $f^{M,L}$ in $\bar{A}(L)$ for every $M \neq G$ and $L \subset G$ with $i^*_{MG} f = \Sigma_{L \subset G} c_{M,L} r^*_{M,L} f^{M,L}$. Using the Combinatorial Lemma we conclude that

$$f = f^d + \sum_{M \neq G} c_M r^*_{GM}(i^*_{MG}(f)) = f^d + \sum_{M \neq G} c_M r^*_{GM} \sum_{L \subset M} c_{M,L} r^*_{ML} f^{M,L}$$

$$= f^d + \sum_{L \neq G} r^*_{GL} \left(\sum_M c_M c_{ML} f^{M,L} \right),$$

where M ranges over the $M \neq G$ which contain L, as required.

To prove that the sum is direct, note that if f^M lies in $\bar{C}(M)$, then by the Geometric Lemma for each Levi subgroup L and ρ in $R(L)$ we have

$$\begin{aligned} \langle i^*_{L,G} r^*_{G,M} f^M, \rho \rangle &= \langle f^M, r_{G,M} \circ i_{L,G}(\rho) \rangle \\ &= \sum_{w \in W(M,L)} \langle f^M, i_{M_w,M} \circ w \circ r_{L,L_w}(\rho) \rangle. \end{aligned} \tag{7.4}$$

If f^M lies in $\bar{A}(M)$ and w contributes a non-zero term in the sum, then $M_w = M$, namely $L \supset w^{-1}Mw$. Consequently (7.4) is zero if L contains no conjugate of M;

and if L is conjugate to M, say $L = s^{-1}Ms$ for some s in $W(M,L)$, then (7.4) is equal to $\langle f^M, s\rho\rangle$. Now, if $f = \Sigma_M r^*_{GM} f^M$ is zero, where the f^M lies in $\bar{A}(M)$, then choose L to be a minimal Levi subgroup (up to conjugation) for which $f^L \neq 0$ in this sum. Then $i^*_{L,G} f = f^L$, and $f = 0$ implies that $f^L = 0$. This contradiction completes the proof of the theorem.

Proof of Proposition. Given $M \neq G$ and f^M in $\bar{C}(M)$, since $\bar{C}(M) = \bigoplus r^*_{M,L}(\bar{A}(L))$ (sum over L in M) by Theorem 7.3, we may assume that $f^M = r^*_{M,N} f^N$ for some Levi N in M and f^N in $\bar{A}(N)$. We claim that the product of $f = r^*_{G,M} f^M = r^*_{G,N} f^N$ by a scalar which depends only on N, M and G, has the properties required by the proposition. Indeed, as in (7.4) for each ρ in $R(N)$ we have

$$\langle i^*_{L,G} r^*_{G,N} f^N, \rho\rangle = \sum_{w \in W(N,L)} \langle f^N, i_{N_w,N} \circ w \circ r_{L,L_w}(\rho)\rangle$$

and it suffices to consider w with $L \supset w^{-1}Nw$, since f^N lies in $\bar{A}(N)$. Hence if L contains no conjugate of M then the sum is empty and $i^*_{L,G} f = f_L$ is zero, as required. If $L = M$, our sum becomes the sum over all w in $W(N, M)$ with $w^{-1}Nw \subset M$ of $\langle f^N, w \circ r_{M,M_w}(\rho)\rangle$. The condition (7.1) implies that each of the summands is equal to $\langle f^N, r_{M,N}(\rho)\rangle = \langle r^*_{M,N} f^N, \rho\rangle = \langle f^M, \rho\rangle$, hence $i^*_{M,G} f$ is equal to f^M up to a multiple by the cardinality of the set of w in $W(N, M)$ with $w^{-1}Nw \subset M$. The proposition follows.

The Proposition implies that a function f^M in $C(M)$ can be "lifted" to a function f in $C(G)$ with the "same" orbital integrals on the regular conjugacy classes of G which intersect M. The orbital integrals of f are not necessarily zero on x in G whose conjugacy class does not intersect M. However, we have

7.5. Corollary. *Suppose that f^M has the property that $F^M(m, f^M)$ is supported on the set of m in M with $|\alpha(m)| \neq 1$ for every root of the split component A_M of the center of M in N. Then f can be chosen to have the property that $F(x, f)$ is zero unless x is conjugate in G to an element of M.*

Proof. Let S_M denote the support of $F^M(f^M)$ in M, and put $S = (S_M)^G = \{g^{-1}sg; g \text{ in } G, s \text{ in } S_M\}$. Then S is open and closed in G. Replace the f obtained in the Proposition by its product with the characteristic function of S to obtain the function f of the corollary.

Finally we recall the germ expansion of orbital integrals of f in $C(G)$. Let $\bar{O}(x)$ be the closure of the conjugacy class $O(x)$ of x in G. It is the disjoint union of the conjugacy classes $O(su_i)$ $(1 \leqq i \leqq r)$ of elements su_i with semi-simple part s, so that (1) $u_1 = 1$, (2) for each t $(1 \leqq t \leqq r)$ the union $O_t = \bigcup_{i=1}^{t} O(su_i)$ is closed, (3) $O(su_t)$ is open in O_t. The (closed) set A_s of elements in G whose semi-simple part is conjugate to s is of the form $\tilde{O}(x)$ for some x, and there are f_i in $C(G)$ with $F(su_j, f_i) = \delta_{ij}$, and $f_i = 0$ on $O(su_j)$ for $j < i$. The germ expansion asserts

7.6. Proposition. (a) *Given f in $C(G)$ and a semi-simple s in G, there exists a neighborhood V_f of s in G so that*

$$F(x, f) = \sum_i F(x, f_i)F(su_i, f) \qquad (\textit{all } x \textit{ in } V_f).$$

(b) *Conversely, given a function $F(x)$ on G such that for each semi-simple s in G there is a neighborhood V of s in G with*

$$F(x) = \sum_i F(x, f_i)F(su_i) \qquad (x \textit{ in } V),$$

there exists f in $C(G)$ with $F(x) = F(x, f)$.

We do not prove this result here, but simply note that it can be deduced from the uniqueness of the Haar measure. A proof is given, e.g., in Vigneras [V].

Chapter II. Comparison

Let F be a local non-archimedean field of characteristic zero. Let G'' be a quasi-split reductive group, defined over F, of the form $G^0 \rtimes \langle\sigma\rangle$ (it is denoted by G in Chapter I).

§1. Stability

The stable conjugacy class of x in G'' is defined in Chapter I, §I.2: x' is stably-conjugate to x if there is y in $\boldsymbol{G}^0$ with $x' = \mathrm{Ad}(y)x$. Put T for the centralizer $Z_{G^0}(x)$ of x in G^0. We shall be interested here only in regular x, and then T is an F-torus. The conjugacy classes within the stable class of x are parametrized by the (finite) set

$$B(T/F) = \ker[H^1(F, T) \to H^1(F, G^0)].$$

If x, x' are stably-conjugate, then $Z_{G^0}(x') = \mathrm{Ad}(y)Z_{G^0}(x)$ is isomorphic to $Z_{G^0}(x)$ over $\bar{F}$. Hence a differential form of maximal degree on $Z_{G^0}(x)$ can be transferred to $Z_{G^0}(x')$, yielding compatible Haar measures on $Z_{G^0}(x)$ and $Z_{G^0}(x')$.

Let $\{\mathrm{Ad}(b)x;\, b \text{ in } B(T/F)\}$ be a set of representatives for the conjugacy classes within the stable conjugacy class of the regular element x of $G^0 \times \sigma$.

Definition. Let F be a function on the regular conjugacy classes in G'' which intersect $G^0 \times \sigma$. The *stable* function F' associated with F is defined by

$$F'(x) = \sum_b F(\mathrm{Ad}(b)x).$$

$F'(x)$ depends only on the stable conjugacy class of the regular x. In particular, for any f in $C(G'')$ (notations of §I.1) we have the stable orbital integral $F'(x, f)$ of f, and the function $\Phi'(f)$.

The stable orbital integrals are introduced for purposes of comparison between the group G'', and a reductive connected F-group G, such that the following holds. Let $G_\sigma^0 = Z_{G^0}(\sigma)$ denote the group of σ-invariant elements in G^0. Let G and H be F-groups. An isomorphism $\psi: G \to H$ over $\bar{F}$ is called an *inner twisting* if for every τ in $\mathrm{Gal}(\bar{F}/F)$ there is g_τ in G such that $(\tau\psi)^{-1} \circ \psi = \mathrm{ad}(g_\tau)$. If such ψ exists then G and H are called inner forms. Suppose that G is an inner form of G_σ^0, and fix an inner twisting ψ. Fix a maximal split torus A in G. It can be identified with a torus A_σ^0 of G_σ^0 via ψ. Each Levi subgroup M of G containing A corresponds by ψ to a Levi subgroup M_σ^0 of G_σ^0 containing A_σ^0. We assume that M_σ^0 is the group of σ-invariant elements in the σ-invariant Levi subgroup M^0 of G^0, obtained as the centralizer in G^0 of the center $Z(M_\sigma^0)$ of M_σ^0. Hence fix a maximally split σ-invariant torus A^0 (containing A_σ^0) in G^0; A^0 is denoted in §I.6 by A. Fix a lattice L as in §I.6, so that each M^0 is of the form $M_\lambda = Z_{G^0}(\lambda)$ for λ in L^-.

In every known comparison situation (base-change, symmetric-square, metaplectic correspondence, inner-twists) there exists a map N which we call a *norm map*, with at least the following propeties. N is a bijection from a subset $'S''$ of the set S'' of stable conjugacy classes of regular elements in G'' with representatives in $G^0 \times \sigma$, to a subset $'S$ of the set S of stable conjugacy classes of regular elements in G, such that (1) $Z_{G^0}(x)$ and $Z_G(Nx)$ are inner forms, (2) $N(x \times \sigma) = \psi^{-1}(x \times \sigma)^l$ for x in A_σ^0 (l is the order of σ), (3) x has a representative in $M^0 \times \sigma$ if and only if Nx has a representative in M, (4) at least one of the subsets $'S$, $'S''$ is the entire set S, S''.

We use (1) to relate measures on the two groups of (1). Fix a norm map N.

Let $W(M, G) = N(M, G)/M$ be the quotient by M of the normalizer $N(M, G)$ of M in G. If $M'' = M^0 \rtimes \langle\sigma\rangle$, let $W(M'', G'')$ be the quotient by M^0 of the normalizer of M'' in G^0. Given f^M in $C(M)$, let ${}^MF(f^M)$ be the conjugacy class function on the set of regular x in G which attains the value 0 unless (a conjugate of) x lies in M when we put

$${}^MF(x, f^M) = \sum_w F^M(wxw^{-1}, f^M) \qquad (w \text{ in } W(M, G)).$$

Similarly, for ϕ^M in $C(M'')$ we put ${}^MF(x, \phi^M) = 0$ for a regular x in $G^0 \times \sigma$ unless (a conjugate of) x lies in $M^0 \times \sigma$, when we put

$${}^MF(x, \phi^M) = \sum_w F^M(wxw^{-1}, \phi^M) \qquad (w \text{ in } W(M'', G'')).$$

In particular ${}^{G}F$ is $F = F^{G}$. Recall that ${}^{M}F'$ indicates the stable function on G associated with ${}^{M}F$.

Definition. The functions ϕ^{M} in $C(M'')$ and f^{M} in $C(M)$ are called *matching* if (1) ${}^{M}f'(s, f^{M})$ is zero for any s in $S - 'S$, (2) ${}^{M}F'(s,\phi^{M})$ is zero for any s in $S'' - 'S''$, (3) ${}^{M}F'(s, \phi^{M})$ is equal to ${}^{M}F'(Ns, f^{M})$ for all s in $'S''$.

§2. Base change

Here we describe a well-known example of a norm map. Let F be a perfect field, l a positive integer, E a cyclic extension of F of degree l, and $\boldsymbol{G}_\sigma$ a reductive connected group over F whose derived group is simply connected. Let $\boldsymbol{G}^0 = \mathrm{Res}_{E/F}\boldsymbol{G}_\sigma$ be the F-group obtained from $\boldsymbol{G}_\sigma$ on restricting scalars from E to F. It can be realized as $\boldsymbol{G}^0 = \boldsymbol{G}_\sigma \times \cdots \times \boldsymbol{G}_\sigma$ (l copies), where $\mathrm{Gal}(\bar{F}/F)$ acts as follows. Fix a generator $\tilde{\sigma}$ of $\mathrm{Gal}(E/F)$, and put $\sigma(x_1, x_2, \ldots, x_l) = (x_2, \ldots, x_l, x_1)$. Then $\tau(x_1, x_2, \ldots) = \sigma^i(\tau x_1, \tau x_2, \ldots)$ if $\tau \mid E = \tilde{\sigma}^i$ $(0 \leqq i < l)$, for any τ in $\mathrm{Gal}(\bar{F}/F)$. With this definition $G^0(E)$ is $G_\sigma(E) \times \cdots \times G_\sigma(E)$ (l copies), and $G^0 = G^0(F)$ is isomorphic to $G_\sigma(E)$. Put $\boldsymbol{G}''$ for the semi-direct product of $\boldsymbol{G}^0$ and the cyclic group $\langle \sigma \rangle$ of order l generated by σ. It is a non-connected reductive group defined over F. It is clear that $x_0 = (y, \tilde{\sigma}y, \ldots)$ and $x_0' = (y', \tilde{\sigma}y', \ldots)$ are σ-conjugate elements of G^0 (namely $x = x_0 \times \sigma$ and $x' = x_0' \times \sigma$ are conjugate by an element of G^0) if and only if y and y' are $\tilde{\sigma}$-conjugate (namely there exists z in $G_0(E)$, with $y' = zy\tilde{\sigma}(z^{-1})$) elements of $G_\sigma(E)$.

To define a norm map take $x = x_0 \times \sigma$, $x_0 = (y, \tilde{\sigma}y, \ldots)$ with y in $G_\sigma(E)$, and consider $z = y\tilde{\sigma}(y) \cdots \tilde{\sigma}^{l-1}(y)$. Since $\tilde{\sigma}(z) = y^{-1}zy$, the conjugacy class of z in $\boldsymbol{G}_\sigma$ is defined over F. Let $\boldsymbol{G}$ be the quasi-split inner form of $\boldsymbol{G}_\sigma$, and fix an inner twisting $\psi: \boldsymbol{G}_\sigma \to \boldsymbol{G}$. It is an isomorphism defined over a finite extension of F. Then the conjugacy class of $\psi(z)$ in $\boldsymbol{G}$ is defined over F, and contains an element Nx in G [Ko], Theorem 4.4. Further, the map $x \to Nx$ induces an injection from the set of stable conjugacy classes in G'' with representatives in $G^0 \times \sigma$, into the set of stable conjugacy classes in G ([Ko], Proposition 5.7). Note that the centralizer $Z_{G^0}(x)$ of $x = x_0 \times \sigma$ in G^0 is isomorphic over F to the $\tilde{\sigma}$-centralizer $Z^{\sigma}_{G^0}(y) = \{g \text{ in } G^0; \tilde{\sigma}_y(g) = g\}$, where $\tilde{\sigma}_y(g) = y\tilde{\sigma}(g)y^{-1}$, of y in G^0. This is an inner form of $Z_G(Nx)$, and in comparing orbital integrals we take compatible Haar measures on $Z_{G^0}(x)$ and $Z_G(Nx)$.

In the special case $l = 1$ we obtain an injection N from the set of stable conjugacy classes in G_σ to the set of stable conjugacy classes in its quasi-split form G.

These definitions are particularly simple in the case of $G = \mathrm{GL}(n)$, where the stable conjugacy class consists of a single conjugacy class. Other norm maps can be defined using the outer automorphism $\sigma(x) = J^{t}x^{-1}J$ of G; see [Sym].

When G'' is connected, and G_{sc} is the simply-connected covering group of the derived group of G'', we have $H^1(F, G_{sc}) = \{0\}$ when F is local non-archimedean. Hence if T_{sc} is the inverse image of T in G_{sc}, $B(T/F)$ (see §1) is the image of $H^1(F, T_{sc})$ in $H^1(F, T)$. On the other hand, $G^0 = \mathrm{Res}_{E/F} G_\sigma$ has $H^1(F, G^0) = H^1(E, G_\sigma)$ and $B(T/F)$ is $H^1(F, T)$ if $H^1(E, G_\sigma) = \{0\}$.

§3. Discrete series

Let F be a local non-archimedean field of characteristic zero, $G'' = G^0 \rtimes \langle\sigma\rangle$ a reductive F-group and G_σ the centralizer $Z_{G^0}(\sigma)$ of σ in G^0. Let π'' be an admissible G''-module of finite length (see §I.2). By a *central exponent* of π'' with respect to a Levi subgroup M'' of G'' (see §I.1) we mean the central character of an irreducible constituent of the module π''_N of coinvariants (see §I.6) of π'' with respect to any parabolic subgroup $P'' = M''N$ with Levi component M''.

Recall (§I.6) that A is a maximally split σ-invariant torus in G^0, and L is a σ-invariant cocompact lattice in A. B is a minimal σ-invariant parabolic subgroup of G^0 containing A, A^- is the set of a in A with $|\alpha(a)| \leqq 1$ for any α in the set Δ of roots of A in B, $L^- = L \cap A^-$. To any semi-simple $t = t_0 \times \sigma$ we associate a in A^- (or λ in L^-), and a subset θ of Δ, consisting of the α with $\alpha(\lambda) = 1$. Given a subgroup H of G^0 we denote by H_σ the group $H \cap G_\sigma$ of σ-invariant elements in H.

Given a σ-invariant λ in L^-, the centralizer $M_\lambda = Z_{G^0}(\lambda)$ of λ in G^0 is σ-invariant, and we denote by $P_\lambda = M_\lambda N_\lambda$ the standard σ-invariant parabolic subgroup of G^0 with Levi component M_λ. The center A_λ of M_λ lies in A. We say that the central exponent ω of π'' with respect to M_λ *decays* if $|\omega(a)| < 1$ for every a in $(A_\lambda)_\sigma$ with (1) $|\alpha(a)| \leqq 1$ for any root α of $(A_\lambda)_\sigma$ in $(N_\lambda)_\sigma$, and (2) $|\alpha(a)| < 1$ for some such α. We say that π'' is *discrete-series* if its central character is unitary, and its central exponents with respect to any proper Levi subgroup M_λ, where λ is any σ-invariant element in L^-, all decay.

In the case where $G'' = G^0$ is connected, namely $\sigma = 1$, Harish-Chandra's criterion for square-integrability ([C′], Theorem 4.4.6; [S], Theorem 4.4.4) asserts that π'' is a discrete-series in the above sense if and only if it is *square-integrable*, in the sense that its matrix coefficients $f(x) = (\pi''(x)v, v')$ are absolutely square-integrable functions of G/Z.

Definition. We say that a discrete-series G''-module π'' satisfies a *trace identity* if there is (1) a set $\{\pi\}$ of G-modules π, which, for each open compact subgroup C in G, contains only finitely many G-modules with a C-fixed vector, (2) positive integers $m(\pi)$ (depending on π'), and a complex number c, so that for all matching ϕ in $C(G'')$ and f in $C(G)$, we have

$$c \operatorname{tr} \pi''(\phi) = \sum m(\pi) \operatorname{tr} \pi(f). \tag{3.1}$$

Assumption. *For any proper Levi subgroup M, and any open compact subgroup C as in Proposition* I.6.2, *there exists ϕ^M in $C(M'')$ matching the characteristic function of $C \cap M$ in $C(M)$.*

Our assumption is tantamount to the following. For any proper Levi subgroup M with center A_M contained in A, and any t_0 in A_M, which we also view as an element in the center of M^0, we have: there exists a function $\phi_{t_0}^M$ in $C(M'')$ matching the function $f_{Nt_0}^M$ in $C(M)$ defined prior to Proposition I.6.2. Here $Nt_0 = (t_0 \times \sigma)^l = t_0^l$, as $\sigma(t_0) = t_0$. Indeed, the function $f_{Nt_0}^M$ is obtained from the characteristic function of $C \cap M$ on translating by the central element Nt_0, and multiplying by a scalar, so that $\phi_{t_0}^M$ can be obtained from ϕ^M on translating by the σ-invariant central element t_0, and multiplying by the same scalar.

§4. Decay

Proposition. *Suppose that the discrete-series G''-module π'' satisfies a Trace Identity, and G satisfies Assumption* 3. *Then all π in* (3.1) *are discrete-series G-modules.*

Proof. Let M be a proper Levi subgroup, C a compact open subgroup of G as in Proposition I.6.2, t_0 in A_M such that $|\alpha(t_0)| \leqq 1$ for all roots α of A_M in the unipotent radical of the standard parabolic subgroup with Levi component M, and $f_{t_0'}^M$ the function of Proposition I.6.2, where t_0' is $Nt_0 = t_0^l$. Proposition I.6.2, and Proposition I.6.3, imply that the function $f_{t_0'}$ on G defined in Proposition I.6.2, which is C-biinvariant, satisfies $F(x, f_{t_0'}) = {}^M F(x, f_{t_0'}^M)$, hence

$$F'(x, f_{t_0'}) = {}^M F'(x, f_{t_0'}^M), \qquad \text{for all } x \text{ regular in } G.$$

As noted following Proposition I.6.2, the function $f_{t_0'}$ is C-biinvariant. Hence $\operatorname{tr} \pi(f_{t_0'}) \neq 0$ only for π with a C-invariant vector. By definition of Trace Identity, there are only finitely many such π in (3.1). On the other hand, if ω_ρ is the central character of the irreducible constituent ρ of the M-module π_N, then $\operatorname{tr} \pi(f_{t_0'}) = \operatorname{tr} \pi_N(f_{t_0'}^M)$ is a sum over ρ of $\omega_\rho(t_0')\delta_P^{1/2}(t_0')n(\rho, C)$, where $n(\rho, C)$ is the non-negative integral multiplicity of the trivial representation of C in ρ (the dimension of the space of C-fixed vectors in ρ).

Assumption 3 asserts that there exists a function $\phi_{t_0}^M$ in $C(M'')$ matching $f_{t_0'}^M$. Proposition 6.3 asserts that there exists a function ϕ_{t_0} in $C(G'')$ with $F'(x, \phi_{t_0}) = {}^M F'(x, \phi_{t_0}^M)$ for all regular $x = x_0 \times \sigma$ in G''. Hence the functions $f_{t_0'}$ on G and ϕ_{t_0} on $G^0 \times \sigma$ are matching. Since π'' appears in the Trace Identity, it is clear that its character $\chi(\pi')$ is a stable function, depending only on the stable conjugacy class of $x = x_0 \times \sigma$ in G''. Using the Weyl integration formula we have

$$\operatorname{tr} \pi''(\phi_{t_0}) = \sum_T{}^s w(T)^{-1} \int_{T/Z} (\Delta\chi(\pi''))(x) F'(x, \phi_{t_0}) d(Nx).$$

The sum is over the stable conjugacy classes of F-tori T in G. $w(T)$ is the cardinality of the quotient $W(T) = N(T)/\boldsymbol{T}$ by $\boldsymbol{T}$ of the group $N(T)$ of x in $\boldsymbol{G}$ such that $\operatorname{ad}(x): T \to T$, $t \to xtx^{-1}$, is defined over F. Recall that

$$F'(x, \phi_{t_0}) = {}^M F'(x, \phi_{t_0}^M) = {}^M F'(x, f_{t_0'}^M).$$

Since t_0' lies in the center of M, we have $M \subset M_{t_0'}$. As we assumed that $|\alpha(t_0')| < 1$ for all roots α of A_M in N_M, we have $M = N_{t_0'}$. But $f_{t_0'}^M$ is supported on a small neighborhood of t_0'. Hence if $F'(x, \phi_{t_0}) \neq 0$, then M_x^0 is conjugate to the Levi subgroup M^0 such that M_σ^0 matches M. Proposition I.6.1 now implies that we have

$$\sum_T w(T)^{-1} \int (\Delta_{M^0}\chi(\pi''_N))(x)\ {}^M F'(x, \phi_{t_0}^M)\, d(Nx).$$

As t_0 lies in the center of $M^0 \rtimes \langle\sigma\rangle$, changing variables $x \to t_0 x$, we obtain

$$\sum_{\rho'' \operatorname{in} \pi''_N} \omega_{\rho''}(t_0) \sum_T w(T)^{-1} \int (\Delta_{M^0}\chi(\pi''_M))(x)\ {}^M F'(x, \phi_1^M)\, d(Nx) = \sum_{\rho''} \omega_{\rho''}(t_0) \operatorname{tr} \rho''(\phi_1^M).$$

ρ'' ranges over the σ-invariant irreducible subquotients of π''_N; $\omega_{\rho''}$ is the central character of ρ''.

We conclude from the trace identity that for any t_0 in the center of $M^0 \rtimes \langle\sigma\rangle$ with $M_t^0 = M^0$ in the notations of §I.6, we have

$$\sum_{\rho''} c(\rho'')\omega_{\rho''}(t_0) = \sum_\rho n(\rho)\omega_\rho(t_0^l).$$

The sum on the left ranges over the constituents ρ'' of π''_N; hence it is finite. The $c(\rho'')$ are complex. On the right the sum is finite, depending on the compact open subgroup C, and the coefficients are positive, so that no cancellation may occur. Linear independence of characters (on the set of t_0 in A_M with $|\alpha(t_0)| < 1$ for the positive roots α) implies that for each ρ there exists ρ'' with $\omega_\rho(t_0^l) = \omega_{\rho''}(t_0)$. Consequently the character ω_ρ decays, where ρ is any constituent of π_N; here M is any proper Levi subgroup of G, and π is any G-module with a C-fixed vector. Since any π has a C-fixed vector for a sufficiently small C, it follows that all π are discrete-series, as required.

Remark. It is clear that if π'' is assumed to be only tempered, then the above proof implies that the π of (3.1) are tempered.

§5. Finiteness

We now continue with the situation and Assumption of §3, but make two additional assumptions.

5.1. Assumption. *Suppose that Φ' is a stable function in $S_e(G'')$ (see §I.5). Then there exists Φ on G in $S_e(G)$ matching Φ'.*

Namely, we suppose that $\Phi'(x \times \sigma) = \Phi'(x' \times \sigma)$ for all stably conjugate regular $x \times \sigma$, $x' \times \sigma$, and assume the existence of a function f in $A(G)$ with ${}'\Phi'(Nx, f) = \Phi'(x)$ on the regular set.

5.2. Assumption. *For any f in $A(G)$ there exists a matching function ϕ in $A(G'')$.*

Using these Assumptions, we conclude

Proposition. *Suppose that the discrete series G''-module π'' satisfies a Trace Identity* (3.1). *Then the set of π is finite.*

Proof. Note that Proposition 4 asserts that the π are all discrete-series. To prove our proposition, note that by the Trace Identity $\operatorname{tr} \pi''(\phi)$ depends only on f, namely on the stable orbital integral of ϕ, hence the character $\chi(\pi'')$ of π'' on $G^0 \times \sigma$ is a stable function. Assumption 5.1 implies that there exists a finite linear combination of G-modules π with complex coefficients $c(\pi)$, so that

$$\sum_{\pi} c(\pi)[\chi(\pi)](Nx) = \chi(\pi'')(x \times \sigma)$$

for any elliptic regular $x \times \sigma$ in G'', and $\Sigma_\pi c(\pi)[\chi(\pi)](y) = 0$ for the elliptic regular y which are not norms. We may assume that all π here are tempered by [K], Proposition 1.1.

Applying the Weyl integration formula we deduce that

$$\operatorname{tr} \pi''(\phi) = \sum_{T}^{s} w(T)^{-1} \int [\Delta\chi(\pi'')](x) F'(x, \phi)\, dx.$$

Only elliptic tori occur since we take ϕ in $A(G'')$. Further, we take ϕ so that it has a matching f, so that $F'(x, \phi) = F'(Nx, f)$. Replacing $\chi(\pi'')$ by our linear combination $\Sigma\, c(\pi)\chi(\pi)$, we obtain

$$\sum_{\pi} c(\pi) \sum_{T}^{s} w(T)^{-1} \int [\Delta\chi(\pi)](x) F'(x, f)\, dx = \sum_{\pi} c(\pi) \operatorname{tr} \pi(f).$$

We deduce from (3.1) the identity $\Sigma_\pi c(\pi) \operatorname{tr} \pi(f) = \Sigma_\pi m(\pi) \operatorname{tr} \pi(f)$. On the left the sum is finite and consists of tempered π. On the right all π are discrete-series. The identity holds for all f in $A(G)$ which have a matching function ϕ. So fix π_0 on the right. By Kazhdan [K], Theorem K, there exists a pseudo-coefficient f_0 in $A(G)$ with $\operatorname{tr} \pi_0(f_0) = 0$ for any tempered irreducible π inequivalent to π_0. But Assumption 5.2 implies that f_0 has a matching function ϕ. Using our identity with $f = f_0$ we conclude that $m(\pi_0) = 0$ for all π_0 on the right which are not equivalent to any of the finitely many π on the left. Consequently, the set of π with $m(\pi) \neq 0$ is finite, as asserted.

Chapter III. Representations of Simple Algebras

§0. Introduction

Let F be a local field, and G an inner form of $\mathrm{GL}(n)$ over F. Thus G is the multiplicative group of a simple algebra A central over F. A is the $m \times m$ matrix algebra $M(m, D)$ over a division algebra D central over F of rank d, with $n = md$. Class field theory (see e.g., [W]; Ch. X) associates with A an invariant $\operatorname{inv} A$ of the form i/d (modulo 1), with i prime to d, and $\operatorname{inv} A = \operatorname{inv} D$ is independent of m. There exists a unique simple algebra A central of rank n over F with invariant i/d (modulo 1) (where $(i, d) = 1$ and d divides n). If $F = \mathbf{C}$ then $d = 1$, if $F = \mathbf{R}$ then $d = 1$ or 2; otherwise d is any positive integer. Put $G = G(F)$ and $G' = \mathrm{GL}(n, F)$, and note that $G(\bar{F}) = \mathrm{GL}(n, \bar{F})$ if $\bar{F}$ is an algebraic closure of F.

A conjugacy class γ in G is called *regular* if its characteristic polynomial p_γ is separable (has distinct roots). If γ, δ are regular and $p_\gamma = p_\delta$ then $\gamma = \delta$. There is an embedding $\gamma \to \gamma'$, defined by $p_{\gamma'} = p_\gamma$, of the set of regular conjugacy classes γ in G into the set of regular conjugacy classes γ' in G'.

Let $C(G)$ denote the convolution algebra of complex valued smooth compactly supported measures f on G. Put $R(G) = R_{\mathbf{Z}}(G) \otimes \mathbf{C}$, where $R_{\mathbf{Z}}(G)$ is the Grothendieck free abelian group generated by the set $\operatorname{Irr} G$ of equivalence classes of smooth (= algebraic) irreducible (hence admissible by [BZ]) G-modules.

If π is an admissible G-module then the convolution operator $\pi(f) = \int_G f(g)\pi(g)$ is of finite rank and its trace is denoted by $\operatorname{tr} \pi(f)$. There exists a complex valued conjugacy invariant smooth function $\chi = \chi(\pi)$ on the regular set of G with $\operatorname{tr} \pi(f) = \int \chi(g) f(g)$ for any f in $C(G)$ which is supported on the regular set of G. It is called the *character* of π, it depends only on the image of π in $R(G)$, and characters of inequivalent irreducible G-modules are linearly independent (namely $\chi \neq 0$ if $\pi \neq 0$ in $R(G)$).

Fix a minimal parabolic subgroup P_0 together with its Levi decomposition M_0N_0 in G (and G''), and denote by i_{MG} (or I_M, or I_M^G) the homomorphism $R(M) \to R(G)$ of unitary induction, for any (standard) Levi subgroup M (thus M is a Levi subgroup, containing M_0, of a parabolic subgroup containing P_0).

An irreducible G-module π whose central character is unitary is called *square-integrable*, or discrete-series, if it has a matrix-coefficient which is square-integrable on G modulo its center. An irreducible G-module π is called *tempered* if there exists a Levi subgroup M and a square-integrable (= discrete-series) M-modules ρ, such that π is a subquotient (necessarily a direct summand) of $i_{MG}\rho$. Put $\nu(x) = |x|$ (x in F), where $|\cdot|$ is the normalized valuation on F, and $\nu(g) = \nu(\det g)$, where $\det g$ is the reduced norm of g in G.

A G-module π is called *relevant* if there is a Levi subgroup of G of the form $M = \Pi_{i=1}^{m} (M_i \times M_i)$ or $M_0 \times M$, where M_i $(0 \leqq i \leqq m)$ are multiplicative groups of simple algebras central over F, and tempered M_i-modules ρ_i $(0 \leqq i \leqq m)$, and distinct positive numbers $s_i < \frac{1}{2}$ $(1 \leqq i \leqq m)$, such that π is

$$i_{M,G}\left[\prod_{i=1}^{m} (\rho_i\nu^{s_i} \times \rho_i\nu^{-s_i})\right] \quad \text{or} \quad i_{M_0\times M,G}\left[\rho_0 \times \prod_{i=1}^{m} (\rho_i\nu^{s_i} \times \rho_i\nu^{-s_i})\right] \quad \text{in } R(G).$$

Local Theorem. (1) *Relevant G-modules are unitary and irreducible*; *in particular, a G-module unitarily induced from a tempered one is irreducible.*

(2) *The relation* $\chi'(\gamma') = (-1)^{n-m}\chi(\gamma)$ *for all matching* $(\gamma \to \gamma')$ *regular conjugacy classes* γ, γ' *in* G, G' *defines a bijection between the set of equivalence classes of square-integrable* (*resp. tempered*; *relevant*) *G-modules* π, *and the set of equivalence classes of square-integrable* G'*-modules* π' (*resp. tempered*; *relevant*, G'*-modules* π' *whose character* χ' *is non-zero on the set of regular* γ' *obtained from* γ *in* G).

The bijection of (2) is called the Deligne–Kazhdan correspondence.

Let F be a global field, and G an inner form of $G' = \mathrm{GL}(n)$ over F. Then G is the multiplicative group of a simple algebra A central over F. A is a matrix algebra $M(m, D)$ of $m \times m$ matrices over a division algebra D central over F of rank d with $n = dm$. Class field theory (see, e.g., [W]; Ch. XI) associates with A the sequence $\{\mathrm{inv}_v A = \mathrm{inv}\, A \otimes_F F_v\}$ of rational numbers modulo one which are almost all zero and whose sum is zero modulo one. Each such sequence $\{i_v/d_v\}$ determines, up to F-isomorphism, a unique division algebra D central over F, and a unique simple algebra A of rank n central over F with these invariants, for any n which is divisible by d_v for all v. Let $G(\mathbf{A})$ be the group of $\mathbf{A}$-points of G, where $\mathbf{A}$ is the ring of adeles of F. Let Z (resp. Z') denote the center of G (resp. G'); then $Z = Z'$ is the multiplicative group. Fix a unitary character ω of $Z(\mathbf{A})/Z(F) = \mathbf{A}^\times/F^\times$. For each place v of F denote by F_v the completion of F at v and by ω_v the restriction of ω to $F_v^\times$.

Let $L(G)$ denote the space of slowly increasing (see, e.g., [BJ]) functions ψ on $G(F)\backslash G(\mathbf{A})$ with $\psi(zg) = \omega(z)\psi(g)$ (z in $Z(\mathbf{A})$). $G(\mathbf{A})$ acts on $L(G)$ by right translation, and any irreducible submodule is unitary and called an *automorphic* $G(\mathbf{A})$-module. $L(G)$ is the direct sum of the discrete spectrum $L_d(G)$, which is the direct sum of irreducible $G(\mathbf{A})$-modules called "discrete-series" $G(\mathbf{A})$-modules, and the continuous spectrum $L_c(G)$, which is a continuous sum. A *cuspidal* $G(\mathbf{A})$-module is an irreducible constituent of the subspace $L_0(G)$, which consists of the ψ in $L(G)$ with $\int_{N(F)\backslash N(\mathbf{A})} \psi(nx)dn$ equals zero for every x in $G(\mathbf{A})$, and for the unipotent radical N of any proper parabolic subgroup of G over F. Each cuspidal ψ is absolutely square-integrable on $G(\mathbf{A})/Z(\mathbf{A})G$. $L_0(G)$ is a sub-$G(\mathbf{A})$-module of $L_d(G)$.

Any cuspidal $G'(\mathbf{A})$-module is non-degenerate, namely (each of its local components) has a Whittaker model (see [BZ]), and it occurs with multiplicity one in $L_0(G)$. An irreducible $G(\mathbf{A})$-module π decomposes as a restricted tensor product $\otimes_v\pi_v$ of irreducible admissible $G_v = G(F_v)$-modules π_v, which are almost all unramified. If $\pi' = \bigotimes\pi'_v$ and $\pi'' = \bigotimes\pi''_v$ are cuspidal $G'(\mathbf{A})$-modules and $\pi'_v \simeq \pi''_v$ for almost all v, then $\pi'_v \simeq \pi''_v$ for all v. All components of a cuspidal $G'(\mathbf{A})$-module are relevant (by [Z], (9.7)), and, as noted above,unitary.

Given G, or D, there is a finite set S of places v of F such that for every v outside S the division algebra D splits, namely $D \otimes_F F_v = M(d, F_v)$. We say that π_v lifts to π'_v if $G_v \simeq G'_v$ (thus $v \notin S$) and $\pi_v \simeq \pi'_v$, or, more generally for arbitrary v, if π_v corresponds to π'_v by the local theorem. An irreducible $G(\mathbf{A})$-module $\pi = \bigotimes_v\pi_v$ *lifts*, or corresponds, to an irreducible $G'(\mathbf{A})$-*module* $\pi = \bigotimes_v \pi'_v$ if π_v lifts to π'_v for all v. An automorphic $G(\mathbf{A})$-module which lifts to a cuspidal $G'(\mathbf{A})$-module will be called *non-degenerate*.

Global Theorem. (1) *All local components of a non-degenerate $G(\mathbf{A})$-module are relevant.*

(2) *Each non-degenerate $G(\mathbf{A})$-module occurs in the discrete spectrum of $L(G)$ with multiplicity one.*

(3) *If $\pi = \bigotimes_v \pi_v$ and $'\pi = \bigotimes_v {}'\pi_v$ are non-degenerate $G(\mathbf{A})$-modules and $\pi_v \simeq {}'\pi_v$ for almost all v, then $\pi = {}'\pi$.*

(4) *Lifting defines a bijection from the set of non-degenerate $G(\mathbf{A})$-modules $\pi = \bigotimes\pi_v$ to the set of cuspidal $G'(\mathbf{A})$-modules $\pi' = \bigotimes\pi'_v$ such that π'_v is obtained by the local correspondence for all v (in S).*

Remark. (1) is motivation for the definition of "relevant" representations. (2) is called "multiplicity one" theorem for the non-degenerate spectrum of G. (3) is called "rigidity" theorem for the non-degenerate spectrum. (4) is called the Deligne–Kazhdan correspondence.

The local theorem is proven below for F of characteristic zero, and the case where F has positive characteristic follows from the Theorem of [K′]. The Global Theorem is proven here only for the subset of the cuspidal $G'(\mathbf{A})$-modules π' with two supercuspidal components, using the simple form of the trace formula proven in Chapter I, Corollary 3. This Corollary I.3 applies to any test function

$f = \otimes f_v$ which has a supercuspidal component f_u, and at a second place u' the component $f_{u'}$ is *any* function whose orbital integrals vanish on the regular non-elliptic set (thus $f_{u'}$ lies in the class $A(G)$ of [K] (see Chapter I, §5.6), which is called the class of discrete functions in [BDK] (see Chapter I, §7)). In particular, $f_{u'}$ can be taken to be a pseudo-coefficient of any square-integrable $G_{u'}$-module. Had we proved Corollary I.3 only for f such that $f_{u'}$ is supported on the elliptic regular set, we would have not been able to prove our Global Theorem, except in the special, more elementary case where the simple algebra underlying G is a division algebra.

The stronger form of the simple trace formula proven in [FK1] makes it possible to prove the global theorem for all π' with at least one supercuspidal component (see [FK1]). In [FK1] we replace the condition at u' by the requirement that $f_{u'}$ be a sufficiently admissible spherical function (a notion defined in [FK1]), and show that this requirement does not restrict the applicability of that trace formula to lifting problems. The trace formula of [FK1] is analogous to — and motivated by — Deligne's conjecture on the Lefschetz fixed point formula (in ètale topology) for finite flat correspondences twisted by a sufficiently high power of the Frobenius. Similar ideas are used in our work with D. Kazhdan (see [FK2], [FK3]) concerning the geometric Ramanujan conjecture for GL(n). The global theorem can be proven for all π' on using Arthur's recent proof of the required trace formulae identity for arbitrary test functions $f = \otimes f_v$ and $f' = \otimes f'_v$ on $G(\mathbb{A})$ and $G'(\mathbb{A})$, but we do not do it here. A simple proof of this trace identity for arbitrary f and f' can possibly be given on using the regular functions of [FK], [Sph], and Chapter IV below, but at the moment we carried it out only for groups of rank one (see [Sym; VI]).

When $n = 2$ the theorem is due to Jacquet–Langlands [JL], when $n = 3$ Flath [Fl] reduced it to the trace identity ([Fl; (6.1)]) proven later in [GL(3); (2.7.3)], and the case of general n, d was treated by Deligne, Kazhdan and Vigneras in [DKV]. Our indebtedness to [DKV] is apparent.

§1. Germs

As an example of the theory of Chapter I, we consider here the case of comparison between $G' = \mathrm{GL}(n)$ and its inner forms G. Our exposition here will be more elementary, as suitable for this introductory case. In particular G and G' are connected, so that $G'' = G' = G^0$ and $\sigma = 1$, and there is no difference between conjugacy and stable conjugacy.

Let F be a local non-archimedean field of characteristic zero, and put $G' = \mathrm{GL}(n, F)$, where n is a positive integer. Let G be an inner form of G'. Fix Haar measures dx' on G' and dx on G. Write $\gamma \to \gamma'$ if γ, γ' are semi-simple elements of G and G' with $p_\gamma = p_{\gamma'}$. If γ, γ' are regular (have distinct eigenvalues), their centralizers in G, G' are tori T, T'; T is isomorphic to T' if $\gamma \to \gamma'$. Haar measures on isomorphic tori are always taken to be equal. The orbital integrals $\Phi(x, f)$ and $\Phi(x', f')$ of f in $C(G)$ and f' in $C(G')$ are defined in §I.1. Proposition I.5.8 implies the following

Corollary. *For every f in $A(G)$ there exists f' in $A(G')$, and for every such f' there is such f, so that ${}'\Phi(\gamma, f) = {}'\Phi(\gamma', f')$ for every elliptic regular γ and γ' with $p_\gamma = p_{\gamma'}$.*

This proves assumptions II.5.1, II.5.2, in our case.

Definition. The functions f in $C(G)$ and f' in $C(G')$ are called *matching* if $\Phi(x, f) = \Phi(x', f')$ for all regular x' in G' and x in G with $p_x = p_{x'}$, and $\Phi(x', f') = 0$ for all regular x' in G' which do not come from G.

We also state the following

Theorem. *For every f in $C(G)$ there exists f' in $C(G')$; and for every f' in $C(G')$ so that $\Phi(f')$ is zero at any regular x in G' which does not come from G, there exists f in $C(G)$; so that f and f' are matching.*

This Theorem will be proven by induction on the Levi subgroup of G. Hence we now assume the validity of the Theorem for every proper Levi subgroup M of G. Consequently we can use Assumption II.3 in our case. The proof is based on the lifting theorem of §5 for tempered local representations; it is completed in §7.

§2. Comparison

Let F be global, $n = md$, $G = \mathrm{GL}(m, D)$ the multiplicative group of the $m \times m$ matrix algebra over the central division algebra D of dimension d^2 over F, and $G' = \mathrm{GL}(n)$. Put $G_v = \mathrm{GL}(m, D_v)$, $G'_v = \mathrm{GL}(n, F_v)$ at any place v of F. Since G is an inner form of G', the groups are isomorphic over an algebraic closure $\bar{F}$ of F, and a differential form of maximal degree on G' rational over F can be transferred to one on G. These define Haar measures dx_v and $d'x_v$ on G_v and G'_v for all v, which we call compatible, and consequently we can choose compatible product measures $dx = \bigotimes dx_v$, $d'x = \bigotimes d'x_v$ on $G(\mathbf{A})$, $G'(\mathbf{A})$.

There is a bijection from the set of conjugacy classes in $D^\times$ (over a local or global field), to the set of elliptic conjugacy classes in $\mathrm{GL}(d, F)$. Similarly, there is a bijection from the set of semi-simple conjugacy classes in $G = \mathrm{GL}(m, D)$ to the set of semi-simple conjugacy classes in $G' = \mathrm{GL}(n, F)$, with an elliptic representative in the Levi subgroup $\Pi_i \mathrm{GL}(da_i, F)$, $\Sigma_i a_i = m$. Globally, if G ramifies at the finite set V of places of F, there is a bijection from the set of conjugacy classes of tori T in G over F, into the set of conjugacy classes of tori T' in G' such that at each v in V the torus T'_v of G'_v is obtained from a C_v-torus T_v. We choose compatible product measures $dt = \otimes dt_v$, $d't = \otimes d't_v$ on the matching tori $T(\mathbf{A})$, $T'(\mathbf{A})$, which are isomorphic over F.

We choose functions $f = \otimes f_v$ on $G(\mathbf{A})$ and $f' = \otimes f'_v$ on $G'(\mathbf{A})$ such that f_v and f'_v are matching for all v. In fact, for v outside V, the groups G_v and G'_v are isomorphic over F_v, and we take f_v, f'_v equal under this isomorphism. For almost all v, we take $f_v = f_v^0 = f'_v$. Corollary 1 and the inductive assumption of Theorem 1 show that there exist sufficiently many matching pairs in $C(G_v)$, $C(G'_v)$ for our purposes.

Proposition I.3 now implies

Proposition. *If f and f' are matching and satisfy (each) the (three) requirements of Proposition I.3, then $\Sigma \operatorname{tr} \pi'(f') = \Sigma\, m(\pi) \operatorname{tr} \pi(f)$. The sums range over the cuspidal spectra of $L^2(G')$ and $L^2(G)$.*

We used the multiplicity one theorem for $L_0^2(G')$ to conclude that the multiplicities $m(\pi')$ on the left are equal to 1.

§3. Existence

Let G be a reductive connected p-adic group, and π' a square-integrable G-module. A *pseudo-coefficient* of π' is a function f in $A(G)$ (see §I.5) with $\operatorname{tr} \pi'(f) = 1$ and $\operatorname{tr} \pi(f) = 0$ for every tempered (irreducible) G-module π inequivalent to π'. If π' is supercuspidal then each of its (normalized) matrix coefficients is a pseudo-coefficient (in fact $\operatorname{tr} \pi(f) = 0$ if π is irreducible and inequivalent to π'). In general, the existence of a pseudo-coefficient is proven in [K], Theorem K (cf. [BDK]).

Let F be a global field, fix a finite set V of non-archimedean places, and three distinct non-archimedean places w, u and u' outside V. Although more general variants of the following Proposition can be proven (cf. Theorem IV.3), for simplicity we now assume that $G = \mathrm{GL}(n)$. Fix a supercuspidal G_u-module π'_u.

Proposition. *Let π'_w be a square-integrable G_w-module. Then there exists a cuspidal $G(\mathbf{A})$-module $\pi = \otimes \pi_v$ such that* (i) $\pi_w \simeq \pi'_w$; (ii) $\pi_u \simeq \pi'_u$; (iii) *for each v in V the component π_v is Steinberg (see* [C'; §8]); (iv) $\pi_{u'}$ *is square-integrable*, (v) π_v *is unramified for each non-archimedean place $v \neq u$, u', w outside V.*

Proof. We use Corollary I.3 with a function $f = \otimes f_v$ chosen as follows:

(i) f_w is a pseudo-coefficient of π'_w;

(ii) f_u is a matrix coefficient of π'_u;

(iii) for each v in V the component f_v is a pseudo-coefficient of the Steinberg G_v-module;

(iv) $f_{u'}$ is supported on the regular elliptic set in $G_{u'}$;

(v) at each finite $v \neq u$, u', w outside V we take spherical (K_v-biinvariant) f_v; $f_v = f_v^0$ for almost all v.

These components can be and are chosen so that $\Phi(x, f) \neq 0$ for some elliptic regular x in G. Since the sum of I.3.1 is finite, we can reduce the support of $f_{u'}$ so that the sum I.3.1 consists of a single entry, hence it is non-zero. Hence there is a cuspidal π with $\operatorname{tr} \pi(f) \neq 0$. This π is non-degenerate, hence each of its local components π_v is non-degenerate. It is easy to check that π has the properties required by the proposition, using the following

Remark. A G_v-module is called *elliptic* if its character is not identically zero on the regular elliptic set of G_v. Theorem 9.7(b) of [Z] implies that every irreducible non-degenerate elliptic G_v-module is square-integrable (in fact, of a "generalized Steinberg" type).

The proposition follows.

In the next lemma, G is a locally compact unimodular topological group with center Z, ω a character of Z of absolute value one, and $f^*(g) = \bar{f}(g^{-1})$. Let $L(G)$ denote the convolution $*$-algebra of complex valued functions on G with $f(zg) = \omega(z)^{-1} f(g)$ (g in G, z in Z) such that $|f(g)|^2$ is integrable on G/Z. For a unitary irreducible G-module π put $\pi(f) = \int_{G/Z} f(g)\pi(g)dg$. Suppose B is a dense $*$-closed subalgebra of $L(G)$, I is a set, $\{\pi, \pi_i (i \text{ in } I)\}$ is a set of irreducible unitary pairwise inequivalent G-modules such that $\pi(f)$, $\pi_i(f)$ (i in I) are Hilbert–Schmidt operators for all f in B, and $\|\cdot\|$ is the norm. Suppose that $\{c_i (i \text{ in } I)\}$ is a set of non-negative real numbers such that $\Sigma_i c_i \|\pi_i(f)\|^2$ is finite for all f in B. Then the remark on page 496 of [JL] asserts that: for each positive ε there exists f in B with $\|\pi(f)\| \neq 0$ and $\Sigma_i c_i \|\pi_i(f)\|^2 \leqq \varepsilon \|\pi(f)\|^2$. We conclude that:

Lemma. *If $\{d_i;\, i \text{ in } I\}$ are complex numbers such that $\Sigma_i d_i \operatorname{tr} \pi_i(f * f^*)$ is absolutely convergent to zero for all f in B, then $d_i = 0$ for all i.*

Proof. Note that $\operatorname{tr} \pi_i(f * f^*) = \|\pi_i(f)\|^2$. If $d_0 \neq 0$, there is f in B such that $\Sigma_{i \neq 0} |d_i| \operatorname{tr} \pi_i(f * f^*)$ is bounded by $\frac{1}{2}|d_0| \operatorname{tr} \pi_0(f * f^*)(\neq 0)$, and we arrive at a contradiction.

§4. Isolation

Let F_w be a local non-archimedean field of characteristic zero, and G_w the multiplicative group of $M(m, D_w)$, where D_w is a central division algebra over F_w or rank d and invariant i/d (modulo one).

Propostion. *For every square-integrable G_w'-module π_w' there exist G_w-modules π_w and positive integers $m(\pi_w)$, such that for all matching f_w' and f_w we have*

$$(-1)^{n-m} \operatorname{tr} \pi_w'(f_w') = \sum m(\pi_w) \operatorname{tr} \pi_w(f_w).$$

If C_w is an open compact subgroup of G_w, then the sum consists only of finitely many π_w with a non-zero C_w-invariant vector.

Proof. Let F be a totally imaginary number field whose completion at some place w is our local field F_w. Choose a set V of $n-m+1$ non-archimedean places including w. We may assume that i is prime to $n = md$. Choose a division algebra D central over F with the following invariants: i/d at w; i/n at each $v \neq w$ in V; 0 outside V. Take $G = D^\times$. Then G_w is our $\mathrm{GL}(m, D_w)$, where $\mathrm{inv}_w D_w = i/d$. Fix three distinct non-archimedean places u, u', u'' of F outside V, a supercuspidal G_u-module π_u, and a matrix coefficient f_u of π_u. If S is any finite set of places of F put $\pi^S = \otimes \pi_v$ and $f^S = \otimes f_v$ (product over v outside S), and $\pi_S = \otimes \pi_v$ and $f_S = \otimes f_v$ (v in S). Denote by ∞ the set of archimedean places of F. Choose a unitary irreducible G_∞-module π_∞. Using Lemma 3 with $B = C(G_\infty)$ we conclude from Proposition 2 that if $f'^\infty = \otimes f'_v$ and $f^\infty = \otimes f_v$ (v outside ∞) and f'_v, f_v are matching for all v, then

$$\sum \mathrm{tr}\, \pi'^\infty(f'^\infty) = \sum m(\pi) \mathrm{tr}\, \pi^\infty(f^\infty). \tag{4.1}$$

Put $\mathbf{A}_f$ for the ring of adeles without archimedean components. On the left, the sum ranges over all $G(\mathbf{A}_f)$-modules π'^∞ such that $\pi' = \pi'^\infty \otimes \pi_\infty$ is a cuspidal $G'(\mathbf{A})$-module; on the right the sum is over the $G(\mathbf{A}_f)$-modules π^∞ so that $\pi = \pi^\infty \otimes \pi_\infty$ appears with positive multiplicity $m(\pi)$ in the (cuspidal) spectrum $L_0(G)$ of G.

Recall the following theorem of Harish-Chandra (see [BJ]).

Lemma. *Let C be an open compact subgroup of $G(\mathbf{A}_f)$. Then there are only finitely many automorphic G-modules π with a non-zero C-fixed vector and a given infinitesimal character at each archimedean place (in particular with the fixed component π_∞ at ∞).*

Let V' be the union of V and $\{u, u', u''\}$. Fix f_v, f'_v for v in V', and let $f_v = f'_v$ be a variable spherical ($K_v = G(R_v)$-biinvariant) function for the finite v outside V'. Then the Lemma implies that the sums in (4.1) are both finite. It is clear from the theory of the Satake transform that: given a finite set $\{\pi_{iv};\ i \geqq 0\}$ of irreducible unramified pairwise-inequivalent G_v-modules, there exists a spherical function f_v with $\mathrm{tr}\, \pi_{iv}(f_v) = 0$ if $i \neq 0$, and $\mathrm{tr}\, \pi_{0v}(f_v) = 1$. We conclude that given an irreducible $G(\mathbf{A}^{V'})$-module $\pi^{V'}$, we have, for all matching f_v, f'_v (v in V'),

$$\sum \mathrm{tr}\, \pi'_{V'}(f'_{V'}) = \sum m(\pi)\, \mathrm{tr}\, \pi_{V'}(f_{V'}). \tag{4.2}$$

On the left the sum is over the irreducible representations $\pi'_{V'}$ of $\Pi\, G'_v$ (v in V') such that $\pi' = \pi'_{V'} \otimes \pi^{V'}$ is cuspidal; by the rigidity theorem of [JS] there exists at most one such π'. We choose $\pi^{V'}$ so that π' of Proposition 3 appears on the left. On the right the sum is over the equivalence classes of irreducible $\pi_{V'}$ such that

$\pi = \pi_{V'} \otimes \pi^{V'}$ is cuspidal, with multiplicity $m(\pi)$. The sum on the right is not finite, *a priori*.

Since f_u is a normalized matrix coefficient of a supercuspidal G_u-module π_u, we have $\operatorname{tr} \pi_u(f_u) = 1$ and $\operatorname{tr} \pi'_u(f'_u) = 1$ for the π, π' which appear in (4.2). At each $v \neq w$ in V, let f_v be the function 1, and f'_v a matching function on G'_v; f'_v exists by Corollary 1. At such v let π_v be the trivial G_v-module, and π'_v the Steinberg G'_v-module. Then $\chi'_v(x') = (-1)^{n-1}\chi_v(x)$ on the elliptic regular set, and $\operatorname{tr} \pi'_v(f'_v) = (-1)^{n-1}$. Moreover, if $v \neq w$ in V and π_v appears on the right of (4.2), then $\operatorname{tr} \pi_v(f_v)$ is 0 or 1. Since $(n-1)(n-m) \equiv n - m$ (modulo 2) we conclude that for all matching f_w, f'_w and for all $f_{u''}$ which vanish on the singular set of $G_{u''}$, we have

$$(4.3) \qquad (-1)^{n-m} \operatorname{tr} \pi'_w(f'_w) \operatorname{tr} \pi'_{u''}(f_{u''}) = \sum m(\pi) \operatorname{tr} \pi_w(f_w) \operatorname{tr} \pi_{u''}(f_{u''}).$$

The sum is over an easily specified set of $(\pi_w, \pi_{u''})$. Note that G splits at u'', hence $f'_{u''} = f_{u''}$; moreover, the place u'' is chosen so that $\pi'_{u''}$ is unramified. The π' of Proposition 3 is cuspidal, hence it has a Whittaker model, and $\pi'_{u''}$ is non-degenerate. Consequently, $\pi'_{u''}$ is equal to an irreducible representation which is induced from an unramified character of the upper triangular subgroup, by [Z], Theorem 9.7(b).

Let $f'_{u''}$ be any function such that $F(f'_{u''})$ is supported on the split regular set of $G_{u''}$, and its restriction to $A(F_{u''})$ is $A(R_{u''})$-invariant. In [FK], [Sph], Chapter IV, [Sym; IV, VI], we call such a function "regular". It is clear that if $F(t, f'_{u''}) \neq 0$ then the Levi subgroup M_t of §I.6 is A, so that $\operatorname{tr} \pi(f'_{u''}) = \operatorname{tr} \pi_N(f'_{u''N})$ for any $G'_{u''}$-module π, where N is the upper triangular unipotent group. The support of $F(f'_{u''})$ is an open closed set; denote by θ its characteristic function, and replace $f'_{u''}$ by its product with θ. This does not change the value of the orbital integral, but assures the vanishing of the compactly (modulo center) supported $f'_{u''}$ on the singular set. Note that Theorem 4.2 of [BZ] implies that if $\operatorname{tr} \pi_N(f'_{u''N}) \neq 0$ then π has a non-zero vector fixed by the first congruence subgroup. By virtue of the Lemma, the sum of (4.3) is finite, uniformly in the $f'_{u''}$ which are considered here. Hence we can apply linear independence of (finitely many) characters on $A_{u''}$. This, together with Frobenius reciprocity, implies that we may consider on the right only $\pi_{u''}$ which are subquotients of, hence equal to, the irreducible $\pi'_{u''}$. The first claim of the proposition follows. The last assertion of the proposition follows from the Lemma.

§5. Lifting

Let F be non-archimedean, $G = \mathrm{GL}(m, D)$ and $G' = \mathrm{GL}(n, F)$. We have an injection $x \to x'$ of conjugacy classes from G to G', and we denote the characters of the G-module π and G'-module π' by χ_π and $\chi_{\pi'}$ (or χ').

Theorem. *The relation $\chi_{\pi'}(x') = (-1)^{n-m}\chi_\pi(x)$ for all matching regular conjugacy classes x, x' in G, G' defines a bijection between the set of equivalence classes of square-integrable (resp. tempered) G-modules π, and the set of equivalence classes of square-integrable (resp. tempered) G'-modules π' (resp. whose character χ' is non-zero on the set of regular x' obtained from x in G).*

Proof. Let π' be a square-integrable G'-module. Proposition 4 (where we now omit the subscript w) establishes the existence of a Trace Identity II.3 for this π'. By virtue of Corollary 1 and the induction assumption of Theorem 1 for $M \neq G$, the Assumptions II.3, II.5.1, II.5.2, are valid. By Proposition II.4 the π of Proposition 4 are square-integrable, and by Proposition II.5 there are only finitely many π in the sum. Since f is an arbitrary function on G, we conclude an identity of characters

$$(-1)^{m-n}\chi_{\pi'}(x') = \sum_\pi m(\pi)\chi_\pi(x)$$

for regular matching classes $x \to x'$. On the right the sum ranges over a finite set of square-integrable G-modules π. Applying the orthonormality relations for square-integrable G and G'-modules of Kazhdan [K], Theorem K, we conclude from $1 = \Sigma_\pi m(\pi)^2$ that the sum consists of a single π with a coefficient $m(\pi) = 1$.

Remark. Another proof for the existence of a square-integrable π to match such π', without using the finiteness result of Proposition II.5, yet using Corollary 1 or the Assumptions II.5.1, II.5.2, is as follows. It is clear that some π appears in the sum of Proposition 4, since we can take f' to be a pseudo-coefficient of π' by Corollary 1. Fixing such π we take f in $A(G)$ with $'\Phi(f) = \chi(\pi)$ on the elliptic regular set; it exists by [K], Theorem K. Then the sum of Proposition 4 is equal to $m(\pi)$. On the other hand, if f' is a matching function (which exists by Corollary 1), then

$$|\operatorname{tr} \pi'(f')|^2 = \left| \int \chi'(x')\,'\Phi(x', f')dx' \right|^2 \leqq \int |\chi'(x')|^2 dx' \int |\chi(x)|^2 dx$$

by Schwarz' inequality. The integrals are taken over the elliptic set of G or G', and we use the fact that $'\Phi(x', f') = '\Phi(x, f) = \chi(x)$; χ, χ' are the characters of π, π'. By the orthonormality relations of [K], Theorem K, we conclude that $m(\pi) \leqq 1$. As $m(\pi)$ is a positive integer, we conclude that $m(\pi) = 1$, and that the Schwarz inequality is an equality in our case, so that $\chi'(x') = c\,'\Phi(x, f) = c\chi(x)$ on the elliptic regular set, where c is a constant with $|c| = 1$. Hence π is the only term in the sum, and $c = (-1)^{m-n}$.

In the opposite direction, given a square-integrable π we take a pseudo-coefficient f in $A(G)$ of π, and a matching function f' in $A(G')$. By Corollary I.4 and the orthonormality relations of [K], there exists a tempered elliptic, hence by

[Z] square-integrable, G'-module π', with $\operatorname{tr} \pi'(f') \neq 0$. By the orthonormality relations on G, the G-module matching π', whose existence was proven above, is our π.

We have now completed the proof of that part of the theorem which concerns square-integrable π, π'. The extension to the case of any tempered π and π' follows once we establish in §6 below that any tempered G-module is equal to an induced G-module from a square-integrable module. This result is well known in the case of the split group G'. In its proof we use that part of the theorem proven above, for square-integrable modules.

Remark. In particular, we completed the proof and hence can use the assertion of Theorem 5 in the case $m = 1$, namely when G is the multiplicative group $D^\times$ of a division algebra D central over F. Indeed, all G-modules in this case are square-integrable and the image of the correspondence here is the set of elliptic tempered, hence square-integrable, G'-modules.

§6. Relevance

Proposition. *Any elliptic tempered G-module is square-integrable.*

Proof. Suppose that the character χ of π is non-zero on the elliptic regular element y. Let f be the characteristic function of a small neighborhood of y (modulo Z), where χ is constant. It is clear from the Weyl integration formula that

$$\langle \chi, {}'\Phi(f)\rangle = \int \chi(x) f(x) dx \qquad (x \text{ in the elliptic set of } G).$$

For our f and χ, we have $\langle \chi, {}'\Phi(f)\rangle \neq 0$. Since f is supported on the regular set, there is a matching f', with ${}'\Phi(x, f) = {}'\Phi(x', f')$ on the elliptic set. As f' lies in $A(G')$, there is a matching function f'' on the multiplicative group G'' of a division algebra of dimension n^2 central over F. Since G'' is compact modulo its center Z, there are finitely many G''-modules π''_i with characters χ''_i, and complex numbers c_i, so that ${}'\Phi(x'', f'') = \Sigma\, c_i \chi''_i(x'')$ on the regular x'' in G''. If χ'_i are the characters of the G'-modules π'_i which correspond to the π''_i, then ${}'\Phi(x', f') = \Sigma\, c_i \chi'_i(x')$ on the elliptic regular set. Since the π'_i are square-integrable, they correspond to square-integrable G-modules π_i with characters χ_i. Hence ${}'\Phi(x, f) = \Sigma\, c_i \chi_i(x)$ on the regular elliptic set. Then $\langle \chi, {}'\Phi(f)\rangle = \Sigma\, c_i \langle \chi, \chi_i\rangle$. Since this is non-zero, we have $\langle \chi, \chi_i\rangle \neq 0$ for some i. But the orthonormality relations for square-integrable G-modules of [K], Theorem K, imply that π is equivalent to the square-integrable π_i, as required.

6.1. Proposition. *Suppose I is a G-module unitarily induced from a square-integable M-module, where M is a Levi component of a proper parabolic subgroup. Then I is irreducible.*

Proof. This is the same as the proof of Proposition 27 of [FK], where the analogous result is proven for the metaplectic group $\tilde{G}$. To obtain a proof for our group G, each symbol $\tilde{x}$ in the proof of [FK], Proposition 27, has to be replaced by the symbol x. Note that this proof is based on Proposition 6, and so on the lifting Theorem 5 for the square-integrable G and G'-modules.

This Proposition, together with parabolic induction, completes the proof of Theorem 5, assuming that Theorem 1 holds for all proper Levi subgroups M of G. In §7 below we use Theorem 5 to prove Theorem 1 by induction on M.

To study lifting, or correspondence, of automorphic $G(\mathbf{A})$-modules, we need an extension of Proposition 6.1, which we now state. Put $\nu(x) = |x|$ for x in $F^\times$, and $\nu(g) = \nu(\det g)$ for g in G, where $\det g$ is the reduced norm of g. Write $\pi\nu^s$ for the G-module $g \to \pi(g) \otimes \nu(g)^s$, where s is a complex number.

6.2. Definition. A G-module π is called *relevant* if there is (i) a Levi subgroup M of G of the form $M_0 \times \Pi_{i=1}^m (M_i \times M_i)$, or of the form $\Pi_{i=1}^m (M_i \times M_i)$, where M_i is a multiplicative group of a simple algebra for each i $(0 \leqq i \leqq m)$, (ii) irreducible tempered M_i-modules ρ_i $(0 \leqq i \leqq m)$, and (iii) distinct positive numbers $s_i < \frac{1}{2}$ $(1 \leqq i \leqq m)$, such that π is equivalent to $I(\rho)$ or $I(\rho_0 \times \rho)$, and ρ is the $\Pi_{i=1}^m (M_i \times M_i)$-module $\Pi_{i=1}^m (\rho_i\nu^{s_i} \times \rho_i\nu^{-s_i})$.

Remark. This definition is analogous to Definition 27.2 of [FK] for the metaplectic group $\tilde{G}$. (Note that the word "Proposition" in [FK], Definition 27.2, should be "Theorem".)

Proposition. *A relevant G-module is irreducible and unitary.*

Proof. This is the same as the proof of Theorem 27.2 in [FK], except that all references to the metaplectic group ought to be replaced by references to our multiplicative group G of a simple algebra. This proof is based on unitarity arguments.

It is now clear that by parabolic induction Theorem 5 extends to hold also for relevant, not only tempered, G- and G'-modules. This completes the proof of the Local Theorem of the introduction.

Remark. (1) The result of Proposition 6.1 and Theorem 5 is due to [DKV], and that of 6.2 is new. (2) Theorem 5 and Propositions 6.1 and 6.2 are proven here for a local field of characteristic zero. The analogous results hold for local fields of positive characteristic on using the Theorem of [K′].

§7. Induction

It remains to complete the proof of Theorem 1, using Theorem 5. In the proof of Theorem 5 we used the induction assumption of Theorem 1, namely the statement of Theorem 1 for all proper Levi subgroups. Our aim is to show that for any f in $C(G)$ there exists a matching f' in $C(G')$, and for any suitable f' in $C(G')$ (thus $\Phi(x', f') = 0$ for any regular x' not obtained from x in G) there is a matching f in $C(G)$. We note:

Lemma. *For every f there exists f' with* $\operatorname{tr} \pi(f) = \operatorname{tr} \pi'(f')$ *for all corresponding tempered π, π', and* $\operatorname{tr} \pi'(f') = 0$ *for the tempered π' which are not obtained by the correspondence.*

Proof. Given f we define the function F on the space of tempered π' by $F(\pi') = \operatorname{tr} \pi(f)$ if π corresponds to π' by Theorem 5, and by $F(\pi') = 0$ if the character of π' is zero on the set of regular x' obtained from x. Then F is in the space F_{good} in the terminology (1.2) of [BDK] (see Chapter II; §7), hence a trace function by the Trace Paley–Wiener Theorem 1.3 of [BDK]. Namely there is an f' with $F(\pi') = \operatorname{tr} \pi'(f')$ for all tempered π', as required.

The same argument implies the existence of f for a given suitable f'.

Proposition. *Suppose that f and f' satisfy* $\operatorname{tr} \pi(f) = \operatorname{tr} \pi'(f')$ *for all corresponding tempered π and π', and* $\operatorname{tr} \pi'(f') = 0$ *for the tempered π' not obtained by the correspondence. Then f, f' are matching.*

Proof. By induction on the Levi subgroup M of the parabolic subgroup $P = MN$ of G. Denote by P', M', N' the corresponding parabolic, Levi, unipotent subgroups of G'. Let δ_P be the modulus homomorphism on P. Thus $d(ab) = \delta_P(a)db$ (a, b in P) for any right Haar measure db on P. For a in the center A of M we have $\delta_P(a) = \Pi\,|\alpha(a)|$; the product ranges over all roots of A in N. As usual, we put

$$f_N(m) = \delta_P(m)^{1/2} \int_K \int_N f(k^{-1}mnk)dndk.$$

Here K is a maximal compact subgroup of G with $G = KP$. For any m in M regular in G we have $F(m, f) = F^M(m, f_N)$, where $F(x, f) = \Delta(x)\Phi(x, f)$ and $\Delta(x) = |\Pi_{i<j}(x_i - x_j)^2/x_ix_j|^{1/2}$ if x has distinct eigenvalues x_i (see [FK], §7). F^M is defined analogously, with respect to M. Analogous notations are employed in the case of G'. Further, we note that if $\pi = I(\rho)$ is the G-module unitarily induced from the M-module ρ, then $\operatorname{tr} \pi(f) = \operatorname{tr} \rho(f_N)$ by a standard evaluation of the character of an induced representation. Consequently, if ρ, ρ' are corresponding tempered M- and M'-modules, we have $\operatorname{tr} \rho(f_N) = \operatorname{tr} \rho'(f'_N)$, and $\operatorname{tr} \rho'(f'_N) = 0$ for tempered ρ' not obtained from any ρ. By induction we have $F(x', f') = F^M(x', f'_N) = F^M(x, f_N) = F(x, f)$ for the regular x' in M' which come from x, and $F(x', f') = 0$ for the regular x' in M' which do not come from G. It remains to show the proposition for elliptic regular x, x'.

Choose matching elliptic regular y, y'. Let U' be a sufficiently small compact neighbourhood of y', and $'f'$ a function on G', supported near $y'Z$, whose orbital integral $'\Phi('f')$ is the characteristic function of $ZU'^{G'}$. Let $'f$ be a matching function on G. Now, $'\Phi('f')$ is a finite linear combination of the characters of square-integrable π_i' with coefficients c_i, by [K], Theorem K. Then $\Phi'('f)$ is the corresponding combination of the characters of the π_i which correspond to the π_i'. Since U' is small, the Weyl integration formula implies that $\int_{T/Z} F(t, 'f')F(t, f')dt$ is equal to $\Sigma\, c_i \operatorname{tr} \pi_i'(f')$; T is the centralizer of y' in G'. The assumption of our proposition implies that this is equal to $\Sigma\, c_i \operatorname{tr} \pi_i(f)$. But this is $\int_{T/Z} F(t, 'f)F(t, f)dt$. We take U' to be so small that both $F(t, 'f')$ and $F(t, 'f)$ are constant on U'. The desired equality $F(y, f) = F(y', f')$ now follows from the choice of $'f$ and $'f'$, which guarantees that $F(t, 'f) = F(t, 'f')$.

§8. Global correspondence

Let now F be a global field, and put $G' = \mathrm{GL}(n, \mathbb{A})$, $G = \mathrm{GL}(m, D_{\mathbb{A}})$, where $D_{\mathbb{A}}$ denotes the adele points of a division algebra D of dimension d^2 central over F, and $n = md$. Also we put $G_v' = \mathrm{GL}(n, F_v)$, $G_v = \mathrm{GL}(m, D_v)$ at each place v of F, where D_v denotes the F_v-points of D. Then $G_v = \mathrm{GL}(m_v, D(v))$, where $D(v)$ is a division algebra of dimension d_v^2 over F_v, and $n = m_v d_v$. Also $G_v \simeq G_v'$ for all v outside a finite set V of places, and we have an injection $x \to x'$ of conjugacy classes from G to G', and from G_v to G_v' for all v. It is a bijection for v outside V, but it is not surjective for v in the set V where D ramifies.

Recall that an irreducible admissible G_v-module π_v is said to lift (or correspond) to a G_v'-module π_v' if their characters χ_v, χ_v' are related by $(-1)^{n-m_v}\chi_v'(x') = \chi_v(x)$ for all regular matching x, x'. At v outside V we have $m_v = n$ and this relation amounts to $\pi_v \simeq \pi_v'$. Our Local Theorem asserts that the map $\pi_v \to \pi_v'$ induces an embedding of the set of (equivalence classes of) tempered (resp. relevant) G_v-modules as a subset of the set of tempered (resp. relevant) G_v'-modules.

A G-module $\pi = \bigotimes \pi_v$ is said to (quasi-) lift to a G'-module $\pi' = \bigotimes \pi_v'$ if π_v lifts to π_v' for (almost) all v. Results about global lifting depend on the form of trace formula which is available. Here we use only Proposition I.3. It implies, on using transfer of orbital integrals (Theorem 1), that any discrete-series (automorphic) G-module π whose components at two places u, u' lift to supercuspidal G_u' and $G_{u'}'$-modules, quasi-lifts to an automorphic (necessarily cuspidal) G'-module with supercuspidal components at u, u'. Further, any automorphic G'-module π' with a supercuspidal component at u, an elliptic component at u', and components π_v' with characters χ_v' which are not identically 0 on the set of regular classes x' obtained from x in G_v for all v in V, is a quasi-lift of a discrete-series G-module. Note that u, u' are not required to be in or out of V.

Using the stronger form of the simple trace formula established in [FK1] we show in [FK1] that all the assertions in this section hold also with no condition at the second place u', namely for automorphic $G'(\mathbf{A})$-modules π' with a supercuspidal component at one place u only, and the corresponding set of $G(\mathbf{A})$-modules π. It will be interesting to extend these results to all π by means of a simple and short proof. This may be afforded by the usage of the regular functions of [FK] and [Sph], but at the moment we have developed this technique only in the case of groups of rank one in [Sym; VI].

Since an automorphic π' with a supercuspidal component is cuspidal, multiplicity one and rigidity theorems for the cuspidal spectrum of $L(G')$ imply that the discrete-series quasi-lift π' of π is unique if it exists. We shall now deal with the notion of lifting, rather than quasi-lifting, and conclude the uniqueness of π too, thereby obtaining multiplicity one and rigidity type theorems for discrete-series G-modules.

Theorem. *Suppose that π' is an automorphic G'-module with supercuspidal components at two places u and u', and components π'_v whose characters are not identically zero on the set of the x' which come from G_v for all v in V. Then there exists a unique automorphic G-module π which quasi-lifts to π'; moreover, π lifts to π'.*

Proof. The condition at u implies that π' is cuspidal. Hence it has a Whittaker model, and its components are all non-degenerate and unitary. Hence, by [Z], Theorem 9.7(b), each π'_v is relevant. The Local Theorem implies that π'_v is the lift of a relevant G_v-module $\tilde{\pi}_v$. The identity of Proposition 2, say in the form (4.2) with π' as the only term on the left and with a sufficiently large but finite set S of places of F (S depends on π'), implies that

$$\prod_{v\in S} \operatorname{tr} \tilde{\pi}_v(f_v) = \prod_{v\in S} \operatorname{tr} \pi'_v(f'_v) = \sum_{\pi} m(\pi) \prod_{v\in S} \operatorname{tr} \pi_v(f_v)$$

for *all* functions f_v on G_v (v in S). The "generalized linear independence" of characters in Lemma 4 implies that the sum on the right consists of a single summand π with $m(\pi) = 1$, and the theorem follows.

Partial results can be obtained also for non-cuspidal discrete-series π', once a suitable form of the traces identity is available. But the conjectural description of such π' has not been proven as yet. Namely it is well known tht the non-cuspidal residual spectrum contains π' whose components are all dual, in the sense of [Z], to generalized Steinberg G'_v-modules, but it has not been shown as yet that these π' exhaust the residual spectrum, and they occur with multiplicity one. Yet, given the identity of trace formulae, the Theorem of [JS] permits working with these exceptional π', and establishing lifting for them. These matters will not be discussed here, but see [FK], §28.

Chapter IV. Automorphic Forms on Compact Unitary Groups

§0. Introduction

Let E/F be a quadratic extension of local non-archimedean fields, $G' = \mathrm{GL}(n, E)$, and G the associated quasi-split unitary group. We show that there is a partition of the set of equivalence classes of irreducible tempered G-modules into finite sets, called *packets*, so that there is a bijection, defined by means of character relations, from the set of packets to the set of equivalence classes of irreducible σ-stable tempered G'-modules. This local result is obtained in §4 using global techniques, in a simple situation.

Let E/F be a quadratic extension of number fields, fix a finite place u of F which splits in E, let G' be the multiplicative group of a division algebra of rank n central over E, ramified above u and split outside u, and G the unitary group associated with G' and an involution σ of the second kind. The quotient $G(F)\backslash G(\mathbf{A})$ is compact; its space of automorphic forms decomposes as a direct sum of irreducible G-modules, and its automorphic representations have a particularly simple, "stable", form in the following sense. We define non-degenerate (σ-invariant) automorphic G'-modules to be those which correspond to cuspidal $\mathrm{GL}(n, E)$-modules by means of the correspondence of Chapter III. We then show that an analogous definition can be made for the set of automorphic G-modules. In §5 we show that there is a partition of the set of non-degenerate automorphic G-modules into packets, which are the restricted products of the local packets, so that there is a bijection from this set of packets to the set of automorphic non-degenerate σ-invariant G'-modules π'. The components of such π' are all σ-stable. In particular we obtain a global rigidity theorem for packets of non-degenerate G-modules.

§1. Theorems

Let E/F be a quadratic extension of number fields; u a finite place of F which splits in E; G' an inner form of $\mathrm{GL}(n)$ over E which is anisotropic at the two places above u, and splits outside u. G' is then the multiplicative group of a division algebra D of rank n central over E. Suppose that σ is an involution of the second kind on D. Namely, σ is an anti-automorphism of order two whose restriction to the center E of D is the non-trivial element of the galois group $\mathrm{Gal}(E/F)$. Then the unitary group G defined by D and σ consists of the x in D with

$\sigma(x)x = 1$. At the place u the completion $G_u = G(F_u)$ is the multiplicative group of a division algebra of rank n central over the completion F_u of F at u. At a place $v \neq u$ of F which splits in E we have $G_v = \mathrm{GL}(n, F_v)$. At a finite non-split v the G_v is a quasi-split unitary group. At a non-split archimedean place v we have $E_v/F_v = \mathbf{C}/\mathbf{R}$, and $G_v = U(i,j)$ is a unitary group of signatures (i,j), where $i+j=n$. Since our theory at the archimedean places is well known, and our main interest is in the non-archimedean cases, to simplify the exposition we assume that each archimedean place of F splits in E.

Our aim is to describe the tempered and automorphic representations of G in terms of those of G'.

Let $\bar{F}$ be an algebraic closure of F, $\boldsymbol{G}$ a reductive connected group defined over F with $G = \boldsymbol{G}(F)$ and $G' = \boldsymbol{G}(E)$. For any extension F' of F we write $\boldsymbol{G}(F')$ for the group of F'-points of $\boldsymbol{G}$. Identify $\boldsymbol{G}$ with $\boldsymbol{G}(\bar{F})$. Let $\boldsymbol{G}'$ be the group $\mathrm{Res}_{E/F}\boldsymbol{G}$ obtained from $\boldsymbol{G}$ upon restricting scalars from E to F. Being the induced galois module $\mathrm{Ind}(\boldsymbol{G}; \mathrm{Gal}(\bar{F}/F), \mathrm{Gal}(\bar{F}/E))$, $\boldsymbol{G}'$ can be realized as follows. As a group, $\boldsymbol{G}' = \boldsymbol{G} \times \boldsymbol{G}$. Denote by σ the non-trivial element of $\mathrm{Gal}(E/F)$, and by $\tilde{\sigma}$ the automorphism $\tilde{\sigma}(x,y) = (y,x)$ of $\boldsymbol{G}'$. τ in $\mathrm{Gal}(\bar{F}/F)$ maps (x,y) to $(\tau x, \tau y)$ if its restriction to E is trivial, and to $\tilde{\sigma}(\tau x, \tau y)$ if $\tau \mid E = \sigma$. Hence $\boldsymbol{G}'(E) = \boldsymbol{G}(E) \times \boldsymbol{G}(E)$, and $G' = \boldsymbol{G}'(F)$ is the group of pairs $(x, \sigma x)$ with x in $\boldsymbol{G}(E)$. Let Z, Z' be the center of G, G'; $\mathbf{A}$, $\mathbf{A}^\times$ and $\mathbf{A}_E$, $\mathbf{A}_E^\times$ the adeles, ideles of F and E; E^1 and $\mathbf{A}_E^1$ the kernel of the norm map from E to F acting on $E^\times$ and $\mathbf{A}_E^\times$. Then $Z'(\mathbf{A}) = \mathbf{A}_E^\times$, $Z' = E^\times$, $Z(\mathbf{A}) = \mathbf{A}_E^1$, $Z = E^1$. Put $C^1 = \mathbf{A}_E^1/E^1$ if E is global, and $C^1 = E^1$ if E is local. Fix a unitary character ω of C^1, and put $\omega'(x) = \omega(x/\bar{x})$; ω' is a character of $\mathbf{A}_E^\times/\mathbf{A}^\times E^\times$ or $E^\times/F^\times$.

Our objects of study are (equivalence classes of) G-modules π and G'-modules π' with central character ω and ω', which are admissible of finite length if F is local, and automorphic if F is global. Let ${}^\sigma\pi'$ be the G'-module ${}^\sigma\pi'(x) = \pi'(\sigma(x))$. We deal only with σ-invariant π', those with $\pi' \simeq {}^\sigma\pi'$. These extend to $G' \rtimes \langle\sigma\rangle$-modules. If F is local, we denote by χ' the restriction of the character of π' to the coset $G' \times \sigma$ (see Chapter I, §5). To simplify the notations, we write $\chi'(x)$ for $\chi'(x \times \sigma)$, where x is in G'. Denote by χ the character of π. We say that x, x' in G (or G') are [*stably*] (σ-) *conjugate* if there is y in G (or G') [resp. $\boldsymbol{G}$ (or $\boldsymbol{G}'$)] with $xy = yx'$ (or $x\tilde{\sigma}(y) = yx'$). A function on G (or G') is called (σ-)*stable* if it is constant on each stable (σ-)conjugacy class. A σ-invariant G'-module π' is called *σ-stable* if its character is σ-stable. In fact, we are interested only in *regular* x in G, those with distinct eigenvalues, and *σ-regular* x in G', those for which $x\tilde{\sigma}(x)$ has distinct eigenvalues. Given a σ-regular x in G', the conjugacy class of $x\tilde{\sigma}(x)$ in $\boldsymbol{G}'$ is defined over F, and contains an F-rational element, giving rise to a bijection N (see [Ko]) from the set of stable σ-conjugacy classes of σ-regular elements in G', to the set of stable conjugacy classes of regular elements in G. If E/F is a local quadratic extension, we have

Local Theorem.* *For each tempered σ-stable irreducible G'-module π' there exists a finite set $\{\pi\}$, named* packet, *of tempered irreducible G-modules π, and positive integers $n(\pi)$, so that $\Sigma_\pi n(\pi)\chi$ is a stable function G and*

$$\chi'(x) = \sum_{\pi} n(\pi)\chi(Nx)$$

for all σ-regular x in G'. Moreover, for each tempered irreducible π there exists a unique π' as above for which the relation holds. If π' is square-integrable, then $\{\pi\}$ consists of a single element.

Namely, there is a partition of the set of equivalence classes of the set of irreducible tempered G-modules into disjoint finite sets $\{\pi\}$, named packets, so that there is a bijection between the set of packets and the set of equivalence classes of tempered σ-stable irreducible G'-modules, defined in terms of characters. Note that in particular we assert that the sum $\Sigma\, n(\pi)\chi$ over $\{\pi\}$, which *a priori* depends on conjugacy classes, in fact depends only on the stable conjugacy class, so that its value at Nx is well-defined. Further, if E/F is unramified, and π is unramified, then so is π'; if π' is unramified there is an unramified π in $\{\pi\}$ with $n(\pi) = 1$. The last claim in the Local Theorem follows at once from the (twisted analogue of the) orthonormality relations ([K], Theorem K) for characters of square-integrable representations, since it follows from our proofs that if π' is square-integrable then $\{\pi\}$ consists of square-integrable G-modules, to which the orthonormality relations of [K] apply.

Let F be global. Denote by $L(G\backslash G(\mathbf{A}))$ the space of smooth functions on $G\backslash G(\mathbf{A})$ which transform under $Z(\mathbf{A})$ by ω. $G(\mathbf{A})$ acts by right translation. An irreducible constituent π is called an automorphic G-module. It is a product $\pi = \otimes\pi_v$, where almost all π_v are unramified. The space $L(G'\backslash G'(\mathbf{A}))$, and automorphic G'-modules $\pi' = \otimes\pi'_v$ which transform under $Z'(\mathbf{A})$ by ω', are defined analogously. If E_v/F_v, π_v and π'_v are unramified, then π_v, π'_v are parametrized by conjugacy classes t_v, t'_v with representatives in the cosets $\mathrm{GL}(n,\mathbf{C})\times\sigma$, $[\mathrm{GL}(n,\mathbf{C})\times\mathrm{GL}(n,\mathbf{C}]\times\sigma$ of the dual groups $\hat{G}_v$ and $\hat{G}'_v$ of G_v and G'_v (see [Sph], §2). We say that π *quasi-lifts* to π', if for almost all v, t_v maps to t'_v by the base-change map

$$\hat{G}_v \to \hat{G}'_v, \qquad t\times\sigma^i \to (t,t)\times\sigma^i \qquad (i = 0, 1).$$

We first show that each automorphic π quasi-lifts to a unique σ-invariant π', and each such π' is a quasi-lift of a π. The correspondence of Chapter III gives a bijection from the set of (equivalence classes of) the automorphic G'-modules π' to the set of automorphic $\mathrm{GL}(n,E)$-modules π'' whose two components above u are elliptic (their character is non-zero on the regular elliptic set). It is defined by

$\pi'_v \simeq \pi''_v$ for all $v \neq u$. We say that π' is *non-degenerate* if the corresponding π'' is cuspidal, and that π is *non-degenerate* if it quasi-lifts to a non-degenerate π'.

The *packet* $\{\pi\}$ of a non-degenerate π is defined to be the set of irreducible $G(\mathbf{A})$-modules $\otimes \tilde{\pi}_v$, where $\tilde{\pi}_v$ lies in the packet $\{\pi_v\}$ of π_v for all v, and $\tilde{\pi}_v$ is equal to π_v for almost all v. Although π_v is not yet known to be tempered, since π is non-degenerate the definition of local packets extends to this case (see §5 below).

Global Theorem.* *Each irreducible $G(\mathbf{A})$-module in a packet of a non-degenerate π is automorphic. The packets define a partition of the set of non-degenerate π. Quasi-lifting defines a bijection from the set of packets of non-degenerate π to the set of σ-invariant non-degenerate π'. If $\{\pi\}$ quasi-lifts to π', then $\{\pi_v\}$ lifts to π'_v in the sense of the Local Theorem, for all v, and all components of π' are σ-stable.*

* Our work consists of reducing the above lifting results to the standard local assumptions of Chapter II concerning stable base-change transfer of orbital integrals, so that the rigidity arguments of Chapter II can be applied. These can be verified for $n = 3$ as in [Sym; I]. In particular our Local and Global Theorems are proven only for $n = 3$ (but not for $n \geqq 4$); however, the proofs apply with any n, to reduce the Theorems to a standard local assumption concerning matching stable orbital integrals, which we do not prove.

Our Theorems generalize those of [U(2)], where the case of $n = 2$ (and arbitrary σ-invariant central character ω' on $\mathbf{A}_E^\times/E^\times$, not necessarily of the form $\omega'(x) = \omega(x/\bar{x})$) was studied.

We use the trace formula, in the case of compact quotient. The usage of this formula depends on the base-change transfer of stable orbital integrals of spherical functions, proven in [Sph] in any stable base change situation. Namely, we use the main theorem of [Sph], which asserts that if ϕ_v and f_v are corresponding spherical functions on G'_v and G_v, then they have matching stable orbital integrals. We also make an extensive use of the work of [BDK] and [K], and their twisted analogues (for the non-connected group $G' \rtimes \langle\sigma\rangle$); see Chapter I; §§6–7.

§2. Approximation

Let E/F be a quadratic extension of global fields, u a finite place of F which splits in E, and G as in §1. The condition on G at u, namely that u splits in E/F and G_u is the multiplicative group of a division algebra, is fundamental in our work. It implies, as we now show, that the part of the trace formula for G (and the twisted trace formula for G') which is given by orbital integrals of regular elements, is stable. This makes it possible to compare these parts of the trace formulae for any functions $f = \otimes f_v$ and $\phi = \otimes \phi_v$ on $G(\mathbf{A})$ and $G'(\mathbf{A})$ of the usual kind which have matching *stable* orbital integrals. As usual, f_v (and ϕ_v) is the unit element f_v^0 (and ϕ_v^0) in the Hecke algebra of K_v- (and K'_v-) biinvariant functions on G_v (and G'_v) for almost all v ($K_v = G(R_v)$, $K'_v = G'(R_v)$, and R_v is the ring of integers of F_v when F_v is non-archimedean). Moreover, $f(zg) = \omega(z)^{-1}f(g)$ (z in $Z(\mathbf{A})$) and $\phi(zg) = \omega'(z)^{-1}\phi(g)$ (z in $Z'(\mathbf{A}) = Z(\mathbf{A}_E)$), and f, ϕ are smooth and compactly supported modulo the center.

Next we state the stability property of the "geometric" part of the trace formula of G, which involves orbital integrals. Given a regular element γ in $G(F)$ denote by $B(\gamma/F)$ a set of representatives for the conjugacy classes in $G(F)$ within the stable conjugacy class of γ. Denote by $B(\gamma/\mathbf{A})$ a set of representatives for the conjugacy classes in $G(\mathbf{A})$ within the stable conjugacy class of γ in $G(\mathbf{A})$. Then the sum $E(\gamma, f) = \Sigma\, \Phi(\delta, f)$ (δ in $B(\gamma/F)$) appears in the elliptic part of the trace formula of G at f. The sum $\Phi'(\gamma, f) = \Sigma\, \Phi(\delta, f)$ (δ in $B(\gamma/\mathbf{A})$) in the product of $\Phi'(\delta, f_v) = \Sigma\, \Phi(\delta, f_v)$ (δ in $B(\gamma/F_v)$) over all places v of F. $\Phi'(\gamma, f)$ and $\Phi'(\gamma, f_v)$ are the stable orbital integrals of f (globally) and f_v (locally) in the notations of Chapter II; §1. Then we have

2.0. Proposition. *$E(\gamma, f)$ is equal to $\Phi'(\gamma, f)$*

Proof. Denote by T the centralizer of the regular element γ in G. It is an elliptic torus. Let T_{sc} be the image of the torus T in the derived group $G_{sc} = \{g$ in G; $\det g = 1\}$ of G (note that this derived group is simply connected). Put $C(T/F) = \mathrm{Im}[H^1(F, T_{sc}) \to H^1(F, T)]$ and $C(T/\mathbf{A}) = \bigoplus C(T/F_v)$ (pointed sum), and $X_*(T) = \mathrm{Hom}(\mathbf{G}_m, T)$ for the group of $\bar{F}$-morphisms from the multiplicative group $\mathbf{G}_m$ to the torus T. Then $X_*(T) \simeq \mathbf{Z}^n$ and $X_*(T_{sc}) = \{(x_i)$ in $\mathbf{Z}^n$; $\Sigma_i x_i = 0\}$. The Tate–Nakayama theory [Ta] implies that $C(T/F)$ embeds in $C(T/\mathbf{A})$, and the quotient $C(T/\mathbf{A})/C(T/F)$ embeds in

$$k(T) = \{\mu \text{ in } X_*(T_{sc});\ N_{K/F}\mu = 0\}/\langle \mu - \tau\mu;\ \tau \text{ in } \mathrm{Gal}(K/F), \mu \text{ in } X_*(T)\rangle,$$

where K is a finite galois extension of F over which T splits, and $N_{K/F}$ is the norm map from K to F. A standard (stabilization) argument (see, e.g., [Sph; §5]) implies that in order to prove the proposition it suffices to show that $k(T)$ is zero.

For this, note that $T_u = T(F_u)$ is the centralizer of the regular element γ in G_u, hence it is isomorphic to an elliptic torus in the split form $\mathrm{GL}(n, F_u)$ of G_u. Let K_u be a finite galois extension of F_u which splits T_u. The galois group $\mathrm{Gal}(K_u/F_u)$ acts on $X_*(T) = \mathbf{Z}^n$ by permutations. Each element of the symmetric group can be expressed as a product of disjoint cycles. Since T_u is elliptic, for each i ($1 \leqq i \leqq n$) there exists τ_i in $\mathrm{Gal}(K_u/F_u)$ which has a cycle $(j_1, j_2, \ldots, j_r)$ with $j_1 = 1$ and $j_s = i$ for some s ($2 \leqq s \leqq r$). In particular, the set $\{\mu - \tau_i^{s-1}\mu;\ 2 \leqq i \leqq n,\ \mu$ in $\mathbf{Z}^n\}$ contains all vectors of the form $x\alpha_i$ (x in $\mathbf{Z}$; $2 \leqq i \leqq n$), where α_i is the n-vector whose non-zero entries are one at the first place and -1 at the ith place. The span of $\{x\alpha_i;\ x$ in $\mathbf{Z}$, $2 \leqq i \leqq n\}$ is $X_*(T_{sc})$. Hence $k(T) = \{0\}$, and the proposition follows.

For the twisted analogue of this result, given a σ-regular element γ in $G'(F) = G(E)$, denote by $\tilde{B}(\gamma/F)$ (resp. $\tilde{B}(\gamma/\mathbf{A})$) a set of representatives for the σ-conjugacy classes in $G'(F)$ (res. $G'(\mathbf{A})$) within the stable σ-conjugacy class of γ in $G'(F)$ (resp. $G'(\mathbf{A})$). The sum $E(\gamma, \phi) = \Sigma\, \Phi(\gamma, \phi)$ (δ in $\tilde{B}(\gamma/F)$) appears in the elliptic part of the twisted trace formula of G' at ϕ, and $\Phi'(\gamma, \phi) = \Sigma\, \Phi(\delta, \phi)$ (δ in $\tilde{B}(\gamma/\mathbf{A})$) is the product of the local stable σ-orbital integrals $\Phi'(\gamma, \phi_v)$ over all v. Then we have

Proposition. *$E(\gamma, \phi)$ is equal to $\frac{1}{2}\Phi'(\gamma, \phi)$.*

Proof. Denote by T the σ-centralizer of γ in $G'(F)$; it is an elliptic torus in $G(F)$ (up to isomorphism). Then $H^1(F, T)$ embeds in $H^1(\mathbf{A}, T) = \bigoplus H^1(F_v, T)$, and with the definitions of [Sph; (5.1)], the quotient of $\tilde{B}(\gamma/\mathbf{A})$ by $\tilde{B}(\gamma/F)$ is isomorphic to the quotient of $H^1(\mathbf{A}, T)$ by $H^1(F, T)$. By [Ta], this last quotient is isomorphic to

$$k'(T) = \{\mu \text{ in } X_*(T);\ N_{K/F}\mu = 0\}/\langle \mu - \tau\mu;\ \tau \text{ in } \mathrm{Gal}(K/F), \mu \text{ in } X_*(T)\rangle.$$

As in the previous Proposition, since T_u is a torus in G_u (isomorphic to an elliptic torus in $\mathrm{GL}(n, F_u)$), the span of $\mu - \tau\mu$ (μ in $X_*(T) = \mathbf{Z}^n$; τ in $\mathrm{Gal}(K_u/F_u)$) contains $X_*(T_{sc}) = \{(x_i) \text{ in } \mathbf{Z}^n;\ \Sigma_i x_i = 0\}$. In addition, any element in $\mathrm{Gal}(K/F)$ whose restriction to E is σ acts by $(x_i) \to (-x_{\varepsilon(i)})$ for some permutation ε of $\{1, \ldots, n\}$. Hence $k'(T) \simeq \mathbf{Z}/2\mathbf{Z}$. This is in sharp contrast with the non-twisted case, where $k(T) = \{0\}$. However, if Z is the center of G, then $X_*(Z) = \{\mu = (x, \ldots, x) \text{ in } X_*(T)\} \simeq Z$, and the quotient $H^1(\mathbf{A}, Z)/H^1(F, Z)$ is isomorphic to $k'(T)$, since it is

$$k'(Z) = \{\mu \text{ in } X_*(Z);\ N_{E/F}\mu = 0\}/\langle \mu - \sigma\mu;\ \mu \text{ in } X_*(Z)\rangle = \mathbf{Z}/2\mathbf{Z}.$$

It can be seen (as in [U(2)], §2) that a set of representatives for the quotient of $\tilde{B}(\gamma/\mathbf{A})$ by $\tilde{B}(\gamma/F)$ is given by $\{\gamma, z\gamma\}$, where z is any element of $\mathbf{A}^\times - F^\times N_{E/F}\mathbf{A}_E^\times$. Since ϕ transforms under the center by the character ω', where $\omega'(z) = \omega(z/\bar{z})$ (z in $\mathbf{A}_E^\times$) is trivial on $\mathbf{A}^\times$, we have $\Phi(\gamma, \phi) = \Phi(z\gamma, \phi)$ for any z in $\mathbf{A}^\times$, and the proposition follows.

Remark 1. It is clear from the proof that on considering ϕ with $\phi(zg) = \omega'(z)^{-1}\kappa(z)\phi(g)$ (z in $Z'(\mathbf{A})$), where ω' is as above and κ is a fixed character of $\mathbf{A}_E^\times/E^\times N_{E/F}\mathbf{A}_E^\times$ whose restriction to $\mathbf{A}^\times$ is non-trivial, an analogous result can be obtained. This point of view is developed in [U(2)] to establish an unstable base-change lifting from $U(2)$ to $\mathrm{GL}(2, E)$, in addition to the stable base-change lifting studied here. This unstable transfer can be developed also in our generality of $U(n)$, but this will not be done here.

Remark 2. The assumption that f_v and ϕ_v have matching orbital integrals means that $\Phi'(\gamma, \phi) = \Phi'(N\gamma, f)$ for every σ-regular γ in $G'(F)$. Since (i) the norm map N from the set of stable σ-conjugacy classes in $G'(F)$ to the set of stable conjugacy classes in $G(F)$ is surjective, and (ii) the quotient of the volumes which appear in the twisted trace formula by the corresponding volumes $|T(\mathbb{A})/T(F)Z(\mathbb{A})|$ in the trace formula is equal to $[\mathbb{A}^\times/F^\times N_{E/F}\mathbb{A}_E^\times]=2$, the elliptic parts of the trace formulae are equal for matching functions f and ϕ.

A standard approximation argument of linear independence of characters of Hecke algebras, see Lemma III.3, based on the main theorem of [Sph] (that corresponding spherical functions on $G(F_v)$ and $G'(F_v)$ are matching, namely have matching stable orbital integrals), implies the following.

Suppose that V is a finite set of finite places of F, containing those which ramify in E. Each v outside V is either split in E, in which case we fix an irreducible G_v-module π_v and the corresonding σ-invariant G'_v-module $\pi'_v=(\pi_v, {}^\sigma\pi_v)$; or is unramified, in which case we fix an irreducible unramified G_v-module π_v. Here π_v is the unique unramified constituent in the composition series of the unramified G_v-module $I(\mu_v)$ induced from the unramified character μ_v of the upper triangular subgroup $B_v=A_vU_v$ (which is trivial on the unipotent radical U_v). Define the character μ'_v of the corresponding subgroup $B'_v=A'_vU'_v$ of G'_v by $\mu'_v(b)=\mu_v(b\sigma(b))$. The induced G'_v-module $I(\mu'_v)$ is σ-invariant and unramified, and we let π'_v be its unique unramified irreducible constituent. π'_v is σ-invariant.

At each place v in V suppose that f_v and ϕ_v are matching functions on G_v and G'_v, namely their stable orbital integrals are equal $\Phi^{st}(x,\phi_v)=\Phi^{st}(Nx,f_v)$ on the regular set. At one place u' in V we further assume that $f_{v'}$ and $\phi_{u'}$ are supported on the regular and σ-regular sets of $G_{u'}$ and $G'_{u'}$. Hence, for the functions $f=\otimes f_v$, $\phi=\otimes\phi_v$ which appear in the trace formula it suffices to consider only orbital integrals at regular conjugacy classes. In these notations, we obtain

2.1. Lemma. *We have*

$$\sum m(\pi')\prod \operatorname{tr}\pi'_v(\phi_v)=\sum m(\pi)\prod \operatorname{tr}\pi_v(f_v). \tag{2.1}$$

The products range over v in V. The sum on the left (resp. right) ranges over the equivalence classes of irreducible automorphic σ-invariant G'-modules π' (resp. G-modules π) whose component at each v outside V is the above π'_v (resp. π_v).

The rigidity theorem for $\mathrm{GL}(n)$ of [JS], and the correspondence of Chapter III, assert that on the left of (2.1) there is at most one term π'. Its multiplicity $m(\pi')$ is 1 if π' is non-degenerate. On the right, $m(\pi)$ denotes the multiplicity of π in $L(G\backslash G(\mathbb{A}))$. It is a non-negative integer. We can clearly assume that V does not contain any places which split in E, since at a split place v we have that π_v lifts to $\pi'_v=(\pi_v, {}^\sigma\pi_v)$ and $\operatorname{tr}\pi'_v(\phi_v)=\operatorname{tr}\pi_v(f_v)$, and we can apply "generalized linear independence" for absolutely convergent sums of characters on the group G_v (see Lemma III.3).

Corollary. *If the sequence $\{\pi_v;$ v outside $V\}$ is such that the sum on the left of* (2.1) *is non-empty, then there exists an automorphic π which quasi-lifts to the π' on the left.*

2.2. Proposition. *Fix a place w in V, which stays prime in E. Fix a unitary character μ_w of the upper triangular subgroup $B_w = A_w U_w$ (which is trivial on the unipotent radical U_w of B_w). Define the corresponding character μ'_w of $B'_w = A'_w U'_w$ by $\mu'_w(b) = \mu_w(b\sigma(b))$. Then* (2.1) *holds for arbitrary matching functions f_v, ϕ_v, provided that the sums are taken over the subsets of π, π' as in Lemma* 2.1 *whose component at w is a subquotient of the induced modules $I(\mu_w)$ and $I(\mu'_w)$, respectively.*

Proof. We use the regular functions of [Sph], §4. Thus, a vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ in $\mathbf{Z}^n$ is called *regular* if $\lambda_i > \lambda_{i+1}$ $(1 \leqq i < n)$ and $\lambda_i + \lambda_{n+1-i} = 0$ $(1 \leqq i < n)$. In particular, $\lambda_{(n+1)/2}$ is zero if n is odd. Fix a local uniformizer $\boldsymbol{\pi}$ of E_w. For such λ, denote by S_λ the set of g in G_w which are conjugate to diagonal elements of the form $a\boldsymbol{\pi}^{-\lambda}$, where a is in $A(R_w)$ and $\boldsymbol{\pi}^{-\lambda} = \operatorname{diag}(\boldsymbol{\pi}^{-\lambda_1}, \boldsymbol{\pi}^{-\lambda_2}, \ldots, \boldsymbol{\pi}^{\lambda_2}, \boldsymbol{\pi}^{\lambda_1})$. A function f_w is called *regular* and associated with μ_w and λ if it in supported on S_λ and the value of $F(a\boldsymbol{\pi}^{-\lambda}, f_w)$ is $\mu_w(a)$ (a in $A(R_w)$). A function ϕ_w is called *regular* and associated with λ and μ_w if ϕ_w is supported on the set of g in G'_w with norm in S_λ, and ϕ_w matches a regular f_w associated with λ and μ_w (thus $F(x, \phi_w) = F(Nx, f_w)$ for all x in G'_w with regular Nx).

Let f_w and ϕ_w be regular functions associated with μ_w and λ. Then, it follows from the Weyl integration formula and the Theorem of [C] (= Proposition I.6.1), that for any irreducible G_w-module π_w, we have that $\operatorname{tr} \pi_w(f_w)$ is zero unless π_w is a subquotient of $I(\eta_w)$, where η_w is a character of A_w with $\eta_w = \mu_w$ on $A(R_w)$. In this case there is a character $'\eta_w$ in the module π_{wU} of coinvariants of π_w with respect to U, and a subset $W(\pi_{wU})$ of $W(A)$, such that

$$\operatorname{tr} \pi_w(f_w) = \sum_\omega \lambda(\omega\, '\eta_w) \qquad (\omega \text{ in } W(\pi_{wU})).$$

Here $\omega\, '\eta_w(a)$ is defined to be $'\eta_w(\omega(a))$, and

$$\lambda(\eta) = \int_{A(R_w)} F(a\boldsymbol{\pi}^{-\lambda}, f_w)\eta(a\boldsymbol{\pi}^{-\lambda})da = \int_{A(R_w)} [\eta(a\boldsymbol{\pi}^{-\lambda})/\mu_w(a)]da$$

is an expression of the form $z_1^{\lambda_1} \cdots z_r^{\lambda_r}$, where $r = [n/2]$. The analogous statement holds for ϕ_w. For any irreducible G'_w-module π'_w, we have $\operatorname{tr} \pi'_w(\phi_w) = 0$ unless π'_w is σ-invariant and there exists a character η_w as above such that π'_w is a subquotient of $I(\eta'_w)$, where $\eta'_w(b) = \eta_w(b\sigma(b))$. In this case there is a σ-invariant character $'\eta'_w$ in the character $\chi(\pi'_{wU})$ of π'_{wU}, and a subset $W(\pi'_{wU})$ of $W(A)$, so that

$$\operatorname{tr} \pi'_w(\phi_w) = \sum_{\omega} \lambda(\omega\,'\eta'_w) \qquad (\omega \text{ in } W(\pi'_{wU})).$$

Note that if $\operatorname{tr} \pi_w(f_w) \neq 0$, then there is an open compact congruence subgroup C_w of G_w such that $I(\eta_w)$ has a non-zero vector fixed under the action of C_w. It depends only on μ_w (clearly). Hence π_w has a non-zero C_w-fixed vector, by the "Iwahori" decomposition of C_w (see [BZ], (3.17)).

Now we fix the components f_v, ϕ_v for $v \neq w$ in V. Then (2.1) attains the form

$$c \sum_{\omega} \lambda(\omega\,'\eta_w) = \sum_{'\eta'_w} c('\eta'_w) \sum_{\omega} \lambda(\omega\,'\eta'_w). \tag{2.3}$$

Since we fixed the archimedean components, and the ramification at all finite places, a theorem of [BJ] (4.3(i), p. 195) asserts that the sum on the right of (2.1) is finite, uniformly in the regular functions f_w, ϕ_w. Namely the sum over $'\eta'_w$ in (2.3) is taken over a finite set which is independent of the regular vector λ in $\mathbf{Z}^n$. Hence we can apply linear independence of finitely many characters, and the proposition follows.

Remark. The regular functions f_w, ϕ_w vanish on the singular set. Hence the condition (of 2.1)) at $u'(=w)$ is met. Since the components π_w, π'_w lie in a finite set, and (2.1) holds for f_w, ϕ_w which vanish on the singular set, (2.1) holds for *any* matching f_w, ϕ_w.

Corollary. *Each π quasi-lifts to a unique σ-invariant π'.*

Proof. If the left side of the Proposition is empty, it suffices to evaluate the right side at a characteristic function of an open compact congruence subgroup C_v for each v in V, to obtain a positive number, and a contradiction. The uniqueness follows from [JS] and Chapter III as noted after Lemma 2.1.

§3. Existence

Let E/F be a quadratic extension of number fields, and G the group of §1.

Lemma. *Each component π'_{1v} of a σ-invariant automorphic G'-module π'_1 with a central character ω' is σ-stable.*

Proof. Note that G_u is anisotropic, hence the component π'_{1u} is stable σ-elliptic (each element of G'_u is σ-elliptic, and each σ-stable conjugacy class consists of a single σ-conjugacy class). We take ϕ with ϕ_u supported on the σ-regular set. By virtue of the second Proposition in 2.0, the twisted trace formula asserts: $\Sigma \operatorname{tr} \pi'(\phi) = \Sigma\, c(x)\Phi'(x, \phi)$; on the right appear only stable σ-orbital integrals of σ-regular elements x in G', and the $c(x)$ are volume factors. Hence the right side vanishes if, at a fixed place w, the component ϕ_w has the property that $\Phi'(\phi_w)$ is zero on the σ-regular set. The approximation argument used in Lemma

2.1 implies that for some finite set V of places including w we have Π tr $\pi'_{1v}(\phi_v) = 0$; the product is over v in V. Since for each $v \neq w$ in V there exists ϕ_v with tr $\pi'_{1v}(\phi_v) \neq 0$, we conclude that tr $\pi'_{1w}(\phi_w) = 0$ for any ϕ_w with $\Phi'(\phi_w) = 0$ on the σ-regular set. A simple application of the Weyl integration formula implies the lemma.

Remark. If G is any (including the quasi-split) form of the unitary group, a similar proof based on Arthur's computations of the trace formula, shows that if π' is a σ-invariant discrete-series automorphic G'-module with a central character $\omega'(x) = \omega(x/\bar{x})$, which has a stable σ-elliptic component π'_u, then each component π'_v of π' is σ-stable. This statement is false if π'_u is not assumed to be σ-elliptic. For simplicity, in the Lemma we proved this statement only in the case specified in §1, which is the only case needed here.

Recall (Chapter II; §3) that an irreducible σ-invariant G_v-module π_v is called σ-discrete-series if each of its σ-invariant exponents (= central characters of the σ-invariant irreducibles in any non-trivial module of coinvariants of π_v) decays.

Proposition. *Suppose that π'_w is a σ-elliptic component of a σ-invariant non-degenerate automorphic $G'(\mathbf{A})$-module π'. Then it is tempered. Moreover, it is σ-discrete-series.*

Proof. (i) As there is nothing to prove when $w = u$, we assume that $w \neq u$. By definition π' lifts by Chapter III to a cuspidal, hence non-degenerate, $\mathrm{GL}(n, \mathbf{A}_E)$-module, hence each component π'_v (for $v \neq u$) of π' is non-degenerate. If v splits in E/F then π'_v is a (generalized) Steinberg G'_v-module by [BZ'], and the proposition follows. Hence we now assume that E_v is a field. Then Theorem 9.7 of [Z] implies that there is a Levi subgroup $M'_v = \Pi_i M'_{iv}$ of G'_v, where $M'_{iv} = \mathrm{GL}(n_i, E_v)$, a square-integrable M'_v-module $\rho'_v = \Pi \rho'_{iv}$ and an unramified positive-valued character $\mu_v = \Pi \mu_{iv}$ of M'_v, so that π'_v is equal to the G'_v-module $I(\rho'_v \otimes \mu_v)$ unitarily induced from $\rho'_v \otimes \mu_v$ on M'_v. Since π'_v is σ-invariant, for each i there is j with ${}^\sigma(\rho'_{iv} \otimes \mu_{iv}) = \rho'_{jv} \otimes \mu_{jv}$, and in particular $\mu_{iv}\mu_{jv} = 1$. To show that π'_v is tempered, we have to prove that $\mu_{iv} = 1$ for all i.

Suppose that there is i for which $\mu_{iv} \neq 1$. Then the corresponding j is not equal to i. Let P denote the standard parabolic subgroup of type $(n_i, n - 2n_i, n_i)$. Then

$$\tau = \rho'_{iv} \otimes \mu_{iv} \times I\left(\prod_{k \neq i,j} (\rho'_{kv} \otimes \mu_{kv})\right) \times \rho'_{jv} \otimes \mu_{jv}$$

is a σ-invariant P-module; it extends to a $P \rtimes \langle\sigma\rangle$-module. Hence the character of the induced representation $\pi'_v = I(\tau; P, G'_v)$ of G'_v (or $G'_v \rtimes \langle\sigma\rangle$) is supported on the conjugacy classes in $G'_v \rtimes \langle\sigma\rangle$ which intersect $P \rtimes \langle\sigma\rangle$. In particular π'_v is not σ-elliptic, contrary to our assumption. Hence $\mu_{iv} = 1$ for all i, and π'_v is tempered, as required.

(ii) It remains to show that π'_w, which we now denote by π (we also write G for G'_w), is σ-discrete-series. Since π is tempered, there is a parabolic subgroup P and discrete-series irreducible $\mathrm{GL}(n_i, E)$-modules γ_i $(1 \leqq i \leqq c)$ such that $\gamma = \gamma_1 \times \cdots \times \gamma_c$ is a P-module and $\pi = I(\gamma; P, G)$. We have to show that for each (σ-invariant, standard) parabolic subgroup $R \neq G$, and each σ-invariant irreducible M_R-module τ in the M_R-module π_R of N_R-coinvariants, where $R = M_R N_R$, the central character of τ decays. We may write τ in the form $\tau = \tau_1 \nu^{s_1} \times \cdots \times \tau_r \nu^{s_r}$, where τ_i are irreducible $\mathrm{GL}(m_i, E)$-modules with unitary central characters, $\nu(x) = |\det x|$, and s_i are real numbers whose sum is zero. Since τ is σ-invariant we have $s_i + s_{r+1-i} = 0$ for all i. Given an r-tuple $(a_1, \ldots, a_r)$ of elements in $E^\times$ with $|a_1| \leqq \cdots \leqq |a_r|$ and $|a_1| < |a_r|$, we put

$$X = \prod_{1 \leqq i \leqq r} |a_i|^{s_i} = \prod_{1 \leqq i \leqq r/2} |a_i/a_{r+1-i}|^{s_i}.$$

Since π is tempered, for each such r-tuple the positive number X is bounded by 1. Hence $s_1 \geqq 0$. We have to show that $X < 1$.

We shall now assume that $X = 1$ for some $(a_1, \ldots, a_r)$, and derive a contradiction. This assumption implies that $s_1 = 0$. Let L be the standard parabolic subgroup of type $(m_1, n - 2m_1, m_1)$. Then $I(\tau; R, L)$ has an irreducible σ-invariant constituent $\alpha = \tau_1 \times \tau' \times \tau_r$ such that α is a subquotient of π_L. Hence π is a subquotient of $I(\alpha; L, G)$. Since π is non-degenerate, so is α. Moreover, since π is tempered and $s_1 = 0$, and the central exponents of τ' are among those of π, it follows that τ' is tempered. To complete the proof it suffices to show that τ_1 is tempered. Indeed, if τ_1 is tempered, then $I(\alpha)$ is irreducible (by [BZ′]), and as explained at the end of (i) the induced representation $\pi = I(\alpha)$ is not σ-elliptic, contrary to our assumption.

To show that τ_1 is tempered, note that it is non-degenerate. Hence it follows from [Z; (9.7)] that there are real numbers t_i and square-integrable $\mathrm{GL}(m_i, E)$-modules ρ_i, such that $\tau_1 = I((\rho_i \nu^{t_i}))$. Since the central character of τ_1 is unitary, we have $\Sigma_i t_i = 0$. If τ_1 is not tempered, then $t_i \neq 0$ for some i, and we may assume that $t_1 < 0$. Let S be the standard parabolic subgroup of type $(m_1, n_1 - m_1, n - 2n_1, n_1 - m_1, m_1)$. Then π_S has a subquotient β (σ-invariant and irreducible), of the form

$$\beta = \rho_1 \nu^{t_1} \times I((\rho_i \nu^{t_i}; i \geqq 2)) \times \tau' \times I(({}^\sigma\rho_i \nu^{-t_i}; i \geqq 2)) \times {}^\sigma\rho_1 \nu^{-t_1}.$$

The absolute value of the value of the central character of β at $(a_1, a_2, a_3, a_4, a_5)$ in $E^{\times 5}$ with $|a_1| < |a_2| = |a_3| = |a_4| < |a_5|$ is $|a_1/a_5|^{t_1} > 1$. This contradicts the assumption that π is tempered. Hence $t_i = 0$, τ_1 is tempered, and as explained above π is σ-discrete-series, as required. The proposition follows.

Let E_w/F_w be a quadratic extension of non-archimedean local fields. For the local theory, fix a quadratic extension E/F of totally imaginary number fields such that at some non-split place w the extension E_w/F_w is as above. Fix a finite place u of F which splits in E and consider the F-group G of §1.

Theorem. *Suppose that π_w^0 is a square-integrable G_w-module. Then there exists an automorphic G-module π whose component at w is π_w^0, whose component at each place $w' \neq w$ of F which ramifies in E is Steinberg, and which is unramified at all finite places other than u, w and the w'.*

Proof. (i) Let f_w be a pseudo-coefficient ([K], Theorem K) of π_w^0. As in [K], note that $f_w(e) \neq 0$. Indeed, the Plancherel formula of Harish-Chandra expresses $f_w(e)$ as an integral

$$f_w(e) = \sum_M c_M \int_{E_2(M)} d(\omega)\cdot \mathrm{tr}(I(\omega))(f_w)\cdot \mu(\omega)d\omega.$$

The sum ranges over conjugacy classes of Levi subgroups; the integral is over the variety $E_2(M)$ of square-integrable M-modules; $I(\omega)$ is the G-module unitarily induced from ω on M, $d(\omega)$ is the Plancherel measure. Here c_M is a constant which is equal to one if $M = G$; moreover, $\mu(\omega) = 1$ if $M = G$. Since f_w is a pseudo-coefficient of π_w^0, all terms associated with $\omega \neq \pi_w^0$ are zero, hence $f_w(e) = d(\pi_w^0)$ is indeed non-zero, as claimed.

Let π_u be a G_u-module with trivial central character which corresponds to a supercuspidal $\mathrm{GL}(n, F_u)$-module by the correspondence of Chapter III, and f_u a matrix coefficient; then again $f_u(e) \neq 0$. At each place w' of F which ramifies in E let $f_{w'}$ be a pseudo-coefficient of the Steinberg $G_{w'}$-module $\pi_{w'}^0$. Then again $f_{w'}(e) \neq 0$. At each finite $v \neq u$, w and the w' let f_v be the characteristic function f_v^0 of the standard maximal compact subgroup K_v of G_v. At each archimedean place v we specify below a component f_v with $f_v(e) \neq 0$. Let ω be a unitary character of $\mathbf{A}_E^1/E^1$ whose component at w is the central character of π_w^0, whose component at each w' is trivial, and its component at each other finite place is unramified. Since E^1 is discrete, hence closed, in $\mathbf{A}_E^1$, and $B = E_w^1 \Pi_{w'} E_{w'}^1 \Pi_v R_v^1$ ($v \neq w, w', \infty$) is compact, E^1B is closed in $\mathbf{A}_E^1$, and it is clear that ω exists. Multiplying ω by a global unitary unramified character we may assume that the component of ω at u is trivial. Note that as usual, our functions f_v are chosen to be smooth if v is archimedean, locally-constant if v is finite, complex-valued, transform under the center via the component ω_v^{-1} of ω^{-1} at v, and are compactly-supported on G_v/Z_v. Put $f = \otimes f_v$. $G(\mathbf{A})$ acts by right translation r on $L(G\backslash G(\mathbf{A}))$. Fix a Haar measure $dx = \otimes dx_v$ on $G(\mathbf{A})/Z(\mathbf{A})$.

Consider the operator $r(f) = \int f(x)r(x)dx$ (x in $G(\mathbf{A})/Z(\mathbf{A})$). It is an integral operator with kernel $K(x, y) = \Sigma_\gamma f(x^{-1}\gamma y)$ (γ in G/Z). Its trace is given by

$\int \Sigma_\gamma f(x\gamma x^{-1})dx$; x ranges over the compact space $G(\mathbf{A})/Z(\mathbf{A})G$. Then $f(x\gamma x^{-1}) \neq 0$ implies that the conjugacy class of γ in G/Z intersects a compact of $G(\mathbf{A})/Z(\mathbf{A})$ depending only on the support of f. Choosing a galois extension K of E which splits G' we can view γ as an element of $\mathrm{GL}(n, K)$; the characteristic polynomial of γ is defined over E. The set S of characteristic polynomials of γ with $f(x\gamma x^{-1}) \neq 0$ lies in the intersection of a compact (depending on f), and a discrete (since $E^\times$ is discrete in $\mathbf{A}_E^\times$) subsets of $(\mathbf{A}_E^{n-1} \times \mathbf{A}_E^\times)/\mathbf{A}_E^\times$ $((za, z^2b, \ldots, z^nc) \equiv (a, b, \ldots, c))$. Hence S is finite. Consequently we can choose the archimedean components to have small support, so that only $\gamma = e$ would contribute a non-zero term to the sum. Hence the trace equals $\int f(e)dx = f(e)|G(\mathbf{A})/Z(\mathbf{A})G|$, and it is non-zero.

(ii) On the other hand, the trace is equal to the sum $\Sigma \operatorname{tr} \pi(f)$ of the traces $\operatorname{tr} \pi(f)$ of the operators $\pi(f) = \int f(x)\pi(x)dx$ over all irreducible constituents π in $L(G \backslash G(\mathbf{A}))$. For each π which appears in the sum we have that its component π_v is unramified at $v \neq u, w, w'$, since $\operatorname{tr} \pi_v(f_v^0) = 1 \neq 0$. Hence

$$(*) \qquad \sum_\pi \operatorname{tr} \pi(f) = \sum_{\{\pi_v;\, v \notin V\}} \left[\sum m(\pi) \prod_{v \in V} \operatorname{tr} \pi_v(f_v) \right].$$

Here we take V to consist of u, w, w' and the archimedean places. The first sum on the right ranges over all sequences $\{\pi_v;\, n \notin V\}$ of unramified G_v-modules. The inner sum is over the set specified in Lemma 2.1. Proposition 2.2 asserts that for *any* choice of components f_v (v in V), and matching ϕ_v, our sum is equal to

$$(**) \qquad \sum \left[\sum m(\pi') \prod_{v \in V} \operatorname{tr} \pi'_v(\phi_v) \right] = \sum_{\pi'} \operatorname{tr} \pi'(\phi).$$

The first sum ranges over the same set as in $(*)$. The inner sum is over the set specified in Lemma 2.1. The component of ϕ at v outside V is the unit element ϕ_v^0. The π' range over all automorphic σ-invariant $G'(\mathbf{A})$-modules.

Consider any π' which appears in $(**)$. Since $\operatorname{tr} \pi'_u(\phi_u) \neq 0$, π'_u lifts to a supercuspidal $\mathrm{GL}(n, E_u)$-module by Chapter III due to the choice of f_u and ϕ_u, hence π' is non-degenerate. Its components at $v = w$ and the w' are σ-elliptic since $\operatorname{tr} \pi'_v(\phi_v) \neq 0$; recall that ϕ_v matches f_v, and the orbital integrals of f_v vanish on the regular non-elliptic set. Hence the components π'_v (for $v = w, w'$) are tempered, in fact σ-discrete-series, by Proposition 3. Consider the identity (2.1) with V being the set of w and the w', where π' is any of the members in $(**)$. We now apply Proposition 2.2 at each of the w', where the character $\mu_{w'}$ there is the trivial character. It is clear from the proof of Proposition 2.2 that the components π_v on the right of (2.1) are square-integrable; indeed, their central exponents decay since π'_v is σ-discrete-series. Moreover, the argument of Chapter II, §3, shows that the component π_w of any π which appears in $(*)$ is also square-integrable.

We can now take any π which appears in $(*)$. We have $\operatorname{tr}\pi(f)\neq 0$ for the f of (i). Hence the component at any finite $v\neq u$, w, w' is unramified (since $\operatorname{tr}\pi_v(f_v^0)=1\neq 0$). As $\pi_{w'}$ is tempered, and $\operatorname{tr}\pi_{w'}(f_{w'})\neq 0$ where $f_{w'}$ is a pseudo-coefficient of the Steinberg $G_{w'}$-module, it follows from the orthogonality relations of [K], Theorem K, that $\pi_{w'}$ is Steinberg. Similarly, we have $\operatorname{tr}\pi_w(f_w)\neq 0$, where π_w is tempered and f_w is a pseudo-coefficient of the square-integrable π_w^0; hence π_w is π_w^0, and the theorem follows.

Corollary. *Given a tempered irreducible G_w-module π_w^0 there exists a tempered σ-stable G_w'-module π_w', finitely many irreducible tempered G_w-modules π_w (including π_w^0), and positive integers $n(\pi_w)$, so that*

$$\operatorname{tr}\pi_w'(\phi_w)=\sum_{\pi_w} n(\pi_w)\operatorname{tr}\pi_w(f_w) \tag{3.1}$$

for all matching functions ϕ_w, f_w.

Proof. (i) Suppose first that π_w^0 is square-integrable. Then the claim follows at once from the proof of the Theorem. Note that the sum is finite by Chapter II, §3, since π_w', which is produced by the Theorem and its proof, is σ-discrete-series.

(ii) In the general case π_w^0 is tempered. Hence there is a Levi subgroup M of G (from now on we omit the index w), and a square-integrable M-module ρ^0, such that π^0 is a direct summand of the G-module $I(\rho^0)$ unitarily induced from ρ^0 on M. By part (i) there exists a σ-stable tempered M'-module ρ', which is σ-square-integrable, finitely many square-integrable M-modules ρ, including ρ^0, and positive integers $n(\rho)$, so that $\operatorname{tr}\rho'(\phi)=\Sigma\, n(\rho)\operatorname{tr}\rho(f)$. Here ϕ, f are matching functions on M', M. A standard computation of characters of induced representations yields the identity of the corollary. The π are the ireducible summands in the composition series of the tempered $I(\rho)$. The π' on the left is the $\mathrm{GL}(n,E)$-module $I(\rho')$, which is unitarily induced from the irreducible tempered module ρ', hence it is irreducible (by [BZ′]).

Let E/F be a quadratic extension of local fields.

Definition. The *packet* $\{\pi\}$ of a tempered G-module π^0 is the set of π which appear in (3.1).

To show that the packets are well-defined, we prove the following

Proposition. *The packets define a partition of the set of tempered G-modules.*

Proof. It suffices to show that if π' and π'' are inequivalent σ-discrete-series and satisfy (3.1), thus $\pi'=\Sigma\, n(\pi)\pi$ and $\pi''=\Sigma\, m(\pi)\pi$, then there is no π which appears in both sums. Since all π here are square-integrable, the orthonormality relations of [K], Theorem K, imply that

$$\left\langle \sum n(\pi)\pi,\ \sum m(\pi)\pi \right\rangle = \sum n(\pi)m(\pi),$$

in the inner product introduced in [K]. On the other hand, the twisted analogue of [K], Theorem G (we do not record here a proof as it follows closely that of [K] in the non-twisted case), asserts that the analogous inner product $\langle \pi', \pi'' \rangle$ vanishes unless π', π'' are relatives in the terminology of [K]. Now $\pi' = I(\rho')$ and $\pi'' = I(\rho'')$, where $\rho' = \bigotimes \rho_i'$, $\rho'' = \bigotimes \rho_i''$ are square-integrable (with ${}^\sigma\rho_i' = \rho_i'$, ${}^\sigma\rho_i'' = \rho_i''$); these are relatives only if they are equivalent. But $\Sigma\, n(\pi)m(\pi) = 0$ implies $n(\pi)m(\pi) = 0$ for all π, as required.

§4. Twisted existence

Let E_w/F_w be a local quadratic extension.

Theorem. *Each tempered σ-stable G_w'-module π_w' satisfies* (3.1).

Proof. By parabolic induction it suffices to deal only with σ-elliptic π_w'. It is of the form $I(\rho')$, $\rho' = \bigotimes \rho_i'$, where the ρ_i' are square-integrable, pairwise inequivalent, and σ-invariant. Using the twisted analogue (Chapter I, §7) of the trace Paley–Wiener theorem of [BDK], since a $\mathrm{GL}(n)$-module which is unitarily induced from a square-integrable one is irreducible (by [BZ′]), we have a function ϕ_w in $C(G_w')$ with $\mathrm{tr}\,\pi_w'(\phi_w) = 1$, and $\mathrm{tr}\,\pi_w''(\phi_w) = 0$ for all tempered π_w'' inequivalent to π_w'. In particular, the orbital integral $\Phi(\phi_w)$ vanishes on the σ-regular non-σ-elliptic subset of G_w'. Since π_w' is σ-stable by our assumption, there is some σ-regular elliptic x_w with $\Phi'(x_w, \phi_w) \neq 0$. Choose a global quadratic extension E/F of which E_w/F_w is a completion, and let G be the quasi-split form of the unitary group, so that $G' = \mathrm{GL}(n, E)$.

Let u be a place of F which splits in E. Let μ_i $(1 \leqq i \leqq n)$ be n unitary characters of $F_u^\times$ such that for each $i \neq j$ the quotient μ_i/μ_j is ramified. As in the proof of Proposition 2.2, we now take a regular function f_u in the sense of [Sph], §4 (see the proof of Proposition 2.2) associated with the character $\mu = \bigotimes \mu_i : (a_{ij}) \to \Pi_i\, \mu_i(a_{ii})$ of the upper triangular subgroup B_u of $G_u = \mathrm{GL}(n, F_u)$, and a regular λ in $\mathbf{Z}^n$ (thus λ is a vector whose n entries are distinct integers in decreasing order). We have that for any G_u-module π_u, the trace $\mathrm{tr}\,\pi_u(f_u)$ vanishes unless there is an unramified character $\bigotimes \nu^{s_i}$ such that π_u is a constituent of the G_u-module $I_u = I(\bigotimes \mu_i \nu^{s_i})$ induced from the character $\bigotimes \mu_i \nu^{s_i}$ of B_u. By [BZ′], our choice of the μ_i guarantees that I_u is irreducible, hence $\pi_u = I_u$. Similarly, if ϕ_u is a function matching f_u then $\mathrm{tr}\,\pi_u'(\phi_u) \neq 0$ implies that π_u' is the lift $(\pi_u, {}^\sigma\pi_u)$ of $\pi_u = I_u$ as above.

Now E/F is a quadratic extension of global fields whose completions at two places w and u are our E_w/F_w and $E_u = F_u \oplus F_u$, and G is the quasi-split form of the unitary group, so that $G' = \mathrm{GL}(n, E)$. Since $\Phi'(\phi_w)$ is a locally constant function on the σ-regular set, there exists a σ-elliptic regular x in $G' = G'(F)$ which is near x_w in G'_w such that $\Phi'(x, \phi_w) \neq 0$ and $\Phi'(x, \phi_u) = \Phi(x, \phi_u) \neq 0$. Moreover, x can be chosen so that its σ-centralier T in G' is related to the σ-centralizer T_w of x_w in G'_w as in the Lemma of Chapter I; §4. Let $\{u'\}$ be a set of places u' of F which stay prime in E, of cardinality larger than the rank of G, excluding w, such that x is σ-elliptic in $G'_{u'}$. For each u' let $\phi_{u'}$ be a function supported on the σ-regular elliptic set, with $\Phi'(x, \phi_{u'}) \neq 0$, such that $\Phi(y, \phi_{u'})$ is a stable function in y.

We now choose a global function $\phi = \bigotimes \phi_v$ whose components at w, u, u' are as above, which satisfies $\Phi'(x, \phi) \neq 0$. Our conditions at u' (we need only one such place) imply (see Corollary 1 in Chapter I; §3) that the side of the trace formula involving σ-conjugacy classes in the group G' takes the form $\Sigma\, c(y)\Phi'(y, \phi)$, the sum ranges over all σ-elliptic regular stable conjugacy classes in G', and $c(y)$ are volume factors. The sum is finite, and we can reduce the support of the components of ϕ (at $v \neq w$), to have that the sum over y consists of x alone. Consequently this sum is equal to $c(x)\Phi'(x, \phi)$, which is non-zero.

It follows that the representation theoretic side of the trace formula is non-zero. Our (sufficiently many) conditions at the places u' guarantee the vanishing of all terms which involve integrals in the expression given by Arthur [A′] for this side of the trace formula; in fact here we use the twisted analogue (Corollary 1 in Chapter I; §3) of Arthur's computations. The terms which are left are of the form $\operatorname{tr} \pi'(\phi)$, with complex coefficients. The construction of the component ϕ_u guarantees that if $\operatorname{tr} \pi'(\phi) \neq 0$, then the component π'_u at u of π' is induced from the subgroup B'_u, of the form $I'_u = (I_u, {}^{\sigma}I_u)$ described above. Now each π' which appears in the trace formula is a quotient of a representation $J' = I(\bigotimes \sigma_i)$ induced from a cuspidal (not necessarily unitary) representation of a Levi subgroup. But if J' is reducible and π' is a proper quotient of it, then π'_u has to be a proper quotient of the component J'_u of J' at u. But $\pi'_u = I'_u$ is an irreducible induced from the upper triangular subgroup B'_u. We conclude that J' is irreducible, hence π' is equal to J', and it is non-degenerate.

We can choose the function ϕ so that its component at a place $u''(\neq w, u, u')$ is a pseudo-coefficient of a Steinberg $G'_{u''}$-module $\pi'_{u''}$. Thus $\operatorname{tr} \pi''_{u''}(\phi_{u''})$ is zero for any tempered $G'_{u''}$-module unless $\pi''_{u''}$ is $\pi'_{u''}$, in which case this trace is equal to one. In particular the orbital integral $\Phi(\phi_{u''})$ vanishes on the σ-regular non-elliptic set. Since our global π' satisfies $\operatorname{tr} \pi'(\phi) \neq 0$, its component at u'' is non-degenerate and σ-elliptic, hence it is tempered, and we conclude that the component of π' at u'' is the Steinberg $\pi'_{u''}$. But this implies that π' is cuspidal, namely that in $J' = I(\bigotimes \sigma_i)$ there is only one σ, which is equal to J'.

It remains to show that the component of π' at w, which is σ-elliptic and tempered, hence it is our π'_w, satisfies the identity (3.1). For that we form the identity (2.1) where our π' is the only term on the left. The set V ranges over all the finite places mentioned above, namely w, u, u', u'', and the functions $\phi_{u'}$ at the places u' have to be supported on the σ-elliptic regular set. However we can take the place u'' duplicated sufficiently many times, so that π' will have several Steinberg components, at the places u''. Proposition 2.2 and its proof imply that each $\pi_{u''}$ which occurs on the right of (2.1) is a Steinberg $G_{u''}$-module. Consequently we can take in (2.1) the functions $\phi_{u''}$, $f_{u''}$ to be supported on the regular elliptic set, and obtain the identity (2.1), where V consists only of the u' in addition to w, but the f_v, ϕ_v are arbitrary matching functions. Hence Chapter II, §3, implies that the sum on the right of (2.1) is finite, consists of square-integrables, and on choosing the $f_{u'}$ to be pseudo-coefficients of square-integrables which occur we obtain the identity (3.1) where our π'_w is on the left, except that the left side of (3.1) takes the form $c\,\mathrm{tr}\,\pi'_w(\phi_w)$, where c is a complex number, necessarily non-zero. Thus we obtain

$$c\,\mathrm{tr}\,\pi'_w(\phi_w) = \sum m(\pi_w)\mathrm{tr}\,\pi_w(f_w).$$

On the other hand, for some π_w^0 in this sum we have the identity (3.1), namely a σ-discrete-series tempered irreducible G'_w-module π''_w, with

$$\mathrm{tr}\,\pi''_w(\phi_w) = \sum n(\pi_w)\mathrm{tr}\,\pi_w(f_w).$$

The $m(\pi_w)$, $n(\pi_w)$ are integers with $n(\pi_w^0)m(\pi_w^0) \neq 0$. The sums are finite, range over square-integrables, and ϕ_w, f_w are arbitrary matching functions. We conclude, using the Weyl integration formula and the orthogonality relations of [K] for characters, that π'_w is π''_w, hence that $c = 1$. The theorem follows.

§5. Minimality

Let E/F be a local quadratic extension, and π a square-integrable G-module. The packet $\{\pi\}$ of π consists of the π which occur in the sum on the right of (3.1), with integral multiplicities $n(\pi)$.

Proposition. *Let $\{\pi\}'$ be a proper non-empty subset of $\{\pi\}$. Then $\Sigma'\, n(\pi)\pi$ is not stable. By Σ' we mean the sum over $\{\pi\}'$.*

Proof. Suppose that $\chi' = \Sigma'\, n(\pi)\chi(\pi)$, where $\chi(\pi)$ denotes the character of π, is stable. Of course $\chi = \Sigma\, n(\pi)\chi(\pi)$ is stable, and so is $\chi'' = \chi - \chi'$. For some positive rational c we have that $\chi' - c\chi''$ is orthogonal to χ, hence to the character of every tempered packet. Let f be a function whose orbital integral $'\Phi(f)$ vanishes on the regular non-elliptic set, and equals $\chi' - c\chi''$ on the regular elliptic set. Let ϕ be a matching function. Then we obtain $\mathrm{tr}\,\pi'(\phi) = 0$ for every tempered σ-invariant G'-module π'. But by Proposition 4 of Chapter I we conclude that all σ-stable orbital integrals of ϕ are zero. Thus the orbital integrals of f vanish, and

$\chi' = c\chi''$, which contradicts the orthogonality relations for characters of square-integrable G-modules. The proposition follows.

Let E/F be a global extension, and G, G' the group of §1.

As in §1 we say that an automorphic G'-module π' is *non-degenerate* if it corresponds to a non-degenerate automorphic $\mathrm{GL}(n, \mathbf{A}_E)$-module π'' by the correspondence of Chapter III. We deal only with such σ-invariant π' from now on. Each component π'_v of $\pi'(v \neq u)$ is non-degenerate. By [Z], Theorem 9.7, π'_v is equal to the induced G'_v-module $I(\otimes_{i=1}^m \rho'_{iv}\nu^{s_i})$, where ρ'_{iv} is a square-integrable $M'_{iv} = \mathrm{GL}(r_i, E_v)$-module, and s_i is a real number with $|s_i| < \frac{1}{2}$, since π'_v is unitary. We may assume that for some m' we have that ${}^\sigma\rho'_{iv} = \rho'_{m-i,v}$ and $s_i = s_{m-i}$ for i in the interval $A = [1, m']$, but $s_i = 0$ and ${}^\sigma\rho'_{iv} = \rho'_{iv} \neq \rho'_{jv}$ for all $i \neq j$ in the interval $B = [m'+1, m-m'-1]$. Put $b = \Sigma\, r_i$ (i in B). Hence we write π'_v in the form

$$I\left(\left(\bigotimes_{i\in A} \rho'_{iv}\nu^{s_i}\right) \otimes I\left(\bigotimes_{i\in B} \rho'_{iv}\right) \otimes \left(\bigotimes_{i\in A} \rho'_{iv}\nu^{-s_i}\right)\right).$$

Since π'_v is σ-stable, we have that $I(\otimes_{i\in B}\rho'_{iv})$ is a σ-stable elliptic $\mathrm{GL}(b, E_v)$-module. Theorem 4.2 asserts that we have the identity

$$\mathrm{tr}\, I\left(\bigotimes_{i\in B} \rho'_{iv}\right)(\phi_v) = \sum_i m(\rho_{iv})\mathrm{tr}\, \rho_{iv}(f_v).$$

The ρ_{iv} are representations of the quasi-split unitary group $U(b)$ in b variables. Then

$$\left(\bigotimes_{i\in A} \rho'_{iv}\nu^{s_i}\right) \otimes \left(\sum_i m(\rho_{iv})\rho_{iv}\right)$$

defines an M_v-module ρ_v, where

$$M'_v = \prod_{i\in A} M'_{iv} \times U(b) \times \prod_{i\in A} M'_{iv}.$$

By parabolic induction we conclude that $\mathrm{tr}\, \pi'_v(\phi_v) = \mathrm{tr}\, I(\rho_v; G_v, M_v)(f_v)$, where on the right $I(\rho_v)$ is the G_v-module induced from ρ_v on M_v. Namely π'_v satisfies the identity (3.1), where on the right we have all irreducible subquotients π_v of $I(\rho_v)$.

If π_v occurs on the right of (3.1) for some π'_v as above we define its *packet* $\{\pi_v\}$ to be the set of irreducible π_v which occur on the right of (3.1). Also we say that π_v, and its packet $\{\pi_v\}$, *lift* to π'_v when (3.1) holds. We noted above that each component π'_v of a non-degenerate π' is a lift of a packet, which is not necessarily tempered, but obtained from a tempered packet on tensoring each element in the packet by the same unramified character and inducing. Note that at u the component π'_u is a unitary representation of an anisotropic group G'_u, hence tempered.

Using this generalized notion of a local packet, a global packet is defined as in §1, and we say that π *lifts* to π' if π_v lifts to π'_v for all v. To complete the proof of the Global Theorem of §1 we show

Theorem. *If π quasi-lifts to a non-degenerate π', then π lifts to π'.*

Proof. Proposition 2.2 implies the first equality in

$$\sum_{\pi} \prod \operatorname{tr} \pi_v(f_v) = \prod \operatorname{tr} \pi'_v(\phi_v) = \prod \left[\sum n(\pi_v) \operatorname{tr} \pi_v(f_v) \right].$$

The second follows from (3.1). The products are over a finite set V, the sums on the right over π_v are finite. As this holds for any $\{f_v; v \text{ in } V\}$, a standard argument using the absolute convergence of the sums, and the unitarity of all representations in the trace formula implies the claim. Note also that since π' is non-degenerate, it appears in the discrete spectrum of $L(G \backslash G(\mathbf{A}))$ with multiplicity one.

The proof has the following obvious

Corollary. *Each irreducible $G(\mathbf{A})$-module in a packet of a non-degenerate π is automorphic.*

This completes the proof of the Global Theorem of §1.

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