A FOURIER FUNDAMENTAL LEMMA FOR THE SYMMETRIC SPACE $GL(n)/GL(n-1)$

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Let π be an irreducible unitarizable admissible representation of $GL(n, F)$, where F is a p-adic local field. Suppose that there is a non zero linear form on π which transforms trivially under $GL(n-1, F)$. Then π is trivial, or there exists an irreducible unitarizable admissible representation ρ of $GL(2, F)$ such that π is normalizedly induced from the representation of the parabolic subgroup of $GL(n, F)$ of type $(n-2, 2)$, which is trivial on the $(n-2)\times(n-2)$ block (and on the unipotent radical), and is ρ on the 2×2 block: $\pi = \mathrm{Ind}_{(n-2,2)}(1 \times \rho)$.

This is Proposition 0 of [F93] (see also Prasad [P93] for $n = 3$). It is proven on using techniques of Bernstein-Zelevinsky [BZ77], Zelevinsky [Z80], Tadic [T86], and there are analogues for finite groups (Thoma [Th71]) and real groups (van Dijk and Poel [DP90]), described in the introduction of [F93].

A global, automorphic, analogue, is proposed in [F93]. It concerns an interpretation of the question of determination of those automorphic forms on $GL(n)$ whose integral over a subgroup $GL(n-1)$, which would have given a natural, "automorphic", $GL(n-1)$-invariant linear form, is non zero. But there are no such cusp forms on $GL(n)$. The approach of [F93] is to develop a Fourier summation formula, analogous to the Selberg trace formula but involving no traces, on integrating the kernel $K_f(x, y)$ of the standard convolution operator on the space of automorphic forms on $GL(n)$, over y in $GL(n-1)$ and x in a suitable unipotent subgroup. The global question becomes that of the determination of the support of the spectral side: do all induced automorphic $\pi = \mathrm{Ind}_{(n-2,2)}(1\times\rho)$, ρ cuspidal on $GL(2)$, occur? Are there any other contributions (due to choice of truncation)?

On the other hand, the geometric side of the summation formula is described in Proposition 1 of [F93]. It has a particularly simple form, as a sum of global orbital integrals, which are products of local orbital integrals of test functions on $PGL(n, F)$, over orbits of the form ug_bh, $h \in GL(n-1, F)$ and u over a certain unipotent subgroup U, against a character $\psi(u)$ of U.

We expect the support of the spectral side of the summation formula to be parametrized by the unitary automorphic forms of $GL(2)$. Hence we may expect the geometric side, of orbital integrals, to be equal to the geometric side of a Fourier summation formula on $GL(2)$. Our assertion here is that such a relation indeed exists, and the corresponding

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formula on the $GL(2)$ side is a Fourier summation formula as in [F91], where the kernel $K_f(x,y)$, multiplied by a character $\psi(x)\psi(y)^{-1}$, is integrated over x and y in the upper unipotent subgroup of $GL(2)$.

This equality is expressed in terms of matching of orbital integrals of corresponding test functions on $GL(n,F)$ and $GL(2,F)$. The purpose of this note is to prove the "fundamental lemma" in this context: The Fourier orbital integrals of the characteristic functions $1_{n,K}$ and $1_{2,K}$ of the standard maximal compact subgroups $K_n = GL(n,R)$ in $GL(n,F)$ and $K_2 = GL(2,R)$ in $GL(2,F)$, are equal, for naturally related orbits. Here R is the ring of integers in F, and the fundamental lemma can also be phrased as: $1_{n,K}$ and $1_{2,K}$ are matching.

To establish the equality of the Fourier summation formulae one needs to verify that corresponding spherical functions are matching (the relation of "corresponding" is expressed in terms of Satake transforms as a statement dual to the morphism $\rho \mapsto \pi = \mathrm{Ind}_{(n-2,2)}(1\times\rho)$ of dual groups). Also one has to show that for each C_c^∞-function on $GL(n,F)$ there is such a function on $GL(2,F)$ and vice versa, so that they have matching orbital integrals.

More importantly, our study of the case $GL(n)/GL(n-1)$ can be viewed as the split case of the more interesting but more complicated case of the symmetric space $U(n)/U(n-1)$ associated with the quasi-split unitary group $U(n)$ in n variables, attached to a quadratic field extension E/F. The case of $n=3$ is studied in [F98]. In this unitary case we determine a family of cusp forms on the unitary group which are parametrized by cusp forms on $GL(2)$. In an unpublished work (to which J.G.M. Mars contributed; see Zinoviev [Zi98] for the relevant spherical fundamental lemma), using a similar technique we construct a family of almost everywhere non tempered cusp forms on the quasi-split group $GSp(4)$ ($\simeq SO(3,2)$) with non zero integrals over some subgroup $SO(3,1)$ (and non zero Fourier coefficients on the Siegel parabolic), parametrized by cusp forms of $GL(2)$. In the interest of clarity and simplicity, we restrict our attention in this note only to the case of $GL(n)/GL(n-1)$.

Our interest in the problem was rekindled recently while writing the orbital integral $\iint 1_{2,K}(xby)\psi(x^{-1}y)dx\,dy$ in the function field case (char $F>0$), as the trace of the Frobenius at the fiber of a perverse sheaf on a suitable orbital variety. Attempting to find an analogous perverse sheaf underlying the orbital integral $\iint 1_{n,K}(ug_bh)\psi(u)du\,dh$, we realized that the fundamental lemma can be proven by simple means. This is then done in this note, for local fields F of any characteristic. A surprising feature is the vanishing of the integrals on $GL(n,F)$ for non square g_b.

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STATEMENT OF THEOREM.

Let F be a local non archimedean field with a ring R of integers, local uniformizer π, residual characteristic $p \neq 2$, and cardinality q of the residual field $k = R/(\pi) = \mathbb{F}_q$. Let dx be the Haar measure on F normalized by assigning R the volume 1. Let $|.|$ be the absolute value $F^\times \to q^{\mathbb{Z}}$, with $|\pi| = q^{-1}$ (and $d(ax) = |a|dx$). Let ψ be a character of F with conductor (maximal subring where ψ is trivial) R, and complex values.

We are interested in the Fourier orbital integral ($c \in R^\times$, $b \in F$)

$$I(b^2c;\psi) = \int_F \int_F 1_{2,K}\big(\big(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\big)\big(\begin{smallmatrix} b^{-1} & 0 \\ 0 & bc/2 \end{smallmatrix}\big)\big(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\big)\big(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\big)\big)\psi(y-x)\,dx\,dy$$

$$= \iint 1_{2,K}\big(\big(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\big)\big(\begin{smallmatrix} bc/2 & 0 \\ 0 & b^{-1} \end{smallmatrix}\big)\big(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\big)\big)\psi(x+y)\,dx\,dy.$$

It is easy to compute:

Proposition 1. *For $|b| \le 1$ we have $I(c;\psi) = 1$ and $I(b^2c;\psi) = \int_{|x|=|b|^{-1}} \psi(x - 2/xb^2c)dx$ if $|b| < 1$; moreover, $I(b^2c;\psi) = 0$ for $|b| > 1$.*

In particular, $I(b^2c;\psi)$ depends only on b^2c, and we write $I(b;\psi) = 0$ if $\mathrm{ord}(b)$ is odd.

Proof. Consider the set of $\big(\begin{smallmatrix} bc/2 & ybc/2 \\ xbc/2 & b^{-1}+xybc/2 \end{smallmatrix}\big)$ in $GL(2,R)$. It is empty (since $c/2$ is a unit), unless $b \in R$. If $b \in R^\times$, then $x \in R$ and $y \in R$, and $I(b^2c;\psi) = 1$. If $|b| < 1$ then $|x|$, $|y| \le |b|^{-1}$ (use the entries $(1,2)$ and $(2,1)$ of our 2×2 matrix), and $|xy + 2/b^2c| \le |b|^{-1} < |b|^{-2}$ (using the entry $(2,2)$). Hence $|x| = |y| = |b|^{-1}$ and $|y + 2/xcb^2| \le 1$. Changing variables $y = \eta - 2/xcb^2$, the proposition follows. □

Remark. For a fixed $\alpha = \mathrm{diag}(a,b)$ in $PGL(2,F)$, the expression $f(n\alpha n')\psi(y-x)$, $n = \big(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\big)$, $n' = \big(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\big)$, for a function f on $GL(2,F)$, is well defined on the orbit $n\alpha n'$ (x, $y \in F$) only if it is equal to $f(\alpha \cdot \alpha^{-1} n\alpha n')\psi(y + xb/a)$ for all x, y, namely $\alpha = \mathrm{diag}(1,-1)$. Then $I_0(\psi) = \int_F 1_{2,K}\big(\big(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\big)\big(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\big)\big)\psi(2x)dx$ is defined. In fact it is a limit value of $I(b^2c/2;\psi)$.

We are also interested in the Fourier orbital integrals

$$J(g;\psi) = \int_{U_g} \int_H 1_{n,K}(ugh)\psi(u)\,du\,dh$$

for g in $G = PGL(n,F)$. Here H is the centralizer of $x_0 = {}^t\varepsilon\varepsilon$ in $G = GL(n,F)$ ($\varepsilon = (1,0,\dots,0,1)$, ${}^tp$ indicates the transpose of p), U is the unipotent subgroup of matrices $u = \left(\begin{smallmatrix} 1 & p & z+p{}^tq/2 \\ 0 & I & {}^tq \\ 0 & 0 & 1 \end{smallmatrix}\right)$, $z \in F$, $p = (p_1,\dots,p_{n-2})$ and $q = (q_1,\dots,q_{n-2})$ in F^{n-2}, $U_g = U \cap gHg^{-1}$, and $\psi(u)$ is $\psi(p_{n-2} + q_{n-2})$. Note that

$$x_0 = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right), \qquad \text{and} \qquad \tau = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & -1 \end{smallmatrix}\right)$$

satisfies

$$H = \Big\{\left(\begin{smallmatrix} a & p & b \\ {}^tq & m & -{}^tq \\ b & -p & a \end{smallmatrix}\right) = \tau^{-1}\left(\begin{smallmatrix} a+b & 0 & 0 \\ 0 & m & {}^tq \\ 0 & 2p & a-b \end{smallmatrix}\right)\tau\Big\} \simeq GL(n-1),$$

as well as

$$w = I - x_0 = \left(\begin{smallmatrix} 0 & 0 & -1 \\ 0 & I & 0 \\ -1 & 0 & 0 \end{smallmatrix}\right) = \tau\left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)\tau^{-1}.$$

Remark. Put $\psi_1(u) = \psi(p_1 + p_{n-2} + q_1 + q_{n-2}) = \psi(wuw^{-1})$, where the non zero entries of $w \in H \cap K$ are $w_{2,2} = 1/2$, $w_{2,n-1} = -1/2$, $w_{n-1,2} = 1$, $w_{n-1,n-1} = 1$, $w_{i,i} = 1$ $(i \neq 2,\, n-1)$. To compare $J(g;\psi)$ with the analogous integral arising from the unitary group, note that

$$J(g;\psi_1) = \int_{U_g}\int_H 1_{n,K}(ugh)\psi_1(u)du\,dh = \int_{U_g}\int_H 1_{n,K}(wuw^{-1}wgw^{-1}h)\psi(wuw^{-1})du\,dh$$

is equal to $\int_{U_{w(g)}}\int_H 1_{n,K}(uw(g)h)\psi(u)du\,dh = J(w(g);\psi)$, since $u \in U_g$ if and only if $w(u) = wuw^{-1} \in U \cap wgHg^{-1}w^{-1} = U_{w(g)}$.

Put $f'(g^t\varepsilon\varepsilon g^{-1}) = \int_H f(gh)dh$ to express our integral in the form

$$\int_{U_g} 1(ug^t\varepsilon\varepsilon g^{-1}u^{-1})\psi(u)du.$$

Here we write 1 for $1'_{n,K}$, and normalize the measure dh to assign $K \cap H$ the volume 1. Note that $g \in K$ and $gh \in K$ implies $h \in K$. Thus f' is a function on the homogeneous space $X = G/H$ of $n \times n$ matrices with rank 1 and trace 2. Put $g_b = \mathrm{diag}(1, I, b)$, $b \in F^\times$.

The union of the U-orbits

$$(ug_b{}^t\varepsilon)(\varepsilon g_b^{-1}u^{-1}) = {}^t(b(p^tq/2 + b^{-1} + z), bq, b)(1, -p, p^tq/2 + b^{-1} - z), \qquad b \in F^\times,$$

is an open subset of X. Note that $U \cap g_bHg_b^{-1} = \{1\}$, and put $J(b;\psi)$ for $J(g_b;\psi)$. Also put $J(0;\psi)$ for $J(g_0;\psi)$.

Proposition 2. *If $J(g;\psi) \neq 0$ then $g \in Ug_bH$ $(b \in F^\times)$ or $g \in Ug_0H$, where g_0 is such that the only non zero entry of $g_0x_0g_0^{-1}$ is 2 at $(n-1, n-1)$.*

The same result holds for $J_f(g;\psi) = \iint f(ugh)\psi(u)dudh$, for any function f for which the integral makes sense.

Proof. At g for which there is u with $J(ug;\psi) = J(g;\psi)$ and $\psi(u) \neq 1$, we have $J(g;\psi) = 0$. Suppose then that $x = g^t\varepsilon\cdot\varepsilon g^{-1} = {}^tvw = {}^t(v_1,\dots,v_n)(w_1,\dots,w_n)$ with $x_{n,1} = 0$ (otherwise $x = (ug_b{}^t\varepsilon)(\varepsilon g_b^{-1}u^{-1})$).

If $v_n \neq 0$ then $w_1 = 0$, and we use u with $q = 0$ and top line $(1, 0, \dots, 0, yv_n, -yv_{n-1})$; it has $u^tv = {}^tv$, $wu^{-1} = w$, and $\psi(u) = \psi(yv_n)$.

If $v_n = 0$ and f $(1 \leq f \leq n-2)$ is the least with $w_f \neq 0$, use u with $p = 0$, and $(z, q) = (0, \dots, 0, -yw_{n-1}, 0, \dots, 0, yw_f)$ ($-yw_{n-1}$ at the fth place), as then $u^tv = v$, $wu^{-1} = w$, and $\psi(u) = \psi(yw_f)$ (y is arbitrary in F). Then $v_n = 0$ and $w_1 = \cdots = w_{n-2} = 0$.

If $v_{n-1} = 0$, use u with $q = 0$, $z = 0$, and $p = (0, \dots, 0, y)$ (then $\psi(u) = \psi(y)$, any y). Then $v_{n-1} \neq 0$.

If $v_f \neq 0$ for f $(2 \leq f \leq n-2)$, use u with $z = 0$ and $q = 0$, and top row $(1, 0, \dots, 0, yv_{n-1}, 0, \dots, 0, -yv_f)$, the yv_{n-1} being at the fth place. Hence $v_2 = \cdots = v_{n-2} = 0$.

We are left with x of the form ${}^t(v_1, 0, \ldots, 0, v_{n-1}, 0)(0, \ldots, 0, w_{n-1}, w_n)$. But all entries of ug_0u^{-1} are 0 except the last two on the top row: these are $(2p_{n-2}, -2p_{n-2}q_{n-2})$, and the last two on the row before last: these entries are $(2, -2q_{n-2})$.

In summary, if $J(g;\psi) \neq 0$ and $x = g^t\varepsilon\varepsilon g^{-1}$ has $x_{n,1} = 0$, then x lies in the U-orbit of $g_0x_0g_0^{-1}$, as required. □

Our main result is the following.

Theorem. *We have* $J(b;\psi) = |b|^{-(n-2)/2}I(b;\psi)$ *for all* b *in* $F^\times$.

To prove this, we simply need to compute the Fourier orbital integrals $J(b;\psi)$, $n \geq 3$. This is done in the following.

Proposition 3. *The integral* $J(b;\psi)$ *is zero unless* $|b| \leq 1$ *and* $\operatorname{ord}(b)$ *is even, in which case it is 1 if* $|b| = 1$ *and it is* $|b|^{-(n-2)/2}I(b;\psi)$ *if* $|b| < 1$. *Recall:*

$$I(b;\psi) = \int_{|p|=|b|^{-1/2}} \psi(p - 2/bp)dp.$$

PROOF OF PROPOSITION 3.

We need to integrate $\psi(p_{n-2} + q_{n-2})$ over $p = (p_1, \ldots, p_{n-2})$ and $q = (q_1, \ldots, q_{n-2})$ in F^{n-2} and over z in F, such that $(ug_b{}^t\varepsilon)(\varepsilon g_b^{-1}u^{-1}) =$

$$\begin{pmatrix} 1 & p & p^tq/2+z \\ 0 & I & {}^tq \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ b \end{pmatrix}(1,0,b^{-1})\begin{pmatrix} 1 & -p & p^tq/2-z \\ 0 & I & -{}^tq \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b(p^tq/2+b^{-1}+z) \\ b^tq \\ b \end{pmatrix}(1, -p, p^tq/2 + b^{-1} - z)$$

$$= \begin{pmatrix} b(p^tq/2+1/b+z) & -bp(p^tq/2+1/b+z) & b(p^tq/2+1/b+z)(p^tq/2+1/b-z) \\ b^tq & -b^tqp & b^tq(p^tq/2+1/b-z) \\ b & -bp & b(p^tq/2+1/b-z) \end{pmatrix}$$

has entries in R. Since the volume of $|x| \leq 1$ is one, if p_i or q_i are in R, we may replace them by zero in our integral. If the entries are in R then $|b| \leq 1$. If $|b| = 1$ then $\|p\| = \max\{|p_i|\}$ and $\|q\|$ are ≤ 1, and the integral $J(b;\psi)$ is equal to one.

Lemma 1. *When* $|b| < 1$, *the contribution to* $J(b;\psi)$ *from the* u *with* $\|p\| \leq 1$ *or* $\|q\| \leq 1$ *is zero.*

Proof. If $p = 0 = q$, we need

$$\begin{pmatrix} b(1/b+z) & 0 & b(1/b+z)(1/b-z) \\ 0 & 0 & 0 \\ b & 0 & b(1/b-z) \end{pmatrix}$$

to be in $M_n(R)$, thus $z = \pm 1/b + \eta$, η in R. The integral over this domain is 2.

If $p = 0$ and $\|q\| > 1$, we need

$$\begin{pmatrix} b(1/b+z) & 0 & b(1/b+z)(1/b-z) \\ b^tq & 0 & b^tq(1/b-z) \\ b & 0 & b(1/b-z) \end{pmatrix}$$

to have entries in R. Using the entry $(1,n)$ we see that $z=\pm 1/b+\eta$. But if $z=-1/b+\eta$ ($\eta\in R$), using the entries (i,n), $1<i<n$, we shall get $\|q\|\le 1$. Hence $z=1/b+\eta$ and $\|q\|\le |1/b|$. The contribution to the integral from this domain is

$$\int_{\|q\|>1}\psi(q_{n-2})dq_{n-2}=\int_{\|q\|\le|1/b|}\psi(q_{n-2})dq_{n-2}-\int_{\|q\|\le 1}\psi(q_{n-2})dq_{n-2}=-1.$$

If $q=0$ and $\|p\|>1$, we need

$$\begin{pmatrix} b(1/b+z) & -b(1/b+z)p & b(1/b+z)(1/b-z)\\ 0 & 0 & 0\\ b & -bp & b(1/b-z)\end{pmatrix}$$

to have entries in R. Using the entry $(1,n)$ we see that $z=\pm 1/b+\eta$, $\eta\in R$, and that z is not $1/b+\eta$ using the top row. Then $z=-1/b+\eta$, $\|p\|\le|1/b|$, and the integral over this domain is again -1, as required. □

We then continue with the contribution to $J(b;\psi)$ from u with $\|p\|>1$ and $\|q\|>1$. There are p_i and q_j with $|p_i|>1$ and $|q_j|>1$. The maximum value of $|b|<1$ such that b^tqp has entries in R is $|\pi|^2$. We consider first this case, of $|b|=|\pi|^2$. Then the non zero entries of p and q are of absolute value $|\pi|^{-1}$ (if $|p_i|\le 1$ or $|q_j|\le 1$ we can and will replace them by zero). Since the entries $(1,2),\dots,(1,n-1)$ lie in R, $|b|=|\pi|^2$ and $\|p\|=|\pi|^{-1}$, we have $|p^tq+2/b+2z|\le|1/\pi|$. Using the entries $(2,n),\dots,(n-1,n)$ we get $|p^tq+2/b-2z|\le|1/\pi|$. Thus $|z|\le|1/\pi|$ and $|\sum_{1\le i\le n-2}p_iq_i+2/b|\le|1/\pi|$. Again, if $|p_i|\le 1$ or $|q_i|\le 1$, then p_iq_i can be removed from the sum.

Lemma 2. *When* $|b|=|\pi|^2$, $J(b;\psi)=q^{(n-2)}I(b;\psi)$.

Proof. The integral $J(b;\psi)$ is the product of $|b|^{-1/2}=q$ (this is the integral over z, $|z|\le|b|^{-1/2}$), and I_n, the integral of $\psi(p_{n-2}+q_{n-2})$ over p, q with $\|p\|=\|q\|=|b|^{-1/2}$ and $|\sum_{1\le i\le n-2}p_iq_i+2/b|\le|1/\pi|$. We write I_n as the sum of two contributions. The first, (i), is over the subset of p, q with $|p_{n-2}|\le 1$ or $|q_{n-2}|\le 1$. (This is the empty set when $n=3$). This part is the sum of: the integral over $|p_{n-2}|\le q$ and $|q_{n-2}|\le 1$ (the integral of $\psi(p_{n-2})$ is zero), the integral over $|p_{n-2}|\le 1$ and $|q_{n-2}|\le q$ ($\int\psi(q_{n-2})=0$), minus the integral over $|p_{n-2}|\le 1$ and $|q_{n-2}|\le 1$. Thus we get

$$-\operatorname{vol}\{(p_1,\dots,p_{n-3};q_1,\dots,q_{n-3});|p_i|\le q,\ |q_i|\le q,\ |\sum_{1\le i\le n-3}p_iq_i+2/b|\le|b|^{-1/2}\}.$$

The second contribution, (ii), is over the subset of p, q with $|p_{n-2}|=|q_{n-2}|=q$. Dividing by p_{n-2} (of absolute value $|b|^{-1/2}$) we get the inequality

$$|q_{n-2}+(\sum_{1\le i\le n-3}p_iq_i+2/b)/p_{n-2}|\le 1,$$

namely $|\sum_{1\le i\le n-3}p_iq_i+2/b|=|1/b|$; we integrate $\psi(p_{n-2}-(\sum_{1\le i\le n-3}p_iq_i+2/b)/p_{n-2})$ over this domain. Writing this domain as the difference of the domains $|*|\le|1/b|$ and

$| * | < |1/b| = q^2$, this second contribution is the difference of $(ii1)$, the integral over $|\sum_{1\leq i\leq n-3} p_iq_i + 2/b| \leq |1/b| = q^2$, namely

$$\int_{|p_{n-2}|=q} [\prod_{1\leq i\leq n-3} \int_{|p_i|,|q_i|\leq q} \psi(-p_iq_i/p_{n-2})dp_idq_i]\cdot\psi(p_{n-2}-2/bp_{n-2})dp_{n-2} = q^{n-3}\cdot I(b;\psi),$$

and the integral $(ii2)$ over $|\sum_{1\leq i\leq n-3} p_iq_i+2/b| \leq q = |b|^{-1/2}$ of $\int_{|p_{n-2}|=q} \psi(p_{n-2})dp_{n-2} = -1$. Then $(i) - (ii2) = 0$, and we conclude that $J(b;\psi)$ is the product of $|b|^{-1/2} = q$ and $q^{n-3} \cdot I(b;\psi)$, as required.

In the evaluation of $(ii1)$ we noted that $\int_{|x|,|y|\leq q} \psi(\pi xy)dx\,dy$ is the sum of the integral over $|x| \leq 1$ and $|y| \leq q$ (this integral is q), and over $|x| = q$ and $|y| \leq q$ (this integral is zero). The lemma follows. □

Lemma 3. *If $|b| < |\pi|^2$, and $J(b;\psi) \neq 0$, then it suffices to restrict the integration which defines $J(b;\psi)$ to $\|p\| = \|q\| = |p_{n-2}| = |q_{n-2}| = |b|^{-1/2}$; in particular,* $\mathrm{ord}(b)$ *is even, $|p^tq + 2/b| \leq |b|^{-1/2}$ and $|z| \leq |b|^{-1/2}$.*

Proof. By Lemma 1, $\|p\| > 1$ and $\|q\| > 1$. By (i,j) we denote, as usual, the (i,j) entry of the $n\times n$ matrix $(ug_b{}^t\varepsilon)(\varepsilon g_b^{-1}u^{-1})$. Considering the entries (i,j), $i,j \neq 1,n$, we conclude that $|b|\|p\|\|q\| \leq 1$ (this implies that $|b|\|p\| < 1$ and $|b|\|q\| < 1$, so no new information is provided by (i,j) $(i = 1,\ 1 < j < n;\ j = n,\ 1 < i < n)$).

The entries $(1,j)$ $(1 < j < n)$ being in R, we conclude that $|b|\|p\| \cdot |p^tq + 2/b + 2z| \leq 1$, namely that $|p^tq + 2/b + 2z| \leq 1/|b|\|p\| < 1/|b|$. The entries (i,n) $(1 < i < n)$ are in R, hence $|b|\|q\| \cdot |p^tq + 2/b - 2z| \leq 1$, thus $|p^tq + 2/b - 2z| \leq 1/|b|\|q\| < 1/|b|$. Together, these imply $|p^tq + 2/b| < |b|^{-1}$ and $|z| < |1/b|$, hence $|p^tq| = |1/b|$. Using (i,j), $i,j \neq 1,n$, we see that there is an i with $|p_i||q_i| = |1/b|$, hence $|b|\|p\|\|q\| = 1$. Then we get no new information from $(1,n) \in R$.

We claim there is no contribution (to the integral of $\psi(p_{n-2}+q_{n-2})$) from the range $1 < \|p\| < \|q\|$. To see this, make the change $q_{n-2} \mapsto q_{n-2}+t$, $|t| \leq |\pi|^{-1}$, and $z \mapsto z+p_{n-2}t/2$. Then $\|q\|$ is not changed. Also, $|tp_{n-2}| \leq \|q\| \leq 1/|b|\|p\|$, so (i,j) $(1 < j < n)$ and (i,n) $(1 < i < n)$ are in R (after the change). Consequently $\int_{1<\|p\|<\|q\|} \psi(p_{n-2} + q_{n-2})dpdqdz$ is equal to $\int_{1<\|p\|<\|q\|} \psi(p_{n-2} + q_{n-2} + t)dpdqdz$ for any $t \in \pi^{-1}R$, namely the integral over the range $1 < \|p\| < \|q\|$ is equal to itself multiplied by $\int_{\pi^{-1}R} \psi(t)dt = 0$.

The same argument applies to the range $1 < \|q\| < \|p\|$. We conclude that we may restrict integration to the range $\|p\| = \|q\| = |b|^{-1/2}$, and hence $\mathrm{ord}(b)$ is even, and $|p^tq + 2/b| \leq |b|^{-1/2}$, $|z| \leq |b|^{-1/2}$.

The range $|p_{n-2}| < \|q\|$ also contributes only zero to our integral, as is seen on applying the same change of variables. The same argument applies to the domain of $|q_{n-2}| < \|p\|$ (making the change $p_{n-2} \mapsto p_{n-2} + t$). We conclude that it suffices to restrict the domain of integration to $|p_{n-2}| = |q_{n-2}| = |b|^{-1/2}$, as required. □

Suppose then that $|b| < |\pi|^2$ and that $\mathrm{ord}(b)$ is even. Our integral $J(b;\psi)$ is the product of $|b|^{-1/2} = \int dz$ and the integral $I_n = \int \psi(p_{n-2} + q_{n-2})dpdq$, over $\|p\| = \|q\| = |p_{n-2}| = |q_{n-2}| = |b|^{-1/2} \geq |p^tq + 2/b|$.

Lemma 4. *If* $|b| < |\pi|^2$ *and* $\mathrm{ord}(b)$ *is even, then* $I_n = |b|^{-(n-3)/2} I(b;\psi)$.

Proof. Dividing $|p^t q + 2/b| \le |b|^{-1/2}$ by p_{n-2}, which has absolute value $|b|^{-1/2}$, we obtain $|q_{n-2} + (2/b + \sum_{1\le i\le n-3} p_i q_i)/p_{n-2}| \le 1$. Since $|q_{n-2}| = |b|^{-1/2}$, an equivalent condition is $|\sum_{1\le i\le n-3} p_i q_i + 2/b| = |b|^{-1}$. Hence $I_n = \int \psi(p_{n-2} - (2/b + \sum_{1\le i\le n-3} p_i q_i)/p_{n-2})$, over $|p_i|, |q_i| \le |p_{n-2}| = |b|^{-1/2}$ $(1 \le i \le n-3)$ and $|\sum_{1\le i\le n-3} p_i q_i + 2/b| = |b|^{-1}$. We write I_n as a sum of two terms, (i) and (ii).

The term (i) is the integral over the domain where $|p_{n-3}| \le 1$ or $|q_{n-3}| \le 1$ (and both are $\le |b|^{-1/2}$). The integral over p_{n-3} and q_{n-3} is $2|b|^{-1/2} - 1$, and in $|\sum p_i q_i + 2/b| = |b|^{-1}$, $p_i q_i$ with $|p_i| \le 1$ or $|q_i| \le 1$ can (and will) be replaced by zero. The term (i) is then the product of $(2|b|^{-1/2} - 1)$ and I_{n-1}.

The term (ii) is the integral over the domain where $|p_{n-3}| > 1$ and $|q_{n-3}| > 1$. Again we express it as a sum of two terms, $(ii1)$ and $(ii2)$. The term $(ii1)$ is the integral over the domain $|p_{n-3}q_{n-3}| < |1/b|$. Then $(ii1)$ is the product of

$$I_{n-1} = \int \psi(p_{n-2} - (2/b + \sum_{1\le i\le n-4} p_i q_i)/p_{n-2})$$

over $|p_i|, |q_i| \le |p_{n-2}| = |b|^{-1/2}$ $(1 \le i \le n-4)$, $|2/b + \sum_{1\le i\le n-4} p_i q_i| = |1/b|$, and

$$\int \psi(-p_{n-3}q_{n-3}/p_{n-2})dp_{n-3}dq_{n-3}, \quad 1 < |p_{n-3}|, |q_{n-3}| \le |b|^{-1/2}, \quad |p_{n-3}q_{n-3}| < |1/b|.$$

The last integral, $\int \psi(xy/p)dxdy$, over $1 < |x|, |y| \le |p| = |b|^{-1/2}$, $|xy| < |b|^{-1}$, is the sum of

$$\int_{|x|=|p|} dx \int_{1<|y|<|p|} \psi(xy/p)dy = -\int_{|x|=|p|} dx$$

and $\int_{1<|x|<|p|} T(x)dx$, where $T(x)$ is $\int_{1<|y|<|p|} \psi(xy/p)dy$

$$= \int_{1<|y|\le|p/x|} dy + \int_{|p/x|<|y|\le|p|} \psi(xy/p)dy = \int_{1<|y|\le|p/x|} dy - \int_{|y|\le|p/x|} dy = -1.$$

That is, it is $-\int_{1<|x|\le|p|} dx = -(|p| - 1) = 1 - |b|^{-1/2}$.

The term $(ii2)$ ranges over the domain where $|p_{n-3}| = |q_{n-3}| = |b|^{-1/2}$. There, $|q_{n-3} + (\sum_{1\le i\le n-4} p_i q_i + 2/b)/p_{n-3}| = |b|^{-1/2}$. This domain is not changed if p_{n-3} is replaced by $p_{n-3} + t$, $|t| \le |\pi|^{-1}$. But the integral of $\psi(-p_{n-3}q_{n-3}/p_{n-2})$ will be multiplied by $\psi(t\eta)$, $\eta = -q_{n-3}/p_{n-2}$ in $R^\times$, and hence be zero.

It follows that $I_n = (i) + (ii1) = |b|^{-1/2} I_{n-1}, = |b|^{-(n-3)/2} I_{n-3}$, as asserted. □

It follows that $J(b;\psi)$ is $|b|^{-1/2} I_n = |b|^{-(n-2)/2} I(b;\psi)$, and the proof of the proposition, and theorem, is complete. □

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