Math. Nachr. (1996),

CUSP FORMS ON $GL(2n)$ WITH $GL(n) \times GL(n)$ PERIODS, AND SIMPLE ALGEBRAS

By Yuval Z. Flicker of Columbus

(Received)

Abstract. The notion of a period of a cusp form on $GL(2,\mathbf{D}(\mathbb{A}))$, with respect to the diagonal subgroup $\mathbf{D}(\mathbb{A})^\times \times \mathbf{D}(\mathbb{A})^\times$, is defined. Here $\mathbf{D}$ is a simple algebra over a global field F with a ring $\mathbb{A}$ of adeles. For $\mathbf{D}^\times = GL(1)$, the period is the value at $1/2$ of the L-function of the cusp form on $GL(2,\mathbb{A})$. A cuspidal representation is called cyclic if it contains a cusp form with a non zero period. It is investigated whether the notion of cyclicity is preserved under the Deligne-Kazhdan correspondence, relating cuspidal representations on the group and its split form, where $\mathbf{D}$ is a matrix algebra. A local analogue is studied too, using the global technique. The method is based on a new bi-period summation formula. Local multiplicity one statements for spherical distributions, and non-vanishing properties of bi-characters, known only in a few cases, play a key role.

1. Statement of Main Result

A central simple algebra over a local or global field F has the form $\mathbf{M}_m(\mathbf{D}_d)$, where $\mathbf{M}_m$ is the algebra of $m\times m$ matrices, and $\mathbf{D}=\mathbf{D}_d$ is a division algebra central of degree d (dimension d^2) over F (see [We]). Denote the multiplicative group of $\mathbf{M}_{2m}(\mathbf{D}_d)$ by $\mathbf{G}$. This is an algebraic group over F, which is an inner form of $\mathbf{G}'=GL(2n)$, $n=md$. When F is global, put $G=\mathbf{G}(F)$, $\mathbb{G}=\mathbf{G}(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of F, as well as $Z=\mathbf{Z}(F)$, $\mathbb{Z}=\mathbf{Z}(\mathbb{A})$, where $\mathbf{Z}$ is the center of $\mathbf{G}$. Denote by $\mathbf{C}$ the (standard) Levi subgroup of the (upper triangular) parabolic subgroup of type (m,m) of $\mathbf{G}$; then $\mathbf{C}$ consists of $h=\operatorname{diag}(A,B)$, $A,B\in GL(m,\mathbf{D}_d)$. As usual, $C=\mathbf{C}(F)$, $\mathbb{C}=\mathbf{C}(\mathbb{A})$. Let η be a character of the idele class group $\mathbb{A}^\times/F^\times$, with $\eta^{2n}=1$. For g in $\mathbb{G}$, put $\eta(g)$ for $\eta(\det g)$ (one could also consider an arbitrary character η of $\mathbb{Z}C\backslash\mathbb{C}$).

This note concerns the integrals $P_\eta(\phi)=\int_{\mathbb{Z}C\backslash\mathbb{C}}\phi(h)\eta(h)^{-1}dh$ of cusp forms ϕ in $L_0^2(\mathbb{Z}G\backslash\mathbb{G})$ on $\mathbb{G}$ (see [BJ]) over the *cycle* $\mathbb{Z}C\backslash\mathbb{C}$. The value of the linear form P_η at the cusp form ϕ is called the *η-period* of ϕ on the cycle $\mathbb{Z}C\backslash\mathbb{C}$. The convergence of

1991 *Mathematics Subject Classification.* Primary 11F70; Secondary 11F71, 11Q25, 20G35.
Keywords and phrases. Cuspidal representations, spherical distributions, periods, bi-period summation formula, simple algebras, Deligne-Kazhdan correspondence.

the integral follows at once from the rapid decay of the cusp form ϕ on $\mathbb{Z}G\backslash\mathbb{G}$, since $\mathbb{Z}C\backslash\mathbb{C}$ has finite volume. Cuspidal (automorphic) representations π (= irreducible submodules of the $\mathbb{G}$-module $L_0^2(\mathbb{Z}G\backslash\mathbb{G})$ of cusp forms in $L^2(G\mathbb{Z}\backslash\mathbb{G})$) which contain a form ϕ with a non-zero η-period are called (here) *η-cyclic*. We say that π is *cyclic* if it is 1-*cyclic* and *η-cyclic*. We put P for P_1.

The interest in such cyclic π originates from studies of arithmetic cohomology, and lifting of automorphic forms. Such studies were initiated by Waldspurger [Wa] using the theory of the Weil representation, in the case of $m = d = n = 1$. Jacquet [J1] introduced a new technique for the study of such cusp forms, which he named the "relative trace formula". It is based on integrating the kernel of the convolution operator $K_f(x,y)$ over x and y in two cycles $\mathbb{Z}C_1\backslash\mathbb{C}_1$ and $\mathbb{Z}C_2\backslash\mathbb{C}_2$. The case of the group $\mathbb{C}\times\mathbb{C}$ and the subgroups $\mathbb{C}_1 = \mathbb{C}_2 = \mathbb{C}$ embedded diagonally, coincides with the standard trace formula.

In general Jacquet's relative trace formula involves no traces; it is a summation formula, equating a geometric with a spectral sums. The case $\mathbb{C}_1 = \mathbb{C}_2$ considered in this note is called here the "bi-period summation formula". Another notable case is introduced in Jacquet [J2] (see also [F3]); there $\mathbb{C}_2$ is a unipotent subgroup, and Fourier coefficients of the cusp forms (in addition to cycles) are obtained. We then refer to this special case of Jacquet's relative trace formula as the "Fourier summation formula". It is my pleasure to use this opportunity to thank H. Jacquet for his interest and influence, in the context of this note and that of other works in this area.

In this note we study a general case of the bi-period summation formula, with arbitrary m, d, n. Naturally, the general case – introduced here – opens up a new area of research, where more open questions than proven results are available. Our main purpose in this note is to point out some of these new notions and questions, as well as to prove the following conditional result.

Denote by V the finite set of F-places v where $\mathbf{D}_d$ does not split, thus for $v \notin V$ the group $G_v = \mathbf{G}(F_v)$ is isomorphic to $G'_v = \mathbf{G}'(F_v)$ (F_v is the completion of F at v).

Theorem 1.1. *Let u, u' be two places of F. Assume that $\mathbf{D}_d$ splits at u ($\mathbf{D}_d(F_u) = \mathbf{M}_d(F_u)$). Let π be a cuspidal $\mathbb{G}$-module whose component π_u at u is supercuspidal. Let π' be the cuspidal $\mathbb{G}'$-module which corresponds to π.*

Suppose that (WH1) and (WH2) hold for π_v ($v \in V \cup \{u'\}$), and that the component $\pi_{u'}$ at u' is bi-elliptic (see below). If π is cyclic then π' is cyclic (namely $P_\eta(\phi'_1) = \int_{\mathbb{Z}C'\backslash\mathbb{C}'} \phi'_1(h)\eta(h)^{-1}dh \neq 0$ and $P(\phi'_2) \neq 0$ for some cusp forms $\phi'_1, \phi'_2 \in \pi' \subset L_0^2(\mathbb{Z}G'\backslash\mathbb{G}')$, $\mathbf{C}' = \{\operatorname{diag}(A,B); A, B \in GL(md)\}$), and the bi-character of π'_v is not identically zero on the set of bi-regular elements of G'_v which come from G_v, for all v.

Suppose that (WH1) and (WH2) hold for π'_v ($v \in V \cup \{u'\}$). If π' is cyclic, $\pi'_{u'}$ is bi-elliptic, and the bi-character of π'_v is not identically zero on the set of bi-regular elements of G'_v which come from G_v ($v \in V$), then π is cyclic.

Proposition 2.1 establishes (WH1) in a special case, and Proposition 4.1 establishes the "bi-period summation formula", our main global tool. The proof of Theorem 1.1 is completed with Propositions 4.6 and 4.7. Then we state and prove Theorem 5.1, which concerns the transfer of the notion of cyclicity from $\mathbb{G}$ to an inner form $\mathbb{G}''$ whose invariants have the same denominators as those of $\mathbb{G}$. The local Theorem

5.2 establishes an analogue of Kazhdan's density theorem [K1], Appendix, for our bi-distributions. Finally Theorems 5.4 and 5.6 are local analogues of Theorem 1.1, dealing with the transfer of the notion of local cyclicity from π_v to the corresponding π'_v. A "quadratic" analogue of our work is carried out in [F4]. It will be interesting to compare the results of [F4] with those of the present note.

The cuspidal $\mathbb{G}$-module $\pi = \otimes\pi_v$ and the cuspidal $\mathbb{G}'$-module $\pi' = \otimes\pi'_v$ *correspond* if $\pi_v \simeq \pi'_v$ for almost all v (where $G_v \simeq G'_v$). It is shown in [FK2] that the cuspidal $\mathbb{G}$-modules π with a supercuspidal component π_u at some place $u \notin V$ occur with multiplicity one in $L^2_0(\mathbb{Z}G\backslash\mathbb{G})$; that they satisfy the rigidity theorem: if $\pi_1 = \otimes\pi_{1v}$ and $\pi_2 = \otimes\pi_{2v}$ have supercuspidal components $\pi_{1u} \simeq \pi_{2u}$, and $\pi_{1v} \simeq \pi_{2v}$ for almost all v, then $\pi_1 \simeq \pi_2$; and that the correspondence defines an embedding of the set of the cuspidal π with a supercuspidal π_u into the set of the cuspidal π' with a supercuspidal π'_u. The image consists of the π' whose local components π'_v are obtained by the local correspondence of relevant representations of G_v to relevant representations of G'_v, for all v. In particular, if π corresponds to π' then $\pi_v \simeq \pi'_v$ for all $v \notin V$.

In fact [FK2] sharpens the work of Bernstein-Deligne-Kazhdan-Vigneras [BDKV] and [F1] Ch. III, where the case of π' with a supercuspidal and in addition another square-integrable component, is dealt with. The global theorem requires in particular establishing the local correspondence not only for tempered local representations, but also for relevant local representations (since the generalized Ramanujan conjecture – asserting that all components of a cuspidal π' are tempered – is merely a conjecture).

The notion of relevant representations (the representations which may be components of a cuspidal $\mathbb{G}$-module) is introduced in [FK1] in a similar context (of an r-fold covering of $GL(n)$), where they are shown to be irreducible and unitarizable. This notion was later used e.g. by Patterson and Piatetski-Shapiro [PPS]. Of course all the main ideas in the proof of the correspondence are due to Deligne and Kazhdan. Their proof in the case of $m = 1$ ($d = n$; i.e. $\mathbf{G}$ is anisotropic) – which is remarkably simple – is explained in [F2].

The proofs of [F2], [F1] Ch. III, and [FK1], are based on the "Deligne-Kazhdan" simple trace formula, and that of [FK2] on a sharper form, the "regular" trace formula, where regular, Iwahori-invariant functions, are used. The proof here does not involve any trace formula, yet we do use some of the ideas which play key roles in the development of the simple trace formula. Our main global tool is a new "bi-period summation formula", obtained on integrating over two copies of $\mathbb{Z}C\backslash\mathbb{C}$ the spectral and geometric expressions for the kernel of the convolution operator $r(f)$ on $L^2(\mathbb{Z}G\backslash\mathbb{G})$, multiplied by the value of η at one of the variables, for a test function f with a supercuspidal component f_u. An observation of Kazhdan implies that $r(f)$ factorizes through the natural projection to the space $L^2_0(\mathbb{Z}G\backslash\mathbb{G})$ of cusp forms.

On the spectral side of our formula we obtain the periods of the cyclic cusp forms. On the geometric side we obtain a new type of bi-orbital integrals. As in [BDKV], [F2], [F1] Ch. III, [FK1], we choose another component – say $f_{u'}$ – of the test function f, and restrict its support to a certain set of "bi-elliptic bi-regular" elements in our bi-periodic sense. This choice of $f_{u'}$ greatly simplifies our study of the geometric side, indeed it makes our study possible. Yet the choice of $f_{u'}$ restricts the applicability of our technique to π and π' with a "bi-elliptic" (a notion presently to be defined) components at u'.

2. Invariant forms

Our proof is based on two statements, (WH1) and (WH2), which we accept here as "working hypotheses". In Proposition 2.1 we prove (WH1) in a special case. We verified (WH2) in some low rank cases; see [F5]. The (WH1) and (WH2) are analogues of similar statements for characters, whose proofs – we hope – are applicable (after some work) in our case too. As noted above, the present note can be viewed also as a motivation to study these hypotheses. Both hypotheses are local. They concern an irreducible admissible G_v-module π_v (see [BZ]), where $G_v = \mathbf{G}(F_v)$, and a complex valued character η_v of $F_v^\times$ (and G_v too, via $\eta_v(g) = \eta_v(\det g)$), with $\eta_v^{2n} = 1$.

Working hypothesis (WH1). *Let π_v be an admissible irreducible G_v-module. Then there exists at most one (up to a scalar multiple) linear form on π_v which transforms under C_v according to $\eta'_v(= 1$ or $\eta_v)$. Thus there is at most a single form $P_{\pi_v,\eta'_v} : \pi_v \to \mathbb{C}$ with $P_{\pi_v,\eta'_v}(\pi_v(h)\xi) = \eta'_v(h)P_{\pi_v}(\xi)$ for all $h \in C_v$ and $\xi \in \pi_v$.*

Alternatively put, $\dim \operatorname{Hom}_{C_v}(\pi_v, \eta'_v) \leq 1$, or: the restriction of π_v to C_v has the quotient η'_v with multiplicity at most one. A G_v-module π_v with $P_{\pi_v,\eta_v} \neq 0$ and $P_{\pi_v} \neq 0$ (we write P_{π_v} for $P_{\pi_v,1}$) is called *cyclic*. Each local component of a cyclic cuspidal π is cyclic, but a cuspidal π whose local components are all cyclic is not necessarily cyclic. Statements similar to (WH1) were established using techniques of Gelfand-Kazhdan [GK] (cf. [BZ], (5.16)-(5.17), (7.6)-(7.10), [R], [NPS]) to prove (existence in the case of $GL(n)$ and) uniqueness of Whittaker models, the uniqueness of a $GL(n, F_v)$-invariant linear form on an irreducible $GL(n, E_v)$-module where E_v/F_v is a quadratic field extension ([F3], p. 163), the uniqueness of a $GL(2, F_v)$-invariant form on a $GL(2, K_v)$-module where K_v is a cubic extension of F_v (Prasad [P], p. 1327), as well as in the cases of such pairs as $(GL(n-1), GL(n))$, $(O(n-1), O(n))$, $(U(n-1), U(n))$ by Bernstein, Piatetski-Shapiro, Rallis. The case of (WH1) where $\mathbf{D}$ is split has recently been treated by Jacquet and Rallis (in fact, after this note was written). I hope a proof of (WH1) in general would then appear soon. Let us verify (WH1) in a special case.

Proposition 2.1. *Let D_v be a division algebra of degree n central over F_v, and put $G_v = GL(2, D_v)$. For any admissible irreducible G_v-module π_v there exists at most one (up to a scalar multiple) linear form on π_v which transforms under C_v according to η_v.*

Proof. On replacing π_v with $\pi_v \otimes \eta_v$, it suffices to deal with the case where $\eta_v = 1$. By a well-known criterion of Gelfand-Kazhdan [GK] (recorded also in [P], p. 1327; [F3], p. 163), it suffices to find an involution $g \mapsto g^\#$ $(g^{\#\#} = g, (gh)^\# = h^\# g^\#)$ on G_v which preserves C_v, such that any bi-C_v-invariant distribution on G_v is fixed by $\#$. We shall check that the involution defined by $g \mapsto g^{-1}$, has this property. For that, note that the group G_v is the disjoint union of the open set

$$\bigcup_{xy\neq 0} C_v \begin{pmatrix} I & x \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & y \\ 0 & I \end{pmatrix} C_v = \bigcup_{\beta\neq 0} C_v \begin{pmatrix} I & I+\beta \\ I & I \end{pmatrix} C_v,$$

the closed set $P_v = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$, and the closed set

$$C_v \begin{pmatrix} I & * \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} C_v \bigcup C_v \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & * \\ 0 & I \end{pmatrix} C_v.$$

A bi-C_v-invariant distribution which is supported on the open set, or on P_v, is invariant under $g \mapsto g^{-1}$, since

$$\begin{pmatrix} I & I+\beta \\ I & I \end{pmatrix}^{-1} = -\beta^{-1} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & I+\beta \\ I & I \end{pmatrix} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

and

$$\begin{pmatrix} I & x \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & x \\ 0 & I \end{pmatrix} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

It suffices then to consider bi-C_v-invariant distributions on the closed set

$$\left[\bigcup_{x \in D_v} C_v \begin{pmatrix} I & x \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} C_v \right] \cup \left[\bigcup_{y \in D_v} C_v \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & y \\ 0 & I \end{pmatrix} C_v \right].$$

Via $f \mapsto \tilde{f}$, where $\tilde{f}(x,0) = f\left(\begin{pmatrix} I & x \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right)$ and $\tilde{f}(0,y) = f\left(\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & y \\ 0 & I \end{pmatrix} \right)$, such a distribution can be viewed as one on $X = \{(x,y) \in D_v \times D_v; xy = 0\}$, which is $D_v^\times$-invariant, where $D_v^\times$ acts by $\tilde{f} \mapsto \tilde{f}^t$, $\tilde{f}^t(x,y) = \tilde{f}(t^{-1}x, yt)$. We need to show that such a $D_v^\times$-invariant distribution on X is fixed by $\tilde{f} \mapsto \tilde{f}^\#$, where $\tilde{f}^\#(x,y) = \tilde{f}(-y,-x)$. The $D_v^\times$-invariant distribution $\delta_0(\tilde{f}) = \tilde{f}(0,0)$ on X is clearly $\#$-invariant.

The evaluation $f \mapsto f((0,0))$ gives rise to an exact sequence

$$0 \to C_c^\infty(D_v^\times) \oplus C_c^\infty(D_v^\times) \to C_c^\infty(X) \to C_c^\infty(\{(0,0)\}) = \mathbb{C} \to 0.$$

For the spaces of distributions we have the dual exact sequence

$$0 \to \mathbb{C} \to C_c^\infty(X)' \to C_c^\infty(D_v^\times)' \oplus C_c^\infty(D_v^\times)' \to 0.$$

Taking invariants under the action of $D_v^\times$ on X by $t : (x,y) \mapsto (t^{-1}x, yt)$ we get

$$0 \to \mathbb{C} \to C_c^\infty(X)'^{D_v^\times} \to \mathbb{C} \oplus \mathbb{C},$$

since any $D_v^\times$-invariant distribution on $D_v^\times$ is a multiple of $f_1 \mapsto \int_{D_v^\times} f_1(x) \frac{dx}{|x|}$ ($|x| =$ absolute value of the reduced norm $D_v^\times \to F_v^\times$). The involution $\#$ acts on this exact sequence by $(x,y) \mapsto (y,x)$, hence trivially on $\mathbb{C}$, and by interchanging the two copies of $\mathbb{C}$ in $\mathbb{C} \oplus \mathbb{C}$.

To prove the proposition we only need to show that the image of $C_c^\infty(X)'^{D_v^\times}$ in $\mathbb{C} \oplus \mathbb{C}$ is fixed under $\#$. Namely we need to show that the image is contained in $\mathbb{C} \cdot (1,1)$, or that for any distribution L on X we have $L(f_1) = L(f_2)$, where f_1 is the characteristic function of $D_v^0 \times 0$, and f_2 is that of $0 \times D_v^0$, in X. Here D_v^0 is the multiplicative group of $D_v^1 = \{x \in D_v; |x| \le 1\}$. For this, let f be the characteristic function of $\{(x,y) \in X; x \in D_v^1 \text{ or } y \in D_v^1\}$. If $\pi \in D_v^1 - D_v^0$ has $|\pi|$ of maximal value, then $L(f^\pi) = L(f)$ and $L(f_1 - f_2) = L(f - f^\pi) = 0$, as required.

I am indebted to D. Prasad for communicating to me the last paragraph, which simplifies my original proof. However, the general case would require using Bernstein's Fourier transform techniques. This will be given elsewhere.

But let us sketch Bernstein's technique in the case of $G_v = GL(2, F_v)(n = 1)$. We need to show that any $F_v^\times$-invariant distribution E on $F_v \times F_v$ ($F_v^\times$ acts by $t : (x,y) \mapsto (tx, yt^{-1})$) which is skew-#-symmetric (where $\# : (x,y) \mapsto (y,x)$), is zero.

First note that the restriction of such E to the complement of the coordinate axis X in F_v^2 is 0. Indeed, on the line (tx, yt^{-1}), $txy \ne 0$, this E – up to a multiple – is $\int_{F_v^\times} f(tx, yt^{-1}) \frac{dt}{|t|}$. Hence $E(f^\#) = \int f(yt^{-1}, tx)\frac{dt}{|t|}$ is $E(f)$, on replacing t by $yt^{-1}x^{-1}$, and it is $-E(f)$ by the skew-#-symmetry, hence it is zero.

There is another action of $F_v^\times$, by $h_t(x,y) = (tx, ty)$. Then $h_t E(f) = E(h_{t^{-1}} f)$, where $h_{t^{-1}} f(x,y) = f(h_t(x,y))$. The distribution E on X is said to be *homogeneous of degree* n if $h_t E = |t|^n E$. For example, δ $(f \mapsto f((0,0)))$ and $\frac{dx}{|x|}$ are homogeneous of degree 0.

The exact sequence

$$0 \to C_c^\infty(X - \{(0,0)\}) \to C_c^\infty(X) \to C_c^\infty(\{(0,0)\}) \to 0$$

gives rise to a dual exact sequence of distributions

$$0 \to C_c^\infty(\{(0,0)\})' \to C_c^\infty(X)' \to C_c^\infty(X - \{(0,0)\})' \to 0.$$

Here $C_c^\infty(\{(0,0)\})'$ is spanned by δ, and $C_c^\infty(X - \{(0,0)\})'$ by $\overline{e}_1 = \frac{dx}{|x|}$, $\overline{e}_2 = \frac{dy}{|y|}$. Hence $C_c^\infty(X)'$ is spanned by δ, and by e_1, e_2, whose images are $\overline{e}_1$, $\overline{e}_2$. Since h_t fixes δ, $\overline{e}_1$, $\overline{e}_2$, it acts as a unipotent transformation on δ, e_1, e_2. Thus $(h_t - 1)^3$ acts as zero on $C_c^\infty(X)'$. Fix a non-trivial character $\psi : F \to \mathbb{C}^\times$, and define the Fourier transform $\mathfrak{F}E$ of E by $\mathfrak{F}E(f) = E(\mathfrak{F}f)$, where $\mathfrak{F}f(x,y) = \int_{F_v}\int_{F_v} f(u,v)\psi(xu + yv)dudv$. Clearly $\mathfrak{F}E$ is $F_v^\times$-invariant and skew-#-symmetric,hence zero outside the coordinate axis. On the other hand, $h_t(\mathfrak{F}f(x,y)) = \int\int f(u,v)\psi(utx + vty)dudv = |t|^{-2}\mathfrak{F}(h_{t^{-1}} f)$. Hence $\mathfrak{F}(h_t f) = |t|^{-2} h_{t^{-1}}(\mathfrak{F}f)$. Then $0 = \mathfrak{F}((h_t - 1)^3 E) = (|t|^{-2} h_{t^{-1}} - 1)^3 \mathfrak{F}E$. But the eigenvalues of $h_{t^{-1}}$ are 1, not $|t|^2$. Hence $\mathfrak{F}E = 0$, and $E = 0$, as required. □

3. Bi-characters

Let $H_v = C_c^\infty(Z_v \backslash G_v)$ denote the convolution algebra (a choice of a Haar measure is implicit) of compactly supported (modulo Z_v) smooth K_v-finite (= locally constant when v is non-archimedean) complex-valued functions on G_v which transform trivially under Z_v. Fix an orthonormal basis $\{\xi_v\}$ in the space of the irreducible admissible G_v-module π_v. Introduce a *bi-period* distribution on H_v by

$$\mathbb{P}_{\pi_v}(f_v) = \mathbb{P}_{\pi_v,\eta_v}(f_v) = \sum_{\xi_v} P_{\pi_v}(\pi_v(f_v)\xi_v)\overline{P_{\pi_v,\eta_v}(\xi_v)}.$$

The linear form P_{π_v,η'_v} lies in the dual π_v^* of π_v. It also defines an element – denoted $P^\vee_{\pi_v,\eta'_v}$ – in the dual $\widetilde{\pi}_v^*$ of the contragredient $\widetilde{\pi}_v$ of π_v – by $P^\vee_{\pi_v,\eta'_v}(\xi_v^\vee) = \overline{P_{\pi_v,\eta'_v}(\xi_v)}$, where$\{\xi_v^\vee\}$ is a basis of $\pi_v^\vee$ dual to $\{\xi_v\}$. Note that$P^\vee_{\pi_v,\eta'_v} = P_{\pi_v^\vee,\eta'^{-1}_v}$, since

$$P^\vee_{\pi_v,\eta'_v}(\pi_v^\vee(h)\xi_v^\vee) = P^\vee_{\pi_v,\eta'_v}((\pi_v(h)\xi_v)^\vee)$$

$$= \overline{P_{\pi_v,\eta'_v}(\pi_v(h)\xi_v)} = \eta_v^{-1}(h)P^\vee_{\pi_v,\eta'_v}(\xi_v^\vee).$$

Put $\langle P_{\pi_v,\eta'_v},\xi_v\rangle = P_{\pi_v,\eta'_v}(\xi_v)$ and $\langle P^\vee_{\pi_v,\eta'_v},\xi_v^\vee\rangle = P^\vee_{\pi_v,\eta'_v}(\xi_v^\vee)$. Then $P^\vee_{\pi_v,\eta'_v}$ decomposes as $P^\vee_{\pi_v,\eta'_v} = \sum_{\xi_v}\langle P^\vee_{\pi_v,\eta'_v},\xi_v^\vee\rangle\xi_v$, and

$$\langle P_{\pi_v},\pi_v(f_v)P^\vee_{\pi_v,\eta_v}\rangle = \sum_{\xi_v}\langle P^\vee_{\pi_v,\eta_v},\xi_v^\vee\rangle\langle P_{\pi_v},\pi_v(f_v)\xi_v\rangle$$

is an alternative expression for $\mathbb{P}_{\pi_v}(f_v)$.

This $\mathbb{P}_{\pi_v}(f_v)$ is clearly independent of the choice of the basis $\{\xi_v\}$ of π_v. If $\pi_{1v},\dots,\pi_{kv}$ are pairwise inequivalent, then $\mathbb{P}_{\pi_{1v}},\dots,\mathbb{P}_{\pi_{kv}}$ are linearly independent (for a proof see the following Remark). Since $\mathbb{P}_{\pi_v}$ is independent of the choice of basis for π_v, it is C_v-η_v-invariant, namely its value at ${}^af_v^b(g) = f_v(a^{-1}gb)$, $(a,b\in C_v)$ is equal to its value at f_v, multiplied by $\eta_v(b)^{-1}$. In particular the distribution $\mathbb{P}_{\pi_v}$ depends on f_v only via the bi-period integral

$$\Xi(\gamma,f_v) = \Xi(\gamma,f_v,\eta_v) = \int_{C_v/C_v\cap\gamma C_v\gamma^{-1}}\int_{C_v/Z_v} f_v(h\gamma h')\eta_v(h')dh\,dh'.$$

The convergence of this bi-orbital integral is obvious when γ is bi-regular (see below). Note that without assuming (WH1), the bi-period distribution $\mathbb{P}_{\pi_v}$ of π_v is not uniquely defined.

Remark 3.1. Let us associate a bi-invariant distribution to any admissible irreducible representation π of a p-adic reductive group G, and prove − along standard lines − that it determines the equivalence class of π. Examples of such distributions are the trace $\operatorname{tr}\pi(f)$, and the bi-period distribution $\mathbb{P}_\pi(f)$ discussed above.

To introduce the bi-invariant distribution, let C_1,C_2 be subgroups of G, and ζ_1,ζ_2 characters of C_1,C_2 into $\mathbb{C}^\times$. Let P_i be non-zero linear forms on π such that $P_i(\pi(h)\xi) = \zeta_i(h)P_i(\xi)$ for all $\xi\in\pi$ and $h\in C_i$. Fix an orthonormal basis $\{\xi\}$ for the space of π, and put

$$p_\pi(f) = \sum_{\{\xi\}} P_1(\pi(f)\xi)\overline{P_2(\xi)} \qquad (f\in C_c^\infty(G)).$$

If ${}^af^b(g) = f(agb^{-1})$ then $p_\pi({}^af^b) = \zeta_1(a)\overline{\zeta}_2(b)p_\pi(f)$ if $a\in C_1, b\in C_2$. The distribution $p_\pi(f)$ is independent of the choice of a basis $\{\xi\}$. Indeed, if $\{\beta\}$ is another such basis, then $\beta = \sum_\xi(\beta,\xi)\xi$, and

$$\sum_{\{\beta\}} P_1(\pi(f)\beta)\overline{P_2(\beta)} = \sum_{\beta,\gamma,\xi}(\beta,\xi)\overline{(\beta,\gamma)}P_1(\pi(f)\xi)\overline{P_2(\gamma)}$$

$$= \sum_{\gamma,\xi} P_1(\pi(f)\xi)\overline{P_2(\gamma)}(\sum_{\beta}(\gamma,\beta)\beta,\xi) = p_\pi(f).$$

The trace distribution $\operatorname{tr}\pi(f)$ can be recovered when G is $H \times H$ and $C_1 = C_2$ is H embedded diagonally, $\zeta_1 = \zeta_2 = 1, \pi = \rho \times \rho^\vee$, and $P_i(\xi \times \xi^\vee) = (\xi, \xi^\vee)$ is the C_i-invariant form, where $(\xi, \xi^\vee)$ is the duality of ρ and its contragredient $\rho^\vee$. If $f = f_1 \times f_2^*, f_2^*(g) = \overline{f}_2(g^{-1})$, then

$$p_\pi(f) = \sum_{\xi_1,\xi_2}(\rho(f_1)\xi_1, \rho^\vee(f_2^*)\xi_2^\vee)\overline{(\xi_1,\xi_2^\vee)} = \sum_{\xi}(\rho(f_2 * f_1)\xi_1, \xi_1^\vee) = \operatorname{tr}\rho(f_2 * f_1),$$

where $\{\xi_1\} = \{\xi_2\} = \{\xi\}$, and $\{\xi_i^\vee\}$ is the basis dual to $\{\xi_i\}$.

The distribution $\mathbb{P}_\pi$ is obtained on taking $C_1 = C_2 = C, \zeta_1 = 1, \zeta_2 = \eta', P_1 = P_\pi$ and $P_2 = P_{\pi,\eta'}$.

Proposition 3.2. *Let $\{\pi_1, \cdots, \pi_n\}$ be a set of pairwise inequivalent irreducible admissible representations of $\mathbb{H} = C_c^\infty(G)$. Then $\{p_{\pi_1}, \cdots, p_{\pi_n}\}$ is a linearly independent set of linear forms on $\mathbb{H}$.*

Proof. Denote by V_i the space of π_i. Let $e \in \mathbb{H}$ be the characteristic function of some sufficiently small compact open subgroup K of G, divided by the volume of K, such that $V_i^K = \pi_i(e)V_i \neq \{0\}$ and P_1, P_2 are non-zero on $V_i^K (1 \leq i \leq n)$. Let $\tilde{\pi}_i$ be the representation of $\mathbb{H}^K = e\mathbb{H}e$ on the finite dimensional space V_i^K. If $\tilde{\pi}_i \simeq \tilde{\pi}_j$ then there is an invertible linear map $A : V_i^K \to V_j^K$ which commutes with the action of $\mathbb{H}^K$. We claim that $\pi_i \simeq \pi_j$.

To show this, choose $v_i \neq 0$ in V_i^K and put $v_j = Av_i$. Then $A(\pi_i(f)v_i) = \pi_j(f)v_j$ defines an isomorphism $\pi_i \tilde{\to} \pi_j$ which commutes with the action of $\mathbb{H}$ provided that $\pi_i(f)v_i = 0$ if and only if $\pi_j(f)v_j = 0$ for all $f \in \mathbb{H}$. But $\pi_i(f)v_i = 0$ iff $\pi_i(e * h)\pi_i(f)v_i = 0$ for all h in $\mathbb{H}$, and $e * h * f * e \in \mathbb{H}^K$.

Consequently the $\tilde{\pi}_1, \cdots, \tilde{\pi}_n$ are inequivalent. It suffices to show that the linear forms $\tilde{p}_1, \cdots, \tilde{p}_n$ on $\mathbb{H}^K$ are linearly independent, where $\tilde{p}_i$ is the restriction of p_{π_i} to $\mathbb{H}^K$. Fix $h \in \mathbb{H}^K$. As $\tilde{\pi}_i$ is irreducible and finite dimensional, $\tilde{p}_i(hf) = 0$ for all $f \in \mathbb{H}^K$ iff $\tilde{\pi}_i(h) = 0$. We claim that for any $h_1, \cdots, h_n \in \mathbb{H}^K$, if $\sum_{i=1}^n \tilde{p}_i(h_i f) = 0$ for all f in $\mathbb{H}^K$, then $\tilde{\pi}_i(h_i) = 0$ for all i. If not, denote by $m (2 \leq m \leq n)$ the least number of indices i with $\tilde{\pi}_i(h_i) \neq 0$ for some choice of h_i's. Rearranging indices, suppose that $\tilde{\pi}_1(h) = 0$ while $\tilde{\pi}_2(h)$ is invertible. But then $\sum_{i=2}^m \tilde{p}_i(h_i h f) = 0$ is a shorter relation of the same type ($\tilde{\pi}_i(h_i h)$ not all zero), contradicting the minimality of m. Thus if $\sum_{i=1}^n \alpha_i \tilde{p}_i(f) = 0$ for all $f \in \mathbb{H}^K$, taking $h_i = \alpha_i e$ (α_i are complex scalars) it follows that $\alpha_i \tilde{\pi}_i(e) = \tilde{\pi}_i(h_i) = 0$, thus $\alpha_i = 0 \ (1 \leq i \leq n)$, as required. □

Working Hypothesis (WH2). *Let π_v be a cyclic admissible irreducible G_v-module. Then there exists a C_v-η_v-invariant ($p(h'gh) = \eta_v(h)p(g); h, h' \in C_v$) complex valued* *function $p(g,\pi_v)$, which is smooth (= locally constant if v is non-archimedean) and not identically zero on a Zariski open (hence dense) subset of G_v (named bi-regular below), such that*

$$\mathbb{P}_{\pi_v}(f_v)=\int_{Z_v\backslash G_v} f_v(g)p(g,\pi_v)dg.$$

In the archimedean case, this has been shown by Sekiguchi [S]. The function $p(g,\pi_v)$ is named here the *bi-character* of π_v. It is analogous to the character $\chi(g,\pi_v)$ or $\chi_{\pi_v}(g)$ of the trace distribution $\operatorname{tr}\pi_v(f_v)=\int f_v(g)\chi(g,\pi_v)dg$, shown (in the p-adic case) by Howe [H] and Harish-Chandra [HC3] to be locally constant on the regular set (which is Zariski open), and moreover (see Harish-Chandra [HC2]) locally integrable on G_v. In fact, the introduction of the character η_v makes our distribution resemble the distribution $\operatorname{tr}(\pi(f)A_\pi)$ investigated by Kazhdan in [K2], p. 211. The proof of [HC3] shows that the restriction of $\mathbb{P}_{\pi_v}$ to the space of functions $f_v^{K_v}(g)=\int_{K_v} f_v(kgk^{-1})dk$ (K_v = good maximal compact subgroup of G_v) is represented by a smooth function on the regular set. Since $\operatorname{tr}\pi_v(f_v)=\operatorname{tr}\pi_v(f_v^{K_v})$, this establishes the result for the trace distribution. It would be interesting to extend this simple proof of [HC3] to apply in our case too.

A similar question is dealt with in [FH], where it is shown − using Howe's orbit method as in [HC2] − that the bi-character exists as a locally constant function on the relatively(=bi)-regular set (introduced there), in the case of $GL(n,D_v)$-invariant distributions on $GL(n,D'_v)$-modules, where D_v is a division algebra central over F_v, while $D'_v=D_v\otimes_{F_v}E_v$, where E_v/F_v is a quadratic field extension. A recent work by Rader and Rallis extends this method to show that the bi-character is locally constant on the bi-regular set in the present case too. The case of a supercuspidal π_v is discussed in the Remark below.

The local integrability ([HC2]) implies that the character is not identically zero on the regular set, in the case of the trace. The bi-character of [FH] is also locally integrable, hence not identically zero on the bi-regular set. This quadratic case is very close to that of Harish-Chandra's group case. But in general, $p(g,\pi_v)$ often fails to be locally integrable on G_v. It may be supported on the closed proper subset of "bi-singular" elements. It will be interesting to determine which π_v satisfy (WH2). We expect all cyclic admissible G_v-modules to satisfy (WH2), in analogy with the archimedean case, see Sekiguchi [S] and Kengmana [Ke]. We have recently shown this (in [F5]) for $n=1$ and $n=2$ − using the germ expansion of the spherical character near the nilpotent cone (due to Rader and Rallis) − and we believe that similar techniques would apply for a general n. However this would require a separate paper, dealing specifically with the local theory.

The relation

$$\begin{aligned}\begin{pmatrix}A & B\\ C & D\end{pmatrix} &= \begin{pmatrix}A & 0\\ 0 & C\end{pmatrix}\begin{pmatrix}I & A^{-1}BD^{-1}C\\ I & I\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & C^{-1}D\end{pmatrix}\\ &= \begin{pmatrix}A & 0\\ 0 & C\end{pmatrix}\begin{pmatrix}I & 0\\ I & I\end{pmatrix}\begin{pmatrix}I & A^{-1}B\\ 0 & C^{-1}D-A^{-1}B\end{pmatrix}\end{aligned}$$

(A,B,C,D in $GL(m,D_v)$, I = identity in $GL(m,D_v)$) shows that on the open dense subset

$$X_v = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}; |A| \neq 0, |C| \neq 0, |D| \neq 0, |C^{-1}D - A^{-1}B| \neq 0 \right\}$$

(where $|A|$ is $\det A$), a set of representatives for $C_v \backslash X_v / C_v$ is given by the matrices $\gamma = \begin{pmatrix} I & \beta + I \\ I & I \end{pmatrix}$, $|\beta| \neq 0$, β determined up to conjugacy in $GL(m, D_v)$. Note that $C_v \cap \gamma C_v \gamma^{-1}$ consists of $Z(\beta) = \{ \operatorname{diag}(A, A); A\beta A^{-1} = \beta \}$.

In analogy with the classical case we say that $g \in G_v$ is *bi-regular* if it is *bi-conjugate* (its product agb on the right and left by a, b in C_v is equal) to $\gamma = \gamma(\beta)$ with regular β (distinct eigenvalues). A $g \in G_v$ is *bi-elliptic* if it is bi-conjugate to $\gamma = \gamma(\beta)$ with an elliptic β. The Zariski open set in (WH2) will be the bi-regular set. A cyclic G_v-module π_v is called *bi-elliptic* if its bi-character is not identically zero on the bi-elliptic bi-regular set. Theorem 1.1 concerns π with a bi-elliptic component $\pi_{u'}$.

Denote by $p_\beta(z) = \det(z - \beta)$ the characteristic polynomial of the conjugacy class in $GL(m, D_v)$ of β. In the case of $m = 1$, the map $\beta \mapsto p_\beta$ is a bijection from the set of regular (necessarily elliptic) conjugacy classes in D_v, to the set of separable irreducible polynomials of degree d over F_v (the same statement holds globally with (F, D) replacing (F_v, D_v)). In general, the map $\beta \mapsto p_\beta$ is a bijection from the set of regular (resp. elliptic regular) conjugacy classes in $GL(m, D_v)$, to the set of separable (resp. irreducible separable) polynomials of degree dm over F_v whose irreducible factors have degrees which are multiples of d.

In particular the set of regular conjugacy classes in $GL(m, D_v)$ embeds as a subset of the set of regular conjugacy classes in $GL(n, F_v)$, $n = md$. A regular conjugacy class in $GL(n, F_v)$ so obtained is said to *come from* $GL(m, D_v)$. The set of regular elliptic conjugacy classes in $GL(m, D_v)$ bijects with the set of regular elliptic conjugacy classes in $GL(n, F_v)$. We say that the bi-regular bi-conjugacy class $a\gamma(\beta)b$ (a, b in C'_v) in G'_v *comes from* G_v if β is regular in $GL(n, F_v)$ and its conjugacy class comes from $GL(m, D_v)$. With this definition, the statement of Theorem 1.1 is now complete.

Remark 3.3. *If π_v is cyclic and supercuspidal, then its bi-character is smooth on the set of the bi-regular bi-elliptic elements, and the set of their transposes.* Indeed, the linear form $\mathbb{P}_{\pi_v}$ is the unique (up to a scalar multiple) non-zero C_v-η_v-invariant linear form on H_v which vanishes on the orthogonal complement of the span of the space of matrix coefficients of π_v. Hence $\mathbb{P}_{\pi_v}(f_v)$ is equal – up to a constant multiple – to

$$\int_{C_v/Z_v} \int_{C_v/Z_v} \langle \pi_v(f_v)\pi_v(h)\xi, \tilde{\pi}_v(h')\xi^\vee \rangle \eta_v(h) dh\, dh'$$
$$= \int_{C_v/Z_v} \int_{C_v/Z_v} \int_{G_v/Z_v} f_v(g) \langle \pi_v(h'gh)\xi, \xi^\vee \rangle dg\, \eta_v(h) dh\, dh',$$

for any vector $\xi \neq 0$ in π_v.

If g is bi-regular bi-elliptic, then it is of the form $g = c'\gamma(\beta)c$, and its bi-centralizer

$$Z_v(g) = \{(h', h) \in C_v \times C_v; h'gh = g\}$$

is equal to

$$\left\{\left(h' = c'\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} c'^{-1}, h = c^{-1}\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}^{-1} c\right); t \in Z_v(\beta)\right\} \simeq Z_v(\beta)$$

where $Z_v(\beta)$ indicates the centralizer of β in $GL(m, D_v)$. This $Z_v(\beta)$ is an elliptic torus – isomorphic to the multiplicative group of the separable extension of F_v of degree n generated by the elliptic regular element β; in particular the volume $|Z_v(g)/Z_v|$ is finite, for such g.

Now suppose that f_v is supported on the bi-regular bi-elliptic set. Then we may change the order of integration, obtaining (equality up to a scalar multiple depending on the choice of ξ):

$$\mathbb{P}_{\pi_v}(f_v) = \int_{G_v/Z_v} f_v(g)|Z_v(g)/Z_v|\Xi(g, c_{\pi_v})dg,$$

where $c_{\pi_v}(g) = \langle \pi_v(g)\xi, \xi^\vee \rangle$ is a matrix coefficient of π_v. In particular the bi-character$p(g, \pi_v)$ of a supercuspidal cyclic π_v is given on the bi-regular bi-elliptic set by $p(g, \pi_v) = |Z_v(g)/Z_v|\Xi(g, c_{\pi_v})$. It is therefore smooth on the bi-regular bi-elliptic set, and on the set of transposes of these elements $-$ which is analogously handled.

However we have not verified that $p(g, \pi_v)$ is not identically zero on the bi-regular bi-elliptic set. In the classical case of characters, it is verified in [HC1] that the characters of the supercuspidal representations are locally integrable functions, and that their restrictions to the elliptic regular subset satisfy orthonormality relations. In particular the character of a supercuspidal representation is not identically zero on the elliptic regular set. It will be interesting to establish an analogue in our case.

Note that the theory of Whittaker models applies with $GL(m, F_v)$ replaced by $GL(m, D_v)$, and a non-trivial character $(u_{ij}) \mapsto \psi(\sum_i \operatorname{tr}_{D_v/F_v} u_{i,i+1})$ on the upper triangular unipotent subgroup (but we have no reference for this analogue). Using this it is clear that any unitarizable irreducible infinite dimensional $GL(2, D_v)$-module π_v is cyclic. Indeed the linear form $\mathbb{P}_{\pi_v}(W) = \int_{D_v^\times} W(\operatorname{diag}(a, 1))d^\times a$ on the Whittaker model $W(\pi_v)$ of π_v is well defined (the integral converges by the asymptotic behaviour of W), it is C_v-invariant, and non zero, since the space of functions $\{a \mapsto W(\operatorname{diag}(a, 1)); W \in W(\pi_v)\}$ contains $C_c^\infty(D_v^\times)$ (with equality if π_v is supercuspidal). Moreover, all elements of $GL(2, D_v)$ are bi-elliptic and bi-regular, or transposes of such, except those in the bi-conjugacy class of the identity. Hence at most one (presumably none) $GL(2, D_v)$-module π_v may have a bi-character which vanishes outside C_v.

4. Proof of Theorem 1.1

The main global tool in the proof of Theorem A is the following *bi-period summation formula*.

Proposition 4.1. *Let $f = \otimes f_v$ be a test function on $\mathbb{G}$ which has a supercuspidal component f_u and a component $f_{u'}$ supported on the bi-elliptic bi-regular set. Then*

$$\sum_\pi \sum_\Phi P(\pi(f)\Phi)\overline{P_\eta(\Phi)} = \sum_{\{\beta\}} |\mathbb{Z}(\beta)/\mathbb{Z}Z(\beta)| \prod_v \Xi(\gamma(\beta), f_v).$$

Here π ranges over the cuspidal cyclic $\mathbb{G}$-modules with a supercuspidal component at u, Φ ranges over an orthonormal basis of smooth vectors in the space of π, and $\{\beta\}$ ranges over a set of representatives of the elliptic regular conjugacy classes in G.

Proof. Let $K_f(x,y)$ be the kernel of the convolution operator $(r(f)\phi)(x) = \int_{\mathbb{Z}\backslash\mathbb{G}} f(g)\phi(xg)dg$ on $L^2(\mathbb{Z}G\backslash\mathbb{G})$. Here $f = \otimes f_v$ is a product over all places v of F of $f_v \in H_v$, such that f_v is the unit element f_v^0 in the convolution algebra $\mathbb{H}_v$ of spherical (bi-K_v-invariant, K_v being the standard maximal compact subgroup of G_v) function in H_v, for almost all v. It is easy to see that $(r(f)\phi)(x) = \int_{\mathbb{Z}\backslash\mathbb{G}} K_f(x,y)\phi(y)dy$, where $K_f(x,y) = \sum_{\gamma\in Z\backslash G} f(x^{-1}\gamma y)$. This is the geometric expansion of the kernel.

We take the component f_u of f to be a supercusp form. A well-known observation of Kazhdan (see [F1] Ch. III) asserts that $r(f)$ then factorizes through the natural projection into the subspace $L_0^2(\mathbb{Z}G\backslash\mathbb{G})$ of cusp forms in $L^2(\mathbb{Z}G\backslash\mathbb{G})$. Then the kernel has the spectral expansion $K_f(x,y) = \sum_\pi \sum_\Phi (\pi(f)\Phi)(x)\overline{\Phi}(y)$. The first sum ranges over the set of cuspidal $\mathbb{G}$-modules π (in fact with a supercuspidal component at u), and Φ ranges over an orthonormal basis of smooth vectors in the space of π. Note that it is π – and not its equivalence class – which occurs here, by virtue of the multiplicity one theorem for such π of ([F1] Ch. III, and) [FK2].

Our formula is obtained on integrating these two expressions for the kernel, multiplied by $\eta(y)$, over x, y in $\mathbb{Z}C\backslash\mathbb{C}$. The integral of the spectral expression is

$$\sum_\pi \sum_\Phi P(\pi(f)\Phi)\overline{P_\eta(\Phi)}, \quad P_{\eta'}(\Phi) = \int_{\mathbb{Z}C\backslash\mathbb{C}} \Phi(h)\eta'(h)^{-1}dh, \qquad \eta' = 1 \,\mathrm{or}\, \eta.$$

The integral over x, y in $\mathbb{Z}C\backslash\mathbb{C}$ of the geometric expression for the kernel is

$$\int_{\mathbb{C}/C\mathbb{Z}} dx \int_{\mathbb{Z}C\backslash\mathbb{C}} \sum_{\gamma\in G/Z} f(x\gamma y)\,\eta(y)dy = \sum_{\gamma\in C\backslash G/C} \int_{\mathbb{C}/\mathbb{Z}\cdot C\cap\gamma C\gamma^{-1}} dx \int_{\mathbb{Z}\backslash\mathbb{C}} f(x\gamma y)\,\eta(y)dy.$$

We take the component $f_{u'}$ of f at u' to be supported on the bi-regular bi-elliptic set. Consequently the rational bi-$\mathbb{C}$-orbits (the set of $x\gamma y$, with x, y in $\mathbb{C}$, and γ in G) on which f is non-zero, are those of the bi-regular bi-elliptic γ, represented by $\gamma = \gamma(\beta) = \begin{pmatrix} I & I+\beta \\ I & I \end{pmatrix}$, where β is an elliptic regular element of $GL(m,D)$. A complete set of representatives of these rational bi-orbits is given by $\gamma(\beta)$, as β ranges over a set of representatives $\{\beta\}$ for the conjugacy classes of elliptic regular elements in $GL(m,D)$.

Note that $a\gamma(\beta)a^{-1} = \gamma(A\beta A^{-1})$, where $a = \operatorname{diag}(A,A)$. Since $C\cap\gamma C\gamma^{-1} = Z(\beta)$, where $Z(\beta)$ is the group of a such that $A\beta A^{-1} = \beta$, our double integral is equal to

$$= \sum_{\{\beta\}} |\mathbb{Z}(\beta)/\mathbb{Z}Z(\beta)| \int_{\mathbb{C}/\mathbb{Z}(\beta)} dx \int_{\mathbb{Z}\backslash\mathbb{C}} f(x\gamma(\beta)y)\eta(y)dy.$$

The double integral here can be expressed as a product, for $f = \otimes f_v$, of local bi-orbital integrals. Thus we obtain

$$= \sum_{\{\beta\}} |\mathbb{Z}(\beta)/\mathbb{Z}Z(\beta)| \prod_v \Xi(\gamma(\beta), f_v),$$

where the sum is finite and the product is absolutely convergent, as required. □

The following is clear.

Lemma 4.2. *Let $f_v \in H_v$ be a function on G_v supported on the bi-regular set. Then $\Xi(\gamma, f_v)$ is a smooth function with compact support on the union of $\gamma(T_v/W(T_v))$ over a set of representatives $\{T_v\}$ of the conjugacy classes of the F_v-tori in $GL(m, D_v)$, where $W(T_v)$ is the Weyl group (normalizer/centralizer) of T_v, and $\gamma(T_v/W(T_v))$ is the set of $\gamma(\beta)$, $\beta \in T_v$, up to $\gamma(w\beta w^{-1}) \sim \gamma(\beta)$ for $w \in W(T_v)$.*

Conversely, given a smooth compactly supported function $\Xi(\gamma)$ on the bi-regular subset of $\cup_{\{T_v\}}\gamma(T_v/W(T_v))$, there exists an $f_v \in H_v$ supported on the bi-regular set, with $\Xi(\gamma) = \Xi(\gamma, f_v)$. Both statements hold with "bi-regular" replaced by "bi-regular and bi-elliptic" throughout, except that now T_v ranges over the classes of elliptic F_v-tori only.

Of course the discussion above holds not only for $\mathbf{G}$ but for any inner form of it, in particular for $\mathbf{G}' = GL(2n)$ (this is the split case, where $d = 1$). To establish the comparison of the Theorem, we compare the geometric sides of the bi-periodic summation formula for $f = \otimes f_v$ on $\mathbb{G}$ and for $f' = \otimes f'_v$ on $\mathbb{G}'$. For this comparison fix a non degenerate differential form of highest degree on $\mathbf{G}$ over F. It defines a Haar measure on G_v and G'_v, hence on $\mathbb{G}$ and $\mathbb{G}'$, in a compatible way. These measures, dg_v, dg, $d'g_v$ and $d'g$, are used to define the bi-period orbital integrals $\Xi(\gamma, f_v)$ and $\Xi(\gamma, f'_v)$, as well as the distributions $P_{\eta'}(\Phi)$ and $P_{\eta'}(\Phi')$.

Definition 4.3. The functions $f_v \in H_v$ and $f'_v \in H'_v$ are called *matching* if $\Xi(\gamma', f'_v)$ is zero on the bi-regular γ' which do not come from G_v, while if γ is a bi-regular element of G'_v which comes from γ in G_v, then $\Xi(\gamma', f'_v) = \Xi(\gamma, f_v)$.

For all $v \notin V$, where V is the finite set of places where D_v does not split, we have that $G_v \simeq G'_v$ and we take f_v and f'_v to correspond to each other under this isomorphism. At the remaining finite number of places v in V, Lemma 4.2 guarantees the existence of f'_v matching any f_v which is supported on the bi-regular set of G_v. This f'_v can be taken to be supported on the bi-regular set of G'_v, in fact on the (open) set of such elements which come from G_v.

Conversely, given any f'_v whose bi-period orbital integrals are supported on the set of bi-regular elements of G'_v which come from G_v, Lemma 4.2 guarantees the existence of an f_v, supported on the bi-regular set of G_v, matching f'_v.

Lemma 4.4. *For any test functions $f = \otimes f_v$ on $\mathbb{G}$ and $f' = \otimes f'_v$ on $\mathbb{G}'$ such that $f_v = f'_v$ for all $v \notin V$, $f_v = f^0_v$ for almost all v, f_u is a supercusp formant $f_{u'}$ supported on the bi-regular bi-elliptic set of $G_{u'}$ ($u \neq u'$, both outside V), and f_v, f'_v matching* *for all $v(\in V)$, we have*

$$\sum_{\pi'}\sum_{\Phi'} P(\pi'(f')\Phi')\overline{P_\eta(\Phi')} = \sum_{\pi}\sum_{\Phi} P(\pi(f)\Phi)\overline{P_\eta(\Phi)}.$$

The sums range over the cuspidal $\mathbb{G}'$-modules π' and cuspidal $\mathbb{G}$-modules π, whose components at u are supercuspidal, and over orthonormal bases of smooth vectors Φ' in π' and Φ in π.

Proof. Our choice of matching f and f', as well as matching measures, guarantees the equality of the geometric sides of the bi-period summation formulae for f on $\mathbb{G}$ and f' on $\mathbb{G}'$ of Proposition 4.1. Hence the spectral sides are equal. □

Lemma 4.5. *Let π be a cuspidal $\mathbb{G}$-module with a supercuspidal component $\pi_u(u \notin V)$, and π' the corresponding cuspidal $\mathbb{G}'$-module. Let S be a finite set of places of F containing V, u, u', and all archimedean places and those where π_v is not unramified. If $f_v \in H_v$ and $f'_v \in H'_v$ are matching $(v \in V)$, $f_u = f'_u$ is a supercusp form, and $f_{u'}, f'_{u'}$ are supported on the bi-regular bi-elliptic sets of $G_{u'}$, $G'_{u'}$, and $f_v = f'_v$ $(v \in S - V)$, then*

$$\sum_{\Phi' \in \pi'^{K'(S)}} P(\pi'_S(f'_S)\Phi')\overline{P_\eta(\Phi')} = \sum_{\Phi \in \pi^{K(S)}} P(\pi_S(f_S)\Phi)\overline{P_\eta(\Phi)}.$$

Here $\mathbb{K}(S) = \prod_{v \notin S} K_v$ $(\simeq \mathbb{K}'(S))$, where K_v is the standard maximal compact $GL(2n, R_v)$ of $G_v \simeq G'_v$; $\pi^{\mathbb{K}(S)}$ is the space of $\mathbb{K}(S)$-fixed vectors in π; Φ ranges over an orthonormal basis of smooth vectors in $\pi^{\mathbb{K}(S)}$. Finally $\pi_S(f_S)$ is $\prod_{v \in S} \pi_v(f_v)$.

Proof. We work with f and f' whose components are spherical (K_v-biinvariant) at each $v \notin S$. Note that $\pi_v(f_v)$ acts as 0 on Φ unless Φ is K_v-invariant, in which case $\pi_v(f_v)$ acts as multiplication by a scalar, denoted again by $\pi_v(f_v)$. Putting $\pi^S(f^S) = \prod_{v \notin S} \pi_v(f_v)$, the identity of Lemma 4.4 can be written as

$$\sum_{\pi'}\sum_{\Phi' \in \pi'^{K'(S)}} \pi'^S(f'^S)P(\pi'_S(f'_S)\Phi')\overline{P_\eta(\Phi')} = \sum_{\pi}\sum_{\Phi \in \pi^{K(S)}} \pi^S(f^S)P(\pi_S(f_S)\Phi)\overline{P_\eta(\Phi)}.$$

A standard argument – originally expressed by Langlands (in the case of $GL(2)$) – used in [F2], [F1], [FK1], [FK2], ..., of "linear independence of characters", based on varying the spherical components of f at the $v \notin S$, using standard unitarity estimates, the Stone-Weierstrass theorem and the absolute convergence of the sums in Lemma 4.4, implies our claim. Of course, we use in the statement of the Lemma multiplicity one theorem for $\mathbb{G}'$ and for $\mathbb{G}$ ([F1] Ch. III, [FK2]), as well as rigidity theorem for $\mathbb{G}'$ and for $\mathbb{G}$ ([F1] Ch. III, [FK2]). □

Proposition 4.6. *Suppose that π is a cuspidal cyclic $\mathbb{G}$-module with a supercuspidal component π_u $(u \notin V)$ and a bi-elliptic component $\pi_{u'}(u' \neq u)$. Suppose that (WH1)* *and (WH2) hold for π_v for all $v \in V$ and $v = u'$. Then the corresponding cuspidal G'-module π' is cyclic, its component at u' is bi-elliptic, and the bi-character of π'_v $(v \in V)$ is not identically zero on the set of bi-regular elements of G'_v which come from G_v.*

Proof. It suffices to show that the side of π in the identity displayed in Lemma 4.5 is non zero. Consider smooth Φ_1, Φ_2 in $\pi^{\mathbb{K}(S)}$ such that $P_\eta(\Phi_1) \neq 0$, and $P(\Phi_2) \neq 0$. In the following proof we regard $\pi^{\mathbb{K}(S)}$ as an abstract representation, rather than in its automorphic realization. Denote by $\xi_0 = \xi_0^S$ the preferred $\mathbb{K}(S)$-fixed vector in $\pi^S = \bigotimes_{v \notin S} \pi_v$, and fix an orthonormal basis $\{\xi_v\}$ of smooth vectors in π_v. Then $\{\xi_0 \otimes (\bigotimes_{v \in S} \xi_v); \xi_v \in \{\xi_v\}, v \in S\}$ is an orthonormal basis of $\pi^{\mathbb{K}(S)}$. Any smooth vector in $\pi^{\mathbb{K}(S)}$ is a finite linear combination of such factorizable vectors.

Expressing the vector $\Phi_i (i = 1, 2)$ as a linear combination of vectors including $\xi_i = \xi_0 \otimes (\bigotimes_{v \in S} \xi_{iv})$ etc., since $P_\eta(\Phi_2) \neq 0, P(\Phi_1) \neq 0$ we may assume that the restriction of P_η to ξ_1 and of P to ξ_2 is non zero. At each $v \in S - V$, $v \neq u$, u', we choose $f_{1v} \in H_v$ such that $\pi_v(f_{1v})\xi_v = 0$ for all $\xi_v \in \{\xi_v\}$, $\xi_v \neq \xi_{1v}$, and $\pi_v(f_{1v})\xi_{1v} = \xi_{2v}$. Such a choice is possible since H_v spans the algebra of endomorphisms of π_v.

In fact this choice can be made also at the place u, where π_u is supercuspidal. Indeed, by the Schur orthogonality relations the matrix coefficient $f_{1u}(x) = (\pi_u(x)\xi_{1u}, \xi_{2u}^\vee)$ acts as zero on any ξ_u orthogonal to ξ_{1u}, and it maps ξ_{1u} to ξ_{2u} (if necessary, we can multiplying f_{1u} by a scalar). Moreover, such a matrix coefficient is a supercusp form (since π_u is supercuspidal), as required to apply Lemma 4.5. With this choice of $f_v = f_{1v}$ $(v \in S - V, v \neq u')$, our sum $\sum P(\pi_S(f_S)\Phi)\overline{P(\Phi)}$ ranges over the vectors Φ whose component outside $V' = V \cup \{u'\}$ is $\xi^{V'} = \xi_0 \otimes (\bigotimes_{v \in S - V'} \xi_{1v})$. Put also $f^{V'} = (\bigotimes_{v \in S - V'} f_{1v}) \otimes (\bigotimes_{v \notin S} f_v^0)$.

The side of π in the identity of Lemma 4.5 can now be expressed as

$$\langle P_{V'}, \pi_{V'}(f_{V'}) P^\vee_{V',\eta} \rangle,$$

where $P_{V',\eta}$ is the restriction of the linear form $\langle P_\eta, \Phi \rangle = P_\eta(\Phi) = \int_{\mathbb{Z}C \backslash \mathbb{C}} \Phi(h)\eta'(h)^{-1} dh$ to $\xi^{V'} \otimes \pi_{V'}$; $P_{V',\eta'}$ lies in the dual $\pi^*_{V'}$ of $\pi_{V'}$. The integral analogously defines a linear form $P^\vee_{\eta'}$ in the dual $\widetilde{\pi}^*$ of the contragredient $\widetilde{\pi}$ of π, which consists of the $\overline{\Phi}$, $\Phi \in \pi$. Namely $\langle P^\vee_{\eta'}, \overline{\Phi} \rangle = \int_{\mathbb{Z}C \backslash \mathbb{C}} \overline{\Phi}(h)\eta'(h) dh$. Denote by $P^\vee_{V',\eta'}$ the restriction of $P^\vee_{\eta'}$ to $(\xi^{V'})^\vee \otimes \widetilde{\pi}_{V'}$. Here $\{\xi_v^\vee\}$ signifies the basis dual to $\{\xi_v\}$, and $\xi_v^{0\vee} = \xi_v^0(\widetilde{\pi}_v)$. Note that $\pi_{V'}(f_{V'})P^\vee_{V',\eta'} \in \pi_{V'}$. Hence $\langle P_{V'}, \pi_{V'}(f_{V'})P^\vee_{V',\eta} \rangle$ is defined. It is equal to the side of π in the identity of Lemma 4.5 as explained when $\mathbb{P}_{\pi_v}(f_v)$ was introduced, before (WH2) was stated. Note that $\pi^{V'}(f^{V'})\Phi$ is a cusp form for each cusp form Φ.

We shall now use (WH1) for π_v $(v \in V')$. It asserts the uniqueness of the form P_{π_v, η'_v} on π_v, up to a scalar multiple. The existence of P_{π_v, η'_v} follows from the cyclicity of π. Since the components of Φ outside $V \cup \{u'\}$ are fixed, there is a constant $c(\pi, \eta')$, depending on these components, such that

$$P_{V',\eta'} = c(\pi, \eta') \bigotimes_{v \in V'} P_{\pi_v, \eta'_v}.$$

Our sum then takes the form

$$c(\pi, 1)c(\pi, \eta) \prod_{v \in V \cup \{u'\}} \mathbb{P}_{\pi_v}(f_v), \quad \mathbb{P}_{\pi_v}(f_v) = \langle P_{\pi_v}, \pi_v(f_v) P^{\vee}_{\pi_v, \eta_v} \rangle.$$

At the place u' we use (WH2). We take $f_{u'}$ which is supported on the bi-elliptic bi-regular set, such that

$$\mathbb{P}_{\pi_{u'}}(f_{u'}) = \int_{Z_{u'} \backslash G_{u'}} f_{u'}(g) p(g, \pi_{u'}) dg$$

is non-zero. The choice of such $f_{u'}$ is clearly possible, since the bi-character $p(g, \pi_{u'})$ of $\pi_{u'}$ is locally constant on the bi-regular set, and is assumed to be non-zero on the bi-regular bi-elliptic set.

Similarly, at each $v \in V$ other than u', we can choose f_v which is supported on the bi-regular set of G_v, with $\mathbb{P}_{\pi_v}(f_v) \neq 0$, again using (WH2): the bi-character is smooth on the bi-regular set, and is not identically zero there. As noted following Lemma 4.2, there are functions f'_v $(v \in V)$ matching the f_v. The matching f'_v will be supported on the set of bi-regular (also bi-elliptic when $v = u'$) elements of G'_v which come from G_v.

With this choice of f_v $(v \in S)$, since π is cyclic, the right side of the identity displayed in Lemma 4.5 is non-zero. Hence the left side is non-zero. This means that π' is cyclic, and $\mathbb{P}_{\pi'_v}(f'_v) \neq 0$ $(v \in S)$ for the matching function f'_v. Since the matching function f'_v is supported on the bi-regular (also bi-elliptic when $v = u'$) elements of G'_v which come from G_v, and $\int_{Z_v \backslash G'_v} f'_v(g) p(g, \pi'_v) dg \neq 0$, the bi-character $p(g, \pi'_v)$ is not identically zero on this set, as asserted. □

Proposition 4.7. *Let π' be a cuspidal cyclic $\mathbb{G}'$-module which corresponds to a cuspidal $\mathbb{G}$-module π. Suppose that π'_u is supercuspidal $(u \notin V)$, that $\pi'_{u'}$ is bi-elliptic, and that for each $v \in V$, the bi-character of π'_v is not identically zero on the set of bi-regular elements which come from G_v. Suppose also that (WH1) and (WH2) hold for π'_v $(v \in V \cup \{u'\})$. Then π is cyclic.*

Proof. The discussion at the places $v \in S - V \cup \{u'\}$, including the case of the supercuspidal component at u, is as in Proposition 4.6. The assumptions at u' and $v \in V$ permit producing matching functions $f_{u'}$ and f_v for functions $f'_{u'}$ and f'_v for which the left side of the identity displayed in Lemma 4.5 is non-zero. The proof then proceeds as that of Proposition 4.6. □

This completes our proof of Theorem 1.1. □

5. Corollaries and analogues

Theorem 1.1 concerns the correspondence from the group $\mathbb{G}$ to its split inner form $\mathbb{G}' = GL(2n, \mathbb{A})$. An analogous discussion can be carried out from $\mathbb{G}$ to any inner form of it. We shall consider next a special – but illuminating – case, of the correspondence from $\mathbb{G}$ to its inner form which has the same ramification, as follows.

Recall that the set of ramification of $\mathbf{G} = GL(2m, \mathbf{D}_d)$ is denoted by V. Thus $\mathrm{inv}_v\mathbf{G} = \mathrm{inv}_v\mathbf{D} = i_v/d_v \in \frac{1}{d}\mathbb{Z}/\mathbb{Z}$, with integral $0 < i_v < d_v$, $(i_v, d_v) = 1$, $d = l.c.m\{d_v; v \in V\}$ (so d_v divides d), and $\sum_{v\in V} \mathrm{inv}_v\mathbf{G} = 0 \pmod{\mathbb{Z}}$. Also $\mathrm{inv}_v\mathbf{G} = 0$ for $v \notin V$.

Now the inner form of $\mathbf{G}$ to be considered is $\mathbf{G}'' = GL(2m, \mathbf{D}''_d)$, specified (see [We]) by: $\mathrm{inv}_v\mathbf{D}'' = 0$ unless $v \in V$, and then $\mathrm{inv}_v\mathbf{D}'' = j_v/d_v \in \frac{1}{d}\mathbb{Z}/\mathbb{Z}$, where $0 < j_v < d_v$ are integral with $(j_v, d_v) = 1$, and $\sum_{v\in V} \mathrm{inv}_v\mathbf{D}'' = 0$ (in $\frac{1}{d}\mathbb{Z}/\mathbb{Z}$).

The work of [FK2] establishes a bijection between the sets of cuspidal representations with a supercuspidal component, of these two groups. In fact, the conjugacy classes in G_v and G''_v are in natural bijection, determined by their characteristic polynomials (in the semi-simple case). The corresponding local components have equal characters under this identification of regular conjugacy classes. In particular, π_u is supercuspidal if and only if the corresponding π''_u is. The proof of Theorem 1.1 can be repeated in this context to establish the following.

Theorem 5.1. *Let π be a cuspidal cyclic $\mathbb{G}$-module with a supercuspidal and a bi-elliptic components (at u, u') such that (WH1), (WH2) are held for π_v $(v \in V \cup \{u'\})$. Then the corresponding cuspidal $\mathbb{G}''$-module π'' is cyclic, so are its components, and $\pi''_{u'}$ is bi-elliptic.*

We can also derive some purely local results. The first will be an analogue of Kazhdan's density theorem for characters (see [K1], Appendix). It does not rely on (WH2), but we do assume that there exists a supercusp form f_u on G_u (an inner form of $GL(2n, F_u)$) with $\Xi(g, f_u) \not\equiv 0$. For example, f_u can be taken to be a coefficient of a cyclic supercuspidal π_u, which we need to assume exists.

Theorem 5.2. *Assume that (WH1) holds for every irreducible admissible (cyclic) representation π_w of the inner form G_w of $GL(2n, F_w)$. Then $\mathbb{P}_{\pi_w}$ is defined. If $f_w \in H_w$ is a test function such that $\mathbb{P}_{\pi_w}(f_w)$ vanishes for all cyclic π_w, then the bi-orbital integral $\Xi(\gamma, f_w)$ is zero on the bi-regular set of γ in G_w.*

Proof. Choose a global field F with completions F_u, F_w, and an inner form $\mathbf{G}$ of $GL(2n)$ over F whose group of points over F_u, F_w is G_u, G_w, and a global character η with the components η_u, η_w. Assume that $\Xi(g, f_w)$ is not identically zero on the bi-regular set of G_w. We shall show that this leads to a contradiction.

Since $\Xi(\gamma, f_u)$, $\Xi(\gamma, f_w)$ are locally constant on the bi-regular sets of G_u, G_w (Lemma 4.2), we can fix a third place u', a bi-elliptic bi-regular global element γ_0 in G, which is bi-elliptic in $G_{u'}$, and $f_{u'} \in H_{u'}$ which is supported on the bi-elliptic bi-regular set in $G_{u'}$, such that $\Xi(\gamma_0, f_v) \neq 0$ $(v = u, w, u')$.

Since $\gamma_0 \in K_v$ for almost all v, and $f_v^0 \geq 0$, the integral $\Xi(\gamma_0, f_v^0)$ is non zero for all v outside some finite set S of places of F. At the remaining finite set of places we choose f_v to be the characteristic function of a small neighborhood of γ_0 in G_v; then $\Xi(\gamma_0, f_v) \neq 0$. It follows that $\Xi(\gamma_0, f) \neq 0$, where $f = \otimes f_v$, and that if γ is rational (in G) with $\Xi(\gamma, f) \neq 0$, then γ is bi-regular bi-elliptic (since it is such in $G_{u'}$).

Since f is compactly supported, such $\gamma = \gamma(\beta)$ lies in a finite set of bi-orbits; indeed, the set of characteristic polynomials of the associated β is both compact – depending on the support of f – and discrete (since β is rational) in the set of polynomials of degree n over $\mathbb{A}(\simeq \mathbb{A}^{n+1})$.

The totally disconnected topology on $G_{u'}$ permits choosing an open closed neighborhood of the orbit of γ_0 which does not intersect the orbits of the other rational γ with $\Xi(\gamma, f) \neq 0$. Replacing $f_{u'}$ by its product with the characteristic function of this neighborhood, we obtain f such that $\Xi(\gamma, f) \neq 0$ for a rational γ implies that γ is in the bi-orbit of γ_0.

We now apply the bi-period summation formula of Proposition 4.1, to our function f on $\mathbb{G}$. The requirements of this Proposition 4.1 are satisfied. Indeed, f_u is supercuspidal, and $f_{u'}$ is supported on the bi-elliptic bi-regular set. Our assumption that $\mathbb{P}_{\pi_w}(f_w)$ vanishes for all π_w implies the vanishing of the spectral (left) side of the summation formula. Hence the geometric side is zero. But it contains a single term, indexed by γ_0. So $\Xi(\gamma_0, f) = 0$, a contradiction to the assumption that $\Xi(g, f_w)$ is not identically zero on the bi-regular set of G_w, as required. □

Remark 5.3. Theorem 5.2 and its proof remain valid if we do not assume (WH1), but instead we assume for all π_w that $\mathbb{P}_{\pi_w}(f_w) = 0$, where $\mathbb{P}_{\pi_w}$ is defined by means of *any* C_w-invariant linear form P_{π_w} on the space of π_w, and any form P_{π_w, η_w}.

Finally we prove a local analogue of Theorem 1.1, assuming that the bi-elliptic part of (WH2) holds for every admissible irreducible representation $\pi_{u'}$ of $GL(2, D_{u'})$, where $D_{u'}$ is a division algebra central of rank n over the local field $F_{u'}$. Namely we assume that the bi-character $p(g, \pi_{u'})$ of any (not only supercuspidal as in the Remark 3.3 following the statement of (WH2)) such $\pi_{u'}$ is locally constant on the (necessarily bi-regular bi-elliptic) set of $G_{u'}$.

Theorem 5.4. *Let π_u be a cyclic supercuspidal G_u-module satisfying (WH1) and (WH2), where G_u is an inner form of $G'_u = GL(2n, F_u)$. Then the corresponding square-integrable G'_u-module π'_u is cyclic.*

Remark 5.5. The local correspondence is defined by means of character relations (see [F1] Ch. III). The corresponding π'_u is square-integrable, but not necessarily supercuspidal.

Proof. Suppose that $\operatorname{inv} G_u = i_u/d_u$, $0 < i_u < d_u$. We shall work with a global field F such that its completions at the places $u_1 = u, \ldots, u_{2d_u}$ are isomorphic to F_u, and at the places $u'_1 = u', \ldots, u'_n$ it is $F_{u'}$, and with an inner form $\mathbf{G}$ of $GL(2n)$ over F with $G_{u_i} \simeq G_u$ $(1 \leq i \leq 2d_u)$ and $G_{u'_i} \simeq G_{u'}$ $(1 \leq i \leq n)$. We shall carry out a comparison with the inner form $\mathbf{G}'$ of $\mathbf{G}$ which is split at the places $u_{d_u+1}, \ldots, u_{2d_u}$, but with $G_v \simeq G'_v$ for all other v.

We compare the bi-period summation formulae for $\mathbb{G}$ and $\mathbb{G}'$ of Proposition 4.1. At the places $u_{d_u+1}, \ldots, u_{2d_u}$ we use matrix coefficients of π_u, while at the places

$u'_1, \dots, u'_u$ we take the test functions to be supported on the bi-elliptic bi-regular set. At the places $u_1, \dots, u_{d_u}$ we take the f_{u_i} and f'_{u_i} to be matching and supported on the bi-regular set (of elements which come from G_{u_i} in the case of f'_{u_i}), as in Lemma 4.2. At all other places, $f_v = f'_v$ under $G_v \simeq G'_v$. Since both $f = \otimes f_v$ and $f' = \otimes f'_v$ have supercuspidal components and components supported on the bi-elliptic bi-regular sets, Proposition 4.1 applies. Since f and f' are matching the geometric parts of these formulae are equal.

Note that f can be chosen so that the geometric side of the bi-period summation formula is non-zero. Indeed, since $\Xi(g, f_v)$ is locally constant on the bi-regular set (Lemma 4.2), and is not identically zero there for $v = u_i$ or u'_i by our assumption on π_u and $f_{u'_i}$, there is some rational bi-regular bi-elliptic element γ_0 with $\Xi(\gamma_0, f_v) \neq 0$ for such v. This relation clearly holds with $f_v = f_v^0$ for almost all v. At the remaining finite set of places we choose f_v supported on a small neighborhood of γ_0, and argue as in the proof of Theorem 5.2 that f can be chosen so that $\Xi(\gamma, f) \neq 0$ for a rational γ implies that γ is in the bi-orbit of γ_0. Applying Proposition 4.1 with such an f we conclude that there exists a cuspidal cyclic $\mathbb{G}$-module π, in fact with the component π_v at $v = u_1, \dots, u_{2d_u}$, and a bi-elliptic component at $u'_1, \dots, u'_n$.

Since we have (WH1) and (WH2) for π_v $(v = u_i)$ by assumption, and also at $v = u'_i$ ((WH1) by Proposition 2.1, (WH2) on the bi-elliptic set by assumption), the proof of Proposition 4.6 implies that the corresponding cuspidal $\mathbb{G}'$-module π' is cyclic. In particular its components, including π'_u, are cyclic, as required. □

An analogous argument establishes a converse to Theorem 5.4. Under the same assumption at u' we have the following.

Theorem 5.6. *Let π_u and π'_u be corresponding supercuspidal G_u- and $G'_u = GL(2n, F_u)$-modules. If π'_u is cyclic, satisfying (WH1) and (WH2), whose bi-character is not identically zero on the set of bi-regular elements which come from G_u, then π_u is cyclic.*

If the existence of a cyclic supercuspidal $G_{u''}$-module $\pi_{u''}$ is assumed – as in Theorem 5.2 – then the same proof establishes Theorem 5.6 with "supercuspidal" replaced by "square-integrable". In particular we have also the local analogue of Theorem 5.1. Let π_u be a supercuspidal G_u-module (square-integrable if the additional assumption at u'' is made) satisfying (WH1/2), and G''_u is an inner form of G_u with $\operatorname{inv} G_u = i_u/d_u$ and $\operatorname{inv} G''_u = j_u/d_u$, $(i_u, d_u) = 1 = (j_u, d_u)$. The corresponding G''_u-module π''_u is supercuspidal (resp. square-integrable) by [F1] Ch. III. If π_u is cyclic then so is π''_u.

Acknowledgements

A Humboldt Fellowship and the hospitality of Rainer Weissauer at Mannheim are gratefully acknowledged.

References

[BZ] J. Bernstein, A. Zelevinskii, Representations of the group $GL(n, F)$ where F is a nonarchimedean local field, Russian Math. Surveys 31 (1976), 1-68.

[BDKV] J. Bernstein, P. Deligne, D. Kazhdan, M.-F. Vigneras, Reprśentations des groupes réductifs sur un corps local, Hermann, Paris (1984).

[BJ] A. Borel, H. Jacquet, Automorphic forms and automorphic representations, Proc. Sympos. Pure Math. 33 (1979), I 189-202.

[F1] Y. Flicker, Rigidity for Automorphic forms, J. Analyse Math. 49 (1987), 135-202.

[F2] Y. Flicker, Transfer of orbital integrals and division algebras, J. Ramanujan math. soc. 5 (1990), 107-122.

[F3] Y. Flicker, On distinguished representations, J. reine angew. Math. 418 (1991), 139-172.

[F4] Y. Flicker, Quadratic cycles on $GL(2n)$ cusp forms, J. Algebra 174 (1995), 678-697.

[F5] Y. Flicker, Orbital integrals on symmetric spaces and spherical characters, J. Algebra (1996).

[FH] Y. Flicker, J. Hakim, Quaternionic distinguished representations, Amer. J. Math. 116 (1994), 683-736.

[FK1] Y. Flicker, D. Kazhdan, Metaplectic correspondence, Publ. Math. IHES 64 (1987), 53-110.

[FK2] Y. Flicker, D. Kazhdan, A simple trace formula, J. Analyse Math. 50 (1988), 189-200.

[GK] I. Gelfand, D. Kazhdan, Representations of $GL(n, K)$ where K is a local field, in Lie groups and their representations, John Wiley and Sons (1975), 95-118.

[HC1] Harish-Chandra, notes by G. van Dijk, Harmonic analysis on reductive p-adic groups, Springer Lecture Notes 162 (1970).

[HC2] Harish-Chandra, Admissible invariant distributions on reductive p-adic groups, Queen's Papers in Pure and Applied Math. 48 (1978), 281-346.

[HC3] Harish-Chandra, A submersion principle and its applications, in Collected Papers, vol. IV, Springer-Verlag, New-York, 1984, 439-446.

[H] R. Howe, Some qualitative results on the representation theory of GL_n over a p-adic field, Pacific J. Math. 73 (1977), 479-538.

[J1] H. Jacquet, Sur un résultat de Waldspurger, Ann. Sci. ENS. 19 (1986), 185-229.

[J2] H. Jacquet, On the nonvanishing of some L-functions, Proc. Indian Acad. Sci. 97 (1987), 117-155.

[K1] D. Kazhdan, Cuspidal geometry on p-adic groups, J. Analyse Math. 47 (1986), 1-36.

[K2] D. Kazhdan, On lifting, in Lie group representations II, Springer Lecture Notes 1041 (1984), 209-249.

[Ke] T. Kengmana, Characters of the discrete series for pseudo Riemannian symmetric spaces, in Representation theory of reductive groups, Proc. Univ. Utah conf. 1982 (P. Trombi, ed), Birkhauser, Boston-Basel (1983), 177-183.

[NPS] M. Novodvorski, I. Piatetski-Shapiro, Generalized Bessel models for a symplectic group of rank 2, Math. USSR Sbornik 19 (1973), 243-255, 275-286, and 21 (1973), 499-509.

[PPS] S. J. Patterson, I. Piatetski-Shapiro, The symmetric-Square L-function attached to a cuspidal automorphic representation of GL_3, Math. Ann. 283 (1989), 551-572.

[P] D. Prasad, Invariant forms for representations of GL_2 over a local field, Amer. J. Math. 114 (1992), 1317-1363.

[R] F. Rodier, Whittaker models for admissible representations of reductive p-adic split groups, in Proc. Sympos. Pure Math. 26, AMS, Providence, (1974), 425-430.

[S] J. Sekiguchi, Invariant spherical hyperfunctions on the tangent space of a symmetric space, in Algebraic Groups and Related Topics, Advanced Studies in Pure Mathematics 6 (1985), 83-126.

[Wa] J.-L. Waldspurger, Correspondence de Shimura, J. Math. Pure Appl. 59 (1980), 1-113; Sur les coefficients de Fourier des formes modulaires de poids demi-entier, J. Math. Pure Appl. 60 (1981), 375-484.

[We] A. Weil, Basic Number Theory, Springer-Verlag, Berlin 1974.

Department of Mathematics
231 W. 18th. Avenue
The Ohio State University
Columbus, OH 43210-1174
USA