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A remark on local-global principles for conjugacy classes

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Abstract. A local–global principle is shown to hold for all conjugacy classes of any inner form of GL(n), SL(n), U(n), SU(n), and for all semisimple conjugacy classes in any inner form of Sp(n), over fields k with vcd(k)$\leq$1. Over number fields such a principle is known to hold for any inner form of GL(n) and U(n), and for the split forms of Sp(n), O(n), as well as for SL(p) but not for SL(n), n non-prime. The principle holds for all conjugacy classes in any inner form of GL(n), but not even for semisimple conjugacy classes in Sp(n), over fields k with vcd(k)$\leq$2. The principle for conjugacy classes is related to that for centralizers.

Introduction

Let k be a perfect field with virtual cohomological dimension (vcd) $\leq$2. (Recall that $\text{vcd}(k)=\text{cd}(k(\sqrt{-1}))$, where $\sqrt{-1}$ is a square root of -1, and cohomological dimension $\text{cd}(k)=\text{cd}(\text{Gal}(\bar{k}/k))$ is defined e.g. in Serre [Se], Sections I.3.1 and II.3.1. We denote by $\bar{k}$ a fixed algebraic closure of k.) For any ordering ξ of k, write k_ξ for a real closure of k (in $\bar{k}$) whose ordering induces ξ on k (such a k_ξ is unique up to a unique isomorphism). An algebraic group G over k is said to satisfy the *real closures principle* if the restriction map

$$H^1(k,G) \longrightarrow \prod_\xi H^1(k_\xi,G)$$

is injective, where ξ ranges over the real spectrum $\text{Sper}\, k$ of k. Note that $\text{Sper}\, k$ is a compact Hausdorff totally disconnected space, consisting of the orderings of k. A basis for the topology of $\text{Sper}\, k$ is given by the sets $\{\xi; a>0 \text{ in } \xi\}$, $a\in k$ (see e.g. Scharlau [Sc], Section 3.5). Such a G satisfies the *strong real closures principle* over k if the map is injective also when ξ ranges only over any dense subset of $\text{Sper}\, k$.

The strong real closures principle holds for any connected linear algebraic group G over any (perfect) k with vcd(k)$\leq$1 (Scheiderer [Sch], Chernousov [C]), and the real closures principle holds for any connected, semisimple, simply connected, linear algebraic group of classical type over any field k with $\mathrm{vcd}(k) \leq 2$ (Bayer-Fluckiger and Parimala [BP]).

A number field k has $\mathrm{vcd}(k) \leq 2$, its real spectrum is the finite set of its real places, and the real closures principle for such a k coincides with the local-global (or Hasse) principle (the restriction map $H^1(k,G) \to \prod_v H^1(k_v, G)$ is injective, where v ranges over all places of the number field k), for any connected, semisimple, simply connected, linear algebraic group G, since for such a G, $H^1(k_v, G)$ is trivial for every non-archimedean v.

When the cohomological dimension $\mathrm{cd}(k)$ of k is finite, the real spectrum is empty and the real closures principle amounts to asserting that the pointed set $H^1(k,G)$ consists of a single element. This is the case of a totally imaginary number field (where $\mathrm{cd}(k)=2$), and a function field of a curve over an algebraically closed field, for example $\mathrm{cd}(\mathbf{C}(x))=1$.

The pointed set $H^1(k,G)$ parametrizes the torsors (right principal homogeneous spaces) over k (see e.g. [Se], Sections I.5.1–2), and the principle implies that a torsor T over k is trivial (is the distinguished element of $H^1(k,G)$)—namely it has a k-point, so $T(k) \neq \emptyset$—if $T(k_\xi) \neq \emptyset$ for all ξ in $\mathrm{Sper}\, k$, or for all ξ in any dense subset of $\mathrm{Sper}\, k$ in the strong case.

Of course the last question can be asked in the context of any variety over k. Conjugacy classes in linear algebraic groups are of special interest in group theory and representation theory, and this note concerns the question of whether an analogous principle holds for conjugacy classes in G. We say that x and y in $G(k)$ are *conjugate* in $G(k)$ if there is a g in $G(k)$ with $\mathrm{Int}(g)x = gxg^{-1}$ equal to y, and we write $x \sim y$ in $G(k)$. If x and y are in $G(k)$, then $x \sim y$ in $G(k)$ implies that $x \sim y$ in $G(k_\xi)$ for each ξ in $\mathrm{Sper}\, k$. We say that the *(strong) real closures principle holds for the conjugacy class of y in $G(k)$* if the converse holds, namely for x and y in $G(k)$, if $x \sim y$ in $G(k_\xi)$ for all ξ in (any dense subset of) $\mathrm{Sper}\, k$, implies that $x \sim y$ in $G(k)$.

The following is a "real closures principle" analogue of Bartels [Ba], Satz 2, which dealt with the number field case.

Theorem. *Let G be a linear algebraic group over a field k with an ordering (thus $\mathrm{Sper}\, k \neq \emptyset$, hence $\mathrm{char}\, k = 0$) for which the (strong) real closures principle holds. Fix y in $G(k)$. The (strong) real closures principle holds for the conjugacy class of y in $G(k)$ if and only if the (strong) real closures principle holds for the centralizer $G_y = Z_G(y)$ of y in G.*

Let $\mathrm{Orb}(y)$ be the conjugacy class of y in G. It is a smooth locally closed k-subvariety of G (Borel [Bo], Proposition II.6.7). Its set $\mathrm{Orb}(y,k)$ of k-points is a disjoint union of $G(k)$-orbits. The (strong) real closures principle for the conjugacy class of y in $G(k)$ can be restated as asserting that: if $x \in \mathrm{Orb}(y,k)$ lies in the $G(k_\xi)$-orbit of y in $\mathrm{Orb}(y,k_\xi)$ for all ξ in (any dense subset of) $\mathrm{Sper}\, k$, then x lies in the $G(k)$-orbit of y in $\mathrm{Orb}(y,k)$.

Examples

While Bartels [Ba] uses Theorem 4.7 of Asai [A] (which Asai attributes to Hijikata (in Japanese)), that the local–global principle holds for conjugacy classes in orthogonal, symplectic and unitary groups over number fields, to conclude that this principle holds for all centralizers G_y in these groups over number fields, one can also use the theorem to recover Asai's theorem from the results of Springer–Steinberg [SS] (used also by Asai [A]), as noted in [Ba] and below. For other fields we shall just use the general results of [SS] on centralizers of elements in reductive groups, to learn about conjugacy classes. We proceed to discuss these examples.

Let G be a connected reductive k-group. Hence the derived group G^{der} of G is semisimple. The centralizer $G_s = Z_G(s)$ of a semisimple element s in $G(k)$ is a reductive k-group. The centralizer G_s is connected when the derived group G^{der} of G is simply connected. This assertion is Corollary 8.5 of Steinberg [S] when G is semisimple; the case of a reductive G follows from the fact that $G_s = (G^{\mathrm{der}})_s \cdot T$ for any maximal torus T containing s. In particular, when $\mathrm{vcd}(k) \leq 1$, the strong real closures principle holds for conjugacy classes of semisimple elements in (any inner form of) the classical groups $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$.

The real closures principle does not hold for a general k with $\mathrm{vcd}(k)=1$ and the non-connected orthogonal groups $\mathrm{O}(n)$, $n \geq 2$, hence neither for centralizers nor for conjugacy classes of semisimple y in $\mathrm{O}(n)$ (in fact,

$$\ker\left[H^1(k,\mathrm{O}(q)) \to \prod_\xi H^1(k_\xi,\mathrm{O}(q))\right]$$

consists of all quadratic forms q' with the same dimension and signatures as the quadratic form q, but q' may have a different determinant than q, so the kernel is in bijection with

$$k^\times_{\mathrm{totpos}}/(k^\times)^2 = \ker\left[H^1(k,\mu_2) \to \prod_\xi H^1(k_\xi,\mu_2)\right],$$

the quotient by the squares of the set of elements in $k^\times$ which are positive for each ξ).

The centralizer G_y of any element y in $\mathrm{GL}(n,k)$ is a connected k-group ([SS], III.3.22). The same holds for any k-form of $\mathrm{GL}(n)/k$, in particular for the unitary group U ([SS], IV.2.26(i)). Hence the strong real closures principle holds for all centralizers G_y, and all conjugacy classes, of $y \in G(k)$, when G is GL(n) or U(n), or any inner form over k thereof, provided vcd(k)≤ 1.

Suppose that k has characteristic zero. Then any unipotent group R over k is an extension of affine lines, hence $H^1(k,R)$ is trivial. Let u be a unipotent element in $G(k)$. Then its centralizer $Z=G_u=Z_G(u)$ in G admits the Levi decomposition CR, where R is the unipotent radical of Z and C is reductive (not necessarily connected), both defined over k, and R is normal in Z ([H], Section 0.2). The exact sequence $1\to R\to Z\to C\to 1$ yields a long exact sequence of cohomology whose 0th part is the short exact sequence

$$1 \longrightarrow R(k) \longrightarrow Z(k) \longrightarrow C(k) \longrightarrow 1,$$

as Z is the semidirect product of C and R. Then $H^1(k,Z)=H^1(k,C)$. Now the group C is a product of groups GL (if G is GL; [SS], Corollary IV.1.8) and also U if G is unitary ([SS], IV.2.25). If G is Sp or O, then C is a product of copies of groups of the form Sp and O ([SS], IV.2.25). Hence the real closures principle does not hold for unipotent conjugacy classes in the groups Sp or O, in general, even when vcd(k)≤ 1.

For general conjugacy classes, note that y in $G(k)$ has the (Jordan) decomposition $y=su=us$ into unique commuting semisimple and unipotent elements s and u in $G(k)$. The centralizer $Z_G(y)$, or G_y, of y in G, is equal to $Z_H(u)$, where $H=Z_G(s)$, by the uniqueness of the Jordan decomposition.

When G is GL, U, Sp or O, the group H is a product of groups of the same type as G, and of GL(n), possibly over field extensions of k. For the real closures principle for these groups we then need to investigate centralizers of the form $Z_G(u)$, where u is a unipotent element in $G(k)$.

When k is a number field, it has vcd(k)=2, and the real closures principle is classically known to hold for the semisimple simply connected groups G, where it coincides with the local–global principle. (The local–global principle holds also for the split groups O(n) (Kneser [Kn], p. 134, since any element of k which is a square in k_v for all v is a square in k; this is the weak Hasse–Minkowski principle for quadratic forms). It does not hold for the orthogonal groups associated with quaternion division algebras ([Kn], p. 138)). Hence the local–global principle holds for all centralizers G_y, $y\in G(k)$, and hence for conjugacy classes of such y, when k is a number field, and G is any inner form of GL(n) or U(n). It holds for semisimple conjugacy classes of the groups U(n), Sp(n) and non-quaternionic O(n).

Indeed, a semisimple group G of type ${}^2A_n$, B_n, C_n and D_n over k is defined by a simple algebra A and an involution $\tau\neq 1$ of A, both defined over k; the center of A is k and τ fixes each element of k except in the case ${}^2A_n$, where the center of A is a quadratic field extension L of k, τ acts on L and its set of fixed points is k. Then $G(k)=\{x\in A; x\tau(x)=1\}$. An element y of $G(k)$ is—by definition—semisimple regular, when its centralizer $E=Z_A(y)$ in A is a semisimple commutative algebra over k in A. Fix a semisimple regular y in $G(k)$. The restriction of τ to E is an involution, clearly non-trivial; put $P=E^\tau$. The centralizer

$$T=G_y=Z_G(y)=\{x\in E\,;x\tau(x)=1\}$$

of y in $G(k)$ is the torus $\mathrm{Res}_{P/k}(\mathrm{Res}^{(1)}_{E/P}\,\mathbf{G}_m)$ (the superscript (1) indicates "norm one" elements). Since $H^1(k,T)=P^\times/N_{E/P}E^\times$, G_y satisfies the local–global principle when k is a number field. Clearly the real closures principle is not satisfied over a general field k with $\mathrm{vcd}(k)=2$.

The local–global principle holds for $\mathrm{SL}(p)$, p prime, but fails even for semisimple conjugacy classes in $\mathrm{SL}(n)$, n not prime. Indeed, a counterexample is constructed in [Ba], p. 196, for the torus of $\mathrm{SL}(4,\mathbf{Q})$ which splits over $L=\mathbf{Q}(\sqrt{13},\sqrt{17})$, as the norm principle fails for $L/\mathbf{Q}$ (thus there is a $\lambda\in\mathbf{Q}^\times$ which is a norm from $L_p=L\otimes_{\mathbf{Q}}\mathbf{Q}_p$ for every rational prime p, but it is not a norm from L).

For general fields k with $\mathrm{vcd}(k)=2$, the real closures principle holds for all inner forms of $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$, but not for $\mathrm{U}(n)$, $\mathrm{SO}(n)$ and $\mathrm{O}(n)$ ([BP]). It holds for conjugacy classes in $\mathrm{GL}(n,k)$. But it does not hold for all semisimple conjugacy classes in the unitary, symplectic or orthogonal groups, as noted in the discussion above in the case of number fields. It will be interesting to increase the set of orderings of such a k to include places (non-discrete valuations?) as those which a number field has, so that a local–global principle would hold.

Proof of the theorem

If x and y in $G(k)$ are conjugate in $G(k_\xi)$ then they are conjugate in $G(\bar{k})$: there is $g\in G(\bar{k})$ with $x=gyg^{-1}$, thus $g^{-1}\sigma(g)\in G_y(\bar{k})$ for every $\sigma\in\mathrm{Gal}(\bar{k}/k)$. This g is uniquely determined by x and y up to right multiplication by $t\in G_y(\bar{k})$, hence the 1-cocycle $\{\sigma\mapsto g^{-1}\sigma(g)\}$ is uniquely determined up to 1-coboundaries, and it defines a cohomology class in $H^1(k,G_y)$ which vanishes in $H^1(k,G)$. This gives a bijection between the conjugacy classes of y in $G(k)$ within its $G(\bar{k})$-conjugacy class and $\ker[H^1(k,G_y)\to H^1(k,G)]$. Since the real closures principle is assumed to hold for G, it holds for G_y if and only if it holds for $\ker[H^1(k,G_y)\to H^1(k,G)]$, in other words, if and only if

$$\ker[H^1(k,G_y)\to H^1(k,G)]\longrightarrow\prod_\xi \ker[H^1(k_\xi,G_y)\to H^1(k_\xi,G)]$$

is injective, where ξ ranges over (any dense subset of) Sper k, as required.

As for the restatement of the real closures principle in terms of the variety $\mathrm{Orb}(y)$, note that the conjugacy class morphism $\mathrm{Int}\colon G\to\mathrm{Orb}(y)$, $g\mapsto gyg^{-1}$, is surjective with kernel G_y, and the induced bijective k-morphism $G/G_y\to\mathrm{Orb}(y)$ is an isomorphism since $\mathrm{char}\,k=0$ ([Bo], Section III.9.1). The long exact sequence associated to

$$1\longrightarrow G_y\longrightarrow G\longrightarrow \mathrm{Orb}(y)\longrightarrow 1,$$

namely

$$1\longrightarrow G_y(K)\longrightarrow G(K)\longrightarrow \mathrm{Orb}(y,K)\longrightarrow H^1(K,G_y)\longrightarrow H^1(K,G)$$

([Se], Section I.5.4, K is any field extension of k), identifies

$$\ker[H^1(K,G_y)\longrightarrow H^1(K,G)]$$

with the set of $G(K)$-orbits in $\mathrm{Orb}(y,K)$.

Stable conjugacy

The local–global principle for conjugacy classes is not the same as the weaker notion of local–global principle for stable conjugacy classes. Stable conjugacy is a notion which plays a key role in the stabilization of the trace formula in the theory of automorphic forms [L], [K], [KS]. The elements x and y in $G(k)$ are said to be $G(\bar{k})$-conjugate if they are conjugate in $G(\bar{k})$. They are *stably conjugate* over k (G being a connected reductive linear algebraic group over k) if they are conjugate in $G(\bar{k})$, thus there is a $g\in G(\bar{k})$ such that $x=\mathrm{Int}(g)y$, and moreover $g^{-1}\sigma(g)\in G_s^0$ for each $\sigma\in\mathrm{Gal}(\bar{k}/k)$, where G_s^0 is the connected component of the identity in the centralizer G_s of the semisimple part s of y. (We always have that $g^{-1}\sigma(g)\in G_y$, hence the second condition means that $g^{-1}\sigma(g)\in G_y^*=G_y\cap G_s^0$.) If the derived group G^{der} of G is simply connected, then the centralizer of any semisimple element is connected: $G_s^0=G_s$ and G_y is equal to G_y^*, hence x and y are stably conjugate if and only if they are conjugate in $G(\bar{k})$. In general, given any z-extension $\alpha\colon H\twoheadrightarrow G$ (H is connected reductive over k with H^{der} simply connected, $\ker\alpha$ is central in H and is isomorphic to a finite product of induced tori $\mathrm{Res}_{k'/k}\,\mathbf{G}_m$, k'/k is a finite extension), x and y in $G(k)$ are stably conjugate if and only if there exist x' and y' in $H(k)$, stably conjugate and such that $\alpha(x')=x$ and $\alpha(y')=y$ ([K], Section 3). Thus if x and y in $G(k)$ are stably conjugate in $G(k_\xi)$, x' and y' are conjugate in $H(\bar{k}_\xi)$, hence in $H(\bar{k})$, hence x and y are stably conjugate in $G(k)$. This requires neither the local–global principle for G, nor the local–global principle for G_y.

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