QUATERNIONIC DISTINGUISHED REPRESENTATIONS

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Let E/F be a quadratic separable extension of global fields, and $\mathbb{A}_E$, $\mathbb{A}$ the corresponding rings of adeles. Fix a character ω' on the group $\mathbb{A}_E^\times/E^\times$ of E-idele classes which is trivial on the F-idele classes, and an irreducible, automorphic discrete-series representation π of $GL(2,\mathbb{A}_E)$ with central character ω' realized (as a closed invariant subspace) in the space of automorphic forms. Then π is said to be $GL(2,\mathbb{A})$-*distinguished* (or cyclic) if there exists a form ϕ in the space of π such that its integral (or period) $\int \phi(x)dx$ over the space (or cycle) $PGL(2,F)\backslash PGL(2,\mathbb{A})$ is non-zero.

One purpose of this paper is to compare the notion of being $GL(2,\mathbb{A})$-distinguished with the notion (defined below) of being distinguished with respect to another subgroup of $GL(2,\mathbb{A}_E)$. Using a "relative trace formula", Jacquet and Lai [JL] carried out such comparisons in certain cases. To extend their results, one could either develop an extensive theory of orbital integrals for the relative trace formula, as is done in [H3], or give a relative version of the Deligne-Kazhdan "simple trace formula," in which this theory simplifies. We adopt the latter approach. Another objective of this work is to consider such a comparison and a "relative trace" (or "bi-period summation") formula in the higher rank case.

Distinguished representations were introduced in a similar context by Waldspurger [Wa], and in our context by Harder, Langlands and Rapoport [HLR] to study Tate's conjectures [T] on algebraic cycles in the case of Hilbert modular surfaces. Then Lai [L] – using the comparisons of distinguished representations in [JL] – extended the results of [HLR] to certain proper Shimura surfaces. Our results can be used to establish Tate's conjectures for some new proper Shimura surfaces. We indicate in an appendix the changes which need to be made to Lai's work to accommodate the surfaces which we consider.

Let $\mathbf{G}$ denote the F-group $GL(2)$ and let $\mathbf{G}'$ denote the F-group $\mathrm{Res}_{E/F}\mathbf{G}$ obtained from $\mathbf{G}$ by restricting scalars from E to F (thus $\mathbf{G}'(F) \simeq \mathbf{G}(E)$). Then π, thought of as a representation of the restricted product $\mathbf{G}'(\mathbb{A})$ of the local groups G'_v, factors over F as $\otimes_v \pi_v$. A local component π_v (or, more generally, an irreducible admissible representation of G'_v) is said to be G_v-*distinguished* if there is a non-zero G_v-invariant linear form on the space of π_v. These representations are classified in [H2] in the case of trivial central character, and in [F8] in general. We say that π is *abstractly*, or *locally,* $\mathbf{G}(\mathbb{A})$-*distinguished* if each of its local components π_v is G_v-distinguished. It is easy to see that if π is $\mathbf{G}(\mathbb{A})$-distinguished then it is abstractly $\mathbf{G}(\mathbb{A})$-distinguished.

It is shown in [F8] that a cuspidal π is $\mathbf{G}(\mathbb{A})$-distinguished if and only if it is the unstable base-change lift of a cuspidal representation of the quasi-split unitary

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group $U(2,E/F)$ in two variables associated with E/F. The analogous local result is also proven. It then follows from the theory of base-change for $U(2,E/F)$ of [F1] that if π is abstractly $\mathbf{G}(\mathbb{A})$-distinguished and at least one of its components is an unstable, but not stable, base-change lift then π is $\mathbf{G}(\mathbb{A})$-distinguished. Such a component is either square-integrable or induced of the form $I(\mu_{1v},\mu_{2v})$ for distinct characters μ_{1v} and μ_{2v} of $E_v^\times/F_v^\times$. On the other hand, there exist cuspidal representations π which are stable base-change lifts, whose components are all of the form $I(\mu_v,\overline{\mu}_v^{-1})$, where $\overline{\mu}_v(x)=\mu_v(\bar{x})$ for $x\in(F_v\otimes E)^\times$ and $\bar{x}$ is the Galois conjugate of x. These local representations are in the image of both stable and unstable maps, and so π is abstractly $\mathbf{G}(\mathbb{A})$-distinguished, but not $\mathbf{G}(\mathbb{A})$-distinguished.

Let $\mathbf{D}$ be an inner form of $\mathbf{G}$. Then $\mathbf{D}(F)$ is the multiplicative group of a quaternion division algebra central over F. The groups $\mathbf{G}$ and $\mathbf{D}$ have isomorphic centers, and it will be convenient to let $\mathbf{Z}$ denote the center of either group. Let V denote the finite set (with even cardinality) of places of F where $\mathbf{D}$ ramifies. Let us assume first that $E^\times$ is contained in $\mathbf{D}(F)$. Equivalently, $E_v=F_v\otimes E$ is a field for each $v\in V$. Then $\mathbf{D}(E)$ is isomorphic to $\mathbf{G}'(F)=\mathbf{G}(E)$. The representation π is said to be $\mathbf{D}(\mathbb{A})$-*distinguished* if there is a form ϕ in the space of π with $\int_{\mathbf{Z}(\mathbb{A})\mathbf{D}(F)\backslash\mathbf{D}(\mathbb{A})}\phi(x)dx\neq 0$.

An irreducible admissible representation π_v of G_v' is said to be D_v-*distinguished* if there is a non-zero D_v-invariant linear form on the space of π_v. If D_v is anisotropic, then an induced representation $I(\mu_1,\mu_2)$ of G_v' is D_v-distinguished if and only if $\mu_2=\overline{\mu}_1^{-1}$ [F8, H2]. On the other hand, the representation $I(\mu_1,\mu_2)$ is G_v-distinguished precisely when $\mu_1\overline{\mu}_2=1$, or $\mu_1\neq\mu_2$ and both characters μ_i of $E_v^\times$ are trivial on $F_v^\times$.

The distinguished "special" representations are classified as follows. Let $sp(\mu)$ denote the square-integrable subrepresentation of the induced representation $I(\mu\nu^{1/2},\mu\nu^{-1/2})$ of G_v', where $\nu(z)=|z|_v$ for $z\in E_v^\times$. Then $sp(\mu)$ is D_v-distinguished if and only if it is G_v-distinguished, and this occurs precisely when the restriction of μ to $F_v^\times$ is the unique nontrivial character of $F_v^\times$ whose kernel is the image of the norm map from $E_v^\times$ into $F_v^\times$ (see Proposition B17, [F8], p. 169, and [H2]).

It is clear that if π is $\mathbf{D}(\mathbb{A})$-distinguished, then each of its components is D_v-distinguished. The square-integrable π_v are those π_v which are special or supercuspidal. The following theorem coincides with Theorem C of D. Prasad [P], which is proven by entirely local means, as a special case of his study of forms on $GL(2,E)\times GL(2,F)$. This theorem is also proven in [H3] on using an extensive analysis of orbital integrals.

0.1 Theorem. *A square-integrable representation π_v of G_v' is D_v-distinguished if and only if it is G_v-distinguished.*

An alternate proof of the above theorem which uses a simpler application of the trace formula appears in this paper. We also prove the following global result:

0.2 Theorem. *An irreducible, automorphic discrete-series representation π of $\mathbf{G}'(\mathbb{A})$ is $\mathbf{D}(\mathbb{A})$-distinguished if and only if it is $\mathbf{G}(\mathbb{A})$-distinguished and its compo-*

nents π_v at $v \in V$ are not of the form $I(\mu_1, \mu_2)$ with μ_i trivial on $F_v^\times$.

We shall in fact prove a more general result, where $\mathbf{D}$ is any inner form of $\mathbf{G}$, where $\mathbf{D}(F)$ does not necessarily contain $E^\times$. To state this, let $\mathbf{D}$ be an inner form of $\mathbf{G}$, and $\mathbf{D}'$ the F-group obtained by restriction of scalars from E to F. Denote by V the set of places of F where $\mathbf{D}$ ramifies, by V' the subset of v in V which stay prime in E, and by V'' the set of v in V which split in E. In particular, when v belongs to V' then the groups D'_v and G'_v are isomorphic. The group $\mathbf{D}'(F) = \mathbf{D}(E)$ is anisotropic exactly when V'' is non-empty.

If $\pi^D = \otimes \pi_v^D$ is an irreducible representation of $\mathbf{D}'(\mathbb{A})$, denote by $\pi = \otimes \pi_v$ the corresponding representation of $\mathbf{G}'(\mathbb{A})$. Thus $\pi_v \simeq \pi_v^D$ for $v \notin V''$, and π_v is the square-integrable representation corresponding to π_v^D for $v \in V''$. If π^D is discrete-series then so is π, and if π^D is $\mathbf{D}(\mathbb{A})$-distinguished then each π_v^D is D_v-distinguished. At the places $v \in V''$, the D_v-distinguished representation π_v^D of $D'_v = D_v \times D_v$ is of the form $\pi_v'^D \otimes \tilde{\pi}_v'^D$. The corresponding representation π_v of $G'_v = G_v \times G_v$ is then of the form $\pi'_v \otimes \tilde{\pi}'_v$ and, in particular, it is G_v-distinguished. We prove:

0.3 Theorem. *Suppose that π is an irreducible automorphic representation of $\mathbf{G}'(\mathbb{A})$ which corresponds to a discrete-series representation π^D of $\mathbf{D}'(\mathbb{A})$. Then π^D is $\mathbf{D}(\mathbb{A})$-distinguished if and only if π is $\mathbf{G}(\mathbb{A})$-distinguished and for each $v \in V'$ the component π_v is not of the form $I(\mu_1, \mu_2)$ with μ_i trivial on $F_v^\times$.*

When V'' is empty, Theorem 0.3 reduces to Theorem 0.2. When V' is empty, Theorem 0.3 coincides with the main theorem of Jacquet-Lai [JL]. Theorem 0.3 is proven in part C of this paper.

In general, we use only the simplest possible expression of the relative trace formula which is suitable for our applications. In particular, matching of orbital integrals needs to be done only on the r-regular set (see A4 and A5). In addition, we show in part D that the r-character is locally constant on the r-regular set (defined below). In dealing with the Eisensteinian contribution to the relative trace formula, we rely on the computations carried out in [JL]. These computations apply to the case when the central character is trivial, but this restriction is removed in [F8], Lemma, p. 156.

The result of [JL] has been generalized in [F2] to the context of $GL(n)$ in the case where V' is empty (as in [JL]) and π has a supercuspidal component. Actually, [F2] requires that π has an additional square-integrable component, but this requirement can perhaps be removed on applying the regular-Iwahori functions as in [F6]. In parts A and B of this paper we shall also work in the context of $GL(n)$, and prove the following generalizations of [F2] and the Theorems 0.1 and 0.3.

Put $\mathbf{G} = GL(n)$ and take $\mathbf{D}$ to be an inner form of $\mathbf{G}$ defined over F. Let $\mathbf{G}'$ and $\mathbf{D}'$ be the groups obtained by restriction of scalars. Fix a non-archimedean place v of F which is inert in E. The notion of being distinguished extends in the obvious fashion to this more general context. In B15 we prove:

0.4 Theorem. *Let π_v^D be an irreducible, admissible representation of D'_v which corresponds (via the Deligne-Kazhdan correspondence; see [F3], III, p. 169) to a square-integrable representation π_v of G'_v. If π_v^D is D_v-distinguished and supercuspidal, then π_v is G_v-distinguished. If π_v is G_v-distinguished and supercuspidal, then π_v^D is D_v-distinguished.*

The archimedean analogue of this can be deduced from well known techniques of Flensted-Jensen, Oshima-Matsuki and Bien, but this will not be done here. Globally we have the following result, as suggested in [F5]. In B10 we prove:

0.5 Theorem. *Let π^D be an irreducible, automorphic cuspidal representation of $\mathbf{D}'(\mathbb{A})$ such that each of its local components is D_v-distinguished, and π the Deligne-Kazhdan ([F3], III, p. 170) corresponding representation of $\mathbf{G}'(\mathbb{A})$. Suppose that π has a supercuspidal component and a square-integrable component at two distinct F-places where D' splits. If π^D is $\mathbf{D}(\mathbb{A})$-distinguished then π is $\mathbf{G}(\mathbb{A})$-distinguished. If π is $\mathbf{G}(\mathbb{A})$-distinguished, and for each $v \in V'$ the r-character of π_v is not identically zero on the set of r-regular $g \in G'_v$ which come from D'_v, then π^D is $\mathbf{D}(\mathbb{A})$-distinguished.*

The condition in 0.4 can be relaxed from "π_v is supercuspidal" to "π_v is a component of a cuspidal representation π of $\mathbf{G}'(\mathbb{A})$ as in 0.5 with a supercuspidal component". The local results 0.1 and 0.4 follow at once from the global results 0.2 and 0.5, on noting that a distinguished supercuspidal representation can be embedded as a component of a cuspidal distinguished representation which has a supercuspidal component at any chosen finite split place, and that any component of a distinguished cuspidal representation is distinguished.

Note that the global theorem of [F3], III, requires in particular establishing the local correspondence not only for tempered local representations, but also for relevant local representations (since the generalized Ramanujan conjecture − asserting that all components of a cuspidal π are tempered − is merely a conjecture). The notion of relevant representations (the representations which may be components of a cuspidal $\mathbf{G}(\mathbb{A})$-module) is introduced in [FK1] in a similar context (of an r-fold covering of $GL(n)$), where they are shown to be irreducible and unitarizable. The proof of the correspondence in the case where $\mathbf{D}$ is anisotropic is remarkably simple, as explained in [F7].

In the proof of 0.4 and 0.5 we use the fact mentioned above that the r-characters of π_v and π_v^D are locally constant on the r-regular set. Consider $v \in V'$. Any infinite dimensional non-square-integrable D_v-distinguished representation of $GL(2, E_v)$ is necessarily of the form $I(\mu, \overline{\mu}^{-1})$. The r-character of such a representation of $GL(2, E_v)$ is identically zero on the set of r-regular elliptic elements in G'_v exactly when μ is trivial on $F_v^\times$; see C14 and [H3]. Using this, we obtain the precise formulation of the special case 0.3 of 0.5, as stated above.

More generally we show in B19, in the context of any reductive group, that normalized parabolic induction respects the notion of being distinguished, and that the r-character of the induced representation is related in a simple manner to the r-character of the inducing representation.

In the case of $\mathbf{G} = GL(n)$, it is conjectured in [F8] that the $\mathbf{G}(\mathbb{A})$-distinguished irreducible cuspidal representations of $\mathbf{G}'(\mathbb{A})$ are obtained by stable (if n is odd) or unstable (if n is even) base change (see [F4]) from the associated quasi-split unitary group, and the conjecture is proven for $n = 2$. In [F10] this conjecture is reduced – by means of a "Fourier summation formula" – to a technical local conjecture concerning "Fourier orbital integrals."

A. Relative conjugacy.

Let E/F be a quadratic separable extension of local or global fields, $\mathbf{D}$ an inner form of $GL(n)$ over F. We denote by D the group $\mathbf{D}(F)$ of F-points on $\mathbf{D}$, and $D' = \mathbf{D}(E)$. Following a common abuse of terminology, we will sometimes say D is an inner form of $GL(n, F)$. Then D is the multiplicative group of a simple algebra of rank n central over F, namely a matrix algebra $M(m, H)$ of $m \times m$ matrices with entries in a division algebra H of rank n/m central over F. Further, $D' = M(m, H \otimes_F E)^\times$. There exists an involutive automorphism $\sigma : D' \to D'$ whose restriction to the center $E^\times$ of D' coincides with the Galois action $z \mapsto \overline{z}$ on E, such that D consists of the fixed points of σ in D'.

Example. When $n = 2$, any anisotropic inner form D of $GL(n, F)$ which is the multiplicative group of a rank 2 central F-algebra containing the field E can be realized as the group D_ϵ of matrices $\begin{pmatrix} a & \epsilon b \\ \overline{b} & \overline{a} \end{pmatrix}$ in $G' = GL(2, E)$, where ϵ is a fixed element of $F - NE$, and a, b range over E. The involutive homomorphism is given by $\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}^{-1}$, where the bar indicates the Galois action of E/F. If we allow $\epsilon \in NE^\times$, then D_ϵ is isomorphic to $GL(2, F)$.

Remark. A division algebra H of rank n central over F contains a cyclic field extension K of F of degree n. Given such a pair $H \supset K$, where K/F is a cyclic Galois extension and τ denotes a generator of $\mathrm{Gal}(K/F)$, then there is some $h \in H$ such that H is the F-algebra generated by the element h and by all $k \in K$, subject to the relations $h^n = 1$ and $hk = \tau(k)h$ for all $k \in K$.

Consider the set

$$S = \{x \in D'; \quad x\sigma(x) = 1\}.$$

To study the structure of the relative conjugacy classes we prove (cf., Proposition I.2.1 of [H1] for the case $n = 2$, and Proposition 10 of [F8] for $GL(n)$):

A1. Lemma. *(1) The map $D'/D \to S$, $x \mapsto x\sigma(x)^{-1}$, is a bijection. It maps the double coset DxD to the orbit $Ad(D)(x\sigma(x)^{-1})$ under the adjoint action of D.* (2) *If $x, y \in S$ are conjugate by an element of D', then they are conjugate by an element of D.*

Proof. (1) It is clear that our map is well-defined and injective. The surjectivity follows at once from the triviality of $H^1(\mathrm{Gal}(E/F), D')$ (see [S], X, §1, Ex. 2): if

$g\sigma(g) = 1$, then $a_\sigma = g$ defines a cocycle, which is a coboundary, namely there is $x \in D'$ with $g = a_\sigma = x\sigma(x)^{-1}$.

(2) Suppose that $g \in D'$ satisfies $xg = gy$. Since $x\sigma(x) = 1$ and $y\sigma(y) = 1$, we have $x\sigma(g) = \sigma(g)y$. Put $a = \frac{1}{2}(g + \sigma(g))$, and $b = (g - \sigma(g))/2\sqrt{\theta}$, where $\theta \in F$ and $E = F(\sqrt{\theta})$. Then $g = a + b\sqrt{\theta}$, $xa = ay$, $xb = by$. Consider the polynomial $p(t) = \det(a + tb)$. Its degree is $\leq n$, and its coefficients lie in F, since $\sigma a = a$ and $\sigma b = b$. It is non-zero since $p(\sqrt{\theta}) = \det g \neq 0$. As long as F has more than n elements, there exists $t \in F$ with $p(t) \neq 0$. With this t, the element $a + tb$ lies in D', in fact in D since $\sigma(a + tb) = a + tb$, and it conjugates x to y.

A2. Corollary. *(1) Given $x \in D'$ there exist $g, h \in D$ with $x^{-1} = g\sigma(x)h$. (2) Let E/F be a quadratic extension of local fields. Then any irreducible admissible representation of D' admits at most one (up to a scalar multiple) D-invariant linear form on its space.*

Proof. (1) The elements $x^{-1}\sigma(x)$ and $\sigma(x)x^{-1}$ lie in S, and they are conjugate by an element in D' (as $x^{-1}\sigma(x) = x^{-1} \cdot \sigma(x)x^{-1} \cdot x$), hence by an element in D (by $A1(2)$), and so $Dx^{-1}D = D\sigma(x)D$ by $A1(1)$. (2) is proven as in [F8], Proposition 11, on taking G', G there to be our D', D.

An element γ of $D' \subset GL(n, \overline{F})$, $\overline{F}$ = an algebraic closure of F containing E, is called *regular* if its eigenvalues are distinct (*singular* otherwise), and *elliptic* if it lies in an anisotropic torus of D'. Thus γ is elliptic regular if and only if it lies in no proper E-parabolic subgroup of D'. As in [F2], we make the

Definition. The element $\gamma \in D'$ is called *r-regular*, or *r-elliptic* if $\gamma\sigma(\gamma)^{-1}$ is regular, or elliptic, in D'. The elements $\gamma, \gamma' \in D'$ are *r-conjugate* if there are $x, y \in D$ with $\gamma' = x\gamma y$; equivalently, $\gamma\sigma(\gamma)^{-1}$ and $\gamma'\sigma(\gamma')^{-1}$ are conjugate by an element of D, in view of Lemma A1.

Here "r-" is an abbreviation for "relatively-". Note that the centralizer of $\gamma\sigma(\gamma)^{-1}$ is defined over F, since $x\gamma\sigma(\gamma)^{-1}x^{-1} = \gamma\sigma(\gamma)^{-1}$ implies

$$\sigma(x)(\gamma\sigma(\gamma)^{-1})^{-1}\sigma(x)^{-1} = (\gamma\sigma(\gamma)^{-1})^{-1}.$$

A3. Corollary. *Let $\{T\}$ denote a set of maximal multiplicative F-subgroups in $\mathbf{D}$ such that $T = \mathbf{T}(F)$ runs through a complete set of representatives for the D-conjugacy classes of (maximal) F-tori in D. Let $T' = \mathbf{T}(E)$ be the group of E-points on $\mathbf{T}$, and $T'^{,r\text{-reg}}$ the set of r-regular elements in T'. Introduce the equivalence relation: $t' \sim t''$ in T' if there are $w = w(t', t'')$ in the Weyl group $W_D(T) = N_D(T)/Z_D(T)$ of T in D, and $t \in T$, such that $wt'w^{-1} = tt''$. Then a set of representatives for the set of r-conjugacy classes of the r-regular elements of D' is given by the union over $\{T\}$ of the $T'^{,r\text{-reg}}/\sim$.*

By a common abuse of language, $\{T\}$ as in $A3$ will be referred to (e.g. in $A4$) as "a set of representatives for the D-conjugacy classes of maximal F-tori in D."

In view of the analytic homeomorphism $x \mapsto x\sigma(x)^{-1}$ from D onto S, we may alternatively describe the set of conjugacy classes (under D, equivalently, by $A1(2)$, under D') in the set $S^{\mathrm{reg}} = S \cap D'^{,\mathrm{reg}}$ of regular elements in S. This is given by the union over $\{T\}$ of $T_S^{\mathrm{reg}}/W_D(T)$, where T_S^{reg} is the set of regular elements in $T_S = T' \cap S$.

Of course, the considerations above apply to any inner form $\mathbf{D}$ of $\mathbf{G}$, where E/F is local or global, and in particular to $\mathbf{G}$ itself. Recall that there is an embedding of the set of D'-conjugacy classes of regular elements γ^D in D', into the set of G'-conjugacy classes of regular elements γ in G'. A class γ^D is determined by its characteristic polynomial (over E), and this determines a conjugacy class γ in G'; however, not every regular conjugacy class in G' is so obtained. Via this map we may embed the set of representatives of conjugacy classes of tori in D', in the set of conjugacy classes of tori in G'.

In view of $A1(2)$, we obtain an embedding of $S_D^{\mathrm{reg}}/Ad(D)$, the set of D-conjugacy classes of regular elements in the set $S = S_D$ defined by D, in the analogous set $S^{\mathrm{reg}}/Ad(G)$. By virtue of $A1(1)$ we obtain an embedding of the set $D\backslash D'^{,r\text{-reg}}/D$ of D-double cosets of r-regular elements in D', into the set $G\backslash G'^{,r\text{-reg}}/G$ of G-double-cosets of r-regular elements in G'. We will say that a double coset DxD *corresponds* to the double coset GyG if the image of DxD under this embedding is GyG.

We shall be concerned with orbital integrals on this double coset space. Let E/F be a quadratic separable extension of local fields. We signify by ω a character of $E^\times$ which is trivial on $F^\times$. Denote by H_D the convolution algebra (implicit is a choice of a Haar measure) of complex-valued locally constant functions f on D' which transform under the center by ω^{-1} and are compactly supported modulo the center. For any t in D' denote by $Z(t)$ the set of $(x,y) \in D \times D$ for which there exists $z = z(x,y) \in Z$ with $xty^{-1} = zt$. If t is r-regular then $x, y \in T = T' \cap D$, where T' is the centralizer of $t\sigma(t)^{-1}$ in D'. Since $H^1(\mathrm{Gal}(E/F), T')$ is trivial, we may assume (on changing x or y) that t lies in T', and so that $xy^{-1} = z \in Z$.

Definition. *For $f \in H_D$ and $t \in D'$ define the r-orbital integral*

$$\Xi(t,f) = \Xi_f(t) = \int\int_{(D\times D)/Z(t)} f(xty^{-1})(dx\,dy).$$

Here dx, dy are Haar measures on D, and $(dx\,dy)$ is the quotient of the product measure by a Haar measure on $Z(t)$. The choice of dx, dy, and the measure on $Z(t)$, is implicit in the notation $\Xi(t,f)$. If t and t' are r-conjugate then $Z(t)$ and $Z(t')$ are isomorphic over F and the measures can – and will – be compatibly chosen.

It is clear that $\Xi(t,f)$ depends only on the double coset DtD of t in D'. Since the map $x \mapsto x\sigma(x)^{-1}$, $D'/D \to S$, is an analytic isomorphism, properties of $\Xi(t,f)$ can be deduced from standard properties of usual orbital integrals $\Phi(t,\phi) = \int \phi(x^{-1}tx)$ on $D'/Ad(D')$ (by $A1(2), S/Ad(D') = S/Ad(D)$). In particular, the integral defining $\Xi(t,f)$ is absolutely convergent on $D'^{,r\text{-reg}}$, and its restriction to the r-regular part $T'^{,r\text{-reg}}$ of T', where $\mathbf{T}$ is any F-torus in $\mathbf{D}$, is locally constant and transforms under Z' via ω^{-1}.

Conversely, given any r-conjugacy invariant function $\Xi(t)$ on D', equivalently a function on the union of $T' = \mathbf{T}(E)$ with T ranging over $\{T\}$, whose restriction to T' vanishes on a neighborhood of the r-singular part of T', and which is locally constant and transforms via ω^{-1} under Z', there exists $f \in H_D$ which is zero in a neighborhood of the r-singular set of D' with $\Xi(t,f) = \Xi(t)$ on D'. This characterization of the integrals $\Xi(t,f)$ for $f \in H_D$ which vanish near $D'^{,r\text{-sing}}$ can be extended to a characterization of the $\Xi(t,f)$ for all $f \in H_D$, but this requires more effort and will not be needed here; see [H3] for a complete characterization for $GL(2)$. Using our characterization we conclude:

A4. Lemma. *Suppose that E/F is local, and D is an inner form of $G = GL(n,F)$. Denote by $\{T_D\}$ and $\{T\}$ a set of representatives for the conjugacy classes of F-tori in D and G, and write $\{T(T_D)\}$ for the subset of $\{T\}$ consisting of the tori T which correspond to tori $T_D \in \{T_D\}$. Then for any $f_D \in H_D$ which is supported on $D'^{,r\text{-reg}}$, there exists $f \in H_G$ which is supported on $G'^{,r\text{-reg}}$ such that $\Xi(t,f) = \Xi(t_D, f_D)$ if t corresponds to $t_D \in D'^{,r\text{-reg}}$, and $\Xi(t,f) = 0$ for all $t \in T'$, where $T \in \{T\} - \{T(T_D)\}$. Conversely, given $f \in H_G$ which is supported on $G'^{,r\text{-reg}}$ with $\Xi(t,f) = 0$ on all $t \in T'$ for all $T \in \{T\}$ not corresponding to any element of $\{T_D\}$, there exists $f_D \in H_D$ which is supported on $D'^{,r\text{-reg}}$, with $\Xi(t_D, f_D) = \Xi(t,f)$ if $t_D \in T'_D$ corresponds to $t \in T(T_D)'^{,r\text{-reg}}$.*

As observed in [JL], at a place v of the ground global field which splits in the quadratic extension, the theory of r-orbital integrals reduces to the theory of usual orbital integrals. We encounter the following situation. Let F be a local field and D an inner form of G, put $E = F \oplus F$, $D' = D \times D$, and $f = (f_1, f_2) \in H_D$ (thus f_i is a smooth compactly supported modulo Z function on D). Write $f_2^\vee(x) = f_2(x^{-1})$, and $h = f_1 * f_2^\vee$ (thus $h(x) = \int f_1(xy) f_2(y) dy$). Clearly,

$$\Xi(t,f) = \int\int f_1(xt_1y) f_2(xt_2y) dx\, dy$$

$$= \int\int f_1(xt_1t_2^{-1}x^{-1}y) f_2(y) dx\, dy = \int h(xt_1t_2^{-1}x^{-1}) dx,$$

and the classification of the $\Xi(t,f)$, with $t = (t_1, t_2)$, reduces to the classification of usual orbital integrals on $D/Ad(D)$, at $t\sigma(t)^{-1} = t_1t_2^{-1}$. The latter theory is well known, and we conclude:

A5. Lemma. *Suppose that $E = F \oplus F$ and $D' = D \times D$ as above, $\{T_D\}$ denotes a set of representatives for the conjugacy classes of F-tori in D, $\{T\}$ the analogous set in G, and $\{T(T_D)\}$ the set of $T \in \{T\}$ corresponding to the $T_D \in \{T_D\}$. Then for each $f_D \in H_D$ there is $f \in H_G$ such that $\Xi(t,f) = \Xi(t_D, f_D)$ if $t \in T(T_D)'^{,r\text{-reg}}$ corresponds to $t_D \in T_D'^{,r\text{-reg}}$, and $\Xi(t,f) = 0$ if t lies in $T'^{,r\text{-reg}}$, $T \in \{T\} - \{T(T_D)\}$. Conversely, given $f \in H_G$ with $\Xi(t,f) = 0$ for all $t \in T'^{,r\text{-reg}}$, $T \in \{T\} - \{T(T_D)\}$, there exists $f_D \in H_D$ with $\Xi(t_D, f_D) = \Xi(t,f)$ for all $t_D \in T'^{,r\text{-reg}}_D$ which correspond to $t \in T'^{,r\text{-reg}}$, $T = T(T_D)$.*

Of course, if f_D is zero on a neighborhood of the r-singular set in D', f can be chosen to vanish on a neighborhood of the r-singular set in G', and vice versa.

Definition. Functions $f_D \in H_D$ and $f \in H_G$ as in A4 and A5, satisfying $\Xi(t,f) = \Xi(t_D, f_D)$ for corresponding $t \in G'$, $t_D \in D'$, and $\Xi(t,f) = 0$ on the $t \in G'$ which do not come from D', are called *r-matching.*

B. Simple relative trace formula.

Let E/F be a quadratic separable extension of global fields, $\mathbf{D}$ an inner form of $\mathbf{G} = GL(n)$ over F, and V the finite set of places where $\mathbf{D}$ ramifies. Denote by F_∞ the product of F_v over the archimedean places v, by $\mathbb{A}_f$ the ring of finite adeles, and by $\mathbb{A}_f^\times$ the finite ideles. At each finite v, denote by R_v the ring of integers in F_v and put $K_v = \mathbf{G}(R_v)$. When v is real let $K_v = O(2,\mathbb{R})$, when v is complex let $K_v = U(2)$ and let $\mathbb{K}$ denote the product of the K_v over all places v of F. Let E_∞, $\mathbb{A}_{E,f}$, $\mathbb{A}_{E,f}^\times$, R'_v, K'_v, $\mathbb{K}'$ denote the corresponding objects with respect to E.

At each finite $v \notin V$, the group $D_v = \mathbf{D}(F_v)$ is isomorphic to $G_v = \mathbf{G}(F_v)$, and K_v is the standard maximal compact subgroup in $D_v \simeq G_v$. A fundamental system of open neighborhoods of 1 in $\mathbf{D}(\mathbb{A})$ consists of the set of $\prod_{v\in S} L_v \times \prod_{v \notin S} K_v$, where $S \supset V$ is a finite set of places of F and L_v is an open subset of D_v containing 1. We have also fixed a character $\omega' : \mathbb{A}_E^\times / E^\times \mathbb{A}^\times \to \mathbb{C}^\times$.

Fix a differential form of maximal degree on the algebraic group $\mathbf{D}/\mathbf{Z}$ over F, hence a Haar measure dx_v on D_v/Z_v such that the product of the volumes $|K_v/K_v \cap Z_v|$ over almost all v converges, and denote by $dx = \otimes dx_v$ the product measure on $\mathbf{D}(\mathbb{A})/\mathbf{Z}(\mathbb{A})$. Similarly, we obtain a measure $dx' = \otimes dx'_v$ with analogous properties on $\mathbf{D}'(\mathbb{A})/\mathbf{Z}'(\mathbb{A})$.

At almost all finite v the component ω'_v is unramified, and we denote by H_v^0 the subalgebra of the convolution algebra $H_v = C_c^\infty(D'_v, \omega_v'^{-1}, dx'_v)$ consisting of the K'_v-biinvariant elements. Denote by f_v^0 the unit element in H_v^0; it is supported on $Z'_v K'_v$. Let $f = \otimes f_v$ be a product of $f_v \in H_v$, with $f_v = f_v^0$ for almost all v. Denote by $\mathbb{H}$ the span of such f. For any $t = (t_v) \in \mathbf{D}(\mathbb{A})$ and $f = \otimes f_v$ in $\mathbb{H}$, put $\Xi(t,f) = \prod_v \Xi(t_v, f_v)$.

Definition. The function f is called *r-discrete* if for every $x, y \in \mathbf{D}(\mathbb{A})$ and $\gamma \in D'$ we have $f(x\gamma y) = 0$ unless γ is r-elliptic regular.

If $\mathbf{T}$ is a maximal multiplicative F-subgroup in $\mathbf{D}$, let $N_D(T)$ denote the normalizer of $T = \mathbf{T}(F)$ in $D = \mathbf{D}(F)$ as in A3, and $W_D(T) = N_D(T)/T$ the Weyl group. The cardinality of the Weyl group is denoted by $w_D(T)$.

B1. Proposition. *If f is r-discrete, then*

$$\int_{D\mathbf{Z}(\mathbb{A})\backslash \mathbf{D}(\mathbb{A})} \int_{D\mathbf{Z}(\mathbb{A})\backslash \mathbf{D}(\mathbb{A})} [\sum_{\gamma \in D'/Z'} f(x^{-1}\gamma y)]\, dx\, dy$$

$$= \sum_{\{T\}_e} |\mathbf{T}(\mathbb{A})/\mathbf{Z}(\mathbb{A})T| \; w_D(T)^{-1} \sum_{\gamma \in T'/TZ'} \Xi(\gamma, f).$$

On the right, T (more precisely $\mathbf{T}$) ranges over a set of maximal multiplicative F-tori in $\mathbf{D}$ such that T runs through a complete set of representatives for the conjugacy classes of elliptic F-tori in D. The inner sum ranges over the r-regular γ in T'/TZ'. Here $T' = \mathbf{T}(E), Z' = \mathbf{Z}(E), D' = \mathbf{D}(E)$.

Proof. The map which associates to $g \in \mathbf{D}'(\mathbb{A})$ the sequence $\{a_1, \ldots, a_n = \det g\}$ of coefficients in the characteristic polynomial $\sum_{i=0}^n a_i x^{n-i}$ of g yields an isomorphism from the set of semisimple conjugacy classes in $\mathbf{D}'(\mathbb{A})/\mathbf{Z}'(\mathbb{A})$ to a subset of the quotient of $\mathbb{A}_E^{n-1} \times \mathbb{A}_E^\times$ by $\mathbb{A}_E^\times$, where $\{a_i\} \sim \{a_i z^i\}$, $z \in \mathbb{A}_E^\times$. If $f(x^{-1}\gamma y) \neq 0$ with $\gamma \in D'$ and $x, y \in \mathbf{D}(\mathbb{A})$, then the image of $x^{-1}\gamma\sigma(\gamma)^{-1}x$ lies in a compact subset of $\mathbb{A}_E^{n-1} \times \mathbb{A}_E^\times/\mathbb{A}_E^\times$, and also in the discrete subset $E^{n-1} \times E^\times/E^\times$, hence in a finite set.

Consequently, only finitely many r-conjugacy classes (of r-elliptic regular) γ contribute to the sum $\sum f(x^{-1}\gamma y)$ over γ in D'/Z', on the left. Rearranging, as in [JL], we have

$$\sum_{\gamma \in D'/Z'} f(x^{-1}\gamma y) = \sum_{\{T\}_e} \sideset{}{'}\sum_{\gamma \in T'/Z'} \sum_{\alpha \in D/T} \sum_{\beta \in N(T)\backslash D} f(x^{-1}\alpha\gamma\beta y)$$
$$= \sum_{\{T\}_e} w_D(T)^{-1} \sideset{}{'}\sum_{\gamma \in T'/TZ'} \sum_{\alpha \in D/T} \sum_{\beta \in Z\backslash D} f(x^{-1}\alpha\gamma\beta y),$$

where $\sum_{\{T\}_e}$ indicates a sum – as in the proposition – over the elliptic F-tori T, and $\sum'_\gamma$ a sum over the r-regular γ. Integrating this finite sum over x, y in $\mathbf{Z}(\mathbb{A})D\backslash\mathbf{D}(\mathbb{A})$, we obtain

$$\sum_{\{T\}_e} |\mathbf{T}(\mathbb{A})/\mathbf{Z}(\mathbb{A})T| \; w_D(T)^{-1} \sideset{}{'}\sum_{\gamma \in T'/TZ'} \int_{\mathbf{D}(\mathbb{A})/\mathbf{T}(\mathbb{A})} dx \int_{\mathbf{Z}(\mathbb{A})\backslash\mathbf{D}(\mathbb{A})} f(x\gamma y) dy,$$

as required.

Remark. We should comment on the convergence of the r-orbital integrals $\Xi(\gamma, f)$. Each of these is a product of local integrals. If v is a place of F which does not split in E, the local integral is

$$\Xi(\gamma, f_v) = \int_{D_v/T_v} dx \int_{D_v/Z_v} f_v(x\gamma y) dy.$$

As noted in A4, this converges. Indeed, if the integrand is non-zero, then $x\gamma y$ lies in a compact, and so does $x\gamma\sigma(\gamma)^{-1}x^{-1}$, hence x is in a compact modulo T' (since $\gamma\sigma(\gamma)^{-1}$ is regular), and so x lies in a compact subset of D_v/T_v. But for such x the function $y \mapsto f_v(x\gamma y)$ is compactly supported on D_v/Z_v, and the integral converges.

At almost all such v the function f_v is f_v^0, the quotient by $|K_v Z_v/Z_v|$ of the characteristic function of $K'_v Z'_v/Z'_v$, E_v/F_v is unramified, $\omega'_v = 1$, $D_v = G_v$, and $\gamma \in K'_v Z'_v$. If $f_v(x\gamma y) \neq 0$ then $x\gamma y \in K'_v Z'_v$, and so is $x\gamma\sigma(\gamma)^{-1}x^{-1}$. Since

$\gamma\sigma(\gamma)^{-1}$ is regular in $K'_v Z'_v$, x lies in $T'_v K'_v \cap D_v$; and this intersection is $T_v K_v$ since E_v/F_v is unramified. Then we may take x in $K_v Z_v$, and conclude that y is in $K_v Z_v$. Hence the integral is equal to the volume $|K_v Z_v/Z_v|/|(K_v Z_v \cap T_v)/Z_v|$ for almost all v where E_v is a field.

If v is a place of F which splits into v' and v'' in E, then $\gamma = (\gamma', \gamma'')$ in $D'_v = D_v \times D_v$, and the r-orbital integral

$$\Xi(\gamma, f_v) = \int_{D_v/T_v} dx \int_{D_v/Z_v} f_{v'}(x\gamma' y) f_{v''}(x\gamma'' y) dx\, dy$$

is equal, as noted in A5, to the usual orbital integral

$$\Phi(\delta, h_v) = \int_{D_v/T_v} h_v(x\delta x^{-1}) dx$$

of $h_v = f_{v'} * \check{f}_{v''}$ at $\delta = \gamma\sigma(\gamma)^{-1} = \gamma'\sigma(\gamma'')^{-1}$ (we embed D_v diagonally in D'_v). The convergence follows, and it is easy to see that at almost all such v the integral is equal again to $|K_v Z_v/Z_v|/|(K_v Z_v \cap T_v)/Z_v|$. We obtain the convergence of each of the global integrals $\Xi(\gamma, f)$ in B1.

To produce discrete functions f, we introduce the local analogue.

Definition. The function $f_v \in H_v$ is called *r-discrete* if for every $x, y \in D_v$ and $\gamma \in D'_v$ we have $f_v(x\gamma y) = 0$ unless γ is r-elliptic regular.

Note that when v is split in E, if $f_v = (f_{v'}, f_{v''})$ is r-discrete then $h_v = f_{v'} * \check{f}_{v''}$ is supported on the elliptic regular set in D_v. The converse is also true, for example, when $f_{v''}$ is supported in $Z_v K'_v$, where K'_v is a small compact open subgroup of G_v and $f_{v'}$ is K'_v-biinvariant.

It is clear that $f = \otimes f_v$ is r-discrete if it has an r-discrete component; an element $\delta \in D'$ is elliptic (resp. regular) if it is elliptic (resp. regular) in D'_v for some v.

Let $L(D') = L_{\omega'}(D' \backslash \mathbf{D}'(\mathbb{A}))$ denote the space of automorphic forms on $\mathbf{D}'(\mathbb{A})$; these are smooth functions on $D' \backslash \mathbf{D}'(\mathbb{A})$ which transform on $\mathbf{Z}'(\mathbb{A})$ according to ω' and are absolutely square-integrable on $\mathbf{Z}'(\mathbb{A}) D' \backslash \mathbf{D}'(\mathbb{A})$. Recall that the function $\phi \in L(D')$ is called *cuspidal* if for each proper parabolic subgroup $\mathbf{P}'$ of $\mathbf{D}'$ over E with unipotent radical $\mathbf{N}'$ we have $\int_{N' \backslash \mathbf{N}'(\mathbb{A})} \phi(ng) dn = 0$ for every $g \in \mathbf{D}'(\mathbb{A})$. The space of cuspidal functions in $L(D')$ is denoted by $L_0(D') = L_{0,\omega'}(D' \backslash \mathbf{D}'(\mathbb{A}))$. Note that G' is the special case of D' with empty set V, hence the definition of $L_0(G')$ is a special case of that of $L_0(D')$.

Denote by r the right representation on $L(D')$, by r_0 its restriction to $L_0(D')$, by $r(f)$ the convolution operator on $L(D')$, and by $r_0(f)$ its restriction to $L_0(D')$. The space $L_0(D')$ decomposes as a direct sum of irreducible, automorphic cuspidal representations of $\mathbf{D}'(\mathbb{A})$. Note that the multiplicity one and rigidity theorems for D' follow from those for G' via the Deligne-Kazhdan correspondence (see [F3], p. 170).

Definition. (1) The function f is called *cuspidal* if for every x, y in $\mathbf{D}'(\mathbb{A})$ and every proper E-parabolic subgroup $\mathbf{P}'$ of $\mathbf{D}$, we have $\int_{\mathbf{N}'(\mathbb{A})} f(xny)dn = 0$, where $\mathbf{N}'$ is the unipotent radical of $\mathbf{P}'$. (2) The function f_v in H_v is called *supercuspidal* if for every x, y in D'_v and every proper E_v-parabolic subgroup P'_v of D'_v, whose unipotent radical is denoted by N'_v, we have $\int_{N'_v} f_v(xny)dn = 0$. Here v is a place of E. If v is a place of F which splits in E then we say that $f_v = (f_{v'}, f_{v''})$ is supercuspidal if $f_{v'}$ or $f_{v''}$ is.

It is easy to see that f is cuspidal if it has a supercuspidal component.

The convolution operator $r(f) = \int_{\mathbf{D}'(\mathbb{A})/\mathbf{Z}'(\mathbb{A})} f(g)r(g)dg$ on $L(D')$ is an integral operator with kernel $K_f(x, y) = \sum_{\gamma \in D'/Z'} f(x^{-1}\gamma y)$. Let $\{\phi\} = \{\phi^\pi\}$ be an orthonormal basis for the space $\pi \subset L_0(D')$. Then $r_0(f)$ is an integral operator on $\mathbf{D}'(\mathbb{A})$ with kernel $K_f^0(x, y) = \sum_\pi \sum_\phi (r(f)\phi)(x)\overline{\phi}(y)$. When f is cuspidal, $r(f)$ factors through the projection on $L_0(D')$, and $K_f^0(x, y) = K_f(x, y)$. Integrating this over x, y in $\mathbf{Z}(\mathbb{A})D\backslash\mathbf{D}(\mathbb{A})$ we obtain:

B2. Proposition. *If f is r-discrete and cuspidal, then*

$$\sum_{\pi \subset L_0(D')} \sum_\phi \int_{\mathbf{Z}(\mathbb{A})D\backslash\mathbf{D}(\mathbb{A})} (r(f)\phi)(x)dx \cdot \int_{\mathbf{Z}(\mathbb{A})D\backslash\mathbf{D}(\mathbb{A})} \overline{\phi}(y)dy$$
$$= \sum_{\{T\}_e} |\mathbf{T}(\mathbb{A})/\mathbf{Z}(\mathbb{A})T| \; w_D(T)^{-1} \sum_{\gamma \in T'/TZ'}{}' \; \Xi(\gamma, f).$$

The sum on the right is as in B1. We proceed to rewrite the left side.

By A2(2), there is at most one (up to a scalar multiple) non-zero D_v-invariant linear form L_{π_v} on the space V of an irreducible admissible representation π_v of D'_v. Let us assume that π_v is D_v-distinguished, so that such a form L_{π_v} exists. Then the contragredient $(\tilde{\pi}_v, \widetilde{V})$ is also D_v-distinguished. This follows, for example, from the result of Gelfand-Kazhdan ([GK], see also [BZ]) that $\tilde{\pi}_v$ is equivalent to the representation $g \mapsto \pi_v({}^t g^{-1})$ on V; cf. proof of Proposition 11 in [F8].

Choose a non-zero D_v-invariant linear form $L_{\tilde{\pi}_v}$ in the space $\widetilde{V}^*$ dual to $\widetilde{V}$. Since $\pi_v(f_v)$ is an operator of finite rank, $\pi_v(f_v)L_{\tilde{\pi}_v}$ lies in the space $\widetilde{\widetilde{V}}$ contragredient to $\widetilde{V}$. But $\widetilde{\widetilde{V}} = V$, and so we can define the linear form $\mathbb{L}_{\pi_v}(f_v) = L_{\pi_v}(\pi_v(f_v)L_{\tilde{\pi}_v})$ on the convolution algebra H_v of the f_v. The linear form $\mathbb{L}_{\pi_v}$ is D_v-biinvariant, that is, if $x, y \in D_v$ and ${}^x f_v^y(g) = f_v(xgy)$ then $\mathbb{L}_{\pi_v}({}^x f_v^y) = \mathbb{L}_{\pi_v}(f_v)$. It depends on π_v only up to equivalence, and if $\pi_{1v}, \dots, \pi_{mv}$ are pairwise inequivalent then the forms $\mathbb{L}_{\pi_{1v}}, \dots, \mathbb{L}_{\pi_{mv}}$ on H_v are linearly independent. We normalize $\mathbb{L}_{\pi_v}$ for an unramified π_v by the requirement that $\mathbb{L}_{\pi_v}(f_v^0) = 1$, where f_v^0 is the unit element in the Hecke algebra H_v^0 of spherical functions.

When v is a place of F which splits in E, π_v is D_v-distinguished precisely when it is of the form $(\rho \otimes \tilde{\rho}, V \otimes \widetilde{V})$ where (ρ, V) is a representation of D_v. Let $\{u_j\}$

denote a basis of V and $\{\tilde{u}_j\}$ the dual basis for $\widetilde{V}$. The canonical pairing $\langle\cdot,\cdot\rangle$ on $V\otimes\widetilde{V}\to\mathbb{C}$ defines a D_v-invariant form on $V\otimes\widetilde{V}$, if D_v is identified with the diagonal of $D_v\times D_v$.

The contragredient of $\rho\otimes\tilde\rho$ is $\tilde\rho\otimes\rho$ and the pairing between the corresponding spaces $V\otimes\widetilde{V}$ and $\widetilde{V}\otimes V$ is given by

$$\langle v\otimes\tilde v,\tilde w\otimes w\rangle=\langle v,\tilde w\rangle\langle w,\tilde v\rangle.$$

We define our invariant forms L_{π_v} and $L_{\tilde\pi_v}$ by $L_{\pi_v}(v\otimes\tilde v)=\langle v,\tilde v\rangle=L_{\tilde\pi_v}(\tilde v\otimes v)$. These linear forms can be regarded as generalized vectors in the dual spaces. For example, L_{π_v} can be identified with the formal sum $\sum_i\tilde u_i\otimes u_i$, and $L_{\tilde\pi_v}=\sum_i u_i\otimes\tilde u_i$. We now compute

$$\begin{aligned}\mathbb{L}_{\pi_v}((f_1,f_2))&=L_{\pi_v}(\pi_v(f_1,f_2)L_{\tilde\pi_v})=\langle\sum_i\rho(f_1)u_i\otimes\tilde\rho(f_2)\tilde u_i,\sum_j\tilde u_j\otimes u_j\rangle\\&=\sum_{i,j}\langle\rho(f_1)u_i,\tilde u_j\rangle\langle u_j,\tilde\rho(f_2)\tilde u_i\rangle=\sum_i\langle\rho(f_1)u_i,\tilde\rho(f_2)\tilde u_i\rangle\\&=\sum_i\langle\rho(f_2^\vee*f_1)u_i,\tilde u_i\rangle=tr\,\rho(f_2^\vee*f_1).\end{aligned}$$

Given a cuspidal representation $\pi=\otimes\pi_v$ of $\mathbf{D}'(\mathbb{A})$ such that each of its components is D_v-distinguished, we can define the form $\mathbb{L}_\pi=\otimes_v\mathbb{L}_{\pi_v}$ on $\otimes_v H_v$. For each $f=\otimes f_v$, we have $f_v=f_v^0$ for almost all v, and so $\mathbb{L}_{\pi_v}(f_v)=1$ for almost all v, and $\mathbb{L}_\pi(f)$ is defined. Consequently π is abstractly distinguished; in particular, $\mathbb{L}_\pi$ is a non-zero $\mathbf{D}(\mathbb{A})$-invariant form on its space. For all other π, we put $\mathbb{L}_\pi\equiv 0$. Note that the definition of $\mathbb{L}_\pi$ on D' includes that of $\mathbb{L}_\pi$ for π on $\mathbf{G}'(\mathbb{A})$, since G' is the special case of D' with empty set V.

If the cuspidal $\pi=\otimes\pi_v$ is distinguished, then the form $A_\pi(\phi)=\int_{\mathbf{Z}(\mathbb{A})D\backslash\mathbf{D}(\mathbb{A})}\phi(g)dg$ is a non-zero $\mathbf{D}(\mathbb{A})$-invariant form on π. Its restriction to π_v is non-zero, implying that each component of π is D_v-distinguished. If $\{\phi\}$ is an orthonormal basis of the cuspidal π, then $\{\overline{\phi}\}$ is a dual basis of the contragredient $\tilde\pi$. The bar denotes complex conjugation. It is easy to check that the distribution

$$\mathbb{A}_\pi(f)=\sum_\phi\int_{\mathbf{Z}(\mathbb{A})D\backslash\mathbf{D}(\mathbb{A})}(\pi(f)\phi)(x)dx\cdot\int_{\mathbf{Z}(\mathbb{A})D\backslash\mathbf{D}(\mathbb{A})}\overline{\phi}(y)dy$$

is bi-$\mathbf{D}(\mathbb{A})$-invariant. It is independent of the choice of the basis $\{\phi\}$, which can and from now on will be chosen to consist of smooth vectors. Since the operators $\{\pi_v(f_v);f_v\in H_v\}$ span the space of endomorphisms of an irreducible admissible π_v, the operators $\{\pi(f);f\in\mathbb{H}\}$ span the space of endomorphisms of the subspace of smooth vectors in the irreducible representation π. Hence there is f with $\mathbb{A}_\pi(f)\neq 0$ if π is $\mathbf{D}(\mathbb{A})$-distinguished. Conversely, if $\mathbb{A}_\pi(f)\neq 0$ then $A_\pi(\phi)\neq 0$ for some ϕ. The local uniqueness result of A2(2) implies:

B3. Lemma. *For any irreducible automorphic cuspidal representation π of $\mathbf{D}'(\mathbb{A})$ which is $\mathbf{D}(\mathbb{A})$-distinguished, there exists a constant $c(\pi) \neq 0$ such that $\mathbb{A}_\pi = c(\pi)\mathbb{L}_\pi$.*

When π is not distinguished, take $c(\pi) = 0$. We refer to the following as the *simple relative trace formula for $\mathbf{D}'(\mathbb{A})/\mathbf{D}(\mathbb{A})$:*

B4. Proposition. *If f is r-discrete and cuspidal, then*

$$\sum_{\pi \subset L_0(D')} c(\pi)\mathbb{L}_\pi(f) = \sum_{\{T\}_e} |\mathbf{T}(\mathbb{A})/T\mathbf{Z}(\mathbb{A})| \; w_D(T)^{-1} \sum_{\gamma \in T'/TZ'}{}' \; \Xi(\gamma, f).$$

This is of course valid for any inner form $\mathbf{D}$ of $\mathbf{G}$, including $\mathbf{G}$ itself. Recall that the set $\{T_D\}_e$ of conjugacy classes of (elliptic) F-tori T_D in D is identified as (a subset $\{T(T_D)\}_e$ of) the corresponding set $\{T\}_e$ of G, and the analogous identification can be made locally too. The functions $f = \otimes f_v \in \mathbb{H}$ and $f^D = \otimes f_v^D \in \mathbb{H}^D$ are r-matching if $\Xi(\gamma, f_v) = \Xi(\gamma^D, f_v^D)$ for all corresponding r-regular $\gamma^D \in T'_{D,v}$ and $\gamma \in T'_v$, $T_v = T(T_{D,v})$, and $\Xi(\gamma, f_v) = 0$ on any r-regular $\gamma \in T'_v$ if $T \in \{T\} - \{T(T_D)\}$.

Let V be the set of F-places where D ramifies. At $v \notin V$ we have $D_v \simeq G_v$, and we take f_v^D to be f_v, via this isomorphism. Let V'' be the set of $v \in V$ which split in E, and V' the set of $v \in V$ which stay prime in E. At each such $v \in V''$, for each $f_v^D = (f_1^D, f_2^D)$ there exists an r-matching $f_v = (f_1, f_2)$, thus $h^D = f_2^{D\vee} * f_1^D$ and $h = f_2^\vee * f_1$ have matching orbital integrals $\Phi(h^D) = \Phi(h)$; and for each f_v with $\Phi(\gamma, h) = 0$ on the regular γ not obtained from D, there exists an r-matching f_v^D, as noted in A5. We conclude:

B5. Proposition. *If f^D and f are r-matching, r-discrete and cuspidal, then*

$$\sum_{\pi^D \subset L_0(D')} c(\pi^D)\mathbb{L}_{\pi^D}(f^D) = \sum_{\pi \subset L_0(G')} c(\pi)\mathbb{L}_\pi(f).$$

Suppose that v is finite, D splits at v and π_v is unramified, namely there is a (unique up to scalar multiples) K_v-fixed non-zero vector in the space of π_v. Then for every w in the space of π_v and $f_v \in H_v^0$, the vector $\pi_v(f_v)w$ is zero unless w is K_v-fixed, in which case the multiple $f_v^\vee(t(\pi_v))w$ of w is obtained. Here $f_v^\vee$ denotes the Satake transform of the spherical function f_v; it is a polynomial in $z_1, z_1^{-1}, \dots, z_n, z_n^{-1}$, invariant under the action of the symmetric group S_n. We put $t(\pi_v) = (z_1, \dots, z_n)$ where z_i are the Hecke eigenvalues of the unramified π_v. Hence for unramified π_v and spherical $f_v \in H_v^0$ we have

$$\mathbb{L}_{\pi_v}(f_v) = L_{\pi_v}(\pi_v(f_v)L_{\tilde{\pi}_v}) = f_v^\vee(t(\pi_v))L_{\pi_v}(\pi_v(f_v^0)L_{\tilde{\pi}_v}) = f_v^\vee(t(\pi_v)),$$

since $\mathbb{L}_{\pi_v}$ takes the value 1 at the unit element f_v^0 in H_v^0, by our normalization.

Let $S \supset V = V' \cup V''$ be a finite set of places containing those which ramify in E/F or are archimedean. Fix an irreducible, unramified G_v-distinguished representation ρ_v of G'_v at each $v \notin S$. There exists at most one cuspidal representation

π of $\mathbf{G}'(\mathbb{A})$ with $\pi_v \simeq \rho_v$ for all $v \notin S$. We put $\epsilon(\pi) = 1$ if π exists and $\epsilon(\pi) = 0$ if not. By the Deligne-Kazhdan correspondence ([F3], p. 170) this rigidity and multiplicity-one theorem applies also to D', and so there exists at most one cuspidal representation π^D of $\mathbf{D}'(\mathbb{A})$ with $\pi_v^D \simeq \rho_v (v \notin S)$; we put $\epsilon(\pi^D) = 1$ if π^D exists, $\epsilon(\pi^D) = 0$ otherwise. A well-known argument of "generalized linear independence of characters" (see [FK2], Theorem 2) implies:

B6. Proposition. *If $f_S^D = \underset{v \in S}{\otimes} f_v^D$ and $f_S = \underset{v \in S}{\otimes} f_v$ are r-matching and have r-discrete and supercuspidal components at two distinct F-places in S, then for any $\{\rho_v; v \notin S\}$ we have*

$$\epsilon(\pi^D)c(\pi^D) \prod_{v \in S} \mathbb{L}_{\pi_v^D}(f_v^D) = \epsilon(\pi)c(\pi) \prod_{v \in S} \mathbb{L}_{\pi_v}(f_v).$$

At $v \in S - V$, we have $\pi_v^D \simeq \pi_v$ and $f_v^D = f_v$ via $D_v \simeq G_v$. If $c(\pi)$ or $c(\pi^D) \neq 0$ then π_v is distinguished, there is some f_v with $\mathbb{L}_{\pi_v}(f_v) \neq 0$, and the place $v \notin S$ can be deleted from the products on both sides of the identity of B6. At the places $v \in V''$ which split in E/F and D is ramified, if $\pi_v^D = \pi_{1v}^D \otimes \pi_{2v}^D$ and $\pi_v = \pi_{1v} \otimes \pi_{2v}$ are distinguished then $\pi_{2v}^D \simeq \tilde{\pi}_{1v}^D$ and $\pi_{2v} \simeq \tilde{\pi}_{1v}$, $\mathbb{L}_{\pi_v^D}(f_v^D) = tr\, \pi_{1v}^D(h_v^D)$ and $\mathbb{L}_{\pi_v}(f_v) = tr\, \pi_{1v}(h_v)$, π_{1v}^D corresponds to π_{1v} (since π^D corresponds to π), and since h_v^D and h_v have matching orbital integrals (by assumption), we have $tr\, \pi_{1v}^D(h_v^D) = tr\, \pi_{1v}(h_v)$, and again v can be deleted from the products of B6.

At two places $v \in S - V'$ we need to use special test functions f_v. At one such place we need to use a supercuspidal function, at another, an r-discrete function. A matrix coefficient f_v of a supercuspidal representation is a supercuspidal function. At a place v which splits E/F, we may choose h_v to be a normalized coefficient of the supercuspidal representation π'_{1v} of G_v, and then $\mathbb{L}_{\pi_v}(f_v)$ is $tr\, \pi_{1v}(h_v)$ if $\pi_v = \pi_{1v} \otimes \tilde{\pi}_{1v}$ (it is 0 otherwise), and this is 1 if $\pi_{1v} \simeq \pi'_{1v}$ and 0 otherwise. Since π'_{1v} is supercuspidal, if $v \in V''$ it corresponds to a supercuspidal $\pi_{1v}^{\prime D}$, and any normalized coefficient h_v^D of $\pi_{1v}^{\prime D}$ matches h_v, and satisfies $\mathbb{L}_{\pi_v^D}(f_v^D) = 1$ if $\pi_v^D \simeq \pi_{1v}^{\prime D} \otimes \tilde{\pi}_{1v}^{\prime D}$, and $= 0$ otherwise.

Consider once more a v which splits and π_v with $\mathbb{L}_{\pi_v}(f_v) = tr\, \pi_{1v}(h_v)$. The character of any admissible π_{1v} is locally constant on the regular set in G_v, and if π_{1v} is square-integrable its character is non-zero on the elliptic regular set (by the orthogonality relations for such characters). Hence there is a discrete h_v with $tr\, \pi_{1v}(h_v) \neq 0$. Such a square-integrable π_{1v} corresponds to a square-integrable π_{1v}^D if $v \in V''$, and $tr\, \pi_{1v}^D(h_v^D) \neq 0$ for a matching discrete h_v^D.

Suppose then that v stays prime in E/F, and π_v^D is D_v-distinguished, where D_v is an inner form of G_v. Since $L_{\tilde{\pi}_v^D} = \sum_{\{\phi\}} L_{\tilde{\pi}_v^D}(\tilde{\phi})\phi$, the bilinear form $\mathbb{L}_{\pi_v^D}$ is given by

$$\mathbb{L}_{\pi_v^D}(f_v^D) = \sum_{\{\phi\}} L_{\pi_v^D}(\pi_v^D(f_v^D)\phi) L_{\tilde{\pi}_v^D}(\tilde{\phi}),$$

where $\{\phi\}$ is a basis of the space of π_v^D, and $\{\tilde{\phi}\}$ is the dual basis in the contragredient $\tilde{\pi}_v^D$. If π_v^D is D_v-distinguished, clearly so is $\tilde{\pi}_v^D$, and there are ϕ' and $\tilde{\phi}''$ with

$L_{\pi_v^D}(\phi') \neq 0$ and $L_{\tilde{\pi}_v^D}(\tilde{\phi}'') \neq 0$. If π_v^D is also supercuspidal, choosing f_v^D to be the coefficient $f_v^D(g) = d(\pi_v^D)(\pi_v^D(g)\phi', \tilde{\phi}'')$ where $d(\pi_v^D)$ is the formal degree of π_v^D, by the Schur orthogonality relations

$$\int_{D_v'/Z_v'} d(\pi_v^D)(\pi_v^D(g)\phi_1, \tilde{\phi}_2)(\pi_v^D(g)\phi_3, \tilde{\phi}_4)dg = (\phi_1, \tilde{\phi}_4)(\phi_2, \tilde{\phi}_3)$$

we obtain that $\pi_v^D(f_v^D)\phi'' = \phi'$ and $\mathbb{L}_{\pi_v^D}(f_v^D) = L_{\pi_V^D}(\phi')L_{\tilde{\pi}_v^D}(\tilde{\phi}'') \neq 0$, while $\mathbb{L}_{\rho_v^D}(f_v^D) = 0$ for all $\rho_v^D \not\simeq \pi_v^D$.

It is clear that both sides of the identity of B6 vanish unless π^D corresponds to π. We conclude:

B7. Proposition. *Suppose that π^D and π are corresponding cuspidal representations which have supercuspidal components at a place $v_1 \notin V'$, and square-integrable components at a place $v_2 \neq v_1$ which splits in E. Suppose that π_v^D is D_v-distinguished and π_v is G_v-distinguished for all $v \notin V'$. Then for any r-matching f_v^D and f_v $(v \in V')$ we have*

$$c(\pi^D) \prod_{v \in V'} \mathbb{L}_{\pi_v^D}(f_v^D) = c(\pi) \prod_{v \in V'} \mathbb{L}_{\pi_v}(f_v).$$

Since the distribution $\mathbb{L} = \mathbb{L}_{\pi_v^D}$ is right D_v-invariant, $\mathbb{L}(f)$ depends only on $\vartheta(g\sigma(g)^{-1}) = \int_{D_v/Z_v} f(gx)dx$. As the left invariance implies the analogous property, it follows that $\mathbb{L}(f)$ depends on f only through its r-orbital integral $\Xi(f)$. In particular, there is an $Ad(D_v)$-invariant distribution Λ on S_D such that $\mathbb{L}(f) = \Lambda(\vartheta)$. Howe [Ho] studied the analytic properties of $Ad(D_v)$-invariant admissible distributions on D_v, in the case where $D_v = GL(n, F_v)$. As is shown in part D, his techniques can be modified to apply also in our case, to yield the smoothness part of

B8. Proposition. *The D_v-biinvariant distribution $\mathbb{L}_{\pi_v^D}$ can be represented by a D_v-biinvariant function on D_v' which is locally constant and not identically zero on the r-regular set in D_v'.*

This function will be called the *r-character* of π_v^D, and denoted by $\Xi_{\pi_v^D}(\gamma) = \Xi(\gamma, \pi_v^D)$. It is common to refer to the distribution $\mathbb{L}_{\pi_v^D}$ represented by the function $\Xi_{\pi_v^D}$ also as the *r-character* of π_v^D. The archimedean analogue of B8 is proven in Sano [Sa], who showed that a generalized spherical function on $G(\mathbb{C})/G(\mathbb{R})$ is locally integrable on $G(\mathbb{C})/G(\mathbb{R})$ and is analytic on the regular set of $G(\mathbb{C})/G(\mathbb{R})$.

Proposition B8 is proven in [H1], pp. 56-61 (III, §1), when $D_v = GL(2, E_v)$, and the central character is trivial; see also [H3]. The case where D_v is anisotropic is trivial. As noted above we delay the proof of B8 to part D. Harish-Chandra showed in [HC2] that the character of an admissible irreducible representation of any p-adic reductive group is locally constant on the regular set. His proof shows that the restriction of $\mathbb{L}_{\pi_v^D}$ to the space of functions $f_v^K(g) = \int_{K_v} f_v(kgk^{-1})dk (f_v \in H_v)$

is locally constant. His techniques may apply to show the smoothness for any H-biinvariant distribution on a p-adic reductive group G, where H is the group of points of G fixed by an involution (the case of usual characters is that where $G = H \times H$, and H embeds diagonally in G, and the involution is $(x, y) \mapsto (y, x)$). In [HC1] Harish-Chandra proved the local integrability of the character in characteristic zero for any reductive connected p-adic group. The analogous result is valid in the case considered in this paper (see [H4]), but it is known to fail for many other general symmetric spaces G/H. The local integrability shows in our case that the r-character is not identically zero on the r-regular set of G.

B9. Proposition. *Suppose that π^D and π are corresponding cuspidal representations which have supercuspidal components at a place $v_1 \notin V'$, and discrete-series components at a place $v_2 \neq v_1$ which splits in E. If π^D is $\mathbf{D}(\mathbb{A})$-distinguished then π is $\mathbf{G}(\mathbb{A})$-distinguished. If π is $\mathbf{G}(\mathbb{A})$-distinguished, and for each $v \in V'$ the r-character Ξ_{π_v} is not identically zero on the set of r-regular $g \in G'_v$ which correspond to elements of D'_v, then π^D is $\mathbf{D}(\mathbb{A})$-distinguished.*

Proof. If π^D is $\mathbf{D}(\mathbb{A})$-distinguished, each π_v^D is D_v-distinguished and we may use B7. At each $v \in V'$ we choose f_v^D which is supported on the r-regular set in D'_v, with $\mathbb{L}_{\pi_v^D}(f_v^D) \neq 0$. Such f_v^D exists by B8. Each such f_v^D has an r-matching function f_v on the r-regular set in G'_v. We shall use the identity of B7 with such functions. Since the left side is non-zero, so is the right side. In particular $c(\pi) \neq 0$ and π is $\mathbf{G}(\mathbb{A})$-distinguished.

In the opposite direction, if π is $\mathbf{G}(\mathbb{A})$-distinguished, each π_v is G_v-distinguished. By assumption on π_v for $v \in V'$, there exists, for each $v \in V'$, a function $f_v \in H_v$ supported on the set of r-regular elements of G'_v which correspond to elements of D'_v, with $\mathbb{L}_{\pi_v}(f_v) \neq 0$. For such f_v, there exists an r-matching function f_v^D on the r-regular set in D'_v, by A4. Consequently we may use the identity of B7 with this choice of local r-matching functions. Since the right side is non-zero, so is the left side. Hence $c(\pi^D) \neq 0$, and π^D is $\mathbf{D}(\mathbb{A})$-distinguished, as required.

Definition. An admissible irreducible D_v-distinguished representation π_v^D of D'_v is called *r-discrete-series* if its r-character is not identically zero on the r-regular elliptic set in D'_v.

The condition at v_2 in B9 can be relaxed.

B10. Proposition. *Suppose that π^D and π are corresponding cuspidal representations which have supercuspidal components at a place $v_1 \notin V'$ (it suffices to require that π_{v_1} be supercuspidal, for then $\pi_{v_1}^D$ is such too). If π^D is $\mathbf{D}(\mathbb{A})$-distinguished and $\pi_{v_2}^D$ is r-discrete-series at $v_2 \neq v_1$, then π is $\mathbf{G}(\mathbb{A})$-distinguished (and π_{v_2} is r-discrete-series). If π is $\mathbf{G}(\mathbb{A})$-distinguished, π_{v_2} is r-discrete-series at $v_2 \neq v_1$, and for each $v \in V'$ the r-character of π_v is not identically zero on the set of r-regular $g \in G'_v$ which come from D'_v, then π^D is $\mathbf{D}(\mathbb{A})$-distinguished (and $\pi_{v_2}^D$ is r-discrete-series).*

Proof. If v_2 splits in E/F, $\pi_{v_2} = \pi_{1v_2} \otimes \tilde{\pi}_{1v_2}$ is r-discrete-series means that π_{1v_2} is discrete-series, and B10 reduces to B9. If v_2 stays prime we may choose r-discrete r-matching f_{v_2} and $f_{v_2}^D$, and then apply B7 as in the proof of B9.

B11. Corollary. *If π_v and π_v^D are components of π and π^D, where π or π^D satisfy the assumptions of B9 or B10, then there exists a non-zero constant $c(\pi_v, \pi_v^D)$ such that $\mathbb{L}_{\pi_v}(f_v) = c(\pi_v, \pi_v^D)\mathbb{L}_{\pi_v^D}(f_v^D)$ for all r-matching functions f_v and f_v^D.*

Proof. If π satisfies the assumption of B9 or B10, then for each $v' \neq v$ in V' there is $f_{v'}$ with $L_{\pi_{v'}}(f_{v'}) \neq 0$. Similar conclusion is obtained if π^D satisfies B9 or B10. The constant $c(\pi_v, \pi_v^D)$ is obtained on fixing $f_{v'}$ and the matching $f_{v'}^D$ (for all $v' \neq v$ in V') in the identity displayed in Proposition B7.

The conclusion here can be restated as asserting that $\Xi_{\pi_v}(\gamma) = c(\pi_v, \pi_v^D)\Xi_{\pi_v^D}(\gamma^D)$ for all pairs (γ, γ^D) of corresponding elements in G_v, D_v. This follows from A4, B8, and the relative Weyl integration formula

$$\int_{D'_v/Z'_v} f_v^D(g)dg = \sum_{\{T_v\}} |T_v/Z_v| w_{T_v^D}^{-1} \int_{T'_v/T_vZ'_v} \Delta_v(t)^2 \Xi(t, f_v^D)dt.$$

The sum ranges over the set of conjugacy classes of F_v-tori in D_v, and f_v is a function on D'_v/Z'_v. This relative formula can be reduced to the standard formula via the isomorphism $D'_v/D_v \to S_{D_v}$ of A1.

B12. Proposition. *If $f_v \in H_v$ is a supercusp form and t is r-regular, then $\Xi(t, f_v)$ is zero unless t is r-elliptic.*

Proof. Write $D'_v = GL(m, A')$ and $D_v = GL(m, A)$, where A is a division algebra central over F_v, and $A' = A \otimes_{F_v} E_v$. We may assume that t lies in the standard Levi subgroup M'_v of a maximal parabolic $P'_v = M'_vU'_v$ in D'_v, and its centralizer is a torus $T'_v = T(E_v) \subset M'_v$, where $T_v = T(F_v)$ is an F_v-torus. By virtue of the Iwasawa decomposition, the integral

$$\Xi(t, f_v) = \int_{D_v/T_v} dx \int_{D_v/Z_v} f_v(xty)dy$$

factorizes through the integral

$$(*) \qquad \int_{U_v} \int_{U_v} f_v(x\mathbf{u}t\mathbf{u}'y)d\mathbf{u}d\mathbf{u}'.$$

If P'_v is of type $(a, b)(a + b = m)$, then $\mathbf{u} = \begin{pmatrix} I & u \\ 0 & I \end{pmatrix}$, $\mathbf{u}' = \begin{pmatrix} I & u' \\ 0 & I \end{pmatrix}$ and $t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$ accordingly, and we need to show that when u, u' range over all $a \times b$ matrices over A, $u' + t_1^{-1}ut_2$ ranges over all $a \times b$ matrices over A'. It suffices to show that the map $u \mapsto t_1^{-1} u t_2 - \overline{t}_1^{-1} u \overline{t}_2$ is injective, and for this we may let u range over $M(a \times b, A')$ (we can also work with the tensor product of A' with a splitting field). Since t_1, t_2 are invertible, we may consider the map $u \mapsto u - \overline{t}_1^{-1} t_1 \cdot u \cdot (\overline{t}_2^{-1} t_2)^{-1}$ instead, and may assume that $\overline{t}_1^{-1} t_1$ and $\overline{t}_2^{-1} t_2$ are diagonal. Since $\sigma(t)^{-1} t = \overline{t}^{-1} t$ is regular in D_v, this vector spaces homomorphism is an isomorphism, and we conclude that

$$(*) = \int_{U'_v} f_v(xt\mathbf{u}y)du.$$

But this is zero since f_v is supercuspidal.

B13. Corollary. *Let π_v^D be a D_v-distinguished supercuspidal representation of D'_v. Then there exists an $f_v^D \in H_v^D$ with $\mathbb{L}_{\pi_v^D}(f_v^D) \neq 0$ and $\mathbb{L}_{\rho_v^D}(f_v^D) = 0$ for all $\rho_v^D \not\simeq \pi_v^D$. Moreover, $\Xi(t, f_v^D)$ is not identically zero on the r-regular elliptic set of D'_v.*

Proof. The first claim is proven in the paragraph prior to B7: we choose f_v^D to be a matrix coefficient of π_v^D. Then f_v^D is a supercusp form, and by B12 the r-orbital integral $\Xi(t, f_v^D)$ vanishes outside the r-elliptic set. Since $\mathbb{L}_{\pi_v^D}(f_v^D) \neq 0$, we have that $\Xi(t, f_v^D)$ is not identically zero on the r-elliptic regular set.

Next we show that local distinguished supercuspidal representations embed as components of global cuspidal distinguished representations.

B14. Proposition. *Given a D_u-distinguished supercuspidal representation ${\pi'}_u^D$ of D'_u, places $v_1, \dots, v_m$ $(\neq u)$ which split in E and D_{v_i}-distinguished supercuspidal representations of D'_{v_i}, there exists a $\mathbf{D}(\mathbb{A})$-distinguished cuspidal representation π^D of $\mathbf{D}'(\mathbb{A})$ whose components at $u, v_1, \dots, v_m$ are the given ones.*

Proof. We use the r-trace formula of B4, with a test function $f^D = \otimes f_v^D$ constructed as follows. At u we take f_u^D to be a function associated to π_u^D as in B13. At v_i we take $f_{v_i}^D$ such that $h_{v_i}^D$ is a coefficient of the given supercuspidals. Let γ_0 be an r-regular elliptic element in D' with $\Xi(\gamma_0, f_v^D) \neq 0$ for $v = u, v_1, \dots, v_m$; it exists since D' is dense in $\prod_{v=u,v_i} D'_v$, and $\Xi(x, f_v^D)$ are locally constant on the r-regular set. We choose $f_v^D (v \neq u, v_i)$ to be almost all $f_v^{0,D}$, and to satisfy $\Xi(\gamma_0, f_v^D) \neq 0$ for all v. Moreover, at some v_0 we require that $f_{v_0}^D$ be supported on the r-regular set. As noted in the proof of B1, $\Xi(\gamma, f^D) \neq 0$ only for finitely many (r-regular elliptic) r-conjugacy classes in D', including that of γ_0.

We can now replace one of the components $f_v^D (v \neq u, v_i)$ by its product with the characteristic function of a small neighborhood (modulo center) of the D_v-double coset of γ_0 in D'_v. The new f^D will have the property that $\Xi(\gamma_0, f^D) \neq 0$, while $\Xi(\gamma, f^D) = 0$ for any $\gamma \in D'$ not in the class of γ_0. For such f^D the sum on the right of B4 reduces to the single term $|\mathbf{T}(\mathbb{A})/T\mathbf{Z}(\mathbb{A})| w(T)^{-1} \Xi(\gamma_0, f^D) \neq 0$, where T is the centralizer of $\gamma_0 \sigma(\gamma_0)^{-1}$.

Consequently the sum on the left of B4 is non zero, and there is a cuspidal $\pi^D \subset L_0(D')$ with $c(\pi^D) \neq 0$ (i.e. π^D is $\mathbf{D}(\mathbb{A})$-distinguished) and $\mathbb{L}_{\pi_v^D}(f_v^D) \neq 0$ for all v. At $v = v_i$, $\mathbb{L}_{\pi_v^D}(f_v^D) = tr\,\pi_{1v}^D(h_v^D)$, where $\pi_v^D = \pi_{1v}^D \otimes \tilde{\pi}_{1v}^D$. This is non-zero only when π_{1v}^D is the supercuspidal whose coefficient is the chosen h_v^D. At $v = u$, B13 implies that $\pi_u^D = \pi'^D_u$, as required.

B15. Proposition. *Suppose that π_v^D and π_v are corresponding representations of D'_v and G'_v. If π_v^D is D_v-distinguished and supercuspidal, then π_v is G_v-distinguished. If π_v is G_v-distinguished and supercuspidal, then π_v^D is distinguished. In this case we have $\Xi_{\pi_v}(\gamma) = c(\pi_v, \pi_v^D)\Xi_{\pi_v^D}(\gamma^D)$ for all pairs of corresponding elements $\gamma \in G'_v$ and $\gamma^D \in D'_v$.*

Proof. This follows from B14 and B11 (only B9, and not B10, is needed to apply B11 here).

When D_v is anisotropic (= multiplicative group of a division algebra), each representation π_v^D of D_v is supercuspidal, and it is clear that the r-character $\Xi(\pi_v^D)$ is locally constant (i.e. B8 is trivially valid). We record this special case of B15 separately as

B16. Corollary. *If D_v is anisotropic and π_v^D is D_v-distinguished, then π_v is G_v-distinguished.*

In the next Proposition we discuss distinguishability with respect to $G = GL(2, F)$, where E/F is a quadratic extension of local fields. Note that by [F8], Proposition 12, the non-supercuspidal infinite dimensional distinguished representations of $GL(2, E)$ are of the form $I(\mu, \overline{\mu}^{-1})$, where $\overline{\mu}(x) = \mu(\overline{x})$, or of the form $I(\mu_1, \mu_2)$, with $\mu_i|NE^\times = 1$, and $\mu_1 \neq \mu_2$, or they are the "special" square-integrable subrepresentation $sp(\mu)$ of $I(\mu\nu^{1/2}, \mu\nu^{-1/2})$, where μ is a character of $E^\times/NE^\times$.

B17. Proposition. *(a) The representation $I_s = I(\mu\nu^s, \overline{\mu}^{-1}\nu^{-s})$ of $GL(2, E)$ is distinguished ($s \in \mathbb{C}$). (b) The representation $I(\mu_1, \mu_2)$, $\mu_1 \neq \mu_2$, is distinguished precisely when $\mu_i|F^\times = 1$. (c) The representation $sp(\mu)$ is distinguished precisely when $\mu|F^\times \neq 1$, but $\mu|NE^\times = 1$.*

Proof. (a) Recall that I_s consists of all smooth functions $\varphi : GL(2, E) \to \mathbb{C}$ satisfying

$$\varphi\left(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} g\right) = \mu(a/\overline{b})|a/b|_E^{1/2+s}\varphi(g) \qquad (a, b \in E^\times; g \in GL(2, E)).$$

We shall construct a $GL(2, F)$ invariant functional L_s on I_s as follows:

$$L_s(\varphi) = \int_{T\backslash G} \varphi\left(\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}\right) dg.$$

We integrate here over the group $G = \{\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}; a, b \in E, a\overline{a} - b\overline{b} \neq 0\}$, which is isomorphic to $GL(2, F)$ (by conjugation in $GL(2, E)$), hence L_s is $GL(2, F)$-invariant, and $T = \{\begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}; a \in E^\times\}$. We shall show that $L_s(\varphi)$ converges for all $s \in \mathbb{C}$ with $\Re(s) > 0$ (not $\Re(s) \geq 0$ as misprinted in [F8], p. 162, ℓ. –7), to a rational function in q^{-s}, where q is the cardinality of the residue field of the ring R of integers in F. For μ such that a singularity occurs for some s we define the $GL(2,F)$-invariant form to be the value at such s of the product of L_s with a suitable linear function in q^{-2s} (or q^{-s}).

In determining the convergence of the integral and the form of the singularity, a certain infinite sum dominates the answer. It is clear that the case of a general μ differs only notationally from the case of $\mu = 1$, so we deal with the case of $\mu = 1$ alone. For simplicity we consider only the case where E/F is unramified. Then q_E, the residual cardinality of E, is q^2. Further it suffices to consider only the unit vector φ_0 in I_s, whose value on the standard maximal compact $K_E = GL(2, R_E)$ of $GL(2,E)$ is 1; the computation of $L_s(\varphi)$ for other φ is similar.

To compute our integral note the measure relation

$$d\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} = \frac{da\, db}{|a\overline{a} - b\overline{b}|_F^2}.$$

Then

$$\int_{T\backslash G} \varphi_0\left(\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}\right) dg = 2q^{-2} + \int_{|b|=1} \varphi_0\left(\begin{pmatrix} 1 & b \\ \overline{b} & 1 \end{pmatrix}\right) |1 - b\overline{b}|_F^{-2} db,$$

since $\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \in K_E$ for $|a| = 1$, $|b| < 1$, or $|a| < 1$, $|b| = 1$, and $\int_{|b|<1} db = q_E^{-1} = q^{-2}$. This equals

$$= 2q^{-2} + \int_{|b|=1} |1 - b\overline{b}|_F^{2s-1} db,$$

since

$$\begin{pmatrix} 1 & b \\ \overline{b} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/\overline{b} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (1-b\overline{b})/\overline{b} & 0 \\ 0 & \overline{b} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1/\overline{b} \\ 0 & 1 \end{pmatrix},$$

and

$$\varphi_0\left(\begin{pmatrix} (1-b\overline{b})/\overline{b} & 0 \\ 0 & \overline{b} \end{pmatrix}\right) = |1 - b\overline{b}|_E^{s+\frac{1}{2}} = |1 - b\overline{b}|^{2s+1}, \quad \text{if } |b| = 1.$$

Lemma. *We have* $\int_{|1-b\overline{b}| \leq q^{-m}} db = q^{-m}(1+q^{-1})$ *for* $m \geq 1$.

Proof of lemma. Write $b = \varepsilon(1 + \pi^m b_1)$ with $b_1 \in R_E$ and $\varepsilon \in R_E^\times/(1 + \pi^m R_E)$, $\varepsilon\overline{\varepsilon} = 1$; here π is a generator of the maximal ideal in the local ring R (and R_E). Then $db = q_E^{-m} db_1$, and our integral is equal to

$$\begin{aligned} &q^{-2m} \cdot \#\{\varepsilon \in R_E^\times/(1+\pi^m R_E); \varepsilon\overline{\varepsilon} = 1\} \\ &= q^{-2m} \cdot \#\{R_E^\times/(1+\pi^m R_E)\}/\#\{R_F^\times/(1+\pi^m R_F)\}. \end{aligned}$$

The last equality follows from Hilbert Theorem 90, asserting that $\varepsilon \in E^\times$ with $\varepsilon\overline{\varepsilon} = 1$ is of the form $\varepsilon = z/\overline{z}$, where $z \in E^\times$ is uniquely determined modulo $F^\times$. This is

$$= q^{-2m}\frac{(1-q_E^{-1})/q_E^{-m}}{(1-q^{-1})/q^{-m}} = q^{-m}(1+q^{-1}).$$

as asserted.

Returning to the proof of the proposition we conclude that

$$L_s(\varphi_0) = 2q^{-s} + \sum_{m=0}^{\infty} q^{-m(2s-1)} \int_{\substack{|b|=1 \\ |1-b\overline{b}|_F = q^{-n}}} db$$

$$= 2q^{-s} + [(1-q^{-2}) - q^{-1}(1+q^{-1})] + \sum_{m=1}^{\infty}(1+q^{-1})(q^{-m}-q^{-m-1})q^{-m(2s-1)}$$

$$= 2q^{-s} + 1 - q^{-1} - 2q^{-2} + (1-q^{-2})q^{-2s}(1-q^{-2s})^{-1} = (1-q^{-1})\frac{1+q^{-2s-1}}{1-q^{-2s}}.$$

Note also that the volume of $T\backslash TK$, where K consists of the $\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$ with $|a| \le 1$, $|b| \le 1$, $|a\overline{a} - b\overline{b}| = 1$, is clearly $2q^{-2} + 1 - q^{-2} - q^{-1} - q^{-2} = 1 - q^{-1}$. Normalizing the measure dg on $T\backslash G$ to assign the volume 1 to $T\backslash TK$, we conclude that

$$L_s(\varphi_0) = \frac{L(2s)}{L(\chi, 2s+1)}.$$

Here χ is the quadratic character of $F^\times/NE^\times$, and $L(s) = (1-q^{-s})^{-1}$, $L(\chi, s) = (1-\chi(\pi)q^{-s})^{-1} = (1+q^{-s})^{-1}$. In conclusion the G-invariant form $L(2s)^{-1}L_s$ is non-zero for all $s \in \mathbb{C}$, and has no poles there. It is defined by a convergent integral on $\Re(s) > 0$, and by analytic continuation for the complementary half s-plane. This completes the proof of the proposition when μ factorizes through $\nu(b) = |b|$ and E/F is unramified. The ramified μ and E/F are similarly handled.

(b) Recall that the representation $I(\mu_1\nu^s, \mu_2\nu^{-s})$ of $H' = GL(2, E)$ consists of all smooth functions $\varphi : H' \to \mathbb{C}$ satisfying $\varphi(ph) = \mu_1(a)\mu_2(b)\delta_{B'}(p)^{1/2+s}\varphi(h)$ ($p = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in B', h \in H'$). We assume as we may that s is real and μ_i are unitary. By [F8], p. 156, ℓ. 2, we have that H' is the disjoint union of $B'H$ and $B'\eta_1 H$ (where $H = GL(2, F)$, and η_1 is η^{-1} of [F8]). Hence any H-invariant linear form on any subspace of I_s must be a linear combination of the forms ℓ_0 and ℓ_1. Here ℓ_0 factorizes through the average of φ on H, namely through the integral of $\varphi(g)dg$ on $B\backslash H$. Since $\varphi(ph) = \mu_1(a)\mu_2(b)\delta_B(p)^{1+2s}\varphi(h)$ ($p \in B, h \in H$), and $dg = \delta_B(p)^{-1}dpdk$, the form $\varphi(g)dg = \mu_1(a)\mu_2(b)\delta_B(p)^{2s}\varphi(h)dpdk$ is left B-invariant only when $s = 0$ and $\mu_i|F^\times = 1$. The form ℓ_1 factorizes through the average of φ on $G = \eta_1 H \eta_1^{-1}$ of (a) above, and since G intersects B' in $T = \{\mathrm{diag}(a, \overline{a}), a \in F^\times\}$, and $\varphi(\mathrm{diag}(a, \overline{a})h) = \mu_1(a)\mu_2(\overline{a})\varphi(h)$, ℓ_1 is 0 unless $\mu_1\overline{\mu}_2 = 1$, namely $\mu_1 = \mu_2$. This completes the proof of (b), and we proceed to prove (c), assuming now that the μ_i are equal, say to μ. So again, ℓ_1 is 0 unless $\mu|NE^\times = 1$.

(c) We conclude from the previous paragraph that when $s > 0$, and $\mu|NE^\times = 1$, the only H-invariant form on I_s, and on any subspace thereof, is the form ℓ_1, which is the same as L_s of the proof of (a). Now the H'-module I_s ($s > 0$) is irreducible except when $s = 1/2$, when its composition series has length two, with quotient $g \mapsto \mu(g)$, and a sub defined by $\int_{B'\backslash H'} \mu(h')^{-1}\varphi(h')dh' = 0$. Since the coset $B\backslash H$ has measure zero (with respect to dh') in $B'\backslash H'$, this last integral is equal to $(\mu(\eta_1)^{-1}$ times) $\int_{T\backslash G} \mu(g)^{-1}\varphi(g\eta_1)dg$. This integral is a multiple of $L_s(\varphi)$ when $\mu|F^\times = 1$. Hence there is no non-zero H-invariant form on $sp(\mu)$ when $\mu|F^\times = 1$ (there is such a non-zero form when $\mu|NE^\times = 1$, $\mu|F^\times \neq 1$, by [F8], Proposition 8). This completes the proof of Proposition B17.

Remark. The first author uses this opportunity to note that the proof of the second half of Theorem 7 in [F8] is too complicated (and incomplete). He adjusts it as follows. On p. 162, ℓ. -3, of [F8], after: "we shall prove that," insert: "(1) the ("special") square-integrable subrepresentation $sp(\mu)$ of $I(\mu\nu^{1/2}, \mu\nu^{-1/2})$ ($\mu|NE^\times = 1$) is not distinguished unless $\mu|F^\times \neq 1$, and $I(\mu_1, \mu_2)$ with $\mu_i : E^\times \to \mathbb{C}^\times$, $\mu_i|NE^\times = 1$, is not distinguished unless $\mu_i|F^\times = 1$, (2)." On ℓ. -2 there, replace "not of the form $I(\mu, \overline{\mu}^{-1})$" by "which is supercuspidal." This (1) is proven in (b) and (c) of Proposition B17 above. The proof of (2) does not require Bernstein's Decomposition Theorem, and so the second half of p. 165 in [F8], as well as the top half of p. 166, including Proposition 13, and the misleading Remark on p. 166, are no longer needed. Simply take ${\pi'_u}^0$ of [F8], Proposition 14, p. 166, to be supercuspidal (in addition to its other properties), and replace: "has infinitesimal ... defined by," in [F8], p. 166, ℓ. 18/19, by "is." The function f'_u of p. 166, ℓ. -10, will be taken to be just a matrix coefficient of ${\pi'_u}^0$, and the f'_{u_i} on ℓ. -6 will similarly be taken to be coefficients of the ${\pi'_{u_i}}^0$ there. In other words, p. 166, ℓ. -10, -9, should be replaced by: "Proof. Let $f'_u \in \mathbb{H}'_u$ be a matrix coefficient of ${\pi'}^0_u$. Since ${\pi'}^0_u$ is generic, distinguished and supercuspidal, f'_u can and is chosen to satisfy $DW_{{\pi'}^0_u,\psi'_u}(f'_u) \neq 0$. This distribution depends on f'_u only through $\Phi(\gamma, f'_u)$. Hence $\Phi(\gamma, f'_u)$ is not identically zero. Let $u_1, \ldots, u_m$." Consequently ℓ. 16 to 18 of [F8], p. 168, should be replaced simply by: "Proof of Theorem 7. By Proposition 14, every supercuspidal distinguished G'_v-module π'_v with central character $\omega'_v\kappa'_v$ is a component of a cuspidal $\mathbb{G}$-distinguished." Note also that throughout [F8], the character κ'^2 should be replaced by $\kappa' = \kappa^2$, for example on p. 144, ℓ. -16, -12, -11; p. 146, ℓ. 10; p. 154, ℓ. -4; p. 155, ℓ. 7; p. 158, ℓ. -5; p. 161, ℓ. 21; p. 167, ℓ. -10.

Proposition 14 of [F8] asserts now that: *each distinguished infinite dimensional supercuspidal representation π_v of $GL(2, E_v)$ can be viewed as a component of a cuspidal distinguished representation π of $GL(2, \mathbb{A}_E)$*, in fact with supercuspidal distinguished components at any prescribed finite set of places. If π_v has trivial central character, π can be chosen to have trivial central character.

It follows from the final Remark (2) in [F8], and from Proposition B17(c), that a Steinberg (=special) representation of $G'_v(= D'_v)$ is G_v-distinguished if and only if it is D_v-distinguished, when $G = GL(2)$. B15 shows this for supercuspidals. It follows from the final Remark (2) in [F8] that an induced representation $I(\mu_1, \mu_2)$ of $G'_v(= D'_v)$ is D_v-distinguished if and only if $\overline{\mu}_2\mu_1 = 1$. Proposition B17 (a) and (b) provides a purely local, and direct, proof of the assertion that $I(\mu_1,\mu_2)$ is G_v-distinguished if and only if either $\mu_1\overline{\mu}_2=1$, or $\mu_1\neq\mu_2$ and both μ_i are trivial on $F_v^\times$. The G_v-distinguished representation $I(\mu_1,\mu_2)$ of G'_v, $\mu_1\neq\mu_2$, $\mu_i|F_v^\times=1$, is the unstable base change lift ([F1]) of a supercuspidal representation of the quasi-split unitary group $U(2,E_v/F_v)$, hence – by [F8] – it is a component of a cuspidal $GL(2,\mathbb{A})$-distinguished representation π of $GL(2,\mathbb{A}_E)$. We can construct π and choose D such that π satisfies the assumptions of B9, provided we assume that the r-character Ξ of $I(\mu_1,\mu_2)$ is not identically zero on the r-elliptic regular set in G'_v. Since $I(\mu_1,\mu_2)$ is G_v-distinguished but not D_v-distinguished, we obtain a contradiction from B11. We proved then the following:

B18. Corollary. *The r-character Ξ of the representation $I(\mu_1,\mu_2)$ of G'_v, $\mu_i: E_v^\times/F_v^\times\to\mathbb{C}^\times, \mu_1\neq\mu_2$, vanishes on the r-elliptic regular set in G'_v ($\Xi(x)\neq 0$ for an r-regular $x\in G'_v$ implies that $x\sigma(x)^{-1}$ is diagonalizable in G'_v).*

A purely local proof of $B18$ in a more general setting, is next.

Let E/F be a quadratic extension of local fields, $\mathbf{G}$ a reductive F-group and $\mathbf{P}$ a parabolic F-subgroup, $G=\mathbf{G}(F)$, $G'=\mathbf{G}(E)$, $P=\mathbf{P}(F)$, and $P'=\mathbf{P}(E)$.

B19. Proposition. *Let $(\pi,V)=I(\rho,V_\rho;G',P')$ be the G'-module normalizedly induced from the admissible irreducible M-distinguished representation (ρ,V_ρ) of a Levi factor M' of P'. Then (π,V) is G-distinguished and its r-character Ξ_π is supported on the subset $GP'G$ of G'; in particular Ξ_π vanishes on the r-elliptic regular set.*

Proof. Recall that V consists of the V_ρ-valued smooth functions ϕ on G' which satisfy $\phi(pg)=\delta_{P'}^{1/2}(p)\rho(p)(\phi(g))$ $(p\in P', g\in G')$. Note that $G'=P'K'$, where K' is the standard maximal compact subgroup of G'. We denote by $\check\pi$ the dual of π, and by $\tilde\pi$ the contragredient of π.

If $\check\ell\in\check\rho$ is a non-zero M-invariant form on (ρ,V_ρ), then $<\check\ell,\phi(pg)>=\delta_{P'}^{1/2}(p)$ $<\check\ell,\phi(g)>$ for $p\in P, g\in G$. Since $\delta_P^2=\delta_{P'}$ and we have the measure decomposition $f(g)dg=f(pk)\delta_P^{-1}(p)dpdk$, the measure $<\check\ell,\phi(g)>dg$ depends only on the projection to the coset space $P\backslash G=K$. We define a non-zero G-invariant form $\check L\in\check\pi$ on (π,V) by $<\check L,\phi>=\int_K<\check\ell,\phi(k)>dk, \qquad \phi\in\pi.$

Similarly, if $\ell\in\check{\tilde\rho}$ is a non-zero M-invariant form on $(\tilde\rho,V_{\tilde\rho})$, then a non-zero G-invariant form $L\in\check{\tilde\pi}$ on $(\tilde\pi,\tilde V)$ is defined by $<L,\tilde\phi>=\int_K<\ell,\tilde\phi(k)>dk$, for $\tilde\phi\in\tilde\pi$.

For any compactly supported smooth function f on G', the vector $\pi(f)L$ lies in $V=\tilde{\tilde V}$ (this is a subspace of $\check{\tilde V}$). The G-invariant distribution attached to π is defined by

$$\mathbb{L}_\pi(f)=<\check L,\pi(f)L>=<\tilde\pi(f^*)\check L,L>,$$

where $f^*(g)=f(g^{-1})$.

Let us compute the V_ρ-valued function $\pi(f)L\in V$ on G'. For that we pair it with any element $\tilde\phi$ in the contragredient representation $\tilde V$; this is a $V_{\tilde\rho}$-valued function on G'. Thus

$$\int_{K'} < \tilde{\phi}(k'), (\pi(f)L)(k') > dk' =< \tilde{\phi}, \pi(f)L >$$
$$=< \tilde{\pi}(f^*)\tilde{\phi}, L >= \int_K < (\tilde{\pi}(f^*)\tilde{\phi})(k), \ell > dk$$
$$= \int_K \int_{G'} f(g^{-1}) < \tilde{\phi}(kg), \ell > dkdg = \int\int f(g^{-1}k) < \tilde{\phi}(g), \ell >$$
$$= \int_K \int_{P'} \int_{K'} f(k'^{-1}p^{-1}k)\delta_{P'}^{1/2}(p) < (\tilde{\rho}(p)\tilde{\phi})(k'), \ell > \delta_{P'}^{-1}(p)$$
$$= \int_{K'} \int_{P'} \int_K \delta_{P'}^{-1/2}(p) f(k'^{-1}p^{-1}k) < \tilde{\phi}(k'), \rho(p^{-1})\ell >$$
$$= \int\int\int \delta_{P'}^{1/2}(p) f(k'^{-1}pk) < \tilde{\phi}(k'), \rho(p)\ell >,$$

for all $\tilde{\phi} \in \tilde{V}$. Hence in V_ρ, for every $k' \in K'$ we have

$$(\pi(f)L)(k') = \int_K \int_{P'} \delta_{P'}^{1/2}(p) f(k'^{-1}pk)\rho(p)\ell \; dpdk.$$

We conclude that $\mathbb{L}_\pi(f) = \int_{G'} f(g)\Xi_\pi(g)dg$ is given by

$$< \tilde{L}, \pi(f)L >= \int_K < \check{\ell}, (\pi(f)L)(k) > dk = \int_K \int_K \int_{P'} \delta_{P'}^{1/2}(p) f(k'pk) < \check{\ell}, \rho(p)\ell > .$$

Hence the r-character Ξ_π is supported on $GP'G = KP'K$, as required.

C. The case of $G = GL(2)$.

The purpose of this section is to remove the restrictions in B9 and B10 in the case of $G = GL(2)$, and to prove the

C1. Theorem. *Suppose that π is an irreducible, automorphic representation of $\mathbf{G}'(\mathbb{A})$ which corresponds to a cuspidal representation π^D of $\mathbf{D}'(\mathbb{A})$. Denote by V' the set of places of F which stay prime in E where D ramifies. Then π^D is $\mathbf{D}(\mathbb{A})$-distinguished if and only if π is $\mathbf{G}(\mathbb{A})$-distinguished, and at each v in V' the component $\pi_v = \pi_v^D$ ($G'_v = D'_v$ at $v \in V'$) is not of the form $I(\mu_1, \mu_2)$ where μ_i are characters of $E_v^\times$ trivial on $F_v^\times$.*

To prove this we can no longer use the simple form B2, B4, of the r-trace formula, since in general the π to be studied may not have a supercuspidal component. We need to use the general form of the r-trace formula, which includes the contribution from the continuous spectrum. Recall that for $f = \otimes f_v$, $f_v \in H_v$, the convolution operator

$$(r(f)\varphi)(g) = \int_{\mathbf{Z}'(\mathbb{A})\backslash\mathbf{G}'(\mathbb{A})} f(h)\varphi(gh)dh = \int_{\mathbf{Z}'(\mathbb{A})G'\backslash\mathbf{G}'(\mathbb{A})} K_f(g,h)\varphi(h)dh$$

on

$$L_{\omega'}(G'\backslash\mathbf{G}'(\mathbb{A}))=\{\varphi:\mathbf{G}'(\mathbb{A})\to\mathbb{C};\varphi(z\gamma g)=\omega'(z)\varphi(g)\ (z\in\mathbf{G}'(\mathbb{A}),\gamma\in G',z\in\mathbf{Z}'(\mathbb{A})),\ \int_{\mathbf{Z}'(\mathbb{A})G'\backslash\mathbf{G}'(\mathbb{A})}|\varphi(g)|^2dy<\infty\}$$

is an integral operator with kernel

$$K_f(g,h)=\sum_{\gamma\in Z'\backslash G'}f(g^{-1}\gamma h).$$

The theory of Eisenstein series decomposes $L(G')=L_{\omega'}(G'\backslash\mathbf{G}'(\mathbb{A}))$ as the direct sum of three mutually orthogonal invariant subspaces: the space $L_0(G')$ of cusp forms, the space $L_1(G')$ of functions $\varphi(g)=\chi(\det g)$ with $\chi^2=\omega'$, and the continuous spectrum $L_c(G')$. Correspondingly

$$K_f(g,h)=K_{f,0}(g,h)+K_{f,1}(g,h)+K_{f,c}(g,h), \tag{1}$$

where

$$K_{f,1}(g,h)=\frac{1}{2}\sum_{\chi^2=\omega'}\chi(\det g)\overline{\chi}(\det h)\int_{\mathbf{Z}'(\mathbb{A})\backslash\mathbf{G}'(\mathbb{A})}f(x)\chi(\det x)dx,$$

and

$$K_{f,c}(g,y)=\frac{1}{4\pi}\sum_{\mu}\sum_{\phi}\int_{-\infty}^{\infty}E(g,I(\mu,it;f)\phi,\mu,it)\overline{E}(y,\phi,\mu,it)dt.$$

The first sum in $K_{f,c}$ ranges over the characters $\mu=(\mu_1,\mu_2)$ of the diagonal subgroup $\mathbf{A}'(\mathbb{A})$ in $\mathbf{G}'(\mathbb{A})$ which satisfy $\mu_1\mu_2=\omega'$, up to the equivalence relation $\mu\sim\mu'$ if $(\mu_1,\mu_2)=(\mu_1\nu^s,\mu_2\nu^{-s})$, $s\in\mathbb{C}$ and $\nu(x)=|x|_E$.

For each μ consider the Hilbert space $H(\mu,s)$ of functions $\phi:\mathbf{G}'(\mathbb{A})\to\mathbb{C}$ which satisfy

$$\phi\left(\begin{pmatrix}a & *\\ 0 & b\end{pmatrix}g\right)=|a/b|_E^{s+1/2}\mu_1(a)\mu_2(b)\phi(g)\quad(g\in\mathbf{G}'(\mathbb{A});\ a,b\in\mathbb{A}_E^\times)$$

and $\int_{\mathbb{K}'}|\phi(k)|^2dk<\infty$. We identify the vector space $H(\mu,s)$ with $H(\mu)=H(\mu,0)$ via the restriction-to-$\mathbb{K}'$ isomorphism, $\phi\mapsto\phi|\mathbb{K}'$. Denote by $\phi(\mu,s)$ the element of $H(\mu,s)$ corresponding to $\phi(\mu)$ in $H(\mu)$. Let $I(\mu,s)$ be the right $\mathbf{G}'(\mathbb{A})$-module structure on $H(\mu,s)$, and introduce the Eisenstein series

$$E(g,\phi,\mu,s)=\sum_{\gamma\in B'\backslash G'}\phi(\gamma g,\mu,s)\quad(\phi=\phi(\mu)\in H(\mu)).$$

This E converges absolutely on $Re(s)>\frac{1}{2}$, and has analytic continuation to $\mathbb{C}$ as a meromorphic function which is holomorphic on $Re(s)=0$. The inner sum in $K_{f,c}$ ranges over an orthonormal basis ϕ of $H(\mu)$.

To obtain an r-trace formula we need to integrate (**1**) over $g, h \in \mathbf{Z}(\mathbb{A})G\backslash\mathbf{G}(\mathbb{A})$. Since K_f and $K_{f,c}$ may not be integrable, we truncate the K_* using the truncation operator T^λ over F. For higher rank G it might be necessary to truncate over E. But in our rank one case the difference between these two truncations goes to zero as λ goes to infinity. We prefer to use the F-truncation here since in our rank one case it leads to a simpler exposition. Given a continuous function Φ on $\mathbf{Z}'(\mathbb{A})G'\backslash\mathbf{G}'(\mathbb{A})$ and $\lambda > 1$, denote by χ_λ the characteristic function of (λ, ∞) in $\mathbb{R}$, and put

$$(\mathbf{2}) \qquad T^\lambda\Phi(g) = \Phi(g) - \sum_{\gamma\in B\backslash G} \Phi_N(\gamma g)\chi_\lambda(H(\gamma g)),$$

where

$$\Phi_N(g) = \int_{N'\backslash\mathbf{N}'(\mathbb{A})} \Phi(ng)dn, \qquad H\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} k\right) = |a/b|_E.$$

Here $\mathbf{N}'$ denotes the upper triangular subgroup of $\mathbf{G}'$. A standard lemma asserts that if $\gamma \in G'$ and $g \in \mathbf{G}'(\mathbb{A})$ satisfy $H(g) > 1$ and $H(\gamma g) > 1$ then $\gamma \in B'$. Hence the sum in (**2**) has at most one term, and if $H(g) > \lambda > 1$ then $\Lambda^T\Phi(g) = \Phi(g) - \Phi_N(g)$. Clearly $\Lambda^T\Phi = \Phi$ if Φ is a cuspidal function. Denoting by $\Lambda_i^{T_i}$ the truncation operator with respect to the ith variable, and noting that the kernel function $K_{f,0}(g,h)$ on the cuspidal spectrum is a cuspidal function in each of its two variables, we conclude (1) in the following

C2. Lemma. *(1) We have* $\Lambda_1^{T_1}\Lambda_2^{T_2}K_{f,0} = K_{f,0}$. *(2) We have*

$$\mathit{lim}_{\lambda_2\to\infty} \lim_{\lambda_1\to\infty} \int_{\mathbf{Z}(\mathbb{A})G\backslash\mathbf{G}(\mathbb{A})} \int_{\mathbf{Z}(\mathbb{A})G\backslash\mathbf{G}(\mathbb{A})} \Lambda_1^{T_1}\Lambda_2^{T_2}K_{f,1}(g,h)dgdh = \int\int K_{f,1}(g,h)dg\,dh.$$

Proof. Recalling the definition of $K_1 = K_{f,1}$ (and Λ^T), we have

$$\begin{aligned}\Lambda_1{}^{T_1}\Lambda_2{}^{T_2}K_1(g,h) = K_1(g,h) &- \sum_\gamma K_1(\gamma g,h)\chi_{\lambda_1}(H(\gamma g)) - \sum_\gamma K_1(g,\gamma h)\chi_{\lambda_2}(H(\gamma g)) \\ &+ \sum_{\gamma,\gamma'} K_1(\gamma g,\gamma' h)\chi_{\lambda_1}(H(\gamma g))\chi_{\lambda_2}(H(\gamma' h));\end{aligned}$$

the Σ's range over $\gamma, \gamma' \in B\backslash G$. Integrate each of the four terms on the right over $\mathbf{Z}(\mathbb{A})G\backslash\mathbf{G}(\mathbb{A})$, and denote the result by $(a)-(b)-(c)+(d)$. For a fixed λ_2, we claim that $(b) \to 0$ as $\lambda_1 \to \infty$. Indeed, (b) is a finite linear combination of integrals of the form

$$\int_{\mathbf{Z}(\mathbb{A})G\backslash\mathbf{G}(\mathbb{A})} \overline{\chi}(\det h)dh \cdot \int_{\mathbf{Z}(\mathbb{A})B\backslash\mathbf{G}(\mathbb{A})} \chi(\det g)\chi_{\lambda_1}(H(g))dg.$$

This is zero unless $\chi = 1$ on $\mathbb{A}^\times$. Otherwise, by the Iwasawa decomposition $\mathbf{G}(\mathbb{A}) = \mathbf{A}(\mathbb{A})\mathbf{N}(\mathbb{A})\mathbb{K}$ we find that this is a scalar multiple of $\int_{\lambda_1}^\infty t^{-2}dt = \lambda_1^{-1}$. The same argument shows that $(d) \to 0$ as $\lambda_1 \to \infty$, and that $(c) \to 0$ as $\lambda_2 \to \infty$; the lemma follows.

C3. Lemma. *If f is r-discrete, then there is $d > 0$ such that for $\lambda_1, \lambda_2 > d$ we have $\Lambda_1^{T_1}\Lambda_2^{T_2}K_f(g,h) = K_f(g,h)$ on $g, h \in \mathbf{G}(\mathbb{A})$.*

Proof. Recall the following well known ([JL], p. 259) facts.

(a) Given a compact-modulo-$\mathbf{Z}'(\mathbb{A})$ subset Ω in $\mathbf{G}'(\mathbb{A})$, there is $d > 0$ such that any $\gamma \in G'$ with $g^{-1}\gamma h \in \Omega$ for some $g, h \in \mathbf{G}(\mathbb{A})$, $H(g) > d$, $H(h) > d$, satisfies $\gamma \in B'$.

(b) Given Ω as in (a), there exists $d > 0$ such that any $\gamma \in G'$ with $g^{-1}\gamma h \in \Omega$ for some $g, h \in \mathbf{G}(\mathbb{A})$ with $H(h) > d$, satisfies $\gamma \in GB'$.

By definition $\Lambda_1^{T_1}\Lambda_2^{T_2}K(g,h)$ is equal to

$$(\mathbf{3}) \qquad K(g,h) - \sum_{\gamma \in B\backslash G} \int_{N'\backslash \mathbf{N}'(\mathbb{A})} \sum_{\delta \in Z'\backslash G'} f(g^{-1}\delta n\gamma h)dn \cdot \chi_{\lambda_2}(H(\gamma h))$$

$$(\mathbf{4}) \qquad - \sum_{\gamma \in B\backslash G} \int_{N'\backslash \mathbf{N}'(\mathbb{A})} \sum_{\delta \in Z'\backslash G'} f(g^{-1}\gamma^{-1}n\delta h)dn \cdot \chi_{\lambda_1}(H(\gamma g))$$

$$(\mathbf{5}) \qquad + \sum_{\gamma,\gamma'} \int\int \sum_{\delta} f(g^{-1}\gamma^{-1}n\delta n'\gamma' h)dn\, dn' \cdot \chi_{\lambda_1}(H(\gamma g))\chi_{\lambda_2}(H(\gamma' h)).$$

By (a) (and (b)) we may choose a sufficiently large $d > 0$ such that for $\lambda_i > d$ the δ in (**5**) is in B'. Then the integration in (**5**) over n' gives 1. Moreover, in (**4**) the δ is in $B'G$, by (b), and in (**3**) the δ is in GB', again by (b). Since f is r-discrete, it vanishes on all element in $\mathbf{G}'(\mathbb{A})$ of the form $g\delta h$ with $g, h \in \mathbf{G}(\mathbb{A})$ and $\delta \in B'\mathbf{N}'(\mathbb{A})$. The lemma follows.

Remark. Recall that $f = \otimes f_v$ will be r-discrete when it has a component f_v which is r-discrete, namely supported on the r-regular r-elliptic set in G'_v.

It remains to examine the effect of the double truncation on the Eisenstein kernel. The intertwining operator $M(\mu, s) : H(\mu) \to H(\tilde{\mu})$, where $\tilde{\mu} = (\mu_2, \mu_1)$, is defined on $Re(s) > \frac{1}{2}$ by

$$(M(\mu,s)\Phi)(g,\tilde{\mu},-s) = \int_{\mathbf{N}'(\mathbb{A})} \Phi(wng,\mu,s)dn \qquad \left(w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right),$$

and by analytic continuation on the entire complex plane. Recall ([F8]) that

$$G' = GB' \cup G\eta B' = B'G \cup B'\eta^{-1}G \qquad \left(\eta = \begin{pmatrix} -\sqrt{\theta} & \sqrt{\theta} \\ 1 & 1 \end{pmatrix},\ E = F(\sqrt{\theta}),\ \theta \in F\right),$$

and put

$$T = G \cap \eta B'\eta^{-1} = \left\{\eta \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \eta^{-1}; a \in E^\times\right\} = G \cap \left\{\begin{pmatrix} \alpha & \beta\theta \\ \beta & \alpha \end{pmatrix}; \alpha, \beta \in F\right\}.$$

If $\mu_1\overline{\mu}_2 = 1$ define on $Re(s) > \frac{1}{2}$

$$J(\mu,s)\Phi = \int_{\mathbf{T}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})} \Phi(\eta^{-1}g,\mu,s)dg;$$

this extends to a meromorphic function on $\mathbb{C}$ by analytic continuation. Put $\mathbb{A}^u = \{a \in \mathbb{A}^\times; \|a\| = 1\}$. Write $\delta(\mu) = 1$ if $\mu_i|\mathbb{A}^\times = 1$, $\delta(\mu) = 0$ otherwise, and $\epsilon(\mu) = 1$ if $\mu_1\overline{\mu}_2 = 1$, $\epsilon(\mu) = 0$ otherwise. As usual, $\chi_{E/F}$ is the unique non-trivial character of $\mathbb{A}^\times$ which is trivial on $E^\times N\mathbb{A}_E^\times$.

C4. Lemma. *(1) The integral of $2\lambda \cdot \Lambda^T E(g,\Phi,\mu,s)$ over $\mathbf{Z}(\mathbb{A})G\backslash\mathbf{G}(\mathbb{A})$ is equal to*

$$|\mathbb{A}^u/F^\times|\delta(\mu)[T^s\int_{\mathbb{K}}\Phi(k)dk - T^{-s}\int_{\mathbb{K}}(M(\mu,s)\Phi)(k)dk] + 2\lambda\cdot\epsilon(\mu)|\mathbb{A}_E^\times/E^\times\mathbb{A}^\times|\cdot J(\mu,s)\Phi.$$

(2) If Φ is $\mathbb{K}$-finite then for some sufficiently large finite set V we have that

$$J_1(\mu,s)\Phi = J(\mu,s)\Phi\cdot L^V(1+2s,\chi_{E/F}\cdot\mu_1|\mathbb{A}^\times)/L^V(2s,\mu_1|\mathbb{A}^\times)$$

is an elementary (i.e. a linear combination of products of rational and exponential) function of s, which is holomorphic on $Re(s) = 0$. Here L^V is the partial (product outside V) Hecke L-function attached to a character of $\mathbb{A}^\times/F^\times$.

(3) The function $\int \Lambda^T E(g,\Phi,\mu,s)dg$ is holomorphic and of polynomial growth on $i\mathbb{R}$.

Proof. This can be extracted from [JL], §8, when $\mu_1\mu_2 = 1$. The general case follows from this on modifying the proof as explained in [F8], Lemma following Proposition 4. Let us recall a proof of (2) patterned on [JL]. Any $\Phi(g,s)$ in $H(\mu,s)$ can be written as

$$\Phi(g,s) = Q(s)L^V(1+2s,\mu_1/\mu_2)^{-1}\mu_1(\det g)|g|_E^{s+\frac{1}{2}}\int_{\mathbb{A}_E^\times}\Psi((0,t)g)(\mu_1/\mu_2)(t)|t|_E^{2s+1}d^\times t,$$

where $Q(s)$ is an elementary function in s and Ψ is a Schwartz function on $\mathbb{A}_E\times\mathbb{A}_E$. Indeed, if P_v' denotes the group of matrices in B_v' with bottom row $(0,1)$, then the map $\Psi_v \mapsto \int_{E_v^\times}\Psi_v((0,t)g)\chi^{-1}(t)d^\times t$ from the space of smooth compactly supported functions on $F_v^2 - \{(0,0)\} \simeq P_v'\backslash G_v'$, to the space of functions Φ on G_v' with $\Phi(hg) = \chi(b)\Phi(g)$, where $h = \begin{pmatrix} a & c \\ 0 & b\end{pmatrix}$, is *surjective* by Bourbaki, Integration, VII, §2, n^o 5 (the point being that integration yields a surjection $C_c^\infty(G) \to C_c^\infty(H\backslash G)$). Moreover, it is easy to see that for almost all places the local factor in the displayed integral above coincides with the L-factor in the denominator. Combining integrations we obtain

$$\int_{\mathbf{T}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})}\Phi(\eta^{-1}g,s)dg = Q(s)L^V(1+2s,\mu_1\circ N)^{-1}\int_{\mathbf{G}(\mathbb{A})}\Psi((1,\sqrt{\theta})g)\mu_1(g)|g|_F^{2s+1}dg,$$

since if $\mu_1\overline{\mu}_2 = 1$ then $\mu_1/\mu_2 = \mu_1\overline{\mu}_1 = \mu_1 \circ N$. By the Iwasawa decomposition $g = ank$:

$$= Q(s)L^V(1+2s, \mu_1 \circ N)^{-1} \int_{\mathbb{A}^\times}\int_{\mathbb{A}^\times} \mu_1(ab)|a|^{2s}|b|^{2s+1}d^\times a d^\times b \int_{\mathbb{A}}\int_{\mathbb{K}} \Psi((a, x+b\sqrt{\theta})k)dx\,dk.$$

After integrating over x and k the resulting function of a and b is a Schwartz function. The local factors of the remaining integrals over a and b are easy to evaluate. Since $L^V(s, \mu_1 \circ N) = L^V(s, \mu_1|\mathbb{A}^\times)L^V(s, \mu_1|\mathbb{A}^\times \cdot \chi_{E/F})$, we obtain that as a function of s our integral is

$$= Q_1(s)L^V(2s, \mu_1|\mathbb{A}^\times)L^V(2s+1, \mu_1|\mathbb{A}^\times)/L^V(2s+1, \mu_1|\mathbb{A}^\times) \cdot L^V(2s+1, \mu_1|\mathbb{A}^\times \cdot \chi_{E/F})$$
$$= \mathbb{Q}_1(s)L^V(2s, \mu_1|\mathbb{A}^\times)/L^V(2s+1, \chi_{E/F} \cdot \mu_1|\mathbb{A}^\times),$$

where $Q_1(s)$ is an elementary function in s, as required.

Denote by $\{\Phi\}$ an orthonormal basis of the space $H(\mu)$.

C5. Lemma. *The integral of $\Lambda_1^{T_1}\Lambda_2^{T_2}K_{f,c}(g,h)$ over $g, h \in \mathbf{Z}(\mathbb{A})G\backslash\mathbf{G}(\mathbb{A})$ has a limit as $\lambda_1 \to \infty$. The resulting function of λ_2 is the sum of a scalar multiple of log λ_2, a term $o(1)$ as $\lambda_2 \to \infty$, and the sum of*

$$(a)\quad c_1 \sum_{\mu_1\overline{\mu}_2=1}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}\frac{d}{dt}[(I(\mu, it; f)\Phi_\beta, \Phi_\alpha) \cdot tJ(\mu, it)\Phi_\alpha \cdot \overline{tJ(\mu, it)\Phi_\beta}]\frac{dt}{t},$$

$$(b)\quad c_2 \sum_{\substack{\mu_i|_{\mathbb{A}^\times}=1\\ \mu_1\neq\mu_2}}\sum_{\alpha,\beta}(I(\mu, 0; f)\Phi_\beta, \Phi_\alpha) \cdot \int_{\mathbb{K}}\Phi_\alpha(k)dk \cdot \frac{d}{dt}|_{t=0}[\int_{\mathbb{K}}(\overline{M(\mu, it)\Phi_\beta})(k)dk],$$

and

$$(c)\quad c_3 \sum_{\substack{\mu_i|_{\mathbb{A}^\times}=1\\ \mu_1=\mu_2}}\sum_{\alpha,\beta}(I(\mu, 0; f)\Phi_\beta, \Phi_\alpha) \cdot \int_{\mathbb{K}}\Phi_\alpha(k)dk \cdot \frac{d}{dt}|_{t=0}[\int_{\mathbb{K}}(\overline{M(\mu, it)\Phi_\beta})(k)dk],$$

for some volume constants c_1, c_2, c_3.

Proof. This is [JL], (9.4), when $\mu_1\mu_2 = \omega'$ is 1; the general case follows on making the modifications alluded to in the proof of C4.

Note that all sums in C5 are finite, depending only on the ramification of f, the function $(I(\mu, it; f)\Phi_\beta, \Phi_\alpha)$ is a Schwartz (rapidly decreasing) function in t on $\mathbb{R}$, and $tJ(\mu, it)\Phi$ is holomorphic in $t \in \mathbb{R}$ and of polynomial growth.

C6. Lemma. *Let F be a Schwartz function on $\mathbb{R}$ with $F(0) = 0$. Then*

$$\lim_{\epsilon\to 0}(\int_\epsilon^\infty + \int_{-\infty}^{-\epsilon})F(x)x^{-2}dx = \lim_{\epsilon\to 0}(\int_\epsilon^\infty + \int_{-\infty}^{-\epsilon})F'(x)x^{-1}dx.$$

Proof. Elementary.

Note that the integral $\int_{-\infty}^{\infty}$ in C5(a) is also an improper integral $\lim_{\epsilon\to 0}(\int_{\epsilon}^{\infty}+\int_{-\infty}^{-\epsilon})$.

Let us summarize what we now have on the r-trace formula for f on $\mathbf{G}'(\mathbb{A})$. Recall that the complex number $c(\pi)$ is defined by Lemma B3. It depends on the choice of the distribution $\mathbb{L}_\pi$.

C7. Proposition. *Given $f=\otimes f_v$, such that $f_v\in H_v$ for all v and f_u is r-discrete at some place u, we have*

$$(i)\qquad \sum_{\pi\subset L_0(G')} c(\pi)\mathbb{L}_\pi(f)+\frac{1}{2}|\mathbf{Z}(\mathbb{A})G\backslash \mathbf{G}(\mathbb{A})|^2\sum_{\substack{\chi^2=\omega'\\ \chi|_{\mathbb{A}_F^\times}=1}} tr\,\pi(\chi;f)$$

$$(ii)\qquad +\sum_{\mu_i|_{\mathbb{A}_F^\times}=1}(c_2\delta(\mu_1\neq\mu_2)+c_3\delta(\mu_1=\mu_2))\sum_{\Phi}\int_{\mathbb{K}}(I(\mu,0;f)\Phi)(k)dk$$

$$\cdot\frac{d}{dt}|_{t=0}\Big(\int_{\mathbb{K}}(M(\mu,it)\Phi)(k)dk\Big)$$

$$(iii)\qquad +c_1\sum_{\mu_1\overline{\mu}_2=1}\int_{-\infty}^{\infty}\sum_{\alpha,\beta}[(I(\mu,it,f)\Phi_\beta,\Phi_\alpha)\cdot tJ(\mu,it)\Phi_\alpha\cdot t\overline{J(\mu,it)\Phi_\beta}]t^{-2}dt$$

$$(iv)\qquad =\sum_{\{T\}_e}|\mathbf{T}(\mathbb{A})/\mathbf{Z}(\mathbb{A})T|w(T)^{-1}\sum_{\gamma\in T'/TZ'}\Xi(\gamma,f),$$

provided that f is chosen to have the property that $[\dots]$ vanishes at $t=0$.

Here $\pi(\chi)$ is the one-dimensional constituent of the full-induced $I(\chi\nu^{1/2},\chi\nu^{-1/2})$, $tr\,\pi(\chi,f)$ is the trace of the convolution operator $(\pi(\chi))(f)$, and we write $\delta(X)=1$ if X happens, and $\delta(X)=0$ otherwise. Of course one can write out the r-trace formula for any function $f=\otimes f_v$, but we prefer to write out only the simplest form which suffices to prove C11.

To prove C1 we need to compare C7 with the analogous r-trace formula for a test function f^D on $\mathbf{D}'(\mathbb{A})$. There are two cases to consider, depending on whether the separable quadratic extension E of F embeds in D, or not. In the first case, referred to below as CASE I, the group D' of E-valued points on D is isomorphic to $G'=GL(2,E)$, while in the second CASE II, D' is an anisotropic form of G', central over E.

If V denotes the set of F-places where D ramifies, V' the subset of $v\in V$ which stay prime in E, and V'' the complement, consisting of the $v\in V$ which split in E, we have that V'' is empty precisely in CASE I. The case of C1 where V' is empty is the Theorem of [JL]. In CASE I, where $D'=G'$, we need to integrate the kernel identity (**1**) over g,h in the compact homogeneous space $\mathbf{Z}(\mathbb{A})D\backslash\mathbf{D}(\mathbb{A})$. In CASE II, since D' is anisotropic we do not have the continuous spectrum, i.e., we set $K_{f,c}=0$ in (**1**), and again integrate over $(\mathbf{Z}(\mathbb{A})D\backslash\mathbf{D}(\mathbb{A}))^2$. In both cases there is no need to truncate. Moreover, at each $v \in V$ the component f_v^D is necessarily r-discrete if it vanishes on the r-singular set.

As in the case of G, if $\mu_1\overline{\mu}_2 = 1$ and $Re(s) > \frac{1}{2}$, we define

$$J(\mu, s)\Phi = \int_{\mathbf{T}(\mathbb{A})\backslash\mathbf{D}(\mathbb{A})} \Phi(g, \mu, s)dg,$$

where

$$\mathbf{T}(\mathbb{A}) = \left\{\begin{pmatrix} a & 0 \\ 0 & \overline{a}\end{pmatrix}\right\} \subset \mathbf{D}(\mathbb{A}) = \mathbf{D}(\mathbb{A})_\epsilon = \left\{\begin{pmatrix} a & b\epsilon \\ \overline{b} & \overline{a}\end{pmatrix}\right\}, \qquad \Phi \in H(\mu), \mu = (\mu_1, \mu_2).$$

The function $J(\mu, s)\Phi$ has analytic continuation to the entire $s \in \mathbb{C}$ plane as a meromorphic function, whose restriction to $Re(s) = 0$ is holomorphic, except at $s = 0$ where it has at most a simple pole, and it has at most polynomial growth in $|t| \to \infty$, on $s = it$. Note that any $\mathbf{D}(\mathbb{A})$ is isomorphic to a $\mathbf{D}(\mathbb{A})_\epsilon$ with some $\epsilon \in F - NE$.

The derivation of the r-trace formula for f^D on $\mathbf{D}'(\mathbb{A})$ is by now routine. In the handling of the integration over $\mathbf{Z}(\mathbb{A})D\backslash\mathbf{D}(\mathbb{A})$ of the Eisensteinian kernel in CASE I, note that $G' = B'D$, and $B'\backslash G' = T\backslash D$, $T = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha}\end{pmatrix}; \alpha \in E^\times\right\}$. We obtain

C8. Proposition. *Given $f^D = \otimes f_v^D$ such that $f_v^D \in H_v^D$ for all v and f_u vanishes on the r-singular set at some place u, we have*

$$(i) \sum_{\substack{\pi^D \subset L_{0,\omega'}(D') \\ \dim \pi^D \neq 1}} c(\pi^D)\mathbb{L}_{\pi^D}(f^D) + \frac{1}{2}|\mathbf{Z}(\mathbb{A})D\backslash\mathbf{D}(\mathbb{A})|^2 \sum_{\substack{\chi^2 = \omega' \\ \chi|_{\mathbb{A}^\times} = 1}} tr\, \pi^D(\chi; f^D)$$

$$(ii) + c_1\delta(I) \sum_{\mu_1\overline{\mu}_2 = 1} \sum_{\alpha,\beta} \int_{-\infty}^{\infty} [(I(\mu, it; f^D)\Phi_\beta, \Phi_\alpha) \cdot tJ(\mu, it)\Phi_\alpha \cdot t\overline{J(\mu, it)\Phi_\beta}]t^{-2}dt$$

$$(iii) = \sum_{\{T_D\}_e} |\mathbf{T}(\mathbb{A})/\mathbf{Z}(\mathbb{A})T|w(T)^{-1} \sum_{\gamma \in T'/TZ'} \Xi(\gamma, f^D),$$

provided that f^D is chosen to have the property that $[\dots]$ vanishes at $t = 0$.

Note that in CASE I, any $\pi^D \subset L_{0,\omega'}(D')$ has $\dim \pi^D = \infty \neq 1$, but in CASE II the π^D with $\dim \pi^D = 1$ are described by the second sum. As usual, $\delta(I) = 1$ in CASE I, and $\delta(I) = 0$ in CASE II.

Propositions C7 and C8 have the immediate

C9. Corollary. *For any r-matching $f = \otimes f_v$ on $\mathbf{G}'(\mathbb{A})$ and $f^D = \otimes f_v^D$ on $\mathbf{D}'(\mathbb{A})$ such that for some $u \in V$ the components f_u and f_u^D are r-discrete we have*

$$(C7(i)) + (C7(ii)) + (C7(iii)) = (C8(i)) + (C8(ii)).$$

Proof. Since $\{T_D\}_e = \{T\}_e$, and by definition of r-matching, since f and f^D are r-matching we have (C7(iv))=(C8(iii)).

To extract C1 from C9 we need to simplify the identity of C9. The first step is to show that (C7(iii))=(C8(ii)), in particular that both are zero in CASE II, for sufficiently many functions f and f^D. We first dispose of the easier case.

C10. Lemma. *In CASE II, that is when V'' is non-empty, we have (C7(iii))=0.*

Proof. (1) The integral $\int_{\mathbf{T}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})} \Phi(\eta^{-1}g,\mu,s)dg$ converges absolutely on $Re(s) > \frac{1}{2}$, and if $\Phi = \otimes\Phi_v$, $\Phi_v \in H(\mu_v, s)$ for all v and $\Phi_v = \Phi_v^0$ for almost all v (Φ_v^0 is the normalized K_v-fixed vector in $H(\mu_v, s)$; it satisfies $\Phi_v^0(k) = 1$ on $k \in K_v$), the integral can be written as a product of the local integrals over all places.

At a place which stays prime in E, the local integral is simply

$$\begin{aligned}&\int_{T_v\backslash G_v} \Phi_v(\eta^{-1}g,\mu_v,s)dg = J(\mu_v,s)\Phi_v\\ &= J_1(\mu_v,s)\Phi_v \cdot L(2s,\mu_{1v}|F_v^\times)/L(1+2s,\mu_{1v}|F_v^\times\cdot\chi_{E_v/F_v}),\end{aligned}$$

and $J_1(\mu_v,s)\Phi_v$ is an elementary function in s.

At a place v of F which splits into v', v'' in E, if $\Phi_v = \Phi_{v'}\times\Phi_{v''}$ we have that the local integral is

$$\int_{T_v\backslash G_v} \Phi_{v'}(\eta^{-1}g,\mu_{v'},s)\Phi_{v''}(\overline{\eta}^{-1}g,\mu_{v''},s)dg.$$

Here η and $\overline{\eta}$ are matrices in G_v with $\eta^{-1}\overline{\eta} = w = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$ and

$$\begin{aligned}\eta^{-1}T_v\eta = \overline{\eta}^{-1}T_v\overline{\eta} = \widetilde{A}_v &= \left\{\begin{pmatrix} a & 0\\ 0 & \overline{a}\end{pmatrix}; a\in E_v^\times\right\} \subset \widetilde{G}_v\\ = \eta^{-1}G_v\eta &= \left\{\begin{pmatrix} a & b\\ \overline{b} & \overline{a}\end{pmatrix}; a,b\in E_v; a\overline{a}\neq b\overline{b}\right\}.\end{aligned}$$

Making the change $g\to\overline{\eta}g$ of variables the integral becomes

$$\int_{A_v\backslash G_v} \Phi_{v'}(wg,\mu_{v'},s)\Phi_{v''}(g,\mu_{v''},s)dg.$$

Since $1 = \mu_{1v}\overline{\mu}_{2v} = (\mu_{1v'},\mu_{1v''})(\mu_{2v''},\mu_{2v'})$, this integral can also be written as

$$\int_{K_v}\int_{N_v} \Phi_{v'}(wnk,\mu_{v'},s)dn\cdot\Phi_{v''}(k,\mu_{v''},s)dk = \langle M(\mu_{v'},s)\Phi_{v'},\Phi_{v''}\rangle.$$

Up to an ϵ-factor (of the form cq^s, = elementary function in s), this is

$$L(2s, \mu_{1v}|F_v^\times)L(2s+1, \mu_{1v}|F_v^\times)^{-1}\langle R(\mu_{v'}, s)\Phi_{v'}, \Phi_{v''}\rangle,$$

where R is the normalized intertwining operator, since $\mu_{1v'}/\mu_{2v'} = \mu_{1v'}\mu_{1v''} = \mu_{1v}|F_v^\times$.

(2) The space $H(\mu)$ is the tensor product over all v of the analogous local spaces $H(\mu_v)$, the operator $R(\mu, s)$ is the tensor product of the local normalized intertwining operators $R(\mu_v, s)$, and the orthonormal basis for $H(\mu)$ can be chosen to be the restricted tensor product of orthonormal bases chosen for the $H(\mu_v)$. The integrand in (C7(iii)), is, up to an elementary function in t, the function

$$\sum_{\Phi} J_1(\mu, it)I(\mu, it; f)\Phi \cdot \overline{J_1(\mu, it)\Phi}.$$

By the above choices, this is a product of local, analogous expressions.

Consider a place v which splits into v', v'' in E. Since $\mu_1\overline{\mu}_2 = 1$, the space $H(\mu_{v'}) = H(\mu_{1v'}, \mu_{2v'})$ and $H(\mu_{v''}) = H(\mu_{1v''}, \mu_{2v''})$ are contragredient, and we can and do choose the basis $\{\Phi_{v''}\}$ on $H(\mu_{v''})$ to be dual to that $\{\Phi_{v'}\}$ on $H(\mu_{v'})$. Put Φ' for $\Phi_{v'}$ and Φ'' for $\Phi_{v''}$. By (1) the local factor is

$$\begin{aligned}
&\sum_{\alpha,\beta}\langle R(\mu_{v'}, it)I(\mu_{v'}, it; f_{v'})\Phi'_\beta, I(\mu_{v''}, it; f_{v''})\Phi''_\alpha\rangle\overline{\langle R(\mu_{v'}, it)\Phi'_\beta, \Phi''_\alpha\rangle}\\
&= \sum_{\alpha,\beta}\langle I(\overline{\mu}_{v''}, -it; \overset{\vee}{f}_{v''})R(\mu_{v'}, it)I(\mu_{v'}, it; f_{v'})\Phi'_\beta, \Phi''_\alpha\rangle\overline{\langle R(\mu_{v'}, it)\Phi'_\beta, \Phi''_\alpha\rangle}\\
&= \sum_{\Phi'}(I(\overline{\mu}_{v''}, -it; \overset{\vee}{f}_{v''})R(\mu_{v'}, it)I(\mu_{v'}, it; f_{v'})\Phi',\ R(\mu'_v, it)\Phi').
\end{aligned}$$

The last equality follows from the fact that for a, b in $H(\mu_{v'})$ we have

$$b = \sum_{\Phi'}\langle b, \Phi''\rangle\Phi',\ \text{hence}\ (a, b) = \langle a, \overline{b}\rangle = \sum_{\Phi''}\overline{\langle b, \Phi''\rangle}\langle a, \overline{\Phi}'\rangle = \sum_{\Phi''}\langle a, \Phi''\rangle\overline{\langle b, \Phi''\rangle}.$$

Here the sum ranges over the orthonormal basis $\{\Phi'\}$, and $\{\Phi'' = \overline{\Phi}'\}$ is the dual basis, of $H(\mu_{v''})$. But $R(\mu_{v'}, it)$ is a unitary intertwining operator. Hence we get

$$= \sum_{\Phi'}(I(\mu_{v'}, it; \overset{\vee}{f}_{v''})I(\mu_{v'}, it; f_{v'})\Phi', \Phi') = tr\, I(\mu_{v'}, it; h_v)$$

where $h_v = f_{v'} * \overset{\vee}{f}_{v''}$.

Finally, since V'' is non-empty there is a place v which splits in E where D ramifies. The corresponding function $f_v = (f_{v'}, f_{v''})$ (to any $f_v^D \in H_v^D$) has the property that $tr\, \pi_v(h_v) = 0$ for any properly induced representation π_v of G_v. Hence the lemma follows.

Our next aim is to show that (C7(iii))=(C8(ii)) in CASE I for sufficiently many functions f (to prove C1). In C7 and C8 we require that f and f^D be chosen so that

[...] in C7(iii) and C8(ii) be zero at $t=0$. We make this choice as follows. Let S be a finite set of places of F containing V, the archimedean places and those which ramify in E. At any $v \notin S$ we take the component $f_v = f_v^D$ to be spherical. Fix $w \notin S$. Note that $f_w * f_w^0 = f_w$ for any spherical f_w. Suppose that the component of f at w is f_w^0, and denote by $f * f_w$ the function obtained from f on replacing its component at w by f_w.

For any $\Phi \in H(\mu)$ we have that $I(\mu, it; f * f_w)$ is the product of $I(\mu, it; f)\Phi$ and the scalar

$$f_w^\vee(\mathrm{diag}(\mu_{1w}(\pi_w)q_w^{-it}, \mu_{2w}(\pi_w)q_w^{it})) \quad \text{if } E_w \text{ is a field,}$$

or

$$h_w^\vee(\mathrm{diag}(\mu_{1w'}(\pi_w)q_w^{-it},\ \mu_{2w'}(\pi_w)q_w^{it})) \quad \text{if } w \text{ splits into } w', w'' \text{ in } E.$$

Here $f_w^\vee$ or $h_w^\vee$ is the Satake transform of f_w or $h_w = f_{w'} * \overset{\vee}{f}_{w''}$. In fact we can and will take $f_{w''} = f_w^0$, and then $h_w = f_{w'}$. As usual π_w is a uniformizer in R_w, and q_w the cardinality of the field $R_w/\pi_w R_w$. Since [...] of C7 and C8 has a zero of order two unless $\mu_i|\mathbb{A}^\times = 1$, we will now assume that $\mu_i|\mathbb{A}^\times = 1$. Since $\mu_1\overline{\mu}_2 = 1$ in C7(iii) and C(ii), we have $\mu_1 = \mu_2$, and $\mu_1\mu_2 = \omega'$, where $\omega'|\mathbb{A}^\times = 1$ (hence there is some ω on $\mathbb{A}_E^1$ with $\omega'(z) = \omega(z/\overline{z})$).

For brevity we now write h for f_w if w stays prime, and for $h_w = f_{w'}$ if w splits. The scalar which appears in [...] is then of the form $h^\vee(\mathrm{diag}(zq^{-it}, zq^{it}))$, $q = q_w$ and $t \in \mathbb{R}$, and $z^2 = \omega(\pi_w)$ ($= 1$ if w stays prime). We need to choose h such that the value at $t = 0$ is zero. Recall that $h^\vee(z_1, z_2)$ is a symmetric polynomial in z_1/z_2, thus

$$h^\vee(z_1, z_2) = \sum_n a_n (a_1/z_2)^n, \qquad a_{-n} = a_n,$$

and any such polynomial is of the form $h^\vee$, for some h. We will choose h such that

$$(\mathbf{6}) \qquad h^\vee(z_1, z_2) = \left(1 - \frac{1}{2}\left(\frac{z_1}{z_2} + \frac{z_2}{z_1}\right)\right)\tilde{h}^\vee(z_1, z_2)$$

for some other spherical $\tilde{h}$.

C11. Proposition. *Fix a place $w \notin V$, where both ω and E/F are unramified, and complex z_1, z_2 with $z_1 z_2 = \omega(\pi_w)$ and $z_1 \neq z_2$. For any $f^D = \otimes f_v^D$ such that f_u^D is r-discrete at some place $u \neq w$, and matching $f = \otimes f_v$ with r-discrete f_u, we have (C7(i))+(C7(ii))=(C8(i)), where the sums over π, $\pi(\chi)$ and $I(\mu, 0)$ range over those automorphic representations whose component π_w at w is unramified with Hecke eigenvalues z_1, z_2 if w stays prime, or of the form $\pi_{w'} \times \tilde{\pi}_{w'}$ (if w splits) with unramified $\pi_{w'}$ having the Hecke eigenvalues z_1, z_2.*

Proof. We shall write the equality of C9 for a test function of the form $f * h$, $h = f_w$ in the non-split case and $h = f_{w'} (= f_{w'} * \overset{\vee}{f}_{w''}$, since $f_{w''}$ is taken above to be f_w^0) in the split case, and h related to $\tilde{h}$ is in (**6**). Following standard lines, the equality of C9 can be written then in the form

$$\sum_{i \geq 0} c_i (1 - \frac{1}{2}(t_i/t_i' + t_i'/t_i))\tilde{h}^\vee(t_i, t_i') = \int_{|t|=1} \tilde{h}^\vee(zt, z/t)(1 - \frac{1}{2}(t^2 + t^{-2}))d(t)dt,$$

where $t_i t_i' = \omega(\pi_w)$, $z^2 = \omega(\pi_w)$, $|t_i| = |t_i'| = 1$ or $(t_i, t_i') = (u_i q_w^{-r_i}, u_i q_w^{r_i})$ with $|u_i| = |u_i'| = 1$ and $-\frac{1}{2} \le r_i \le \frac{1}{2}$, and

$$\sum_i |c_i(1 - \frac{1}{2}(t_i/t_i' + t_i'/t_i))| < \infty, \qquad \int_{|t|=1} |d(t)||dt| < \infty.$$

A standard application of the Stone-Weierstrass theorem (e.g., as in [FK2], Proposition, p. 198) implies that the set of polynomials $\tilde{h}^\vee$ is dense in the space of continuous functions on the compact set consisting of the t in $\mathbb{C}$ with $|t| = 1$ of $q_w^{-1} \le t \le q_w$. Choosing a suitable $\tilde{h}$ we conclude that $c_i = 0$ for all i, and the proposition follows.

Remark. Note that the requirement in C7 and C8 that "f has the property that [...] vanishes at $t = 0$" forces us to introduce the factor $1 - \frac{1}{2}(t/t' + t'/t)$, which vanishes at $t = t'$, hence the requirement in C11 that $z_1 \ne z_2$.

C12. Corollary. *For any corresponding cuspidal π^D and π, such that $\dim \pi^D > 1$, and $\pi_v \simeq \pi_v^D$ is distinguished for all $v \notin V$ and for any r-matching f_v^D and f_v $(v \in V)$, we have*

$$c(\pi^D) \prod_{v \in V} \mathbb{L}_{\pi_v^D}(f_v^D) = c(\pi) \prod_{v \in V} \mathbb{L}_{\pi_v}(f_v).$$

Proof. Let $S \supset V$ be a set such that ω, E/F and π are unramified outside S. The identity of C11 applies with $f = \otimes f_v$ where at any $v \notin S$ we may use any spherical f_v. A standard approximation argument used – as mentioned above – in [FK2], Theorem 2, implies that the identity (C7(i))+(C7(ii))=(C8(i)) remains true if we sum only over those π, $\pi(\chi)$, $I(\mu)$, π^D and $\pi^D(\chi)$ whose component at any $v \notin S$ is (equivalent to) π_v, *and* at some place $w \notin S$ the Hecke eigenvalues z_1, z_2 of π_w are distinct (this last requirement appears in C11).

By rigidity (see [JS2]) and multiplicity one theorems for $GL(2)$, π is the only automorphic representation of $\mathbf{G}'(\mathbb{A})$ whose components are equivalent to π_v for almost all v. Hence there is only one term in the sum of (C7(i))+(C7(ii)), indexed by π. The analogous theorems for $\mathbf{D}'(\mathbb{A})-$ which follow from the correspondence from the set of automorphic representations of $\mathbf{D}'(\mathbb{A})$ to those of $\mathbf{G}'(\mathbb{A})-$ imply that π^D is the only automorphic representation of $\mathbf{D}'(\mathbb{A})$ whose components are equivalent to π_v for almost all v. Hence there is only one term in the sum of (C8(i)), indexed by π^D. The identity of C12 follows, but only for π and π^D which satisfy the requirement at w.

Moreover, the restriction at w can be dropped. Indeed, suppose that at each v outside S the Hecke eigenvalues z_{1v}, z_{2v} of π_v are equal. Consider the symmetric-square lifting $\prod$ of π (see [GJ] or [F9]). This is a cuspidal representation of $GL(3, \mathbb{A}_E)$, since π is cuspidal and not of the form $\pi(\theta)$ for any character $\theta : \mathbb{A}_L^\times / L^\times \to \mathbb{C}^\times$ of any quadratic extension L of E ($\pi \ne \pi(\theta)$ since the Hecke eigenvalues of π are equal outside S). On the other hand, the Hecke eigenvalues of $\prod$

outside S are $(z_{1v}/z_{2v}, 1, z_{2v}/z_{1v})$, namely $(1,1,1)$. Consequently the cuspidal $\prod$ has the same Hecke eigenvalues (at almost all v) as the representation $I(1,1,1)$ of $GL(3,\mathbb{A}_E)$ normalizedly induced from the trivial character of the Borel subgroup. This is impossible by rigidity theorem for $GL(3)$ (see [JS2]), implying that $z_{1w} \neq z_{2w}$ for some $w \notin S$. We apply C11 with this w, and the corollary follows for all matching π^D and π.

Note that at any $v \in S - V$ we have $\pi_v \simeq \pi_v^D$ and $f_v = f_v^D$, hence $\mathbb{L}_{\pi_v}(f_v) = \mathbb{L}_{\pi_v^D}(f_v^D)$. We may choose this f_v to vanish on the r-singular set in G'_v, and to satisfy $\mathbb{L}_{\pi_v}(f_v) \neq 0$. The corollary follows.

Remark. At the places $v \in V'' \subset V$ which split in E, we have

$$\mathbb{L}_{\pi_v^D}(f_v^D) = tr\,\pi_{1v}^D(h_v^D) = tr\,\pi_{1v}(h_v) = \mathbb{L}_{\pi_v}(f_v)$$

if $\pi_v = \pi_{1v} \times \tilde{\pi}_{1v}$ and $\pi_v^D = \pi_{1v}^D \times \tilde{\pi}_{1v}^D$ are distinguished, π_{1v} and π_{1v}^D are corresponding, and h_v^D and h_v are matching. We choose (as we may) h_v with $tr\,\pi_{1v}(h_v) \neq 0$. Hence the products in C12 can be taken to range only over V', assuming that (π_v and) π_v^D are D_v-distinguished for all $v \notin V'$.

C13. Corollary. *Suppose that D_u is an anisotropic inner form of G_u, E_u/F_u is a quadratic extension, and π_u is a square-integrable D_u-distinguished representation of G'_u; note that $G'_u = D'_u$. Then there exists a non-zero constant $c(\pi_u)$ such that*

$$\mathbb{L}_{\pi_u}(f_u^D) = c(\pi_u)\mathbb{L}_{\pi_u}(f_u) \qquad (\mathbf{7})$$

for any r-matching function f_u and f_u^D. Alternatively put, for any r-regular-elliptic r-corresponding γ^D and γ in D_u and G_u, we have

$$\Xi_{\pi_u}^D(\gamma^D) = c(\pi_u)\Xi_{\pi_u}(\gamma),$$

where $\Xi_{\pi_u}^D$ is the r-character of π_u with respect to D_u, and Ξ_{π_u} is the r-character of π_u with respect to G_u.

Proof. Consider a global quadratic separable extension E/F which is the given local extension at the place u, and denote by $u' \neq u$ a finite place which stays prime at E. Let D be the multiplicative group of a quaternion algebra central over F which ramifies precisely at u and u'. Then $D' = GL(2,E)$, and $V' = \{u, u'\}$; V'' is empty. If π_u is supercuspidal, B14 implies that there exists a cuspidal representation π^D of $\mathbf{D}'(\mathbb{A})$ which is $\mathbf{D}(\mathbb{A})$-distinguished, whose component at u is the given one, and whose component at u' is supercuspidal (there are $D_{u'}$-distinguished supercuspidal $GL(2,E_{u'})$-modules by [F8]). If π_u is special then we can construct a cuspidal representation of the unitary group in two variables associated to E/F which is anisotropic at u, u', whose component at u is special, and whose component at u' is supercuspidal. As in [F8] we deduce that the unstable lift π^D to $\mathbf{D}'(\mathbb{A})$ is cuspidal and $\mathbf{D}(\mathbb{A})$-distinguished, with the required components at u, u'. Applying C12 with this π^D we obtain (**7**). The r-character relation follows from (**7**) on using the r-Weyl integration formula.

In particular, if π_u is a distinguished square-integrable representation of G'_u, there exists an r-discrete function f_u (which has an r-matching r-discrete function f_u^D) with $\mathbb{L}_{\pi_u}(f_u) \neq 0$. To prove one of the sides of C1, we need this property also for D_u-distinguished non-square-integrable representations of G'_u.

C14. Proposition. *The r-character of the induced representation $I(\mu, \overline{\mu}^{-1})$ of G'_u is not identically zero on the r-elliptic-regular set in G'_u precisely when the restriction $\mu|F_u^\times$ of μ to $F_u^\times$ is nontrivial.*

Proof. This is proven in [H3]; the vanishing of the r-character on the r-elliptic-regular set when $\mu|F_u^\times = 1$ is shown in $B19$.

Proof of C1. Suppose that π^D is $\mathbf{D}(\mathbb{A})$-distinguished, namely that $c(\pi^D) \neq 0$. Then each π_v^D is D_v-distinguished. By B8 we may choose r-regular f_v^D with $\mathbb{L}_{\pi_v^D}(f_v^D) \neq 0$. By A4 there exists an (r-regular) r-discrete f_v which r-matches f_v^D. Applying C12, since the left side is non-zero, so is the right, and $c(\pi) \neq 0$, implying that π is $\mathbf{G}(\mathbb{A})$-distinguished.

In the opposite direction, suppose that π is $\mathbf{G}(\mathbb{A})$-distinguished, and π_v is D_v-distinguished at every place $v \in V'$, but not of the form $I(\mu, \overline{\mu}^{-1})$ with $\mu|F_v^\times = 1$ for any $v \in V'$. The last supposition means that in addition to being G_v-distinguished, at each $v \in V'$ the representation π_v of G'_v is either square-integrable or of the form $I(\mu, \overline{\mu}^{-1})$ with $\mu|F_v^\times \neq 1$, but it is not of the form $I(\mu_1, \mu_2)$, $\mu_i : E_v^\times / F_v^\times \to \mathbb{C}^\times$, $\mu_1 \neq \mu_2$. By C13 and C14 there exist r-discrete f_v with $\mathbb{L}_{\pi_v}(f_v) \neq 0$ for every v in V'. By A4 there exist r-matching r-discrete f_v^D on $D'_v = G'_v$. Applying C12, we have $c(\pi) \neq 0$ by assumption, and for this choice of f_v, the right side is non-zero. The same is true for the left side. Hence $c(\pi^D) \neq 0$ and π^D is $\mathbf{D}(\mathbb{A})$-distinguished, as required.

Remark. (1) The proof of C12 can be adapted in an obvious fashion to imply that C7(ii) is zero. In fact, for f_w as in C11, the part corresponding to $\mu_1 = \mu_2$ in C7(ii) is zero by the choice of f_w. The case where V'' contains at least two elements is discussed by local means in [JL], (9.5), pp. 305/6.

(2) The proof of C12 can also be adapted to show that (C7(i))=(C8(i)), further that $tr\, \pi(\chi; f) = tr\, \pi^D(\chi; f^D)$ for all $\chi : \mathbb{A}_E^\times / \mathbb{A}^\times E^\times \to \mathbb{C}^\times$ with $\chi^2 = \omega'$, and that $tr\, \pi_v(\chi_v, f_v) = tr\, \pi_v^D(\chi_v; f_v^D)$ when D_v is an anisotropic form of G_v. For this local statement, note that given a local character χ_u of $E_u^\times / F_u^\times$ and a place u' which splits in E, there is a global character χ with this component at u and such that χ is unramified outside u and u'. The character identity at a split place, for example u', is easy to prove.

(3) An alternative proof of C1 – but only in the case where V'' is empty – can be given on working out an analogue of [F8], in the context of an inner form of the unitary group $G = U(2, E/F)$ of that paper, and comparing this analogue with the results of [F8] in the quasi split case. All technical difficulties have already been overcome in [F8]. Interesting identities of "Whittaker-Period" distributions (DW_{π_v, ψ_v} of [F8], p. 168) will follow, instead of the identity (**7**) of C13. We need

V'' to be empty since $\mathbf{D}'(\mathbb{A})$ must contain the unipotent upper triangular subgroup for the Fourier summation formula of [F8] to exist.

(4) In [H3] it is shown that for each unitary $\mu_v : E_v^\times \to \mathbb{C}^\times$ with $\mu_v = \overline{\mu}_v$ there is $c > 0$ such that for r-corresponding r-regular γ^D and γ, and $t \in \mathbb{R}$, we have $\Xi_t^D(\gamma^D) = -c\Xi_t(\gamma)$ where Ξ_t^D is the r-character of $I(\mu_v\nu_v^{it}, \overline{\mu}_v^{-1}\nu_v^{-it})$ with respect to D_v, while Ξ_t is its r-character with respect to G_v.

D. Smoothness of the r-character.

Let E/F be a quadratic extension of non-archimedean local fields, H a division algebra with center F, and $H' = H \otimes_F E$. Then $H' = M(m, H'')$ for some m and some division algebra H'' with center E. For simplicity we assume that the residual characteristic of F is not two. Fix a positive integer n and let (π, V) be an irreducible, admissible representation of $G' = GL(n, H')$. Assume that there exists a non-zero linear form $\tilde{L}$ on V which is invariant under $G = GL(n, H)$. Then there also exists a non-zero linear form L on the space of the contragredient $\tilde{\pi}$. Up to scalars, these forms are unique and in this section the exact normalizations are irrelevant. Consider the G-biinvariant distribution defined by

$$\mathbb{L}_\pi(f) = \langle \pi(f)L, \tilde{L}\rangle$$

for $f \in C_c^\infty(G')$, where $\langle\ ,\ \rangle$ is the canonical pairing on $V \times \tilde{V}$. We will prove:

D1. Proposition. *There exists a locally constant G-biinvariant function Ξ_π on the set of r-regular semisimple elements of G' such that*

$$\mathbb{L}_\pi(f) = \int_{G'} f(g)\Xi_\pi(g)\ dg$$

whenever $f \in C_c^\infty(G')$ is supported on the set of r-regular semisimple elements.

The case of $GL(2)$ has been discussed in [H1], following closely Howe's ideas in [Ho]. Our proof is similar.

Let R_F, R_E, R, R' and R'' denote the maximal compact subrings of F, E, H, H' and H'', respectively. Thus $R' = M(m, R'')$. Let $d^2 = [H'' : E]$ and let e be the ramification index of E/F. According to Proposition 5 of I.4 in [W], we can choose local uniformizers $\pi_F \in R_F$, $\pi_E \in R_E$, $\pi_0 \in R$ and $\pi \in R''$ such that $\pi_E = \pi^d$, $\pi_F = \pi_E^e$ and $\pi^e = \pi_0^m$. When $r \in \mathbb{Z}$, we take $L_r = \pi^r M(n, R') = \pi^r M(mn, R'')$. When r is positive, $K_r = 1 + L_r$ defines a group called *the r-th congruence subgroup of* $GL(mn, R'')$. The set of equivalence classes of irreducible, unitary representations of K_r is denoted by $\hat{K}_r$. Fix, for the remainder of this section, a distinguished representation (π, V), as above. Fix also a positive integer r_0. If $\delta \in \hat{K}_{r_0}$, we let V_δ denote the corresponding isotypic component of V. Let E_δ be the projection of V onto V_δ which commutes with δ. Define a function $\Xi_\delta(g) = \langle E_\delta\pi(g)L, \tilde{L}\rangle$ on G'.

Fix a Cartan subgroup T of G. The centralizer T' of T in G' is a Cartan subgroup of G'. Fix a compact, open subset X of T' consisting of regular elements. If there exists a matrix g in G' such that $g\bar{g}^{-1} \in X$ and $\Xi_\delta(g) \neq 0$ then we will say that δ *contributes* to Ξ_π. This notion depends on the choices of X and r_0.

D2. Proposition. *Only a finite number of δ contribute to Ξ_π for fixed X and r_0.*

Let us quickly show how this proposition implies Proposition D1. Let Y be the finite set of δ which contribute to Ξ_π. Given $g \in G'$ such that $g\bar{g}^{-1} \in X$, we can choose a compact, open subgroup $K \subseteq K_{r_0}$ such that $kg\bar{g}^{-1}\bar{k}^{-1} \in X$ and $\pi(k)\tilde{v} = \tilde{v}$ for all $k \in K$ and $\tilde{v} \in \oplus_{\delta \in Y} \tilde{V}_\delta$. Then $\Xi_\delta(kg) = \Xi_\delta(g)$ for all $\delta \in Y$ and $k \in K$. Now if f is supported on the set of g such that $g\bar{g}^{-1} \in X$, then $\mathbb{L}_\pi(f) = \langle \pi(f)L, \tilde{L}\rangle$ is equal to $\sum_{\delta \in \hat{K}_{r_0}} \langle E_\delta \pi(f)L, \tilde{L}\rangle = \sum_{\delta \in Y} \int f(g)\Xi_\delta(g)dg$, and Proposition D1 would follow.

Let us make a further reduction. If $\delta \in \hat{K}_{r_0}$ then the *conductor* of δ is the subgroup K_r with r minimal such that K_r is contained in the kernel of δ.

D3. Proposition. *There exists a positive integer n_1, depending only on X and r_0, such that if δ has conductor r and δ contributes to Ξ_π then $r < n_1$.*

If this is so, then δ will be a representation of the finite group K_{r_0}/K_r. Hence only a finite number of δ can contribute. We are therefore reduced to finding such an n_1.

Let $M = M(n, H)$ and $M' = M \otimes_F E = M(mn, H'')$. Fix an additive character ψ_F of F with conductor R_F. If A is a closed additive subgroup of M', define

$$A^* = \{x \in M' \mid \psi_F(tr_{M'/F}(xy)) = 1 \text{ for all } y \in A\}.$$

Pontryagin duality implies that $A^{**} = A$ and $(A_1 \cap A_2)^* = A_1^* + A_2^*$, when A, A_1 and A_2 are closed subgroups of M'.

D4. Lemma. *If r is a rational integer, then $L_r^* = L_{-r-d+1}$.*

Proof. We note, first of all, that the character $\psi_E = \psi_F \circ tr_{E/F}$ of E has conductor R_E. The condition which $x \in M'$ must satisfy in order to lie in L_r^* is equivalent to $\psi_E(tr_{M'/E}(\pi^r xy)) = 1$ for all $y \in M(mn, R'')$. Our claim now follows from Corollary 1 to Proposition 5 of X.2 in [W].

D5. Lemma. *The set M^* consists of all $x \in M'$ such that $\bar{x} = -x$.*

Proof. In order for $x \in M'$ to belong to M^*, it is necessary and sufficient that $\psi_F(tr_{M'/F}(xy)) = 1$ for all $y \in M$. Equivalently, $\psi_F(tr_{M/F}((x + \bar{x})y)) = 1$ for all $y \in M$, but this is the same as $x + \bar{x} = 0$.

D6. Corollary. *If r is a rational integer, then $(L_r \cap M)^* = L_{-r-d+1} + M^*$.*

Assume that X and r_0 are fixed as above and fix $\delta \in \hat{K}_{r_0}$ which contributes to Ξ_π. Choose g such that $g\bar{g}^{-1} \in X$ and $\Xi_\delta(g) \neq 0$. It is easily shown that if r and s are positive integers such that $r \leq s \leq 2r$, then $x \mapsto 1 + x$ defines an isomorphism of groups $L_r/L_s \simeq K_r/K_s$. In particular, K_r/K_s is abelian. Let K_{r_2} denote the conductor of δ and let $r_1 = \max(r_0, [(r_2+1)/2])$, where $[x]$ is the greatest integer $\leq x$. Then K_{r_1}/K_{r_2} is abelian. Consequently, the restriction of δ to K_{r_1} decomposes as a direct sum of characters ψ. The non-zero vector $v_0 = E_\delta \pi(g) L$ in V_δ has a corresponding decomposition $v_0 = \sum v_\psi$. There exists a character ψ_0 such that $\langle v_{\psi_0}, \tilde{L} \rangle \neq 0$.

Imitating Howe's definition (in section 2 of [Ho]) of the "dual blob" of ψ_0, we take

$$\beta(\psi_0) = \{x \in M' \mid \psi_0(1+y) = \psi_F(tr_{M'/F}(xy)) \text{ for all } y \in L_{r_1}\}.$$

Given $x, y \in \beta(\psi_0)$, then $x - y \in L^*_{r_1}$. It follows that $\beta(\psi_0)$ is a coset of the form $x + L_{-r_1-d+1}$. Similarly, one can define the dual set $\beta(\psi)$ for each character ψ occurring in the restriction of δ to K_{r_1}. There is a "coadjoint" action of K_{r_0} on the characters of K_{r_1} defined by $Ad^*(h)\psi(k) = \psi(h^{-1}kh)$.

D7. Lemma. *The group K_{r_0} acts transitively on the set of characters occurring in the restriction of δ to K_{r_1}.*

Proof. Suppose ψ_1 and ψ_2 are two such characters. The irreducibility of δ implies the existence of $h \in K_{r_0}$ such that $E_1 \delta(h) E_2 \neq 0$, where E_i is the projection onto the space of ψ_i. Then $E_1 \delta(h) E_2$ must intertwine $Ad^*(h)\psi_2$ and ψ_1.

The previous lemma implies that the conductor of any ψ occurring in $\delta | K_{r_1}$ must be identical to the conductor K_{r_2} of δ. Moreover, the dual sets $\beta(\psi_1)$ and $\beta(\psi_2)$ of any two of these characters must be conjugate by any element of K_{r_0}.

Now let $\mathcal{N}$ denote the set of nilpotent elements in M'. The next result has been proven by Howe in the context of $GL(n)$ (Lemma 2.4 in [Ho]). The same proof works for the more general case which we consider.

D8. Lemma. *For every integer r, $Ad\, G'(L_r) \subseteq L_r + \mathcal{N}$.*

This is needed for the following:

D9. Lemma. *There exists a positive number n_2, depending only on r_0 and X, such that if $r_2 \geq n_2$ then $\beta(\psi_0)$ contains a nilpotent element.*

Proof. Fix, independently of the choice of δ, another representation $\delta' \in \hat{K}_{r_0}$ which occurs in π. Let $K_{r'}$ be its conductor. The irreducibility of π implies that there exists $h \in G'$ such that $E_\delta \pi(h) E_{\delta'} \neq 0$. Then $E_\delta \pi(h) E_{\delta'}$ intertwines the restriction of δ to $K_{r_0} \cap h K_{r'} h^{-1}$ with the trivial representation. There is a character ψ of K_{r_1} which occurs in δ and is trivial on $K_{r_1} \cap h K_{r'} h^{-1}$. Now let y belong to $\beta(\psi)$. Then $\psi_F(tr_{M'/F}(xy)) = 1$ for all $x \in L_{r_1} \cap h L_{r'} h^{-1}$. Hence $y \in (L_{r_1} \cap h L_{r'} h^{-1})^* = L_{-r_1-d+1} + h L_{-r'-d+1} h^{-1} \subseteq L_{-r_1-d+1} + L_{-r'-d+1} + \mathcal{N}$. If $r_2 \geq 2r'$ then $r_1 \geq r'$, since $r_1 \geq [(r_2+1)/2]$. In this case, $y \in L_{-r_1-d} + \mathcal{N}$. Thus $\beta(\psi) \cap \mathcal{N}$ is not empty. But $\beta(\psi_0)$ must also contain a nilpotent element since it is conjugate to $\beta(\psi)$. Thus $n_2 = 2r'$ satisfies our needs.

This allows us to reduce to the case where $\beta(\psi_0)$ contains a nilpotent element when we prove Proposition D3. The following lemmas will also be useful. Recall that g is introduced after Corollary D6.

D10. Lemma. *The character ψ_0 is trivial on $K_{r_1} \cap G$ and on $K_{r_1} \cap gGg^{-1}$.*

Proof. First suppose that $k \in K_{r_1} \cap G$. Then $\langle v_{\psi_0}, \tilde{L}\rangle = \langle \pi(k) v_{\psi_0}, \tilde{L}\rangle = \psi_0(k)\langle v_{\psi_0}, \tilde{L}\rangle$. Thus $\psi_0(k) = 1$. Now suppose that $k \in K_{r_1} \cap gGg^{-1}$. Then $\pi(k)v_0 = E_\delta \pi(k)\pi(g)L = E_\delta \pi(g)\pi(g^{-1}kg)L = v_0$. Therefore $\psi_0(k)v_{\psi_0} = v_{\psi_0}$ and our claim follows.

D11. Lemma. *If $x \in \beta(\psi_0)$ then $x + \bar{x} \in L_{-r_1-ed-1} \cap M$.*

Proof. An element $x \in M'$ belongs to $\beta(\psi_0)$ precisely when $\psi_0(1+y) = \psi_F(tr_{M'/F}(xy))$ for all $y \in L_{r_1}$. For such x, we have $\psi_F(tr_{M'/F}(xy)) = 1$ for all $y \in L_{r_1} \cap M$, since ψ_0 is trivial on $K_{r_1} \cap G$. Equivalently, $\psi_F(tr_{M/F}((x+\bar{x})y)) = 1$ for all $y \in \pi_0^{-[-mr_1/e]} M(n,R)$. Corollary 1 to Proposition 5 of X.2 in [W] implies that $\pi_0^{-[-mr_1/e]}(x+\bar{x})$ lies in $\pi_0^{1-md} M(n,R)$. Our claim now follows from the fact that $\pi_0^m = \pi^e$.

For each $x \in M'$, we define $ord(x)$ to be the unique integer r such that $x \in L_r - L_{r+1}$. According to the next lemma, giving an upper bound for r_2 is equivalent to giving a lower bound for $ord(x)$ when $x \in \beta(\psi_0)$.

D12. Lemma. *If $x \in \beta(\psi_0)$ then $ord(x) = -r_2 - d + 1$.*

Proof. Suppose $x \in \beta(\psi_0)$. Then r_2 is the smallest integer such that $\psi_F(tr_{M'/F}(xy)) = 1$ for all $y \in L_{r_2}$. That is, r_2 is the smallest integer such that $x \in L^*_{r_2} = L_{-r_2-d+1}$. Hence $ord(x) = -r_2 - d + 1$.

We now proceed to prove Proposition D3. For this, we may as well assume that $\beta(\psi_0)$ contains a nilpotent element ν, according to D9. Otherwise $r_2 < n_2$. Lemma D10 implies that there exists $\zeta \in gM^*g^{-1} \cap \beta(\psi_0)$. Put $\mu = \nu - \zeta$. Then $\mu \in L_{-r_1-d+1}$. We have $Ad(\bar{g}g^{-1})\nu + \bar{\nu} = Ad(\bar{g}g^{-1})\mu + \bar{\mu}$, or equivalently

$$Ad(\bar{g}g^{-1})\nu - \nu = -(\nu + \bar{\nu}) + Ad(\bar{g}g^{-1})\mu + \bar{\mu}$$

We can certainly choose a positive integer l such that $Ad(x^{-1})L_r \subseteq L_{r-l}$ for all $x \in X$ and all r. Therefore D11 implies $Ad(\bar{g}g^{-1})\nu - \nu \in L_{-r_1-d_1}$, where $d_1 = \max(d+l-1, ed+1)$. On the other hand, we can choose an integer b, as in [Ho], so that for any $\eta \in \mathcal{N}$ and $x \in X$ we have $ord(Ad(x^{-1})\eta - \eta) \le ord(\eta) + b$. Consequently,

$$-r_2 - d + 1 + b = ord(\nu) + b \ge ord(Ad(\bar{g}g^{-1})\nu - \nu) \ge -r_1 - d_1.$$

Suppose $r_1 = r_0$. Then r_2 is bounded above by $r_0 + d_1 - d + 1 + b$. Otherwise $r_1 = [(r_2+1)/2]$ and $r_2 \le 2(d_1 - d + 3 + b)$. This completes the proof of D3. As explained above, D1 and D2 follow from D3.

Appendix. Algebraic cycles.

Theorem 0.3 can be used to establish Tate's conjecture [T] on algebraic cycles for some new Shimura surfaces, following the reduction of Lai [L] to the work of Harder-Langlands-Rapoport [HLR]. In this appendix, we state the result and indicate the changes which have to be made in [L]; we do not record a comprehensive exposition to this proof. Our only contribution is representation theoretic, asserting that given a $GL(2,\mathbb{A})$-distinguished cuspidal representation of $GL(2,\mathbb{A}_F)$ there exists a suitable – in the sense that the proofs of [HLR] and [L] apply – inner form $\mathbf{D}^{(p)}$ of $GL(2)$ over Q, such that the corresponding representation of $\mathbf{D}^{(p)}(\mathbb{A}_F)$ exists and is $\mathbf{D}^{(p)}(\mathbb{A})$-distinguished.

We first introduce the Shimura surface in question. Let F be a real quadratic field extension of the field $\mathbb{Q}$ of rational numbers, and $\mathbf{G}$ an anisotropic inner form of $GL(2)$ over F which splits at the two real places of F, and which has the property that for every finite prime p in $\mathbb{Q}$ we have $\sum_v \mathrm{inv}_v \mathbf{G}(F_v) = 0$, where the sum ranges over all places of F over p. Thus $\mathbf{G}(F)$ is the multiplicative group of a quaternion division algebra M central over F which splits at the archimedean places, and $\sum_{v|p} \mathrm{inv}_v M = 0$ for all primes p; here inv_v denotes the invariant of M at v (see Weil [W]). Thus $M = D \otimes_Q F$, where D is a division algebra over Q which splits at the archimedean place. Again, $\mathbf{G}$ is ramified only at finite places which split in $F/\mathbb{Q}$, and then the ramification occurs at both places above the $\mathbb{Q}$-prime in question. Denote by $\mathbf{G}'$ the algebraic group obtained from $\mathbf{G}$ on restricting scalars from F to $\mathbb{Q}$.

Let $\mathbb{A}$ denote the ring of $\mathbb{Q}$-adeles. Then $\mathbf{G}'(\mathbb{Q}) = \mathbf{G}(F)$, $\mathbf{G}'(\mathbb{R}) = GL(2,\mathbb{R}) \times GL(2,\mathbb{R})$, $\mathbf{G}'(\mathbb{A}) = \mathbf{G}(\mathbb{A}_F)$, and $\mathbf{G}'(F) = \{(x,\overline{x}); x \in \mathbf{G}(F)\}$ where $x \mapsto \overline{x}$ denotes the action of the non-trivial element of $\mathrm{Gal}(F/\mathbb{Q})$. Let $h : \mathbb{C}^\times \to \mathbf{G}'(\mathbb{R})$ be the $\mathbb{R}$-monomorphism which maps $i = \sqrt{-1}$ to $\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)$. Let K_∞ be the centralizer in $\mathbf{G}'(\mathbb{R})$ of the image of h. Let K be a sufficiently small compact open subgroup of $\mathbf{G}'(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the ring of finite $\mathbb{Q}$-adeles. The data $(\mathbf{G}', h, K)$ defines (see Deligne [D]) a proper smooth ("Shimura") surface S_K over $\mathbb{Q}$ whose space of complex points is

$$S_K(\mathbb{C}) = \mathbf{G}(F)\backslash\mathbf{G}'(\mathbb{A})/K_\infty K.$$

We shall be concerned with Tate's conjecture for the surface S_K, and the (fixed) absolutely irreducible finite dimensional representation (ξ, V) of $\mathbf{G}'$ over $\mathbb{Q}$. Note that the conjecture in [T] is stated with $\xi = 1$ only , but we follow the exposition of [HLR], see p. 66. The representation (ξ, V) defines an ℓ-adic sheaf $V_\xi(\mathbb{Q}_\ell)$ on the étale site $S_K \times_{\mathbb{Q}} \overline{\mathbb{Q}}$, and one has the associated ℓ-adic cohomology vector spaces

$$H^j = H^j(S_K \times_{\mathbb{Q}} \overline{\mathbb{Q}}, V_\xi(\mathbb{Q}_\ell)).$$

Here $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$, and we fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\overline{\mathbb{Q}}$, hence on $\mathrm{Spec}\,\overline{\mathbb{Q}}$, on $S_K \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ (via the second factor), and on the cohomology spaces. Denote the action on H^j by ρ^j. Note that $H^j = 0$ unless $j = 0, 2, 4$, and put $\rho = \rho^0 \oplus \rho^2 \oplus \rho^4$. Following [D] and [HLR] (but not [L]), in the definition of the canonical model we choose the reciprocity law homomorphism of class field theory which associates to the Frobenius substitution Fr_p the inverse p^{-1} of a local uniformizer. Denote by $\alpha : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\sqrt[\ell^\infty]{1})/\mathbb{Q}) \simeq \mathbb{Z}_\ell^\times$ the cyclotomic character corresponding to the absolute value character $\nu(x) = |x|$ of the idele class group $\mathbb{A}^\times/\mathbb{Q}^\times$.

Let d be a rational integer with $\xi(q) = q^{2d}$ for $q \in \mathbb{Q}^\times \subset \mathbf{G}(F)$. Let ω be a character of finite order of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Introduce the space

$$T(\omega) = \{x \in H^2(S_K \times_{\mathbb{Q}} \overline{\mathbb{Q}}, V_\xi(\mathbb{Q}_\ell));\ \rho^2(\tau)x = \alpha^{-1-d}(\tau)\omega^{-1}(\tau)x,\ \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}$$

of *Tate-cycles*, as in [HLR], (2.5), p. 66. The $\mathbb{Q}_\ell$-dimension $t(\omega)$ of $T(\omega)$ occurs in Tate's conjecture stated below.

The next ingredient in Tate's conjecture is the $\mathbb{Q}_\ell$-dimension $a(\omega)$ of the space $A(\omega) = A \cap T(\omega)$ of ω-*algebraic cycles*, which we proceed to define. Let E be an abelian extension of $\mathbb{Q}$, and $A^1(S_K \times_{\mathbb{Q}} \overline{\mathbb{Q}}, E)$ the $\mathbb{Q}_\ell$-span of the curves in $S_K \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ which are defined over E. Put $A^1 = \cup_E A^1(S_K \times_{\mathbb{Q}} \overline{\mathbb{Q}}, E)$, where the union ranges over all abelian extensions of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$. Then A is the image of the cycle map $A^1 \to H^2(S_K \times_{\mathbb{Q}} \overline{\mathbb{Q}}, V_\xi(\mathbb{Q}_\ell))$.

The last ingredient in the statement of the Tate conjecture concerns the L-function

$$L(s, S_K, \omega) = \prod_{p \notin S_0} \det[1 - p^{-s}\omega(Fr_p)\rho^2(Fr_p)]^{-1},$$

where the product ranges over all primes outside a finite set S_0 which contains all places where ω or ρ^2 ramify, and ∞. This product converges in some half plane $\mathrm{Re}(s) >> 0$, and has analytic continuation as a meromorphic function to a neighborhood of $s = d + 2$. Denote by $p(\omega)$ the *order of pole* at $s = d + 2$; see [HLR], (2.6), p. 66. It is independent of the set S_0.

Theorem. *For any* $\mathbf{G}$, h, K, (ξ, V), ω *as above, we have* $a(\omega) = t(\omega) = p(\omega)$.

This is the same as the conjecture of Tate [T] for the scheme S_K over $\mathbb{Q}$, in the case where $\xi = 1$. Let V'' be the set of $\mathbb{Q}$-places which split in F and where $\mathbf{G}$ is ramified. In the case where V'' has even cardinality, the Theorem coincides with Theorem 2.7 of [L]. The work of [L] consists of reducing the proof of [L], (2.7), to the proof in [HLR] of the analogous conjecture for the Shimura variety S_K associated with the group $GL(2)/F$, rather than with its inner forms.

Since the scheme of [HLR] is no longer proper, [HLR] work instead with intersection ℓ-adic cohomology. In the case considered in [L], where V'' is an even set of places of $\mathbb{Q}$ which split in F, let D be a quaternion division algebra central over $\mathbb{Q}$ which ramifies precisely at the places in V''. Then $\mathbf{G}(F) = (D \otimes_Q F)^\times$, and the main tool used in [L], to reduce the proof of [L], (2.7), to that of [HLR], is Lemma 4.5 of [L], which is the same as the Theorem of [JL], and also the same as the special case where V' is empty in our Theorem 0.3. The multiplicative group of this D is denoted by H' in [L], §4.

To prove the remaining case of the Theorem, where the set V'' has odd cardinality, let p be a finite $\mathbb{Q}$-prime which stays prime in F, and put $V^{(p)} = V'' \cup \{p\}$. Note that $p \notin V''$ since V'' consists of $\mathbb{Q}$-places which split in F. As in [L], (8.3), fix an inner form $\mathbf{D}^{(p)}$ of $GL(2)$ over $\mathbb{Q}$ which is ramified precisely at the places of $V^{(p)}$. Then $\mathbf{D}^{(p)}(F) = \mathbf{G}(F)$. We can work with $\mathbf{D}^{(p)}(F)$, for any p which does not split in F, instead of the H' of [L], §4, and the proof there is easily adjustable to rephrase [L], Corollary 4.4, as asserting that the Hirzebruch-Zagier number $Z(\omega,\pi')$ is positive if and only if the automorphic representation π' of $\mathbf{G}'(\mathbb{A})$ in [L], Corollary 4.4, is distinguished with respect to $\mathbf{D}^{(p)}(\mathbb{A})$ and ω, for some p (i.e. $\mathcal{T}_{\pi'} \neq 0$ where $\mathcal{T}_{\pi'} f$ is defined in the lines prior to [L], Lemma 4.3, with H' replaced by $\mathbf{D}^{(p)}(F)$).

The π' of [L], §4, is denoted by $\pi^{D^{(p)}}$ in Theorem 0.3, and both [L], Lemma 4.5, and Theorem 0.3, denote the corresponding automorphic representation of $GL(2,\mathbb{A}_F)$ by π. Theorem 0.3 asserts that $\pi^{D^{(p)}}$ is $\mathbf{D}^{(p)}(\mathbb{A})$-distinguished if and only if π is $GL(2,\mathbb{A})$-distinguished, and the component $\pi_p(\simeq \pi_p^{D^{(p)}})$ of π at p is not of the form $I(\mu_1,\mu_2)$ with characters μ_i of $F_p^\times$ trivial on $\mathbb{Q}_p^\times$. Note that only a finite number of automorphic representations π' occur in the decomposition ([L], §2.4) of the cohomology space H^j, since the representation ξ fixes the infinitesimal character and the compact open subgroup K fixes the ramification at all finite places.

We need to find a prime p_0 which stays prime in F for which Lemma 4.5 of [L] remains true provided that H' is replaced by $\mathbf{D}^{(p_0)}$. Given π' (which occurs in H^j), if such p_0 does not exist then at almost all places p of $\mathbb{Q}$ which stay prime in F the component π'_p of π would be of the form $I(\mu_1,\mu_2)$, where μ_i are characters of $F_p^\times$ which are trivial on $\mathbb{Q}_p^\times$. At almost all p the component π'_p is unramified, namely the μ_i are unramified, and consequently $\mu_1=\mu_2=1$. At almost all places p which split in $F/\mathbb{Q}$, since the component is distinguished (and unramified), it is of the form $I(\mu_1,\mu_2)\times I(\mu_2^{-1},\mu_1^{-1})$. To show that p_0 does exist, we will now show the following:

Lemma. *No cuspidal π' has components as described above.*

Proof. Consider first the (partial) twisted tensor L-function $L(t,\pi',r)$ of [F5]. At almost all p, the local factor is

$$\left(1-p^{-t}\right)^{-2}\left(1-\frac{\mu_1}{\mu_2}p^{-t}\right)^{-1}\left(1-\frac{\mu_2}{\mu_1}p^{-t}\right)^{-1}$$

if p splits and $(1-p^{-t})^{-2}(1-p^{-2t})^{-1}$ if p stays prime in $F/\mathbb{Q}$.

Consider also the symmetric square L-function $L(t,\pi',Sym^2)$ of [GJ] or [F9]. At the places which split and $Sym^2\,\pi'_p = I(\mu_1/\mu_2,1,\mu_2/\mu_1)\times I(\mu_1/\mu_2,1,\mu_2/\mu_1)$ is unramified, the local factor is

$$\left(1-p^{-t}\right)^{-2}\left(1-\frac{\mu_1}{\mu_2}p^{-t}\right)^{-2}\left(1-\frac{\mu_2}{\mu_1}p^{-t}\right)^{-2}.$$

At almost all primes where p stays prime the local factor associated with $Sym^2\,\pi'_p = I(1,1,1)$ is $(1-p^{-2t})^{-3}$.

Hence the quotient

$$\frac{L(t,\pi',r)^2}{L(t,\pi',Sym^2)}$$

is the product of $(1-p^{-t})^{-2}$ over almost all p which split, and of $(1-p^{-t})^{-4}(1-p^{-2t})$ over almost all p which stay prime in $F/\mathbb{Q}$. This can be expressed as a product over almost all p as follows:

$$\prod_p (1-p^{-t})^{-3} \cdot \prod_p (1-\chi(p)p^{-t}).$$

Here χ is the quadratic character of $\mathbb{A}^\times/\mathbb{Q}^\times$ associated with the quadratic extension $F/\mathbb{Q}$, so that $\chi(p)=1$ if p splits and $\chi(p)=-1$ if p stays prime. Consequently, we have

$$(*) \qquad L(t,\pi',r)^2 L(t,\chi) = \zeta(t)^3 L(t,\pi', Sym^2).$$

The ζ function on the right has a simple pole at $t=1$. The representation $Sym^2\,\pi'$ of $GL(3,\mathbb{A})$ is cuspidal, or is induced from a cuspidal representation of a Levi subgroup of a maximal parabolic of the form $\chi_1 \times \pi(\theta/\bar{\theta})$, where $\chi_1 \neq 1$ is a quadratic character of $\mathbb{A}^\times/F^\times$ associated with a quadratic extension F_1/F, and θ is a character of $\mathbb{A}_{F_1}^\times/F_1^\times$ (see [F9]). In any case, by [JS1] and [JS2] the function $L(t,\pi',Sym^2)$ has neither poles nor zeroes on $\mathrm{Re}(t)=1$. The function $L(t,\chi)$ is entire, and has no zeroes on $\mathrm{Re}(t)=1$ by [JS1]. By [F5], the twisted tensor L-function $L(t,\pi',r)$ has at most a simple pole at $t=1$, since ∞ splits in F. We obtain a contradiction to $(*)$, which asserts that $L(t,\pi',r)^2$ has a pole of order 3 at $t=1$. The lemma follows.

It follows from the Lemma that the required p_0 does exist, in fact there is an infinite number of such p_0's. For any such p_0, Lemma 4.5 of [L] remains true provided that H' is replaced by $D^{(p_0)}$. With this clarified, the proof of [L] establishes also our Theorem. Indeed, by [HLR] and [L], (5.1), in the notations of [L], (2.6), we have $B(\omega,\pi^{D^{(p)}}) = B(\omega,\pi) \leq 1$ with equality if and only if π is $GL(2,\mathbb{A})$-distinguished. The same conclusion holds by [L], (4.7), with B replaced by C (in the notations of [L], (2.6)). Further, if $\pi^{D^{(p)}}$ is $\mathbf{D}^{(p)}(\mathbb{A})$-distinguished for some p then $Z(\omega,\pi^{D^{(p)}})>0$, and by the Lemma if π is $GL(2,\mathbb{A})$-distinguished then there exists a p such that $\pi^{D^{(p)}}$ is $\mathbf{D}^{(p)}(\mathbb{A})$-distinguished.

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