ORBITAL INTEGRALS ON SYMMETRIC SPACES AND SPHERICAL CHARACTERS

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Abstract. In the first Section of this paper we obtain an asymptotic expansion near semi simple elements, of orbital integrals $\mu_{\tilde{x}}(\tilde{f})$ of C_c^∞-functions $\tilde{f}$ on symmetric spaces G/H. Here G is a reductive p-adic group, and H is the group of fixed points of an involution σ on G. This extends the germ expansion of Shalika [Sh] and Vigneras [V] in the group case.

The main part of the paper studies examples of groups G with involution σ, which have the property that the spherical characters associated with its spherical admissible representations are not identically zero on the regular set of G/H. These include $G = GL(n+m)$, $H = GL(n) \times GL(m)$ for $n = m = 1$ or 2, and $n = 1$, $m \geq 3$. More general results had been obtained by Sekiguchi [S1] in the case of real symmetric spaces, generalizing Harish-Chandra's work in the group case, over archimedean and non archimedean fields. Our interest is in the p-adic case. There the techniques are entirely different from Sekiguchi's. In fact we use the recent work of Rader-Rallis [RR] who showed that the spherical character is smooth on the regular set, and has asymptotic expansion in terms of Fourier transforms of invariant distributions on the nilpotent cone, as found by Harish-Chandra [HC1] in the group case.

Our study of the non vanishing of some spherical characters uses a construction of an explicit basis of the space of invariant distributions on the nilpotent cone. This is done on regularizing spherical orbital integrals, and taking suitable linear combinations. This local work is motivated by concrete applications to the theory of Deligne-Kazhdan lifting of spherical automorphic representations [F2], [F2$'$]. In some other examples, concerning $G = GL(3n)$ and $H = GL(n) \times GL(2n)$, and $G = O(3,2)$, $H = O(2,2)$, we explicitly construct invariant distributions on the nilpotent cone which are equal to their Fourier transform. Such examples do not exist in Harish-Chandra's group case.

In the last two Sections, following Harish-Chandra's simple proof in the group case [HC2], we show that $\mu_{\tilde{x}}(\tilde{f})$ is locally constant on the regular set of $\tilde{x}$, uniformly in $\tilde{f}$, in some cases. Following Kazhdan's proof of his density theorem [K;Appendix], we show that an $\tilde{f}$ which annihilates all spherical characters has $\mu_{\tilde{x}}(\tilde{f}) = 0$ on the regular elliptic set.

Introduction. Let $\mathbf{G}$ be a reductive group defined over a non archimedean local field F. As in [SS], [St], $\boldsymbol{\sigma}$ denotes an involution (automorphism of order two) of $\mathbf{G}$ over F. Let $\mathbf{H} = \mathbf{G}^+$ be the group of fixed points of $\boldsymbol{\sigma}$ in $\mathbf{G}$. Put $G = \mathbf{G}(F)$, $H = G^+ = \mathbf{H}(F)$ for the corresponding groups of F-points, and σ for the induced involution of G. Then G, H, are ℓ-groups in the terminology of Bernstein-Zelevinski [BZ]. To simplify this introduction the centers of the groups are ignored. For any ℓ-space X one has the space $C_c^\infty(X)$ of complex valued locally constant compactly supported functions on X, and the dual space $C_c^\infty(X)^*$ of distributions on X. When A is a group acting on X, $C_c^\infty(X)^{*A}$ denotes the space of A-invariant distributions on X.

Following the work of Jacquet-Lai [JL] on automorphic forms with non zero periods (a brief discussion of this motivation is postponed to the end of this introduction), there is a

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considerable interest currently in $H\times H$-invariant distributions on G. These occur as orbital integrals in the geometric side of Jacquet's "relative trace formula" (a more descriptive title is "bi-period summation formula" for the case at hand), and as spherical characters on the spectral side of this formula.

Denote the image of the map $G/H \to G$, $g \mapsto \tilde{g} = g\sigma(g)^{-1}$, by $\tilde{G}$. Given $f \in C_c^\infty(G)$, put $\tilde{f}(\tilde{g}) = \int_H f(gh)dh$. If $H \times H$ acts on G by right and left multiplication, and H acts on $\tilde{G}$ by conjugation, then $C_c^\infty(G)^{*H\times H} = C_c^\infty(\tilde{G})^{*H}$. Examples of elements in the space $C_c^\infty(\tilde{G})^{*H}$ of interest are given by orbital integrals. A semi-simple (in G) element $\tilde{\gamma}$ of $\tilde{G}$ is called σ-regular if its centralizer $Z_H(\tilde{\gamma})$ in H is a torus. Using the fact that H and $Z_H(\tilde{\gamma})$ are unimodular, and [BZ], one has a unique Haar measure on the orbit $\mathcal{O}(\tilde{\gamma}) = \mathrm{Int}(H)\tilde{\gamma} \simeq H/Z_H(\tilde{\gamma})$. Since this orbit is closed (by Richardson [Ri]), this extends to an H-invariant distribution on $\tilde{G}$ supported on the orbit $\mathcal{O}(\tilde{\gamma})$, denoted by $\mu_{\tilde{\gamma}}(\tilde{f}) = \int_{H/Z_H(\tilde{\gamma})} \tilde{f}(\mathrm{Int}(h)\tilde{\gamma})dh$ ($\tilde{f} \in C_c^\infty(\tilde{G})$), and named the "spherical" orbital integral of $\tilde{f}$ at $\tilde{\gamma}$. Thus $\mu_{\tilde{\gamma}}$ is an element of $C_c^\infty(\tilde{G})^{*H}$. In Section 1 we study the asymptotic behavior of the distributions $\mu_{\tilde{\gamma}}$ in the vicinity of a semi-simple (in G) element $\tilde{s}$ in $\tilde{G}$, which is not necessarily regular.

Before describing this, note that our setting of symmetric spaces G/H, reduces to the classical "group" case of a reductive group G acting on itself by conjugation, when one takes $G = H \times H$ and $\sigma(x,y) = (y,x)$. The asymptotic behavior of the orbital integrals $\mu_\gamma(f) = \int_{G/Z_G(\gamma)} f(\mathrm{Int}(g)\gamma)dg$ in the vicinity of a semi-simple element s of G has been studied by Shalika [Sh] and Vigneras [V]. Section 1 extends their work from the group case $H \times H/H$, to that of a general symmetric space G/H. This extension is needed both for the study of Jacquet's summation formula mentioned above, as well as for the study of the spherical characters.

A preliminary result in the study of these orbital integrals is geometric. Fix a semi-simple (in G) element $\tilde{s}$ in $\tilde{G}$, and denote by $\tilde{G}_{\tilde{s}}$ the set of $\tilde{x}$ in $\tilde{G}$ whose semi-simple part lies in $\mathrm{Int}(H)\tilde{s}$. Then H acts on $\tilde{G}_{\tilde{s}}$ by conjugation, and by Richardson [Ri], the set $\tilde{G}_{\tilde{s}}$ is closed and consists of finitely many H-orbits. Proposition 1.1 shows that the dimension of the complex vector space $C_c^\infty(\tilde{G}_{\tilde{s}})^{*H}$ is finite, bounded by the number of H-orbits in $\tilde{G}_{\tilde{s}}$.

The contrast between the proof of this Proposition in the symmetric space and the group cases is interesting. In the group case, the centralizer $Z_G(x)$ of any x in G is unimodular (e.g., Springer-Steinberg [SS]), hence the orbital integral $\mu_{\mathcal{O}(x)}$ exists on $C_c^\infty(\mathcal{O}(x))$ (by [BZ]). By the theorem of Rao [R] (and Bernstein [Be] for $GL(n)$ and fields of positive characteristic), it extends to the closure $\overline{\mathcal{O}(x)}$ of the orbit $\mathcal{O}(x)$=$\mathrm{Int}(G)x$ of x, and to G (by [BZ]). These results do not extend to the general symmetric space case, but of course one still has the "closed orbit lemma" (e.g. Borel [Bo]), and the finite dimensionality can be established; yet no natural basis exists.

Using the "closed orbit lemma" one can choose a basis $\{\Lambda\}$ for the space $C_c^\infty(\tilde{G}_{\tilde{s}})^{*H}$, and functions $,_\Lambda(\tilde{x})$ on the regular set of $\tilde{x}$ in $\tilde{G}$, called germs of orbital integrals, and the asymptotic expansion takes the following form.

For every $\tilde{f}$ in $C_c^\infty(\tilde{G})$ and semi-simple $\tilde{s}$ in $\tilde{G}$, there exists an H-invariant open and closed neighborhood $V_{\tilde{f}}$ of $\tilde{s}$ in $\tilde{G}$ such that for every regular $\tilde{x}$ in $V_{\tilde{f}}$ one has $\mu_{\tilde{x}}(\tilde{f}) = \sum_\Lambda ,_\Lambda(\tilde{x})\Lambda(\tilde{f})$.

Conversely, any H-invariant function on the regular set of $\tilde{G}$ which is compactly supported on $\tilde{G}/\mathrm{Int}(H)$ and has such asymptotic behavior, is an orbital integral of some $\tilde{f}$.

It is important to note that in the group case, Harish-Chandra [HC1], Theorem 10, shows that $\mu_x(f)$ is zero for all x in G if it vanishes for all regular elements x in G. The proof of this relies on [HC1], Lemma 8, which uses the local integrability of characters of supercuspidal representations. This had been proven by Harish-Chandra [HCD]. The analogues in the symmetric space case – the spherical characters – are rarely locally integrable, even for spherical supercuspidal representations. The proof of [HC1], Theorem 10, does not extend to the spherical case, and indeed the vanishing of $\mu_{\tilde{x}}(\tilde{f})$ for all regular $\tilde{x}$ in $\tilde{G}$ does not imply in general that $\Lambda(\tilde{f})$ is zero for all Λ in $C_c^\infty(\tilde{G})^{*H}$. Some examples and counter-examples have been studied in the case of rank one symmetric spaces by van Dijk [D] in the real case, and Rader-Rallis [RR] in the non archimedean case.

To describe the contents of the main part, Sections $2-10$, of this paper, let us recall the notion of a spherical character of an irreducible admissible G-module π which is H-spherical. The adjective H-spherical means that the dual $\pi^* = \mathrm{Hom}_{\mathbb{C}}(\pi, \mathbb{C})$ of π contains a non zero H-invariant element L. Fix also an H-invariant $L' \neq 0$ in the dual of the contragredient $\tilde{\pi}$ of π. Then for every $f \in C_c^\infty(G)$ the vector $\pi(f)L'$ lies in the smooth part π of the dual of $\tilde{\pi}$. The spherical character of π is defined to be $\mathbb{L}_\pi(\tilde{f}) =< L, \pi(f)L' >$. It is an H-invariant distribution on $\tilde{G}$. In the group case, where $G = H \times H$ and $\sigma(x, y) = (y, x)$, it coincides with the trace distribution.

Harish-Chandra [HC1], Theorem 5, described the asymptotic expansion of the character χ_π, where $\mathrm{tr}\ \pi(f) = \int \chi_\pi(g)f(g)dg$, near any semi-simple element of the group. The description extends to the spherical situation, as shown by Hakim [H] in the case where E/F is a quadratic field extension, $G = \mathbf{G}(F)$ and $H = \mathbf{G}(F)$ (σ is the Galois action), and Rader-Rallis [RR] in the general case of G/H ([RR] considers the expansion only near the identity, but the description extends to any semi-simple element by the arguments of [H]). Their combined result is as follows.

Given s in G such that $\tilde{s} = s\sigma(s)^{-1}$ is semi-simple, let M be the connected component of the centralizer $Z_G(\tilde{s})$ of $\tilde{s}$ in G, and $M^+ = M \cap G^+$, where $G^+ = H$ is the group of fixed points of σ in G. Denote by $\mathfrak{g}$ the Lie algebra of G, by $\mathfrak{g}^+$ the algebra of σ-fixed points in $\mathfrak{g}$, and by $\mathfrak{g}^-$ the -1 eigenspace of σ in $\mathfrak{g}$. Let $\mathfrak{M}$ be the centralizer $Z_{\mathfrak{g}}(\tilde{s})$ of $\tilde{s}$ in $\mathfrak{g}$, and $\mathfrak{M}^\pm = \mathfrak{M} \cap \mathfrak{g}^\pm$. Let $\{\Lambda\}$ be a basis for the space of $\mathrm{Ad}(M^+)$-invariant distributions on $\mathfrak{M}^-$ which are supported on the nilpotent set $\mathfrak{M}^-_{\mathrm{nilp}}$ of $\mathfrak{M}^-$. One defines as usual the Fourier transform $\hat{f}$ of $f \in C_c^\infty(\mathfrak{M}^-)$ (by $\hat{f}(Y) = \int_{\mathfrak{M}^-} \psi(B(Y, X))f(X)dX$, see [HC1]). The Fourier transform $\hat{T}$ of the distribution T in $C_c^\infty(\mathfrak{M}^-)^*$ is the element of $C_c^\infty(\mathfrak{M}^-)^*$ defined by $\hat{T}(f) = T(\hat{f})$. Then the asymptotic expansion of Hakim [H] and Rader-Rallis [RR] can be expressed as follows.

Given the H-spherical irreducible admissible G-module π, and the basis $\{\Lambda\}$ of the finite dimensional space $C_c^\infty(\mathfrak{M}^-_{\mathrm{nilp}})^{*M^+}$, there are unique complex numbers $c_\Lambda(\pi)$ such that for every $f \in C_c^\infty(\mathfrak{M}^-)$ which is supported in a sufficiently small neighborhood of zero, we have the asymptotic expansion $\mathbb{L}_\pi(f \circ \exp) = \sum_\Lambda c_\Lambda(\pi)\hat{\Lambda}(f)$.

Here the exponential map $\exp : \mathfrak{M}^- \to \tilde{M} = \{\tilde{m} = m\sigma(m)^{-1}; m \in M\}$ is a homeomorphism on a sufficiently small neighborhood of zero in $\mathfrak{M}^-$, extending under conjugation by M^+ to the entire nilpotent set of $\mathfrak{M}^-$. Moreover, on the regular set of $\tilde{G}$ the distribution $\mathbb{L}_\pi$ is represented by a locally constant function.

The last assertion is proven in Harish-Chandra [HC1], Theorem 3, in the group case (a simple proof is given in Harish-Chandra [HC2]), in [FH] in the special (quadratic) symmetric space case considered in [FH], and in general in Rader-Rallis [RR]. It will be interesting to extend the simple proof of [HC2] to the general spherical case.

In the group case, Harish-Chandra [HC1], Theorem 3, using [HCD], has shown that the character of a G-module π is a locally integrable function. In particular it is not identically zero on the regular set of G. The analogous local integrability statement holds when E/F is quadratic, $G = \mathbf{G}(F)$, $H = \mathbf{G}(F)$, by Hakim [H]. Hence $\mathbb{L}_\pi$ is again not identically zero on the regular set of $\tilde{G}$. However, for a general symmetric space G/H, the H-invariant distribution $\mathbb{L}_\pi$ is not locally integrable. Moreover, we show (in Sections 7 and 8) that there exist H-invariant distributions Λ on the nilpotent set of $\mathfrak{g}^-$ whose Fourier transform on $\mathfrak{g}^-$ is also supported on the nilpotent set of $\mathfrak{g}^-$. This suggests that there might be admissible G-modules π (perhaps only in the Grothendieck group), such that $\mathbb{L}_\pi$ is supported on the nilpotent cone of $\mathfrak{g}^-$.

In the case where the base field F is the field of real numbers, Sekiguchi [S1] has given a condition which – when satisfied – implies that $\mathbb{L}_\pi$ is not identically zero on the regular set. Further he listed several cases where his condition is satisfied. The techniques employed in the archimedean and non archimedean cases are very different, but the final result are similar. Sections 2-10 of this paper are then concerned with showing in some cases that $\mathbb{L}_\pi$ is not identically zero on the regular set, and that in some cases there are self-dual (equal to their Fourier transform) distributions supported on the nilpotent cone. This we do in the p-adic case. Our examples are consistent with those of Sekiguchi [S1].

Our examples concern the group $G = GL(n+m, F)$, and the involution σ given by conjugation with $J = \operatorname{diag}(I_n, -I_m)$; thus $H = G^+$ is isomorphic to $GL(n,F) \times GL(m,F)$. We show that when $m = 1, n > 2$ (see Section 4), or $n = m = 1$ (see Section 3), or $n = m = 2$ (see Sections 7-10), for any admissible irreducible H-spherical G-module π, the distribution $\mathbb{L}_\pi$ is not identically zero on the regular set. However, in Section 5 we show that if $n = 2m, m \geq 1$, then there are H-invariant distributions which are supported on the nilpotent set of $\mathfrak{g}^-$ which are self dual. Section 6 constructs another example of such invariant distributions on the nilpotent cone which are equal to their Fourier transforms, on the pair $G = O(3,2)$, $H = O(2,2)$ of quasi-split orthogonal groups. We conjecture that the non vanishing result holds for all $n = m \geq 1$.

Our proof uses the asymptotic expansion result of [H] and [RR] stated above. We construct an explicit basis of invariant distributions on the relevant nilpotent cone on extending orbital integrals to the closure of the orbit by viewing them as principal values of regularized integrals, and taking linear combinations which are H-invariant. Then we compute their Fourier transforms, and compare their behavior. In the vanishing case, we construct explicit examples.

The study of the spherical characters is not an idle extension of Harish-Chandra's work to symmetric spaces. The initial motivation has been the paper of Jacquet-Lai [JL], which dealt with $GL(2)$, a quadratic field extension E/F, and a quaternion algebra which is split at each place where E splits over F. The relevant comparison of automorphic forms has applications to the study of Shimura surfaces. But the splitting assumption does not give rise to local symmetric spaces different from the group case.

In [FH] the splitting assumption is removed, and the work is extended to $GL(n)$. It is observed that the ideas underlying the Deligne-Kazhdan simple trace formula can be used to carry out the work without elaborating too much on the asymptotic expansion of integrals and characters mentioned above. But in the recent work [F2′] (see [F2] for a quadratic analogue), which concern the group $G = GL(2n, F)$, and the involution σ which is given by conjugation with $J = \mathrm{diag}(I_n, -I_n)$, thus $H = G^+$ is isomorphic to $GL(n, F) \times GL(n, F)$, the property of non vanishing of the spherical character on the regular set came to play a prominent role. This motivates and underlies our interest in the questions considered here.

Section 11 here is an attempt to extend the techniques of Harish-Chandra's short and beautiful paper [HC2] to the context of symmetric spaces. In some cases, including $(G, H) = (\mathbf{G}(E), \mathbf{G}(F))$, E/F a quadratic extension, we show that the orbital integral $\mu_{\tilde{x}}(\tilde{f})$ is locally constant on the σ-regular set, uniformly in f. Surely this should be provable in general by the techniques of [HC1]. But the simplicity of the "submersion" principle of [HC2] appealed to us. [HC2] gives also a simple proof that characters are locally constant on the regular set. It will be interesting to extend this proof to the spherical case.

Section 12 concerns a spherical analogue of Kazhdan's density theorem [K], who dealt with the group case. Very briefly, Harish-Chandra's density mentioned above, and Bernstein's localization principle ([BZ], [Be]), show that the vanishing of all regular orbital integrals implies the vanishing of all invariant distributions in the group case. Kazhdan [K] shows in this case that the vanishing of all characters of irreducible admissible representations implies the vanishing of all regular orbital integrals, hence of all invariant distributions. Kazhdan's proof is global. Section 12 states and proves an analogue in the spherical situation. This essentially says that the vanishing of all spherical characters implies the vanishing of all spherical orbital integrals on the regular elliptic set. The passage from the elliptic to general elements is a trivial change of variables formula in the group case. But I do not know to carry it out in the spherical case. In the elliptic case, our proof is global, as is Kazhdan's, and requires developing a bi-period summation formula, as well as basic Galois cohomology. Bernstein [F4] has given an entirely different, local proof of Kazhdan's density theorem, but this too would not extend to the spherical case, since (in particular) the singular orbital integrals are not determined by the regular ones.

Sections 1, 11, 12 can be read independently of each other, and so can be Sections 3, 4, 5, 6, 7 – 10 which depend on Section 2.

Much of the material here I learnt from J. Bernstein, D. Kazhdan, J.G.M. Mars, and especially S. Rallis, who suggested for example the case of Section 6. The paper was written up while I benefitted from the hospitality (and support) of J. Soto-Andrade at Santiago (and the NSF "Americas Program") and R. Weissauer at Mannheim (and a DAAD grant). I am grateful for their help and encouragement.

1. Asymptotic behaviour of orbital integrals. Let F be a non archimedean local field, $\mathbf{G}$ a reductive group defined over F, $\boldsymbol{\sigma}$ an involution (automorphism of order two) of $\mathbf{G}$ over F, $\mathbf{G}^+ = \mathbf{H} = \mathbf{G}^\sigma$ the group of fixed points of $\boldsymbol{\sigma}$ in $\mathbf{G}$, $\mathbf{Z}$ the center of $\mathbf{G}$, $\mathbf{Z_H} = \mathbf{Z} \cap \mathbf{H}$, and put $G = \mathbf{G}(F)$, $H = \mathbf{H}(F)$, $Z = \mathbf{Z}(F)$, etc., for the corresponding groups of F-points, and σ for the induced involution of G (then $H = G^\sigma$). The groups G, H, … are ℓ-groups in the terminology of Bernstein-Zelevinski [BZ;(1.1)], namely they are Hausdorff, locally compact, and there is a fundamental system of neighborhoods of the unit element consisting of open compact subgroups. By abuse of terminology we refer to G, H, … as reductive groups, or reductive F-groups, if $\mathbf{G}$, $\mathbf{H}$, … are. Note that $\mathbf{H}$ is reductive (see Steinberg [St], Theorem 8.1,and Richardson [Ri], p. 288).

Definition. For any ℓ-space ([BZ;(1.1)]) X, denote by $C_c^\infty(X)$ the space of complex valued locally constant compactly supported functions on X, and by $C_c^\infty(X)^* = \mathrm{Hom}_{\mathbb{C}}(C_c^\infty(X), \mathbb{C})$ the space of *distributions* on X (linear complex valued functions on $C_c^\infty(X)$) ([BZ;(1.7)]).

In particular $C_c^\infty(G/Z)$ is the space of locally constant functions on G which transform trivially under Z and are compactly supported on G mod Z. More generally, we could fix a closed subgroup Z_0 of Z such that Z/Z_0 has finite volume, and a character ω of Z_0, and consider $C_{c,\omega}^\infty(G/Z_0)$, the space of locally constant functions on G which transform under Z_0 via ω, and are compactly supported on G mod Z_0. But this would complicate the notations, so we restrict attention to $Z_0 = Z$, $\omega = 1$.

The group $H \times H$ acts on G by right and left translation: $(h, h')g = hgh'^{-1}$, hence on $C_c^\infty(G/Z)$ by $(h, h')f(g) = f(h^{-1}gh')$, and on $C_c^\infty(G/Z)^*$ by $((h, h')D)(f) = D((h, h')^{-1}f)$. Denote by $C_c^\infty(X)^{*A}$ the space of A-invariant distributions on X, where A is a group which acts on X. We are interested in the space $C_c^\infty(G/Z)^{*H\times H}$ of $H \times H$-invariant distributions on G/Z. If H acts by right translation on G, then the map $C_c^\infty(G/Z) \to C_c^\infty(G/HZ)$, $f \mapsto \tilde{f}$, $\tilde{f}(g) = \int_{H/Z_H} f(gh)dh$, is surjective, with kernel generated by the differences $f - h\cdot f$ ($f \in C_c^\infty(G/Z), h \in H, h \cdot f(g) = f(gh)$). The dual map $C_c^\infty(G/HZ)^* \to C_c^\infty(G/Z)^*$ is then an injection onto the space of H-invariant distributions on G/Z. Thus, as noted in Prasad [P], Lemma 4.1, $C_c^\infty(G/Z)^{*H} = C_c^\infty(G/HZ)^*$. Denote the image of the map $G/HZ \to G/Z_H$, $g \mapsto \tilde{g} = g\sigma(g)^{-1}$, by $\tilde{G}$. Then $C_c^\infty(G/Z)^{*H} = C_c^\infty(\tilde{G})^*$. Similarly, $C_c^\infty(G/Z)^{*H\times H} = C_c^\infty(\tilde{G})^{*H}$, where H acts on $\tilde{G}$ by conjugation: $h \cdot \tilde{g} = (hg)\tilde{} = \mathrm{Int}(h)\tilde{g} = h\tilde{g}h^{-1}$.

Examples of elements in the space $C_c^\infty(\tilde{G})^{*H}$ of interest are given by orbital integrals.

Definition. An element $\tilde{\gamma}$ of $\tilde{G}$ will be called *σ-regular* if it is semi-simple in G and its centralizer $Z_H(\tilde{\gamma})$ in H is a torus.

So if $\tilde{\gamma}$ is regular in G (its centralizer $Z_G(\tilde{\gamma})$ in G is a torus), then it is σ-regular. A reductive group is unimodular ([BZ;(1.19)]). Hence H is unimodular, and if $\tilde{\gamma}$ is σ-regular, so is the torus $Z_H(\tilde{\gamma})$. By [BZ;(1.21)], there exists a unique (up to a scalar multiple) H-invariant distribution on $H/Z_H(\tilde{\gamma})$ (H acts by left translation). It is a *measure* (it takes positive values at $f \neq 0$, $f \geq 0$). The orbit of $\tilde{\gamma}$ is denoted by $\mathcal{O}(\tilde{\gamma})$=$\mathrm{Int}(H)(\tilde{\gamma}) = \{\mathrm{Int}(h)\tilde{\gamma}; h \in H\}$. Then $H/Z_H(\tilde{\gamma}) \simeq \mathcal{O}(\tilde{\gamma})$ by $h \mapsto \mathrm{Int}(h)\tilde{\gamma}$, and we have:

Definition. The *orbital integral* $\mu_{\tilde{\gamma}}(\tilde{f}) = \int \tilde{f}(\mathrm{Int}(h)\tilde{\gamma})d_{H/Z_H(\tilde{\gamma})}(h)$ is the unique (up to a positive multiple) H-invariant measure on $\mathcal{O}(\tilde{\gamma})$; thus $\mu_{\tilde{\gamma}}$ spans $C_c^\infty(\mathcal{O}(\tilde{\gamma}))^{*H}$.

Since $\tilde{\gamma}$ is semi simple, its H-orbit $\mathcal{O}(\tilde{\gamma})$ is closed in $\tilde{G}$ (Richardson [Ri], Theorem 7.5, p. 287). We shall use the following observation of [BZ;(1.8),(1.9)]. If Y is an open subset in an ℓ-space X, and $Z = X - Y$, i_Y is the extension by zero and p_Z is the restriction, then the sequence

$$(i) \qquad 0 \to C_c^\infty(Y) \xrightarrow{i_Y} C_c^\infty(X) \xrightarrow{p_Z} C_c^\infty(Z) \to 0$$

is exact, and so is its dual

$$(ii) \qquad 0 \to C_c^\infty(Z) \xrightarrow{p_Z^*} C_c^\infty(X)^* \xrightarrow{i_Y^*} C_c^\infty(Y)^* \to 0.$$

Hence for a σ-regular $\tilde{\gamma}$, the distribution $\mu_{\tilde{\gamma}}$ on the closed subset $\mathcal{O}(\tilde{\gamma})$ of $\tilde{G}$ extends to a measure on $\tilde{G}$, which is supported ([BZ;(1.10)]) on $\mathcal{O}(\tilde{\gamma})$, hence it is H-invariant. Namely

$\mu_{\tilde{\gamma}}$ extends to an element of $C_c^\infty(\tilde{G})^{*H}$. Our aim is to study the asymptotic behaviour of the distributions $\mu_{\tilde{\gamma}}$ in the vicinity of a semi simple (in G) element $\tilde{s}$ of $\tilde{G}$, which is not necessarily σ-regular.

Example. Before describing this, we note that our setting $-$ of symmetric spaces G/H $-$ reduces to the classical situation of a reductive group G acting on itself by conjugation, in the case where we take $G = H \times H$, $\sigma(x,y) = (y,x)$. This case is referred to as the *group case.* Then G^σ is H, embedded diagonally $(x \mapsto (x,x))$ in G. Moreover, $C_c^\infty(G/Z)$ is spanned by $f = (f_1, f_2)$, where $f(a,b) = f_1(a)f_2(b)$, $f_i \in C_c^\infty(H/Z_H)$, and, putting $\gamma = (\gamma_1, \gamma_2)$, $\tilde{\gamma} = (\gamma_1\gamma_2^{-1}, \gamma_2\gamma_1^{-1}))$, we have

$$\begin{aligned}\int \tilde{f}(\mathrm{Int}(x)\tilde{\gamma}) &= \iint f(x\gamma y) = \iint f_1(x\gamma_1 y) f_2(x\gamma_2 y)\\ &= \iint f_1(x\gamma_1\gamma_2^{-1}x^{-1}y)f_2(y) = \int (f_1 * f_2^*)(x\gamma_1\gamma_2^{-1}x^{-1}),\end{aligned}$$

where $f_2^*(g) = f_2(g^{-1})$, and $f_1 * f_2(g) = \int_{H/Z_H} f_1(gx)f_2(x^{-1})dx$. Then $\tilde{f}(u,u^{-1}) = (f_1 * f_2^*)(u)$, and $\mu_{\tilde{\gamma}}(\tilde{f})$ coincides with the orbital integral $\mu_{\gamma_0}(f_0)$ of f_0 at γ_0, if $\tilde{\gamma} = (\gamma_0, \gamma_0^{-1})$ and $f_0(u) = \tilde{f}(u,u^{-1})$. The asymptotic behaviour of the orbital integrals $\mu_{\gamma_0}(f_0) = \int_{G/Z_G(\gamma_0)} f_0(\mathrm{Int}(g)\gamma_0)dg$ in the vicinity of a semi simple element s of G has been studied by Shalika [Sh], Vigneras [V], and others. This section merely extends their work to the situation of general symmetric spaces G/H, not only the case of $H \times H/H$.

Let $x = su = us$ be the Jordan decomposition of $x \in G$ as a product of a semi simple element s and a unipotent element u. As noted in [Ri], p. 287, x lies in $\tilde{G}$ if and only if both s and u lie in $\tilde{G}$. Fix a semi simple element $\tilde{s}$ in $\tilde{G}$. Denote by $\tilde{G}_{\tilde{s}}$ the set of $\tilde{x}$ in $\tilde{G}$ whose semi simple part lies in $\mathrm{Int}(H)\tilde{s}$. Then H acts on $\tilde{G}_{\tilde{s}}$ by conjugation. By [Ri], (9.11), p. 303, $\tilde{G}_{\tilde{s}}$ is closed and it consists of finitely many H-orbits $\mathcal{O}$.

1. Proposition. *The dimension of the complex vector space $C_c^\infty(\tilde{G}_{\tilde{s}})^{*H}$ is finite, bounded by the number of H-orbits in $\tilde{G}_{\tilde{s}}$.*

Remark. Before starting the proof, note that in the group case (1) the centralizer $Z_G(x)$ of any element $x \in G$ is unimodular (see Springer-Steinberg [SS], III, (3.27b), p. 234), hence the orbital integral $\mu_{\mathcal{O}(x)}$ exists on $C_c^\infty(\mathcal{O}(x))$, where $\mathcal{O}(x)$=$\mathrm{Int}(G)x$ is the orbit of x, and (2) the theorem of Rao [R] shows that $\mu_{\mathcal{O}(x)}$ extends as a G-invariant distribution to the closure $\overline{\mathcal{O}(x)}$ of $\mathcal{O}(x)$, and by (ii) to G, yielding, for each orbit $\mathcal{O}$, a unique up to a scalar multiple and measures supported on the complement $\overline{\mathcal{O}} - \mathcal{O}$ of $\mathcal{O}$ in $\overline{\mathcal{O}}$, G-invariant measure $\mu_{\mathcal{O}}$ whose support is the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$. Then these $\mu_{\mathcal{O}}$ $-$ where $\mathcal{O}$ ranges over the G-orbits in $G_s = \{x \in G$ with semi simple part in $\mathrm{Int}(G)s\}$ $-$ give a basis of $C_c^\infty(G_s)^{*G}$.

In our symmetric space situation there are $\tilde{x} \in \tilde{G}_{\tilde{s}}$ such that $Z_H(\tilde{x})$ is not unimodular. Then there is no H-invariant distribution on the H-orbit $\mathcal{O}(\tilde{x})$ of $\tilde{x}$. Moreover, there can be orbits $\mathcal{O} = \mathcal{O}(\tilde{x})$ such that $Z_H(\tilde{x})$ is unimodular, but the H-invariant measure $\mu_{\mathcal{O}}$ on $C_c^\infty(\mathcal{O})$ may not extend to an H-invariant distribution on $\overline{\mathcal{O}}$, and $\tilde{G}$.

Proof. We proceed as follows. The "Closed Orbit Lemma" (see, e.g., Borel [Bo], (I.1.8), and [HCD], Lemma 31, p. 71), implies that every orbit $\mathcal{O}$ in $\tilde{G}_{\tilde{s}}$ is open in its closure, and $\dim \mathcal{O}' < \dim \mathcal{O}$ for every orbit $\mathcal{O}' \neq \mathcal{O}$ in $\overline{\mathcal{O}}$. We shall number these H-orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots$, in such a way that $\dim \mathcal{O}_i \leq \dim \mathcal{O}_{i+1}$, $\mathcal{O}^j = \cup_{i\leq j}\mathcal{O}_i$ is closed, $\mathcal{O}_j$ is open in $\mathcal{O}^j$, and $\mathcal{O}^1 = \mathcal{O}_1=\mathrm{Int}(H)\tilde{s}$. Having listed $\mathcal{O}_1,\dots,\mathcal{O}_i$, we list by $\mathcal{O}_{i+1},\dots,\mathcal{O}_j$ the smallest number of orbits $(= j-i)$ subject to the conditions above, such that the closed set $\mathcal{O}^j$ supports an H-invariant distribution Λ'_j. In particular, $\Lambda'_j|\mathcal{O}_j$ is the H-invariant distribution on the (open in $\mathcal{O}^j$) orbit $\mathcal{O}_j$ of some $\tilde{x}_j$ in $\tilde{G}_{\tilde{s}}$, hence $Z_H(\tilde{x}_j)$ is unimodular. For $i < m < j$ either $\mathcal{O}_m = \mathcal{O}(\tilde{x}_m)$ and $Z_H(\tilde{x}_m)$ is not unimodular, or $\mu_{\mathcal{O}_m}$ exists on $C_c^\infty(\mathcal{O}_m)$, but it does not extend as an H-invariant distribution on $C_c^\infty(\mathcal{O}^m)$. Denote by $\Lambda_1,\dots,\Lambda_k$ the basis of $C_c^\infty(\tilde{G}_{\tilde{s}})^{*H}$ so obtained. Then Λ_i is supported on $\mathcal{O}^{j_i}$, where $1 \leq j_i < j_{i+1} \leq k' =$ number of H-orbits in $\tilde{G}_{\tilde{s}}$. This proves the proposition, and fixes a basis of the space of the proposition (but the basis is not canonical). Indeed, we claim that given Λ in $C_c^\infty(\tilde{G}_{\tilde{s}})^{*H}$ there are complex numbers $c_i (1\leq i\leq k)$ such that $\Lambda = \sum_{1\leq i\leq k} c_k\Lambda_k$. These c_i are chosen by descending induction. Having chosen $c_k,\dots,c_{i+1}$ such that $\Lambda - \sum_{i+1\leq u\leq k} c_u\Lambda_u$ lies in $C_c^\infty(\mathcal{O}^{j_i})^{*H}$, there is c_i such that $\Lambda - \sum_{i\leq u\leq k} c_u\Lambda_u$ lies in $C_c^\infty(\mathcal{O}^{j_i-1})^{*H}$, hence it must lie in $C_c^\infty(\mathcal{O}^{j_{i-1}})^{*H}$ by the definition of Λ_i and j_i. □

For every $i (1 \leq i \leq k)$, let $\tilde{f}'_i$ be an element of the subspace $C_c^\infty(\mathcal{O}_{j_i})$ of $C_c^\infty(\mathcal{O}^{j_i})$ such that $\Lambda_i(\tilde{f}'_i) = 1$. Then $\Lambda_j(\tilde{f}_i) = 0$ for $j < i$. Extend $\tilde{f}'_i$ to $\tilde{f}''_i$ in $C_c^\infty(\tilde{G})$ (or even just $C_c^\infty(\tilde{G}_{\tilde{s}})$). Define $\tilde{f}_i$ in $C_c^\infty(\tilde{G})$ inductively to satisfy $\Lambda_j(\tilde{f}_i) = \delta_{ij} (1 \leq i,j \leq k)$ as follows. Put $\tilde{f}_k = \tilde{f}''_k$. Having defined $\tilde{f}_k,\dots,\tilde{f}_{i+1}$, let $\tilde{f}_i$ be $\tilde{f}''_i - \sum_{j>i}\Lambda_j(\tilde{f}''_i)\tilde{f}_j$. For any σ-regular element $\tilde{x}$ in $\tilde{G}$, define the complex number $,{}_i(\tilde{x})$ to be $\mu_{\tilde{x}}(\tilde{f}_i) = \int \tilde{f}_i(\mathrm{Int}(h)\tilde{x}) d_{H/Z_H(\tilde{x})}(h)$. The complex valued function $,{}_i(\tilde{x})$ on the σ-regular set in $\tilde{G}$ is $\mathrm{Int}(H)$-invariant, and is called a *germ* of orbital integrals on $\tilde{G}$. Then one has the following asymptotic expansion.

2. Theorem. *For every $\tilde{f}$ in $C_c^\infty(\tilde{G})$ and semi simple $\tilde{s}$ in $\tilde{G}$, there exists an H-invariant open and closed neighborhood $V_{\tilde{f}}$ of $\tilde{s}$ in $\tilde{G}$ such that for every σ-regular $\tilde{x}$ in $V_{\tilde{f}}$ we have $\mu_{\tilde{x}}(\tilde{f}) = \sum_{1\leq i\leq k} ,{}_i(\tilde{x})\Lambda_i(\tilde{f})$.*

Namely the distributions $\mu_{\tilde{x}}$ in $C_c^\infty(\tilde{G})^{*H}$ are given by the distribution $\sum_i ,{}_i(\tilde{x})\Lambda_i$ in $C_c^\infty(\tilde{G}_{\tilde{s}})^{*H}$, but only for σ-regular $\tilde{x}$ in a neighborhood of $\tilde{s}$ depending on the $\tilde{f}$ at which $\mu_{\tilde{x}}$ is evaluated. Alternatively, fixing $\tilde{f}$, the function $\mu_{\tilde{x}}(\tilde{f})$ of $\tilde{x}$ in $\tilde{G}/\mathrm{Int}(H)$ has asymptotic behaviour near $\tilde{s}$ which is controlled by the functions $,{}_i(\tilde{x})$.

Proof. Put $\tilde{f}' = \tilde{f} - \sum_i \Lambda_i(\tilde{f})\tilde{f}_i$. Then $\Lambda_i(\tilde{f}') = 0$ for all i. Hence for all Λ in $C_c^\infty(\tilde{G}_{\tilde{s}})^{*H} = (C_c^\infty(\tilde{G}_{\tilde{s}})/C_c^\infty(\tilde{G}_{\tilde{s}})_0)^*$ we have $\Lambda(\tilde{f}') = 0$. Here $C_c^\infty(\tilde{G}_{\tilde{s}})_0$ is the span of $\overline{\tilde{\phi}} - h\cdot\overline{\tilde{\phi}}$, where $\overline{\tilde{\phi}}$ denotes the restriction of $\tilde{\phi} \in C_c^\infty(\tilde{G})$ to $\tilde{G}_{\tilde{s}}$, and $h\cdot\overline{\tilde{\phi}}(\tilde{x}) = \overline{\tilde{\phi}}(\mathrm{Int}(h^{-1})\tilde{x})$ $(h \in H)$. It then follows that the restriction $\overline{\tilde{f}}'$ of $\tilde{f}'$ to $\tilde{G}_{\tilde{s}}$ lies in $C_c^\infty(\tilde{G}_{\tilde{s}})_0$, namely there exist finitely many $\phi_i \in C_c^\infty(\tilde{G})$ and $h_i \in H$ such that $\overline{\tilde{f}}' = \sum_i(\overline{\tilde{\phi}}_i - h_i\cdot\overline{\tilde{\phi}}_i)$. We conclude that $\tilde{f} - \sum_i \Lambda_i(\tilde{f})\tilde{f}_i - \sum_i(\tilde{\phi}_i - h_i\cdot\tilde{\phi}_i)$ lies in $C_c^\infty(\tilde{G}-\tilde{G}_{\tilde{s}})$. Since $\tilde{G}_{\tilde{s}}$ is an H-invariant closed subset in $\tilde{G}$, and this function is compactly supported, there is some H-invariant open and closed set $V_{\tilde{f}}$ in $\tilde{G}$, containing $\tilde{G}_{\tilde{s}}$, such that $\tilde{f} = \sum_i \Lambda_i(\tilde{f})\tilde{f}_i + \sum_i(\tilde{\phi}_i - h_i\cdot\tilde{\phi}_i)$ on $V_{\tilde{f}}$. Then for any σ-regular $\tilde{x}$ in $V_{\tilde{f}}$, the H-orbit of $\tilde{x}$ lies in $V_{\tilde{f}}$, and $\mu_{\tilde{x}}(\tilde{f}) = \sum_{1\leq i\leq k}\Lambda_i(\tilde{f})\mu_{\tilde{x}}(\tilde{f}_i)$, as required. □

Conversely, this germ expansion characterizes the orbital integrals.

3. Theorem. *Let $\mu(\tilde{x})$ be a function on the σ-regular set of $\tilde{G}$, which is H-invariant and its support is compact on $\tilde{G}/\operatorname{Int}(H)$, such that for each semi simple element $\tilde{s}$ of $\tilde{G}$ there is an open and closed H-invariant neighborhood $V_{\tilde{s}}$ of $\tilde{s}$ in $\tilde{G}$, such that $\mu(\tilde{x}) = \sum_i ,_i^{\tilde{s}}(\tilde{x})\Lambda_i^{\tilde{s}}$ for all σ-regular $\tilde{x}$ in $V_{\tilde{s}}$, where $\Lambda_i^{\tilde{s}}$ are complex numbers, and $,_i^{\tilde{s}}(\tilde{x})$ are the finitely many H-invariant functions introduced above. Then there is $\tilde{f}$ in $C_c^\infty(\tilde{G})$ such that $\mu(\tilde{x}) = \mu_{\tilde{x}}(\tilde{f})$ for all σ-regular $\tilde{x}$ in $\tilde{G}$.*

Proof. The assumption at the semi simple $\tilde{s}$ implies that for $\tilde{f}_{\tilde{s}} = \sum_i \Lambda_i^{\tilde{s}} \cdot \tilde{f}_i^{\tilde{s}}$ we have that $\mu(\tilde{x}) = \mu_{\tilde{x}}(\tilde{f}_{\tilde{s}})$ for all σ-regular $\tilde{x}$ in $V_{\tilde{s}}$ (since $\mu_{\tilde{x}}(\tilde{f}_i^{\tilde{s}}) = ,_i^{\tilde{s}}(\tilde{x})$). We may assume that $\tilde{f}_{\tilde{s}}$ is supported on $V_{\tilde{s}}$. Since $\operatorname{supp}\mu$ is compact in $\tilde{G}/\operatorname{Int}(H)$, and it is covered by $\cup V_{\tilde{s}}$, union over all semi simple $\tilde{s}$ in $\tilde{G}$, there is a finite subcover, which we may assume to be disjoint since the $V_{\tilde{s}}$ are open and closed. Put $\tilde{f} = \sum \tilde{f}_{\tilde{s}}$ (finite sum). Then $\mu(\tilde{x}) = \mu_{\tilde{x}}(\tilde{f})$ for all σ-regular $\tilde{x}$ in $\tilde{G}$, as required. □

Remark. In the group case, if $\mu_x(f) = \int f(\operatorname{Int}(g)x)d_{G/Z_G(x)}(g)$ is zero for all regular elements x in G, then $\mu_x(f) = 0$ for all x in G, by [HC1], Theorem 10. The proof of this relies on [HC1], Lemma 8, which uses the local integrability of characters of supercuspidal representations. This had been proven in [HCD]. The symmetric space analogues – the spherical characters, which are defined below (as in [FH]) – are rarely locally integrable, even for spherical supercuspidal representations. The proof of Theorem 10 in [HC1] does not extend to the spherical case, and indeed the vanishing of $\mu_{\tilde{x}}(\tilde{f})$ for all σ-regular $\tilde{x}$ in $\tilde{G}$, for a fixed $\tilde{f}$ in $C_c^\infty(\tilde{G})$, does not imply that $\Lambda(\tilde{f}) = 0$ for all Λ in $C_c^\infty(\tilde{G})^{*H}$.

2. Spherical characters on the regular set. Let G be an ℓ-group ([BZ]), σ an involution of G, $H = G^\sigma$ the group of fixed points, $Z = Z(G)$ the center of G, $\tilde{G}$ the image of the map $G/H \to G$, $g \mapsto \tilde{g} = g\sigma(g)^{-1}$. Let π be an irreducible admissible G-module, which is H-*spherical*, thus the dual $\pi^* = \operatorname{Hom}_{\mathbb{C}}(\pi, \mathbb{C})$ of π contains H-invariant non zero elements. Fix $L = L_\pi \neq 0$ in $\pi^{*H} = \operatorname{Hom}_H(\pi, \mathbb{C})$. Then the contragredient $\tilde{\pi}$ of π is also H-spherical: given an orthonormal basis $\{\xi\}$ of π, define $\hat{L} = L_{\tilde{\pi}}$ in $\tilde{\pi}^{*H} = \operatorname{Hom}_H(\tilde{\pi}, \mathbb{C})$ by $< L_{\tilde{\pi}}, \hat{\xi} >=< L_\pi, \xi >$, where $\{\hat{\xi}\}$ is the basis of $\tilde{\pi}$ dual to $\{\xi\}$. For simplicity assume that the restriction of π to the center Z is trivial. As in [FH], [H], [Ke], [RR], we make:

Definition. For every $f \in C_c^\infty(G/Z)$, the vector $\pi(f)\hat{L}$ lies in the smooth part $(\tilde{\pi})^*_{\mathrm{sm}} = \tilde{\tilde{\pi}} = \pi$ of π^*. Hence the *spherical character* $\mathbb{L}_\pi(\tilde{f}) =< L, \pi(f)\hat{L} >$ of π is a well-defined H-invariant distribution on $\tilde{G}/\tilde{Z}$, where $\tilde{f}(\tilde{g}) = \int_H f(gh)dh$.

Example. In the group case, where $G = H \times H$ and $\sigma(x, y) = (y, x)$, it coincides with the trace distribution. Indeed, a G-module $\pi = \pi_1 \times \pi_2$ is H-spherical when $\operatorname{Hom}_H(\pi_1 \times \pi_2, \mathbb{C})$ is non empty. Then $\pi_2 = \tilde{\pi}_1$, and an H-invariant form $L : \pi_1 \times \tilde{\pi}_1 \to \mathbb{C}$ is given by $L(\eta_1 \times \hat{\eta}_2) =< \eta_1, \hat{\eta}_2 >$. In fact $L = \sum_{\xi \in B} \hat{\xi} \times \xi$, where ξ ranges over an orthonormal basis of π (check $L(\eta_1 \times \hat{\eta}_2)$ on basis elements). Moreover $\hat{L} = \sum \xi \times \hat{\xi}$, and for $f(x, y) =$

$f_1(x)f_2(y) \in C_c^\infty(G/Z)$, we have

$$\mathbb{L}_\pi(\tilde{f}) =< L, \pi(f)\hat{L} >= \sum_{\xi_1,\xi_2\in B} < \hat{\xi}_1 \times \xi_1, \pi_1(f_1)\xi_2 \times \tilde{\pi}_1(f_2)\hat{\xi}_2 >$$

$$= \sum_{\xi_1,\xi_2} < \hat{\xi}_1, \pi_1(f_1)\xi_2 >< \xi_1, \tilde{\pi}_1(f_2)\hat{\xi}_2 >= \sum_{\xi_2} < \sum_{\xi_1} < \hat{\xi}_1, \pi_1(f_1)\xi_2 > \xi_1, \tilde{\pi}_1(f_2)\hat{\xi}_2 >$$

$$= \sum_{\xi_2} < \pi_1(f_2^*)\pi_1(f_1)\xi_2, \hat{\xi}_2 >= \mathrm{tr}\ \pi_1(f_2^* * f_1) = \mathrm{tr}\ \pi_1(f_1 * f_2^*).$$

Here $f_2^*(x) = f_2(x^{-1})$, and $(f_1 * f_2)(x) = \int_{H/Z} f_1(xy^{-1})f_2(y)dy$ is the convolution of f_1 and f_2. Moreover, $\tilde{f}(xy^{-1}, yx^{-1}) = \int_H f(xu, yu)du = (f_1 * f_2^*)(xy^{-1})$.

Harish-Chandra [HC1], Theorem 5, describes the asymptotic expansion of the character χ_π, where $\mathrm{tr}\ \pi(f) = \int \chi_\pi(g)f(g)dg$, near any semi simple element s of the group. This description extends to the spherical situation, as shown by Hakim [H] in the case where E/F is a quadratic field extension, $G = \mathbf{G}(E)$ and $H = \mathbf{H}(F)$ (σ is the galois action; see [H], Theorem 2), and Rader-Rallis [RR] in the general case of G/H ([RR] considers the expansion only near the identity, but the description extends to any semi simple element by the arguments of [H], §2). Their combined result is as follows.

Given s in G such that $\tilde{s} = s\sigma(s)^{-1}$ is semi simple, let M be the connected component of the centralizer $Z_G(\tilde{s})$ of $\tilde{s}$ in G, and $M^+ = M \cap G^+$, where $G^+ = H$ is the group of fixed points of σ in G. Denote by $\mathfrak{g}$ the Lie algebra of G, by $\mathfrak{g}^+$ the algebra of σ-fixed points in $\mathfrak{g}$, and by $\mathfrak{g}^-$ the -1 eigenspace of σ in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$, and $\mathfrak{g}^+$ is the Lie algebra of G^+. Let $\mathfrak{M}$ be the centralizer $Z_{\mathfrak{g}}(\tilde{s})$ of $\tilde{s}$ in $\mathfrak{g}$, and $\mathfrak{M}^\pm = \mathfrak{M} \cap \mathfrak{g}^\pm$. Let $\{\Lambda\}$ be a basis for the finite dimensional space of $\mathrm{Ad}(M^+)$-invariant distributions on $\mathfrak{M}^-$ which are supported on the nilpotent set $\mathfrak{M}^-_{\mathrm{nilp}}$ of $\mathfrak{M}^-$. Since M is reductive there is an F-valued symmetric non degenerate σ and M-invariant bilinear form B on $\mathfrak{M}$. Fix an additive character $\psi \neq 1$ of F, and a Haar measure dX on $\mathfrak{M}^-$. The *Fourier transform* $\hat{f}$ of $f \in C_c^\infty(\mathfrak{M}^-)$ is the element of $C_c^\infty(\mathfrak{M}^-)$ defined by $\hat{f}(Y) = \int_{\mathfrak{M}^-} \psi(B(Y,X))f(X)dX$. The *Fourier transform* $\hat{T}$ of the distribution T in $C_c^\infty(\mathfrak{M}^-)^*$ is the element of $C_c^\infty(\mathfrak{M}^-)^*$ defined by $\hat{T}(f) = T(\hat{f})$. Then the asymptotic expansion of the spherical character $\mathbb{L}_\pi(\tilde{f})$ can be expressed as follows.

1. Theorem ([H],[RR]). *Given the H-spherical irreducible admissible G-module π, and the basis $\{\Lambda\}$ of the finite dimensional space $C_c^\infty(\mathfrak{M}^-_{\mathrm{nilp}})^{*M^+}$, there are unique complex numbers $c_\Lambda(\pi)$ such that for every $f \in C_c^\infty(\mathfrak{M}^-)$ which is supported in a sufficiently small neighborhood of zero, we have $\mathbb{L}_\pi(f \circ \exp) = \sum_\Lambda c_\Lambda(\pi)\hat{\Lambda}(f)$.*

Here the exponential map $\exp : \mathfrak{M}^- \to \tilde{M} = \{\tilde{m} = m\sigma(m)^{-1}; m \in M\}$ is a homeomorphism on a sufficiently small neighborhood of zero in $\mathfrak{M}^-$, extending under conjugation by M^+ to the entire nilpotent set $\mathfrak{M}^-_{\mathrm{nilp}}$ of $\mathfrak{M}^-$. Moreover, on the σ-regular set of G the distribution $\mathbb{L}_\pi$ is represented by a locally constant function.

The last assertion is proven in Harish-Chandra [HC1], Theorem 3, in the group case (see also the simple proof in [HC2]), in [FH] in the special (quadratic) case considered in [FH], and in general in Rader-Rallis [RR]. It will be interesting to extend the simple proof of [HC2] to the general spherical case.

In the group case, Harish-Chandra [HC1], Theorem 3, using [HCD], has shown that the character χ_π of a G-module π is a locally integrable function. In particular χ_π is not identically zero on the regular set of G. The analogous local integrability statement holds when E/F is quadratic, $G = \mathbf{G}(E), H = \mathbf{H}(F)$, by Hakim [H], Theorem 1. Hence $\mathbb{L}_\pi$ is not identically zero on the σ-regular set of G.

However, for a general symmetric space G/H, the H-invariant distribution $\mathbb{L}_\pi$ is not locally integrable. Moreover, there are sometimes H-invariant distributions Λ on the nilpotent set $\mathfrak{g}^-_{\text{nilp}}$ of $\mathfrak{g}^-$ whose Fourier transform on $\mathfrak{g}^-$ is also supported on the nilpotent subset of $\mathfrak{g}^-$. This suggests that there are – sometimes – admissible G-modules π (perhaps only in the Grothendieck group), such that $\mathbb{L}_\pi$ is supported on the nilpotent cone $\mathfrak{g}^-_{\text{nilp}}$.

In the general case where the base field F is the field $\mathbb{R}$ of real numbers, Sekiguchi [S1] has given a condition which – when satisfied – implies that $\mathbb{L}_\pi$ is not identically zero on the σ-regular set. Further, he listed several cases where his condition is satisfied. The techniques employed in the archimedean case are very different from those of the non archimedean case. But the final results are often similar.

The purpose of this paper is to consider several examples, where we show that $\mathbb{L}_\pi$ is not identically zero on the σ-regular set, and some where there are self dual distributions supported on the nilpotent cone $\mathfrak{g}^-_{\text{nilp}}$. Our examples are consistent with the investigation of Sekiguchi [S1].

Our main example concerns the group $G = GL(n+m, F)$, convolution σ given by conjugation with $J = \operatorname{diag}(I_n, -I_m)$, thus $H = G^+ = G^\sigma$ consists of $\operatorname{diag}(A, B), A \in GL(n, F), B \in GL(m, F)$, $\mathfrak{g} = M((n+m) \times (n+m), F)$, $\mathfrak{g}^+ = \{\operatorname{diag}(A, B); A \in M(n \times n, F), B \in M(m \times m, F)\}$, $\mathfrak{g}^- = \{\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right); A \in M(n \times m, F), B \in M(m \times n, F)\}$.

2. Theorem. *When $m = 1, n > 2$, or $n = m = 1$, or $n = m = 2$, for any admissible irreducible H-spherical G-module π, the distribution $\mathbb{L}_\pi$ is not identically zero on the σ-regular set. However, if $n = 2m, m \geq 1$, then there are H-invariant distributions on $\mathfrak{g}^-$ which are supported on $\mathfrak{g}^-_{\text{nilp}}$ and are equal to their Fourier transform.*

The case where $m = 1$ is considered (from other points of view) in ([DP] and) [F3]. The case where $n = m$ is considered in [F2′]. There it is shown that the non vanishing assertion of Theorem 2 has applications to the theory of liftings of admissible and automorphic representations between $GL(2n)$ and some of its inner forms. In fact, Theorem 2 is our attempt to settle the Working Hypothesis II of [F2′], in the case where $n = m = 1$ and $n = m = 2$. Our approach will be to explicitly construct a basis $\{\Lambda\}$ for $C_c^\infty(\mathfrak{g}^-_{\text{nilp}})^{*H}$, and show that no $\hat{\Lambda}, 0 \neq \Lambda \in C_c^\infty(\mathfrak{g}^-_{\text{nilp}})^{*H}$, can vanish identically on the σ-regular set of $\mathfrak{g}^-$.

Even in the case of $n = m = 2$ the computations are non trivial. They might be pursuable in the case of $n = m = 3$, but perhaps new (combinatorial) techniques need to be introduced to deal with the general case of $n = m$. In any case, the example of $n = m = 2$ is already interesting. Note that when $\tilde{s}$ is semi simple, $M = Z_G(\tilde{s})$ is a product of $GL(n_i, E_i)$, and σ acts trivially on all of them, except perhaps on one of them. Hence induction can be applied, and we shall restrict our attention to $\tilde{s} = 1$ in G.

Let us begin by describing the $H = G^+$-orbits on $\mathfrak{g}^-_{\text{nilp}}$, where $G = GL(n+m, F), H = GL(n, F) \times GL(m, F)$, following Sekiguchi [S1], Lemma 6.7, p. 113, (and [S2], §3.3).

3. Proposition. *Let $\eta = (p_1, p_2, \dots)$ be a partition of $n+m$, thus $p_1 \geq p_2 \geq \cdots \geq p_k > 0, p_1 + \cdots + p_k = n+m$. Put $q_i = [p_i/2]$ for $p_i \geq 2$, and let $k'(0 \leq k' \leq k)$ be the integer such that $p_{k'} > 1 \geq p_{k'+1}$. Write $\{i; 1 \leq i \leq k, p_i$ even $\}$ as a disjoint union*

$I_1 \cup I_2$. Write $\{i; 1 \le i \le k, p_i$ odd $\}$ as $I_3 \cup I_4$, disjoint union with $|I_3| = |I_4|$, and put $I_3' = \{i \in I_3; p_i \ge 3\}$, $I_4' = \{i \in I_4; p_i \ge 3\}$. Put $M(a \times b)$ for the algebra of $a \times b$ matrices over F, $J_d = (\delta_{i,i+1}) \in M(d \times d)$, $K_d = {}^t(I_d; 0) \in M((d+1) \times d)$ (I_d is the identity in $M(d \times d)$, ${}^t g$ is the transpose of g), $L_d = (0, I_d)$ in $M(d \times (d+1))$. Then a set of representatives for the $H = GL(n) \times GL(m)$-orbits in $\mathfrak{g}^-_{\mathrm{nilp}}$ ($\subset \mathfrak{g}^- \subset \mathfrak{g} = M((n+m) \times (n+m))$ is given by

$$X_\eta = \begin{pmatrix} 0 & \mathrm{diag}(X_1, X_2, \dots, X_{k'}) \\ \mathrm{diag}(Y_1, Y_2, \dots, Y_{k'}) & 0 \end{pmatrix},$$

where (X_i, Y_i) is (I_{q_i}, J_{q_i}) if $i \in I_1$, (J_{q_i}, I_{q_i}) if $i \in I_2$, (K_{q_i}, L_{q_i}) if $i \in I_3'$, (L_{q_i}, K_{q_i}) if $i \in I_4'$.

Proof. Let $X \in \mathfrak{g}^-_{\mathrm{nilp}}$ have the Jordan canonical form $J_\eta = \mathrm{diag}(J_{p_1}, J_{p_2}, \dots)$, where η is a partition of $n+m$ as in the proposition. Write the $(n+m)$-dimensional row space $V = F^{n+m}$ as a direct sum of the subspaces $V_1 = \{(x_1, \dots x_n, \dots 0)\} \simeq F^n$ and $V_2 = \{(0, \dots 0, y_1, \dots y_m)\} \simeq F^m$. Then we can choose v_i in V_1 or V_2 such that $v_{ij} = v_i X^{j-1} (1 \le i \le k, 1 \le j \le p_i)$ make a basis of V. The elements of this basis will be ordered in chunks $v_i, v_i X^2, v_i X^4 \dots$ or $v_i X, v_i X^3 \dots$, each chunk lies entirely in V_1 or V_2. We go through $i = 1, 2, \dots, k$, indexing the elements of the new chunk by not yet used j's $\le n$, or by j's $> n$, if the chunk lies in V_1, or in V_2, respectively. Then X has the form X_η with respect to this basis. □

Example. (1) If $n = 1 = m, k = 1, p_1 = 2$, then X is $U_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $U_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. If $k = 2, p_1 = p_2 = 1$, then $X = 0$.
(2) If $n = 2 = m, k = 1$: $X = \begin{pmatrix} 0 & I_2 \\ J_2 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & J_2 \\ I_2 & 0 \end{pmatrix}$. Put $U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $U_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. If $k = 2, (p_1, p_2) = (3, 1)$: $X = \begin{pmatrix} 0 & U_1 \\ U_2 & 0 \end{pmatrix}$ (which is conjugate under the reflection (34) to $\begin{pmatrix} 0 & U_2 \\ U_4 & 0 \end{pmatrix}$), or $\begin{pmatrix} 0 & U_2 \\ U_1 & 0 \end{pmatrix}$. If $k = 2, (p_1, p_2) = (2, 2), X = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & U_1 \\ U_4 & 0 \end{pmatrix}$ (which is conjugate under the action of the transposition (34) to $\begin{pmatrix} 0 & U_2 \\ U_2 & 0 \end{pmatrix}$), or $\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$.
If $k = 3, (p_1, p_2, p_3) = (2, 1, 1)$, $X = \begin{pmatrix} 0 & U_1 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ U_1 & 0 \end{pmatrix}$. If $k = 4$, $X = 0$.
Note that the centralizers of the $X_\eta, \eta = (2, 1, 1)$, are not unimodular. Hence there are no invariant measures on these two orbits.
(3) If (n, m) has $m = 1 (n \ge 2)$, then $X^3 = 0$ ($V_1 X \subset V_2, V_2 X^2 = 0$). If $p_1 = 1$, then $X = 0$. If $p_1 = 2$ then X is the matrix whose only non zero entry is 1 at the top right or bottom left entry. If $p_1 = 3$, X has top row $(0, \dots, 0, 1)$ and bottom row $(0, 1, 0, \dots, 0)$ (this matrix is conjugate under the reflection (12) to the matrix with second row $(0, \dots, 0, 1)$ and bottom row $(1, 0, \dots, 0)$) and its other entries are zero.

3. The case of $G = GL(2), H = GL(1) \times GL(1)$. The restriction of an H-invariant distribution on $\mathfrak{g}^-_{\mathrm{nilp}}$ to an orbit $\mathcal{O}$ in $\mathfrak{g}^-_{\mathrm{nilp}}$ (namely to $C_c^\infty(\mathcal{O})$) is a multiple of the unique (up to a scalar multiple) H-invariant measure $\mu_{\mathcal{O}}$ on $\mathcal{O}$, or it is zero if $\mu_{\mathcal{O}}$ does not exist. Our strategy in the proof of Theorem 2.2 will then be to determine which orbits carry an invariant measure, examine whether $\mu_{\mathcal{O}}$ extends to $\mathfrak{g}^-_{\mathrm{nilp}}$, and if not, determine a linear combination of measures in which it occurs, such that this linear combination is a distribution on $\mathfrak{g}^-_{\mathrm{nilp}}$.

Let us consider first the example where $G = GL(2, F), \sigma(g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Our aim is to prove the following.

1. Theorem. *The space* $C_c^\infty(\mathfrak{g}^-_{\mathrm{nilp}})^{*H}$ *is two dimensional, with basis* $\delta_0(f) = f(0)$ *and*

$$\mu(f) = \int_F f(yU_2)\psi(xy)\ln|x|dxdy + \int_F f(yU_3)\psi(xy)\ln|x|dxdy.$$

Proof. If $\mathcal{O}_+ = \mathrm{Ad}(H)U_2 = F^\times U_2, U_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and $a' = \mathrm{diag}(a,1)$, then

$$\mu_{\mathcal{O}_+}(f) = \int_{F^\times} f(a'U_2a'^{-1})d^\times a = \int_{F^\times} f(aU_2)d^\times a$$

is a linear form on $C_c^\infty(\mathcal{O}_+)$ which does not extend to $\mathfrak{g}^-_{\mathrm{nilp}} = \{xU_2 + yU_3; xy = 0\}$.

We claim that on $C_c^\infty(\mathcal{O}_+)$, $\mu_{\mathcal{O}_+}$ at f is equal to the *principal value of* $d^\times x$ distribution $PV(d^\times x)$ at $f^+(x) = f(xU_2)$. Recall that $PV(d^\times x)(\phi)$, $\phi \in C_c^\infty(F)$, is defined to be the constant term $c_0(\phi)$ in the Laurent expansion at $t = 0$ of

$$\int_{F^\times} \phi(x)|x|^t d^\times x = t^{-1}c_{-1}(\phi) + c_0(\phi) + tc_1(\phi) + \dots.$$

This constant term $c_0(\phi)$ can be computed using the Hecke-Tate functional equation of the zeta function (see Jacquet [J], whose notations we follow here):

$$\begin{aligned}
&\int_{F^\times} \phi(x)|x|^t d^\times x = Z(\phi,t) = \varepsilon(1-t,\psi)L(t)L(1-t)^{-1}Z(\hat{\phi},1-t) \\
&= \varepsilon(1-t,\psi)(1-q^{t-1})(1-q^{-t})^{-1}\int_{F^\times}\hat{\phi}(x)|x|^{1-t}d^\times x \\
&= (t^{-1}\alpha_{-1} + \alpha_0 + \dots)(\phi(0) - t\int_F \hat{\phi}(x)\ln|x|dx + \dots) \\
&= \alpha_{-1}t^{-1}\phi(0) + (\alpha_0\phi(0) - \alpha_{-1}\int_F \hat{\phi}(x)\ln|x|dx) + \dots.
\end{aligned}$$

Here $dx = |x|d^\times x$, and $\hat{\phi}(x) = \int_F \phi(y)\psi(xy)dy$ is the Fourier transform of ϕ with respect to the character $\psi \neq 1$ of F. We used the Laurent expansion $|x|^{-t} = 1 - t\ln|x| + \frac{1}{2}t^2(\ln|x|)^2 + \dots$. Clearly $\int_F \hat{\phi}(x)\ln|x|dx$ converges for $\phi \in C_c^\infty(F)$. Thus

$$PV(d^\times x)(\phi) = \alpha_0\phi(0) - \alpha_{-1}\int_F \hat{\phi}(x)\ln|x|dx.$$

Put

$$\mu^+(f) = \int_F (f^+)\hat{}(x)\ln|x|dx = \int_F f(yU_2)\psi(xy)\ln|x|dydx.$$

Then $\mu^+(f)$ extends (a multiple of) $\mu_{\mathcal{O}_+}$ from $C_c^\infty(\mathcal{O}_+)$ to $C_c^\infty(\mathfrak{g}^-_{\mathrm{nilp}})$.

Similarly, $\mu^-(f) = \int_F \hat{f^-}(x)\ln|x|dx$ extends $\mu_{\mathcal{O}_-}$, where $\mathcal{O}_- = \mathrm{Ad}(H)U_3 = \{xU_3; x \neq 0\}$, and $f^-(x) = f(xU_3)$, to $C_c^\infty(\mathfrak{g}^-_{\mathrm{nilp}})$.

However, μ^+ and μ^- are not H-invariant. Let us examine how they fail to be invariant. For $a \in F^\times$, put $a' = \operatorname{diag}(a,1)$, and $(a^{-1}f^+)(x) = f^+(ax)$. Then

$$(\operatorname{Ad}(a')\mu^+)(f) = \mu^+(\operatorname{Ad}(a')^{-1}f) = \int_F ((a^{-1}f^+)\check{})(x)\ln|x|dx.$$

Since $(a^{-1}f^+)\check{}(x) = \int f^+(ay)\psi(xy)dy = |a|^{-1}(f^+)\check{}(a^{-1}x)$, using the Fourier inversion formula we obtain

$$(\operatorname{Ad}(a')\mu^+)(f) = \mu^+(f) + f^+(0)\ln|a| = \mu^+(f) + \delta_0(f)\ln|a|,$$

$\delta_0(f) = f(0)$. Similarly $(\operatorname{Ad}(a')\mu^-)(f) = \mu^-(f) - \delta_0(f)\ln|a|$.

In conclusion, neither μ^+ nor μ^- is H-invariant. Any two extensions of $\mu_{\mathcal{O}_+}$ to the closure $\overline{\mathcal{O}_+}$ of $\mathcal{O}_+$ will differ by a multiple of δ_0. Then $\mu^+ + \mu^-$ is the unique H-invariant extension of $\mu_{\mathcal{O}_+}$ and of $\mu_{\mathcal{O}_-}$ to $\mathfrak{g}^-_{\text{nilp}}$, up to δ_0. Hence $C_c^\infty(\mathfrak{g}^-_{\text{nilp}})^{*H}$ has the basis δ_0 and $\mu = \mu^+ + \mu^-$, as asserted. □

The Fourier transform of δ_0 is $\hat{\delta_0}(f) = \hat{f}(0) = \int_{\mathfrak{g}^-} f(X)dX$, and that of μ is

$$\begin{aligned}\hat{\mu}(f) &= \int f(\left(\begin{smallmatrix} 0 & u \\ v & 0\end{smallmatrix}\right))\psi(\operatorname{tr}[\left(\begin{smallmatrix} 0 & u \\ v & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & y \\ 0 & 0\end{smallmatrix}\right)])\psi(xy)\ln|x|dudvdydx \\ &+ \int f(\left(\begin{smallmatrix} 0 & u \\ v & 0\end{smallmatrix}\right))\psi(\operatorname{tr}[\left(\begin{smallmatrix} 0 & u \\ v & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ y & 0\end{smallmatrix}\right)])\psi(xy)\ln|x|dudvdydx \\ &= \int f(\left(\begin{smallmatrix} 0 & u \\ x & 0\end{smallmatrix}\right))\ln|x|dxdu + \int f(\left(\begin{smallmatrix} 0 & x \\ u & 0\end{smallmatrix}\right))\ln|x|dxdu.\end{aligned}$$

Consequently, no linear combination of $\hat{\delta_0}$ and $\hat{\mu}$ can vanish on any neighborhood of zero in $\mathfrak{g}^-_{\text{nilp}}$, and so there is no admissible π such that $\mathbb{L}_\pi$ be supported on $\mathfrak{g}^-_{\text{nilp}}$, the complement of the σ-regular set in $\mathfrak{g}^-$. This proves Theorem 2.2 for $n = m = 1$. □

A related – yet different – question, is as follows. For $f \in C_c^\infty(\mathfrak{g}^-)$, and $X \in \mathfrak{g}^-$, the H-orbit $\mathcal{O}$=Ad$(H)X$ lies in $\mathfrak{g}^-$. If the centralizer $Z_H(X) = \{h \in H; \operatorname{Ad}(h)X = X\}$ of X in H is unimodular, one can introduce the orbital integral $\mu_{\mathcal{O}}(f) = \int_{H/Z_H(X)} f(\operatorname{Ad}(h)X)dh$.

Definition. A *Cartan subspace* $\mathfrak{c}$ of $\mathfrak{g}^-$ is defined in Kostant-Rallis [KR], p. 754, or [S1], (1.4), to be a maximal abelian subspace consisting of semi simple elements. The element $X \in \mathfrak{g}^-$ is called *σ-regular* if it is semi simple and the centralizer $Z_{\mathfrak{g}^-}(X)$ is a Cartan subspace.

Question. Following Harish-Chandra [HC1], Theorem 10, who gave a positive answer to the question in the group case, one may ask whether – given f – the vanishing of $\mu_{\mathcal{O}}(f)$ for all σ-regular H-orbits $\mathcal{O}$ in $\mathfrak{g}^-$, implies the vanishing of $\Lambda(f)$ for every H-invariant distribution Λ on the σ-singular set of $\mathfrak{g}^-$.

Although a general discussion will be interesting, we shall consider only the easiest

Example. Put $\mathfrak{g} = M(2,F) = 2\times 2$ matrices over F, $\sigma(X) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right) X \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right)$, $G = GL(2,F)$, $H = \{\operatorname{diag}(a,b); a,b \in F^\times\}$, $\mathfrak{g}^- = \{uU_2 + vU_3; u,v \in F\}$. Then a semi simple element $X = uU_2 + vU_3$ of $\mathfrak{g}^-$ is one with $uv \neq 0$. Its centralizer is $F \cdot X$, which is a Cartan subspace. The σ-singular H-orbits are represented by U_2, U_3, and 0.

Put $\mu_2(f) = \int_{H/Z_H(U_2)} f(\mathrm{Ad}(h)U_2)dh$, define $\mu_3(f)$ on replacing U_2 by U_3 in the definition of $\mu_2(f)$, and put $\delta_0(f) = f(0)$. Although μ_2, μ_3 converge for $f \in C_c^\infty(\mathrm{Ad}(H)U_2)$, $C_c^\infty(\mathrm{Ad}(H)U_3)$, they clearly do not converge for a general f in $C_c^\infty(\mathfrak{g}^-)$. Each of μ_2, μ_3 can be extended to the closure of $\mathrm{Ad}(H)U_2$ or $\mathrm{Ad}(H)U_3$ in $\mathfrak{g}^-$, but the extension – which is unique up to a multiple of δ_0 – is not H-invariant. On the other hand, $\mu = \mu_2 + \mu_3$ is an H-invariant distribution on the closed subset $\mathfrak{g}^-_{\mathrm{nilp}}$ of $\mathfrak{g}^-$. Then μ and δ_0 make a basis of $C_c^\infty(\mathfrak{g}^-_{\mathrm{nilp}})^{*H}$, and we show:

2. Theorem. *Suppose $f \in C_c^\infty(\mathfrak{g}^-)$ has $\mu_{\mathcal{O}}(f) = 0$ for every σ-regular H-orbit $\mathcal{O}$ in $\mathfrak{g}^-$. Then $\Lambda(f) = 0$ for all Λ in $C_c^\infty(\mathfrak{g}^-)^{*H}$.*

Proof. Let R be the ring of integers in F. Put $f(x,y)$ for $f(xU_2 + yU_3)$, and let f_0 be the characteristic function of $R \times R$ in $F \times F$. Then $C_c^\infty(\mathfrak{g}^-) = \mathbb{C} \cdot f_0 \oplus C_c^\infty(\mathfrak{g}^- - \{0\})$, since $f = f - f(0)f_0 + f(0)f_0$. If $\mathcal{O}=\mathrm{Ad}(H)\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$, $uv \neq 0$, then

$$\mu_{\mathcal{O}}(f) = \int_{F^\times} f(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix})d^\times a = \int_{F^\times} f(\begin{pmatrix} 0 & au \\ v/a & 0 \end{pmatrix})d^\times a$$

$$= \int f(\begin{pmatrix} 0 & a \\ uv/a & 0 \end{pmatrix})d^\times a \to \int f(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix})d^\times a = \mu_2(f)$$

as $uv \to 0$, if $f \in C_c^\infty(F \times F^\times)$,

$$= \int f(\begin{pmatrix} 0 & auv \\ 1/a & 0 \end{pmatrix})d^\times a \to \int f(\begin{pmatrix} 0 & 0 \\ 1/a & 0 \end{pmatrix})d^\times a = \mu_3(f)$$

as $uv \to 0$, if $f \in C_c^\infty(F^\times \times F)$.

Any $f \in C_c^\infty(\mathfrak{g}^- - \{0\})$ can be written as $f = f_2 + f_3$, $f_2 \in C_c^\infty(F \times F^\times)$, $f_3 \in C_c^\infty(F^\times \times F)$, uniquely up to the replacement of (f_2, f_3) by $(f_2 + f_1, f_3 - f_1)$, $f_1 \in C_c^\infty(F^\times \times F^\times)$. Since $\mu_2(f_1) = \mu_3(f_1)$, the distribution $\mu(f) = \mu_2(f_2) + \mu_3(f_3)$ on $C_c^\infty(\mathfrak{g}^- - \{0\})$ is well defined. In fact it is supported on $\mathfrak{g}^-_{\mathrm{nilp}}$, but not on $\{0\}$, hence μ is equal to the sum of the principal values extensions of μ_2 and μ_3, up to a scalar multiple and a multiple of δ_0. Further,

$$\mu_{\mathcal{O}}(f_0) = \int_{|uv| \leq |a| \leq 1} d^\times a = 1 + \mathrm{val}(uv) \qquad (|uv| = q^{-\mathrm{val}(uv)}).$$

Hence $\mu_{\mathcal{O}}(f) = \mu_{\mathcal{O}}(f - f(0)f_0) + \mu_{\mathcal{O}}(f(0)f_0)$, and $\mu_{\mathcal{O}}(f)/\mathrm{val}(uv) \to \delta_0(f) = f(0)$ as $uv \to 0$. If $\delta_0(f) = 0$ then $\mu_{\mathcal{O}}(f) \to \mu(f)$ as $uv \to 0$, and the theorem follows. $\square$

4. The case of $G = GL(n), H = GL(n-1)$. We now prove Theorem 2.2 for $m = 1$. Thus assume $n \geq 3, G = GL(n,F), \sigma(g) = JgJ, J = \mathrm{diag}(I_{n-1}, -1)$. Then $H/Z = GL(n-1,F), \mathfrak{g}^- = \{\begin{pmatrix} 0 & {}^tX \\ Y & 0 \end{pmatrix}\}$, where X, Y are row $(n-1)$-vectors, the σ-regular set is the complement of $\mathfrak{g}^-_{\mathrm{nilp}} = \{Y\,{}^tX = 0\}$. The H-orbits in $\mathfrak{g}^-_{\mathrm{nilp}}$ are represented by 0, $X_+ =$ matrix whose only non zero entry is at the top right corner, $X_- =$ matrix whose only non zero entry is at the bottom left corner, $X_1 =$ matrix with non zero rows $(0, \ldots, 0, 1)$ at top and $(0,\dots,0,1,0)$ at the bottom. The stabilizers are H, $\left(\begin{smallmatrix}1&*\\0&*\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&0\\*&*\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&*&*\\0&*&*\\0&0&1\end{smallmatrix}\right)$, where the "1"s are scalars. Hence the stabilizers of X_+ and X_- are not unimodular.

To compute the integral $\mu_1=\mu_{\mathrm{Ad}(H)X_1}$ on the orbit of X_1, use the decomposition $h=kuam, a=\mathrm{diag}(\alpha,I_{n-2},\beta), m=\mathrm{diag}(1,m',1), u$ in the unipotent radical of the upper triangular parabolic subgroup of type $(1,n-3,1)$. The corresponding measure decomposition is $dh=|\alpha/\beta|^{n-2}dkdudadm$. Then

$$\mu_1(f)=\int_{F^\times}\int_{F^\times}f^K(\begin{pmatrix}0&0&\alpha\\0&0&0\\0&\beta^{-1}&0\end{pmatrix})|\alpha/\beta|^{n-2}d^\times\alpha d^\times\beta$$
$$=\int_{F^\times}\int_{F^\times}f^K(\begin{pmatrix}0&0&\alpha\\0&0&0\\0&\beta&0\end{pmatrix})|\alpha\beta|^{n-2}d^\times\alpha d^\times\beta,$$

where $f^K(X)=\int_K f(\mathrm{Ad}(k)X)dk$.

When $n\geq 3$ the integral which defines $\mu_1(f)$ is convergent. Hence μ_1 extends to a distribution on the closed set $\mathfrak{g}^-_{\mathrm{nilp}}$ and hence on $\mathfrak{g}^-$. Its Fourier transform is

$$\hat\mu_1(f)=\iint f^K(\begin{pmatrix}0&{}^tX\\Y&0\end{pmatrix})\psi(\mathrm{tr}\,[\begin{pmatrix}0&0&\alpha\\0&0&0\\0&\beta&0\end{pmatrix}\begin{pmatrix}0&{}^tX\\Y&0\end{pmatrix}])dX\,dY\cdot|\alpha\beta|^{n-2}d^\times\alpha d^\times\beta$$
$$=\int f^K(\begin{pmatrix}0&0&\hat 0\\0&0&\hat b\\\hat a&\hat 0&0\end{pmatrix})|ab|^{n-2}d^\times a d^\times b.$$

Here a,b are scalars. A hat denotes the Fourier transform with respect to the indicated parameter.

When $n=3$, $\hat\mu_1(f)=\mu_1(f)$. But when $n\geq 4$, the support of $\hat\mu_1$ is not concentrated in $\mathfrak{g}^-_{\mathrm{nilp}}$. The Fourier transform of δ_0 is $\hat\delta_0(f)=\int f(\left(\begin{smallmatrix}0&{}^tX\\Y&0\end{smallmatrix}\right))dX\,dY=f(\left(\begin{smallmatrix}0&\hat0\\\hat0&0\end{smallmatrix}\right))$.

The distributions $\hat\delta_0$ and $\hat\mu_1$ have different degrees of homogeneity when $n\geq 4$. Indeed, for $t\in F^\times$, put $(tf)(X)=f(t^{-1}X)$. Then

$$\hat\mu_1(tf)=|t|^{2(n-1)-2(n-2)}\hat\mu_1(f)=|t|^2\hat\mu_1(f),\qquad \hat\delta_0(tf)=|t|^{2(n-1)}\hat\delta_0(f).$$

Consequently no element of the form $\hat\Lambda, 0\neq\Lambda\in C_c^\infty(\mathfrak{g}^-_{\mathrm{nilp}})^{*H}$, can be zero on the σ-regular set, since no linear combination of $\hat\mu_1$ and $\hat\delta_0$ can be concentrated on $\mathfrak{g}^-_{\mathrm{nilp}}$. This completes the proof of theorem 2.2 for $m=1$. □

5. The case of $G=GL(3m), H=GL(2m)\times GL(m)$. Self dual invariant distributions which are supported on the σ-singular set exist also in the higher rank situation, where $G=GL(3m,F),\sigma(X)=JXJ, J=\mathrm{diag}(I_{2m},-I_m)$, so that $H=GL(2m,F)\times GL(m,F)$. This example generalizes the one from the previous section, where $m=1$. It suggests that there are admissible H-spherical G-modules π such that $\mathbb{L}_\pi$ is supported on the σ-singular set, in fact on the unipotent set in $\tilde G$. Yet π may be reducible, and in any case we have not proven its existence.

1. Theorem. *The orbital integral μ of $X_0=\begin{pmatrix}0&0&I\\0&0&0\\0&I&0\end{pmatrix}$ is a self dual element in $C_c^\infty(\mathfrak{g}^-_{\mathrm{nilp}})^{*H}$. Here 0 and I are the zero and identity $m\times m$ matrices.*

Proof. Put $H_0=GL(2m,F)\times I_m\subset H$, and note that $Z_{H_0}(X_0)=\{u=\left(\begin{smallmatrix}I&B\\0&I\end{smallmatrix}\right)\}$, hence the orbital integral μ exists. Use the Iwasawa decomposition $dh=|AC^{-1}|^m dkdadu$ in H_0, where $a = \mathrm{diag}(A, C)$, and $|X|$ denotes $|\det X|$. Then for $f \in C_c^\infty(\mathfrak{g}^-)$, putting $f^K(X) = \int f(\mathrm{Ad}(k)X)dk$, and noting that for $A \in GL(m, F)$ and $X \in M(m, F)(= m \times m$ matrices over $F)$ the measures are related by $dA = |X|^{-m}dX$, we obtain

$$\mu(f) =\mu_{\mathrm{Ad}(H)X_0}(f) = \iint_{GL(m,F)^2} f^K(\begin{pmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ 0 & C^{-1} & 0 \end{pmatrix})|AC^{-1}|^m dAdC$$
$$= \iint_{M(m,F)^2} f^K(\begin{pmatrix} 0 & 0 & X \\ 0 & 0 & 0 \\ 0 & Y & 0 \end{pmatrix})dXdY.$$

Then

$$\hat{\mu}(f) =\mu(\hat{f}) = \int \hat{f}(\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & X \\ 0 & 0 & 0 \\ 0 & Y & 0 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}^{-1})dkdXdY$$
$$= \int f(\begin{pmatrix} 0 & 0 & X_1 \\ 0 & 0 & X_2 \\ Y_2 & Y_1 & 0 \end{pmatrix})\psi(\mathrm{tr}\,[\begin{pmatrix} 0 & 0 & X_1 \\ 0 & 0 & X_2 \\ Y_2 & Y_1 & 0 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & X \\ 0 & 0 & 0 \\ 0 & Y & 0 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}^{-1}])$$
$$= \int f^K(\begin{pmatrix} 0 & 0 & X_1 \\ 0 & 0 & X_2 \\ Y_2 & Y_1 & 0 \end{pmatrix})\psi(\mathrm{tr}\,[X_2Y + Y_2X])dX_1dX_2dY_1dY_2dXdY$$
$$= \int f^K(\begin{pmatrix} 0 & 0 & X_1 \\ 0 & 0 & 0 \\ 0 & Y_1 & 0 \end{pmatrix})dX_1dY_1 = \mu(f),$$

and μ is indeed equal to its Fourier transform, and is supported on the nilpotent cone. □

Alternative proof. Consider the distribution $\Lambda(f) = \int f(\begin{pmatrix} 0 & 0 & X \\ 0 & 0 & 0 \\ 0 & Y & 0 \end{pmatrix})dXdY$, $X, Y \in M(m, F)$. For $p = \begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{pmatrix}$ in $P \subset H(= GL(2m, F) \times GL(m, F))$, we have

$$\Lambda(pf) = \int f(\mathrm{Ad}(p^{-1})\begin{pmatrix} 0 & 0 & X \\ 0 & 0 & 0 \\ 0 & Y & 0 \end{pmatrix})dXdY$$
$$= \int f(\begin{pmatrix} 0 & 0 & a^{-1}Xd \\ 0 & 0 & 0 \\ 0 & d^{-1}Yc & 0 \end{pmatrix})dXdY = |ac^{-1}|^m\Lambda(f).$$

But the modular function on P is $\delta_P(p) = |ac^{-1}|^m$. Hence the function $\lambda_f(g) = \Lambda(gf)$ on G lies in the space of the induced representation $\mathrm{ind}_P^H(\delta_P)$. Frobenius reciprocity ([BZ], (2.29)): $\mathrm{Hom}_H(\mathrm{ind}_P^H(\delta_P), 1) = \mathrm{Hom}_P(\delta_P^{-1}\delta_P, 1)$, implies that there exists (a unique up to a scalar multiple) $\ell \neq 0$ in $\mathrm{Hom}_H(\mathrm{ind}_P^H(\delta_P), 1)$. It is given by $\ell(\phi) = \int_K \phi(k)dk$. Put $\mu(f) = \ell(\lambda_f)$. Since $\lambda_{hf}(g) = \Lambda(ghf) = \lambda_f(gh) = h\lambda_f(g)$, we have $\mu(hf) = \ell(\lambda_{hf}) = \ell(h\lambda_f) = \ell(\lambda_f) = \mu(f)$. Then μ is a non zero H-invariant measure supported on $\mathfrak{g}^-_{\mathrm{nilp}}$, which is equal to its Fourier transform, as seen above. □

6. The case of $G = O(n + 1, n), H = O(n, n), n = 2$**.** Here we show the following.

1. Theorem. *There exists a non zero H-invariant measure on $\mathfrak{g}^-$ which is supported on the nilpotent cone $\mathfrak{g}^-_{\mathrm{nilp}}$, which is equal to its Fourier transform (which is then also supported on the nilpotent cone).*

Proof. Put $w = (\delta_{i,2n+1-i})$ in $GL(2n, F)$, $J = \mathrm{diag}(w, 1)$ in $GL(2n + 1, F)$, and fix the quasi-split orthogonal group G in $2n + 1$ variables to be $O(n + 1, n) = \{g = J{}^tg^{-1}J \in$

$GL(2n+1,F)\}$. The involution σ is taken to be conjugation by $\mathrm{diag}(I_{2n},-1)$, thus $H=\{\mathrm{diag}(h,1); h=w{}^th^{-1}w \text{ in } GL(2n,F)\}$. Further, $\mathfrak{g}=\{X=-J{}^tXJ\}$, $\mathfrak{g}^+=\{\mathrm{diag}(X,0); X=-w{}^tXw\}$, and $\mathfrak{g}^-$ consists of $\tilde{X}=\begin{pmatrix} 0 & X \\ -{}^tXw & 0\end{pmatrix}$, $X={}^t(x_1,\dots,x_{2n})$. The nilpotent cone $\mathfrak{g}^-_{\mathrm{nilp}}$ of $\mathfrak{g}^-$ consists of those elements with ${}^tXwX=0$, thus $\sum_i x_ix_{2n+1-i}=0$. It consists of two H-orbits, $\{0\}$ and $\mathrm{Ad}(H)({}^t(I,0))\tilde{}$, where $I=(1,\dots,1)$ is a row vector of length n.

Consider the linear form $\ell(\phi)=\int\phi({}^t(X_n,0))dX_n$ on $(\phi\in)C_c^\infty(F^{2n})$, where $X_n=(x_1,\dots,x_n)$, and $dX_n=\prod dx_i (1\le i\le n)$. The "Siegel" parabolic subgroup P_H of H consists of $p=mu$, $m=\mathrm{diag}(a,w{}^ta^{-1}w)$, $u=u(X)=\begin{pmatrix} I & X\\ 0 & I\end{pmatrix}$, $X=-w{}^tXw$. Clearly $\ell(p\cdot\phi)=\int\phi({}^t(a^{-1}X_n,0))dX_n=|\det a|\ell(\phi)$, and $\delta_{P_H}(m)=|\det a|^{n-1}$. Hence $\lambda_\phi(g)=\ell(g\phi)$ lies in $\mathrm{ind}_{P_H}^H(\delta_{P_H}^{1/(n-1)})$. By Frobenius reciprocity ([BZ, (2.29)]), the space $\mathrm{Hom}_H(\mathrm{ind}_{P_H}^H(\delta_{P_H}^{1/(n-1)}),1)=\mathrm{Hom}_{P_H}(\delta_{P_H}^{1/(n-1)}\delta_{P_H}^{-1},1)$ is zero if $n\ne 2$, and it is $\mathbb{C}$ if $n=2$. Fix $n=2$. Then $L_1(\lambda)=\int_K\lambda(k)dk$, where K is a maximal compact open subgroup of H with $H=KP_H$, is a non zero element of the space on the left.

The measure $\mu_{\mathcal{O}}(\phi)=L_1(\lambda_\phi)=\int_K\lambda_\phi(k)dk=\int\phi(k{}^t(X_n,0))dX_ndk$ is non zero and is supported on the nilpotent cone $\mathfrak{g}^-_{\mathrm{nilp}}$ of $\mathfrak{g}^-$, but not on the zero orbit. It is H-invariant. Indeed, $\mu_{\mathcal{O}}(h\phi)=L_1(\lambda_{h\phi})=L_1(h\lambda_\phi)=L_1(\lambda_\phi)=\mu(\phi)$ since $\lambda_{h\phi}(g)=\ell(gh\phi)=\lambda_\phi(gh)=h\lambda_\phi(g)$. Moreover, it is equal to its Fourier transform. Indeed

$$\begin{aligned}\hat{\mu}_{\mathcal{O}}(\phi)=&\mu_{\mathcal{O}}(\hat{\phi})=\int\hat{\phi}(k\ {}^t(X_n,0))dX_ndk\\ =&\int_K dk\int_{F^2}dX_n\int_{F^4}\phi(Y)\psi(\mathrm{tr}\,[\tilde{Y}(k\ {}^t(X_n,0))\tilde{}\,])dY\\ =&\int_K\int_{X_n}\int_Y\phi(kY)\psi(-y_4x_1-y_3x_2)dX_ndYdk\\ =&\int_K dk\int_{F^2}\phi(k\ {}^t(y_1,y_2,0,0))dy_1dy_2=\mu_{\mathcal{O}}(\phi),\end{aligned}$$

where $Y={}^t(y_1,y_2,y_3,y_4)$ and $X_n=(x_1,x_2)$. The theorem follows. □

7. The case of $G=GL(4), H=GL(2)\times GL(2)$. Our next aim is to construct a basis for the space $C_c^\infty(\mathfrak{g}^-_{\mathrm{nilp}})^{*H}$ in the case where $G=GL(n+m,F)$, $J=\mathrm{diag}(I_n,-I_m)$, $n=m=2$, and to deduce Theorem 2.2 in this case. Our technique is straightforward. Having listed (see Example (2), end of §2) the (finitely many) H-orbits $\mathcal{O}$ in $\mathfrak{g}^-_{\mathrm{nilp}}$, and erased those which do not support an H-invariant measure (since the centralizer of their elements are not unimodular), we need to check for which orbits $\mathcal{O}$ the measure in $C_c^\infty(\mathcal{O})^{*H}$ extends to an H-invariant distribution on a closed H-invariant subset of $\mathfrak{g}^-_{\mathrm{nilp}}$ which contains $\mathcal{O}$. It turns out in some examples below – as was the case for $n=m=1$ in §3 – that this closed set is not the closure of $\mathcal{O}$, but rather the closure of the union of several (two in our case) orbits of the same dimension. For other orbits $\mathcal{O}$, the orbital integral is not the restriction to $\mathcal{O}$ of any H-invariant distribution on $\mathfrak{g}^-_{\mathrm{nilp}}$.

1. Theorem. *A basis for the space $C_c^\infty(\mathfrak{g}^-_{\mathrm{nilp}})^{*H}$ of H-invariant distributions on the nilpotent cone $\mathfrak{g}^-_{\mathrm{nilp}}$ of $\mathfrak{g}^-$ is given by the four distributions:* $\mu_0(f)=\delta_0(f)=f(0)$,

$$\mu_1(f)=\iint f(U_+(Y))\psi(\mathrm{tr}\ XY)\ln|X|dYdX+\iint f(U_-(Y))\psi(\mathrm{tr}\ XY)\ln|X|dYdX,$$

where X, Y *range over* $M(2\times 2, F)$,

$$\mu_4(f)=\int f^K(\begin{pmatrix} 0 & U\\ V & 0\end{pmatrix})dudv \qquad (U=\begin{pmatrix} 0 & u\\ 0 & 0\end{pmatrix}, V=\begin{pmatrix} 0 & v\\ 0 & 0\end{pmatrix}, f^K(X)=\int_K f(\mathrm{Int}(k)X)dk),$$

where K *is the standard maximal compact open subgroup of* H, *and*

$$\mu_5(f)=\int f^K(\begin{pmatrix} 0 & U\\ V & 0\end{pmatrix})\psi(xy)\ln|x|dxdydudv \qquad (U=\begin{pmatrix} 0 & u\\ 0 & y\end{pmatrix}, V=\begin{pmatrix} 0 & v\\ 0 & 0\end{pmatrix})$$
$$+\int f^K(\begin{pmatrix} 0 & U\\ V & 0\end{pmatrix})\psi(xy)\ln|x|dxdydudv \qquad (U=\begin{pmatrix} 0 & u\\ 0 & 0\end{pmatrix}, V=\begin{pmatrix} 0 & v\\ 0 & y\end{pmatrix}).$$

Their Fourier transforms are:

$$\hat\mu_0(f)=\iint f(\begin{pmatrix} 0 & X\\ Y & 0\end{pmatrix})dXdY, \qquad \hat\mu_4(f)=f^K(\begin{pmatrix} 0 & U_0\\ U_0 & 0\end{pmatrix}), U_0=\begin{pmatrix} \hat 0 & \hat 0\\ 0 & \hat 0\end{pmatrix},$$

$$\hat\mu_1(f)=\iint f(\begin{pmatrix} 0 & Z\\ X & 0\end{pmatrix})\ln|X|dZdX+\iint f(\begin{pmatrix} 0 & X\\ Z & 0\end{pmatrix})\ln|X|dZdX,$$

and

$$\hat\mu_5(f)=\int f^K(\begin{pmatrix} 0 & U_x\\ U_0 & 0\end{pmatrix})\ln|x|dx+\int f^K(\begin{pmatrix} 0 & U_0\\ U_x & 0\end{pmatrix})\ln|x|dx \qquad (U_x=\begin{pmatrix} x & \hat 0\\ 0 & \hat 0\end{pmatrix}).$$

In this section we consider the invariant distribution associated with the H-orbit of $U_+(I)$, where $U_+(X)=\begin{pmatrix} 0 & X\\ 0 & 0\end{pmatrix}$. This can and will be discussed in the generality of $n=m\geq 1$, thus X is an $n\times n$ matrix over F, and $I=I_n$ is the identity in $GL(n,F)$. The H-invariant measure on the H-orbit $\mathrm{Ad}(H)U_+(I)$ is the value at $t=0$ of the following distribution. Put $d(A)$ for $\mathrm{diag}(A,I)$, A in $GL(n,F)$; recall that $|A|$ denotes $|\det A|$.

$$Z^+(f,t)=\int f(\mathrm{Ad}(d(A))U_+(I))|A|^t dA=\int f(U_+(A))|A|^t dA$$
$$=\int f^+(A)|A|^t dA=Z(f^+,t), \qquad (f^+(A)=f(U_+(A))).$$

Although this integral converges at $t=0$ for $f\in C_c^\infty(\mathrm{Ad}(H)U_+(I))$, it does not converge for a general f in $C_c^\infty(\mathfrak{g}^-)$. However the function $Z(f^+,t)$ is the well known Zeta function on the algebra $M(n\times n,F)$ of $n\times n$ matrices. Its analytic properties of use for us are studied in Jacquet [J]. In particular it has a simple pole at $t=0$, and the H-invariant measure on $\mathrm{Ad}(H)U_+(I)$ can be extended as a measure on the closure of this orbit as the principal value of $Z^+(f,t)=Z(f^+,t)$ at $t=0$. Let us explicitly compute this principal value $Z_0^+(f)=Z_0(f^+)$.

Let $\phi(=f^+$, denoted by Φ in [J]) be a C_c^∞-function on $M(n\times n,F)$. Then our $Z(\phi,t)$ coincides with $Z(\Phi,s,f)$ of [J], (1.1.3), with $\Phi=\phi, s=t$, and $f=1$, and π of [J] is taken to be the trivial representation $\mathbf{1}$ of $GL(n,F)$. Proposition 1.2(3) of [J] asserts that

$Z(\phi,t)$ satisfies a functional equation ($Z(\Phi,s,f)$ should be $Z(\Phi,s+\frac{1}{2}(n-1),f)$ there) which according to [J], (1.3.7), is

$$Z(\phi,s+\frac{1}{2}(n-1))=\varepsilon(s,\psi)^{-1}L(s,\mathbf{1})L(1-s,\mathbf{1})^{-1}Z(\hat{\phi},\frac{1}{2}(n+1)-s).$$

Put $\nu(x)=|x|$. Then Theorem 3.4 of [J] (and (1.3.11) there) asserts that $L(s,J(\tau_1\nu^{t_1}\times\cdots\times\tau_r\nu^{t_r}))=\prod_i L(s+t_i,\tau_i)$, where J denotes the Langlands quotient as described in [J], (3.3) (P should be Q in (3.3.3) and on the following line, ξ should be η on the line before (3.3.4)). Since $\mathbf{1}$ is J with $\tau_i=1$ and $(t_1,\dots,t_n)=((n-1)/2,(n-3)/2,\dots,-(n-1)/2)$, we have (by [J], (3.1.1)) that $L(s,\mathbf{1})=\prod_{0\le i<n}(1-q^{-s-\frac{1}{2}(n-1)+i})^{-1}$. If $t=s+\frac{1}{2}(n-1)$, then

$$\begin{aligned}Z(\phi,t)=&\varepsilon_\psi(t-\frac{1}{2}(n-1))^{-1}L(t-\frac{1}{2}(n-1),\mathbf{1})L(\frac{1}{2}(n+1)-t,\mathbf{1})^{-1}Z(\hat{\phi},n-t)\\ =&\varepsilon_\psi(t-\frac{1}{2}(n-1))^{-1}\prod_{0\le i<n}[(1-q^{t-n+i})/(1-q^{-t+i})]Z(\hat{\phi},n-t)\\ =&(1-q^{-t})^{-1}[(1-q^{t-n})\varepsilon_\psi(t-\frac{1}{2}(n-1))^{-1}(-1)^{n-1}\prod_{1\le i<n}q^{t-i}]Z(\hat{\phi},n-t).\end{aligned}$$

In particular, $Z(\phi,t)$ has a simple pole at $t=0$, with residue which is a multiple of

$$Z(\hat{\phi},n)=\int_{GL(n,F)}\hat{\phi}(A)|A|^n dA=\int_{M(n\times n,F)}\hat{\phi}(X)dX=\phi(0)$$

by the Fourier inversion formula. Moreover,

$$\begin{aligned}&Z(\hat{\phi},n-t)=\int_{GL(n,F)}\hat{\phi}(A)|A|^{n-t}dA=\int_{M(n\times n,F)}\hat{\phi}(X)|X|^{-t}dX\\ &=\int\hat{\phi}(X)dX-t\int\hat{\phi}(X)\ln|X|dX+\dots+(-1)^m(t^m/m)\int\hat{\phi}(X)(\ln|X|)^m dX+\dots\\ &=\phi(0)-tZ_0(\phi)+\dots,\qquad Z_0(\phi)=\int\hat{\phi}(X)\ln|X|dX=-PV\{t^{-1}Z(\hat{\phi},n-t)\}.\end{aligned}$$

Then

$$Z(\phi,t)=(t^{-1}c_{-1}+c_0+\dots)(\phi(0)-tZ_0(\phi)+\dots)=t^{-1}c_{-1}\phi(0)+(c_0\phi(0)-c_{-1}Z_0(\phi))+\dots.$$

In particular, the distribution $Z_0(\phi)$ extends (up to a constant multiple) the H-invariant distribution $Z(\phi,0)$ on the $\phi\in C_c^\infty(GL(n,F))$, to $C_c^\infty(M(n\times n,F))$.

Yet $Z_0(\phi)$ is not H-invariant. Recall that $B=\operatorname{diag}(B_1,B_2)$ acts on the $\phi(=f^+)$ by $B\cdot\phi(A)=\phi(B_1^{-1}AB_2)$. Put $|B|=|B_1B_2^{-1}|$. Then

$$\begin{aligned}Z_0(B\phi)=&\int(B\phi)\hat{}(X)\ln|X|dX=\iint(B\phi)(Y)\psi(\mathrm{tr}\ XY)\ln|X|dYdX\\ =&Z_0(\phi)-\phi(0)\ln|B|\qquad(Y\mapsto B_1YB_2^{-1},X\mapsto B_2XB_1^{-1}).\end{aligned}$$

In summary, $Z_0^+(f) = Z_0(f^+) = -PV\{t^{-1}Z((f^+)\check{}, n-t)\}$ is a distribution on the closure of $\mathrm{Ad}(H)U_+(I)$, namely on $U_+(M(n\times n, F))$, extending the orbital integral on $\mathrm{Ad}(H)U_+(I) = \{U_+(X), \mathrm{rk}(X) = n\}$, which satisfies $Z_0^+(Bf) = Z_0^+(f) - f(0)\ln|B|$.

One can repeat this construction for the orbit $\mathrm{Ad}(H)U_-(I)$, where $U_-(X) = \left(\begin{smallmatrix} 0 & 0 \\ X & 0\end{smallmatrix}\right)$. Put $f^-(A) = f(\left(\begin{smallmatrix} 0 & 0 \\ A & 0\end{smallmatrix}\right))$. Then the orbital integral on this orbit is the value at $t=0$ of

$$Z^-(f,t) = \int f(\mathrm{Ad}(d(A))U_-(I))|A|^t dA = \int f^-(A)|A|^{-t}dA = Z(f^-, -t).$$

The distribution $Z_0^-(f) = Z_0(f^-) = -PV\{t^{-1}Z((f^-)\check{}, n+t)\}$ extends $Z^-(f,0)$ from $f \in C_c^\infty(\mathrm{Ad}(H)U_-(I))$ to $f \in C_c^\infty(U_-(M(n\times n, F)))$. It transforms under the action of H by $Z_0^-(Bf) = Z_0^-(f) + f(0)\ln|B|$. Of course $Bf(X) = f(B^{-1}XB)$, and in particular $Bf^-(A) = f^-(B_2^{-1}AB_1)$.

We conclude that $Z_0(f) = Z_0^+(f) + Z_0^-(f)$ is an H-invariant distribution on $\mathfrak{g}^-$ which is supported on $\mathfrak{g}^-_{\mathrm{nilp}}$, which coincides with the orbital integral on the H-orbits of $U_+(I)$ and $U_-(I)$ when restricted to these orbits. Explicitly

$$\begin{aligned} Z_0(f) &= \int (f^+)\check{}(X)\ln|X|dX + \int (f^-)\check{}(X)\ln|X|dX \\ &= \iint f(U_+(Y))\psi(\mathrm{tr}\ XY)\ln|X|dYdX + \iint f(U_-(Y))\psi(\mathrm{tr}\ XY)\ln|X|dYdX. \end{aligned}$$

In particular

$$\begin{aligned} Z_0(\hat{f}) &= \iint\iint f(\left(\begin{smallmatrix} 0 & Z \\ T & 0\end{smallmatrix}\right))\psi(\mathrm{tr}\,[\left(\begin{smallmatrix} 0 & Z \\ T & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & Y \\ 0 & 0\end{smallmatrix}\right)])\psi(\mathrm{tr}\ XY)\ln|X|dZdTdYdX + \dots \\ &= \iint f(\left(\begin{smallmatrix} 0 & Z \\ X & 0\end{smallmatrix}\right))\ln|X|dZdX + \iint f(\left(\begin{smallmatrix} 0 & X \\ Z & 0\end{smallmatrix}\right))\ln|X|dZdX. \end{aligned}$$

Note also that for $t\in F^\times$, if $(tf)(X) = f(t^{-1}X)$, then $Z_0(tf) = Z_0(f) - n\ln|t|[f(U_+(\hat{0})) + f(U_-(\hat{0}))]$ and $\hat{Z}_0(tf) = |t|^{2n^2}\hat{Z}_0(f) + 2n(\ln|t|)|t|^{2n^2}\hat{f}(0)$ (the hat indicating Fourier transform with respect to the specified location only, in $\mathfrak{g}^-$).

8. A 4-dimensional orbit. Consider the 4 dimensional H-orbit of $X_4 = \left(\begin{smallmatrix} 0 & U_2 \\ U_2 & 0\end{smallmatrix}\right)$, $U_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right)$. The stabilizer $Z_H(X_4)$ consists of all $\mathrm{diag}(A,B)$ with $AU_2 = U_2B$ and $BU_2 = U_2A$, thus $(A,B) = (\left(\begin{smallmatrix} a_1 & a_2 \\ 0 & a_4\end{smallmatrix}\right), \left(\begin{smallmatrix} a_4 & b_2 \\ 0 & a_1\end{smallmatrix}\right))$. Hence $\dim Z_H(X_4) = 4$, and as $\dim H = 8$, the dimension of $\mathrm{Ad}(H)X_4$ is indeed 4. The measure decompositions $dA = (dk'dn'da' =)|a_1'/a_2'|dk'da'dn'$, and $dB = (dkdnda =)|a_1/a_2||a_2'/a_1'|dkd(aw(a'))dn$, where $w(\mathrm{diag}(u,v)) = \mathrm{diag}(v,u)$, corresponding to the Iwasawa decompositions $A = k'a'n'$, $B = kaw(a')n$, are used to rewrite

$$\mu_4(f) = \int_{H/Z_H(X_4)} f(\mathrm{Ad}(h)X_4)dh = \int_{F^\times}\int_{F^\times} f^K(\left(\begin{smallmatrix} 0 & a_2^{-1}U_2 \\ a_1U_2 & 0\end{smallmatrix}\right))|a_1/a_2|d^\times a_1 d^\times a_2,$$

which is the expression of the theorem.

Note that $\mu_4(f)$, originally computed for $f \in C_c^\infty(\mathrm{Ad}(H)X_4)$, is defined by an integral which converges for all $f \in C_c^\infty(\mathfrak{g}^-)$. Hence μ_4 extends to an H-invariant distribution on $\mathfrak{g}^-$ which is supported on $\mathfrak{g}^-_{\mathrm{nilp}}$, with no modifications. Its Fourier transform is

$$\hat{\mu}_4(f) = \int f^K(\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix})\psi(\mathrm{tr}\,[\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}\left(\begin{smallmatrix} 0 & uU_2 \\ vU_2 & 0 \end{smallmatrix}\right)])dXdYdudv$$
$$= \int f^K(\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix})\psi(x_3v + y_3u)dXdYdudv = f^K(\left(\begin{smallmatrix} 0 & U_0 \\ U_0 & 0 \end{smallmatrix}\right))$$

$(X = \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right), Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix})$.

9. The 5-dimensional orbits. Consider the 5-dimensional H-orbits. There are two of these. Their analysis is similar to that of the 4 dimensional H-orbits of $U_+(I)$ and $U_-(I)$. Namely each of the two orbital integrals can be extended as a principal value of a regularized integral from the orbit to all of $\mathfrak{g}^-$, but the extension is not H-invariant. On the other hand, the sum of the two extensions is H-invariant.

As usual, we put $U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, U_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, U_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, U_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. A set of representatives for the two 5 dimensional H-orbits is given by $X_{5,1} = \begin{pmatrix} 0 & U_4 \\ U_2 & 0 \end{pmatrix}$ (which is mapped to $X^*_{5,1} = \begin{pmatrix} 0 & U_2 \\ U_1 & 0 \end{pmatrix}$ under the action of the reflection (12) in H), and $X_{5,2} = \begin{pmatrix} 0 & U_2 \\ U_4 & 0 \end{pmatrix}$ (which is mapped to $X^*_{5,2} = \begin{pmatrix} 0 & U_1 \\ U_2 & 0 \end{pmatrix}$ under the action of the reflection (34) in H).

The centralizer $Z_H(X_{5,1})$ consists of $\mathrm{diag}(A, B)$ with $AU_4 = U_4B$ and $BU_2 = U_2A$. Matrix multiplication implies that $(A, B) = (\begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix}, \begin{pmatrix} a_4 & b_2 \\ 0 & a_4 \end{pmatrix})$. Hence $\dim Z_H(X_{5,1}) = 3$, and as $\dim H = 8$, the dimension of $\mathrm{Ad}(H)X_{5,1}$ is indeed 5. The measure decompositions $dA = dk'dr'da'$, and $dB = (dkdrda =)|a_1/a_2|dkdadr$, corresponding to the Iwasawa decompositions $A = k'r'a', a' = \mathrm{diag}(a'_1, a'_2)$, and $B = ka'_2ar, a = \mathrm{diag}(a_1, a_2)$, are used to rewrite

$$\mu'_{5,1}(f) = \int f^K(\begin{pmatrix} 0 & r'U_4a^{-1} \\ aU_2 & 0 \end{pmatrix})|a_1/a_2|d^\times a_1 d^\times a_2 dr'$$
$$= \int f^K(\begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix})d^\times a_2 da_1 du$$

$(U = \begin{pmatrix} 0 & u \\ 0 & a_2 \end{pmatrix}, V = \begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix})$. This integral converges for F in $C_c^\infty(\mathrm{Ad}(H)X_{5,1})$. But the integration on $a_2 \in F^\times$ by $d^\times a_2$ shows that $\mu'_{5,1}$ does not extend to $\mathfrak{g}^-$.

Then we need to take the principal value at 0 in this variable, using the formula

$$PV(d^\times x)(\phi) = PV_{t=0}(\int \phi(x)|x|^t d^\times x)$$
$$= \alpha_0\phi(0) - \alpha_{-1}\int \phi(y)\psi(xy)\ln|x|dydx.$$

We deduce that the distribution

$$\mu_{5,1}(f) = \int f^K(\begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix})\psi(xy)\ln|x|dydudvdx$$

$(U = \begin{pmatrix} 0 & u \\ 0 & y \end{pmatrix}, V = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix})$ in $\mathfrak{g}^-$ is supported on the closure $\mathrm{Ad}(H)X_{5,1} \cup \mathrm{Ad}(H)X_4$ of $\mathrm{Ad}(H)X_{5,1}$ in $\mathfrak{g}^-_{\mathrm{nilp}} \subset \mathfrak{g}^-$, and its restriction to the H-orbit of $X_{5,1}$ is a multiple of $\mu'_{5,1}$. Reversing the Iwasawa decompositions, this distribution can also be expressed in the form

$$\mu_{5,1}(f) = \int f(\mathrm{Ad}(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}) \begin{pmatrix} 0 & U_4 \\ U_2 & 0 \end{pmatrix})\psi(xa'_2/a_2) \ln |x| dAdBdx,$$

where a_2, a'_2 are defined by $A = k'u'a', a' = \mathrm{diag}(a'_1, a'_2), B = kua, a = \mathrm{diag}(a_1, a_2)$.

Note that taking the representative $X^*_{5,1}$ instead of $X_{5,1}$, we would get the extension $\mu^*_{5,1}$ of $\mu'_{5,1}$, which is defined as $\mu_{5,1}$ is, but with $U = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}, V = \begin{pmatrix} y & v \\ 0 & 0 \end{pmatrix}$. This extension differs from $\mu_{5,1}$ by a multiple of the distribution μ_4 which is supported on the boundary of the orbit $\mathrm{Ad}(H)X_{5,1}$=$\mathrm{Ad}(H)X^*_{5,1}$.

To describe the orbital integral of $X_{5,2}$, note that $Z_H(X_{5,2})$ consists of $\mathrm{diag}(A, B)$ with $AU_2 = U_2B$, $BU_4 = U_4A$. Thus $(A, B) = (\begin{pmatrix} a_1 & a_2 \\ 0 & a_1 \end{pmatrix}, \begin{pmatrix} b_1 & 0 \\ 0 & a_1 \end{pmatrix})$. The measure decompositions $dA = |a'_1/a'_2|dk'da'dr'$, and $dB = dkdrda$, corresponding to the Iwasawa decompositions $A = k'r'a', a' = \mathrm{diag}(a'_1, a'_2)$, and $B = ka'_2ar, a = \mathrm{diag}(a_1, a_2)$, are used to rewrite this orbital integral in the form

$$\mu'_{5,2}(f) = \int f^K(\begin{pmatrix} 0 & a'U_2 \\ rU_4a'^{-1} & 0 \end{pmatrix})|a'_1/a'_2|da'dr = \int f^K(\begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix})dudvd^\times y$$

$(U = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & v \\ 0 & y \end{pmatrix})$, for $f \in C_c^\infty(\mathrm{Ad}(H)X_{5,2})$.

The presence of $d^\times y$ indicates that $\mu'_{5,2}$ does not extend to $\mathfrak{g}^-$. But taking the principal value with respect to this parameter we obtain the distribution

$$\mu_{5,2}(f) = \int f^K(\begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix})\psi(xy) \ln |x| dydudvdx$$

on $\mathfrak{g}^-$. It is supported on the closure of $\mathrm{Ad}(H)X_{5,2}$ in $\mathfrak{g}^-$, which is the union of the H-orbits of $X_{5,2}$ and X_4 (both in $\mathfrak{g}^-_{\mathrm{nilp}}$). The restriction of $\mu_{5,2}$ to the orbit of $X_{5,2}$ is a multiple of $\mu'_{5,2}$.

If we worked with the representative $X^*_{5,2}$ for the orbit of $X_{5,2}$, we would have obtained an expression $\mu^*_{5,2}$ such as $\mu_{5,2}$, but in which $U = \begin{pmatrix} y & u \\ 0 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$. This would differ from $\mu_{5,2}$ by a multiple of μ_4, which is supported on the boundary $\mathrm{Ad}(H)X_4$ of the orbit of $X_{5,2}$. Reversing the Iwasawa decomposition we can write $\mu_{5,2}$ in the form

$$\mu_{5,2}(f) = \int f(\mathrm{Ad}(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}) \begin{pmatrix} 0 & U_2 \\ U_4 & 0 \end{pmatrix})\psi(xa_2/a'_2) \ln |x| dAdBdx,$$

where a_2, a'_2 are defined exactly as in the case of $\mu_{5,1}$ from A, B.

The extensions $\mu_{5,1}$ and $\mu_{5,2}$ of the orbital integrals $\mu'_{5,1}$ and $\mu'_{5,2}$ to the closures of the orbits are unique up to a scalar multiple and a distribution supported on the boundary of the orbit (which is the orbit of X_4). However, these distributions are not H-invariant. Indeed, replacing f by $h \cdot f : X \mapsto f(\mathrm{Ad}(h^{-1})X)$, and writing the quotient "a'_2/a_2" of

$h^{-1}\mathrm{diag}(A,B)$ as a product of $z=z(h)$ (which is independent of a_2 and a_2'), and a_2'/a_2, we obtain

$$\begin{aligned}\mu_{5,1}(hf)=&\int f(\mathrm{Ad}(h^{-1}\begin{pmatrix}A&0\\0&B\end{pmatrix})\begin{pmatrix}0&U_4\\U_2&0\end{pmatrix})\psi(z^{-1}xza_2'/a_2)\ln|x|dAdBdx\\=&\int f(\mathrm{Ad}(\begin{pmatrix}A&0\\0&B\end{pmatrix})\begin{pmatrix}0&U_4\\U_2&0\end{pmatrix})\psi(xa_2'/a_2)(\ln|x|+\ln|z|)dAdBdx\\=&\mu_{5,1}(f)+\int f(\mathrm{Ad}(\begin{pmatrix}A&0\\0&B\end{pmatrix})\begin{pmatrix}0&U_2\\U_2&0\end{pmatrix})\ln|z|dAdB.\end{aligned}$$

In the second equality we changed variables $a_2'\mapsto z^{-1}a_2'$, and $x\mapsto zx$. In the last equality we applied the Fourier inversion formula in the variables a_2'/a_2 and x. Similarly for $\mu_{5,2}$ we obtain

$$\begin{aligned}\mu_{5,2}(hf)=&\int f(\mathrm{Ad}(h^{-1}\begin{pmatrix}A&0\\0&B\end{pmatrix})\begin{pmatrix}0&U_2\\U_4&0\end{pmatrix})\psi(zxz^{-1}a_2/a_2')\ln|x|dAdBdx\\=&\int f(\mathrm{Ad}(\begin{pmatrix}A&0\\0&B\end{pmatrix})\begin{pmatrix}0&U_2\\U_4&0\end{pmatrix})\psi(xa_2/a_2')(\ln|x|-\ln|z|)dAdBdx\\=&\mu_{5,2}(f)-\int f(\mathrm{Ad}(\begin{pmatrix}A&0\\0&B\end{pmatrix})\begin{pmatrix}0&U_2\\U_2&0\end{pmatrix})\ln|z|dAdB.\end{aligned}$$

However, the sum $\mu_5=\mu_{5,1}+\mu_{5,2}$ is an H-invariant distribution (on $\mathfrak{g}^-$, supported on $\mathfrak{g}^-_{\mathrm{nilp}}$, in fact on the closure of the union of the orbits of $X_{5,1}$ and $X_{5,2}$, which is the union of the orbits of $X_{5,1}$, $X_{5,2}$ and X_4). It restricts to a multiple of $\mu_{5,1}$ (resp. $\mu_{5,2}$) on the orbit of $X_{5,1}$ (resp. $\mu_{5,2}$).

The Fourier transform of μ_5 is given by

$$\hat{\mu}_{5,1}(f)=\int f^K(\begin{pmatrix}0&X\\Y&0\end{pmatrix})\psi(a)\psi(xy)\ln|x|+\int f^K(\begin{pmatrix}0&X\\Y&0\end{pmatrix})\psi(b)\psi(xy)\ln|x|,$$

where X,Y are as in §10,

$$a=\mathrm{tr}\,[\begin{pmatrix}0&X\\Y&0\end{pmatrix}\begin{pmatrix}0&U\\V&0\end{pmatrix}]=x_3v+y_3u+y_4y,\qquad U=\begin{pmatrix}0&u\\0&y\end{pmatrix},V=\begin{pmatrix}0&v\\0&0\end{pmatrix},$$

and

$$b=\mathrm{tr}\,[\begin{pmatrix}0&X\\Y&0\end{pmatrix}\begin{pmatrix}0&U\\V&0\end{pmatrix}]=x_3v+x_4y+y_3u,\qquad U=\begin{pmatrix}0&u\\0&0\end{pmatrix},V=\begin{pmatrix}0&v\\0&y\end{pmatrix}.$$

This is equal to

$$\int f^K(\begin{pmatrix}0&U_0\\U_x'&0\end{pmatrix})\ln|x|dx+\int f^K(\begin{pmatrix}0&U_x'\\U_0&0\end{pmatrix})\ln|x|dx$$

$$=\int f^K(\begin{pmatrix}0&U_x\\U_0&0\end{pmatrix})\ln|x|dx+\int f^K(\begin{pmatrix}0&U_0\\U_x&0\end{pmatrix})\ln|x|dx,$$

$U_0=\begin{pmatrix}\hat{0}&\hat{0}\\0&\hat{0}\end{pmatrix}$, $U_x=\begin{pmatrix}x&\hat{0}\\0&\hat{0}\end{pmatrix}$, $U_x'=\begin{pmatrix}\hat{0}&\hat{0}\\0&x\end{pmatrix}$.

10. The 6-dimensional orbits. According to the classification of the H-orbits in Example (2), end of §2, there are two more H-orbits, of dimension 6. We claim:

1. Theorem. *The orbital integrals on the* 6 *dimensional H-orbits do not contribute to* $C_c^\infty(\mathfrak{g}^-_{\mathrm{nilp}})^{*H}$.

Proof. One of these orbits is represented by $X_6^+ = \begin{pmatrix} 0 & I \\ U_2 & 0 \end{pmatrix}$, whose centralizer $Z_H(X_6^+)$ is $\{\mathrm{diag}(zn, zn)\}$, z is a scalar, n is a unipotent upper triangular matrix. To write out the orbital integral of X_6^+ we express the variable in the form $h = \mathrm{diag}(AB, B)$, and decompose $B = kau$, so that $dB = |a_1/a_2| dk da du (a = \mathrm{diag}(a_1, a_2))$. Write

$$f^+(A) = \int_K f^K(\begin{pmatrix} 0 & A \\ xU_2A^{-1} & 0 \end{pmatrix})dx,$$

and note that if A lies in a compact in $GL(2, F)$ depending on f, and so does xU_2A^{-1}, then x lies in a compact of F.

For any f in $C_c^\infty(\mathrm{Ad}(H)X_6^+)$, the orbital integral of f on $\mathrm{Ad}(H)X_6^+$ is

$$\begin{aligned}\mu_6^+(f) &= \int f(\mathrm{Ad}(\begin{pmatrix} AB & 0 \\ 0 & B \end{pmatrix}))X_6^+)dAdB \\ &= \int f(\begin{pmatrix} 0 & A \\ xkU_2k^{-1}A^{-1} & 0 \end{pmatrix})dxdAdk = \int_{K_1\backslash GL(2,F)} f^+(A)dA.\end{aligned}$$

Decompose $dA = dk_1 du da^*, a^* = \mathrm{diag}(a, b), u = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, to obtain

$$= \int f(\begin{pmatrix} 0 & ua^* \\ b^{-1}xU_2 & 0 \end{pmatrix})dxdud^\times ad^\times b = \int f(\begin{pmatrix} 0 & V \\ xU_2 & 0 \end{pmatrix})dxdyd^\times ad^\times b,$$

$V = \begin{pmatrix} a & y \\ 0 & b \end{pmatrix}$, where in the last equality we changed $x \mapsto xb, y \mapsto y/b$.

Recall that $\int_F \int_F \phi(x)\psi(xy) \ln|y| dxdy$ extends the distribution $\int_{F^\times} \phi(x) d^\times x$ from $F^\times$ to F. Applying this to the two variables a and b, we get the distribution

$$\mu_{6,1}(f) = \int f^K(\begin{pmatrix} 0 & V \\ xU_2 & 0 \end{pmatrix})\psi(a\alpha + b\beta) \ln|\alpha| \ln|\beta| dxdydadbd\alpha d\beta$$

on the closure of the H-orbit of X_6^+, hence on $\mathfrak{g}^-_{\mathrm{nilp}}$ and on $\mathfrak{g}^-$. This $\mu_{6,1}$ extends μ_6^+, and the extension is unique up to a linear combination of $\mu_{5,1}$, $\mu_{5,2}$, μ_4.

Similarly

$$\mu_{6,2}(f) = \int f^K(\begin{pmatrix} 0 & xU_2 \\ V & 0 \end{pmatrix})\psi(a\alpha + b\beta) \ln|\alpha| \ln|\beta| dxdydadbd\alpha d\beta$$

is a distribution on the closure of the H-orbit of $X_6^- = \begin{pmatrix} 0 & U_2 \\ I & 0 \end{pmatrix}$, whose restriction to the orbit of X_6^- is a multiple of the orbital integral on this orbit. This extension of the orbital integral of X_6^- is unique up to a linear combination of $\mu_{5,1}$, $\mu_{5,2}$, μ_4.

Now, concerning H-invariance properties of $\mu_{6,1}$ and $\mu_{6,2}$, for $t \in F^\times$, define $f^t(X)$ to be $f(\mathrm{Ad}(\begin{pmatrix} t & 0 \\ 0 & I \end{pmatrix}))X)$. Then

$$\begin{aligned}\mu_{6,1}(f^t) =& \mu_{6,1}(f) + (\mu_{5,1}(f) + \mu^*_{5,2}(f)) \ln |t| + \mu_4(f)(\ln |t|)^2 \\ =& \mu_{6,1}(f) + \mu_5(f) \ln |t| + p_1(\ln |t|)\mu_4(f),\end{aligned}$$

and

$$\begin{aligned}\mu_{6,2}(f^t) =& \mu_{6,2}(f) - (\mu^*_{5,1}(f) + \mu_{5,2}(f)) \ln |t| + \mu_4(f)(\ln |t|)^2 \\ =& \mu_{6,2}(f) - \mu_5(f) \ln |t| + p_2(\ln |t|)\mu_4(f),\end{aligned}$$

where p_1 and p_2 are quadratic polynomials with leading coefficient 1. Consequently no linear combination of $\mu_{6,1}$, $\mu_{6,2}$, $\mu_{5,1}$, $\mu_{5,2}$, μ_4, Z_0^+, Z_0^- and μ_0, which depends non trivially on $\mu_{6,1}$ and $\mu_{6,2}$, can be H-invariant. □

This completes the construction of the basis of $C_c^\infty(\mathfrak{g}^-_{\mathrm{nilp}})^{*H}$. Clearly the Fourier transforms of this basis, $\hat{\mu}_0$, $\hat{\mu}_1$, $\hat{\mu}_4$, $\hat{\mu}_5$, are linearly independent on any neighborhood of zero in $\mathfrak{g}^-$, and Theorem 2.2 follows. □

11. Uniform smoothness of orbital integrals. We shall now consider the behaviour of the $G^+ = H$-invariant distributions $\mu_{\tilde{x}}(\tilde{f})$ as $\tilde{x}$ varies over the σ-regular set of $\tilde{G}$. Under an assumption on the group – presently to be stated – we shall show that $\mu_{\tilde{x}}(\tilde{f})$ is a locally constant function of $\tilde{x}$ (in $\tilde{G}' = \sigma$-regular set of $\tilde{G}$), uniformly in $\tilde{f}$. Our main interest is simply to extend Harish-Chandra's submersion principle [HC2] to the spherical settings.

Assumption. Let (P, A) be a σ-invariant minimal parabolic pair in G, $P = MN$ the corresponding Levi decomposition, K a maximal compact subgroup with $G = PK$, and put $X^+ = X \cap G^+$ for $X \subset G$. Then (P^+, A^+) is a minimal parabolic pair in G^+, and there is a maximal compact subgroup K^+ in G^+ with $G^+ = P^+K^+$. Denote by M_- the set of m in M^+ with $\beta(m) \leq 1$ for every root β of A^+ in P^+. Then $G^+ = K^+M_-K^+$, and M^+/A^+ is compact. Put $A_- = A^+ \cap M_-$. Denote by $\mathfrak{g}$ and $\mathfrak{P}$ the Lie algebras of G and P, by $\mathfrak{g}^+$ and $\mathfrak{P}^+$ the sets of fixed points of σ (these are the Lie algebras of G^+ and P^+), and by $\mathfrak{g}^-$ and $\mathfrak{P}^-$ the -1 eigenspaces of σ in $\mathfrak{g}$ and $\mathfrak{P}$ (these are the Lie algebras of $\tilde{G}$ and $\tilde{P}$). Our assumption is that for any compact open subgroup K_0 in K there is a compact open subgroup P_0 in P such that $\mathrm{Int}(a)(\tilde{P} \cap P_0) \subset K_0$ for all a in A_-. Recall that a lattice L in $\mathfrak{g}$ is a compact open R-submodule of $\mathfrak{g}$, where R is the ring of integers of F. Then our assumption can be put as saying that for any lattice L in $\mathfrak{g}$ there is a lattice L_1 in $\mathfrak{g}$ such that $\mathrm{Int}(a)(\mathfrak{P}^- \cap L_1) \subset L$ for all a in A_-.

Example. This assumption holds in the group case, where $\mathfrak{g} = \{(X, Y); X, Y \in \mathfrak{H}\}$, $\sigma(X, Y) = (Y, X)$, $\mathfrak{g}^+ = \{(X, X)\}$, $\mathfrak{g}^- = \{(X, -X)\}$, and in the case where E/F is a quadratic extension of fields, $\mathfrak{g} = \mathfrak{H}(E)$, $\sigma =$ galois action, $\mathfrak{g}^+ = \mathfrak{H}(F)$, $\mathfrak{g}^- = i\mathfrak{g}^+$, where i generates E over F and has trace zero. However, this assumption does not hold for example when $\mathfrak{g}$ is the algebra $M(2n, F)$ of $2n \times 2n$ matrices over a field F, σ is given by conjugation by $\mathrm{diag}(I_n, -I_n)$, and $\mathfrak{P}$ is the algebra of upper triangular matrices. In this case, taking A to be the diagonal subgroup, A_- consists of $\mathrm{diag}(a_1, \ldots, a_{2n})$, $a_i \in F^\times$, $|a_i| \leq |a_{i+1}|$ for $1 \leq i < n$ and $n < i < 2n$, but not for $i = n$, and $\mathfrak{P}^- = \{\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\}$, and $\tilde{P} = I + \mathfrak{P}^-$.

For pairs (G, σ) satisfying the assumption above, following Harish-Chandra [HC2], we prove:

1. Theorem. *Let K_0 be a compact open subgroup of G. Then for every σ-regular $\tilde{\gamma}_0$ in $\tilde{G}'$ there is a neighborhood $\tilde{\omega}$ of $\tilde{\gamma}_0$ in $\tilde{G}'$, such that $\mu_{\tilde{x}}(\tilde{f})$ is constant in $\tilde{x} \in \tilde{\omega}$ for every $f \in C_c(K_0\backslash G)$ (here $\tilde{f}(\tilde{g}) = \int_{G^+} f(gx)dx, g \in G$).*

Following [HC2], the proof consists of several steps, the assumption will be used only in the last step, while the first is the following "submersion" result. Recall (e.g., Serre [S]) that an analytic map $a : X \to Y$ of analytic manifolds is called *submersive* if its differential is surjective. The group G acts on itself, and on $\tilde{G}$, by $\sigma\text{-Int}(g)\gamma = g\gamma\sigma(g)^{-1}$. We write $A/_\sigma B$ for the quotient of A by B under this σ-conjugacy.

2. Proposition. *Let P be a σ-invariant parabolic subgroup of G. Fix a σ-regular $\tilde{\gamma}$ in $\tilde{G}'$. Then the map $\psi_{\tilde{\gamma}} : G^+ \to \tilde{G}/_\sigma\tilde{P}$, $\psi_{\tilde{\gamma}}(x) = (\text{Int}(x)\tilde{\gamma})/_\sigma\tilde{P}$, is submersive everywhere.*

Proof. Since $\psi_{\tilde{\gamma}}(xy) = \psi_{\text{Int}(y)\tilde{\gamma}}(x)$ for all x, y in G^+, it suffices to show that $\psi_{\tilde{\gamma}}$ is submersive at $x = 1$. Then $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$, $\mathfrak{P} = \mathfrak{P}^+ \oplus \mathfrak{P}^-$, $\mathfrak{g}^+ = \mathfrak{g}^\sigma$, $\mathfrak{P}^+ = \mathfrak{P}^\sigma$, $\mathfrak{g}^-$, $\mathfrak{P}^-$, are the Lie algebras of G, P, $G^+ = G^\sigma$, $P^+ = P^\sigma$, $\tilde{G} = G^-$ and $\tilde{P}$. We have to show that $\mathfrak{g}^- = \mathfrak{P}^- + (d\psi_{\tilde{\gamma}})\mathfrak{g}^+$. To compute the differential $d\psi_{\tilde{\gamma}}$ of $\psi_{\tilde{\gamma}}$ near zero, write

$$\begin{aligned}\psi_{\tilde{\gamma}}(e^{tX}) &= \gamma \cdot \gamma^{-1} e^{tX} \gamma \cdot \sigma(\gamma^{-1}) e^{-tX} \sigma(\gamma) \cdot \sigma(\gamma^{-1}) \\ &= \gamma e^{t \cdot \text{Ad}(\gamma^{-1})X} e^{-t \cdot \text{Ad}\sigma(\gamma)^{-1}X} \sigma(\gamma)^{-1}.\end{aligned}$$

Then $d\psi_{\tilde{\gamma}}(X) = (\text{Ad}(\gamma^{-1}) - \text{Ad}\sigma(\gamma^{-1}))X$ on $X \in \mathfrak{g}^+$. Since G is reductive, there exists an F-valued symmetric non degenerate G- and σ-invariant bilinear form B on $\mathfrak{g}$. Then $\mathfrak{g}^+$ is orthogonal to $\mathfrak{g}^-$ with respect to B. Let $\mathfrak{T} = \ker[(\text{Ad}(\tilde{\gamma}) - 1)|\mathfrak{g}]$ denote the Lie algebra of the torus $T = Z_G(\tilde{\gamma})$ in G. Then the orthogonal complement of $d\psi_{\tilde{\gamma}}(\mathfrak{g}^+)$ in $\mathfrak{g}^-$ is

$$[(\text{Ad}(\gamma^{-1}) - \text{Ad}\sigma(\gamma^{-1}))\mathfrak{g}^+]^\perp_{\mathfrak{g}^-} = \ker[(\text{Ad}(\gamma) - \text{Ad}\sigma(\gamma))|\mathfrak{g}^-] = \mathfrak{g}^- \cap \text{Ad}(\sigma(\gamma)^{-1})\mathfrak{T}.$$

Fix a split σ-invariant component of P, and a corresponding Levi decomposition $P = MN$ of P, with Lie algebras $\mathfrak{P} = \mathfrak{M} + \mathfrak{N}$. Then $(\mathfrak{P}^-)^\perp_{\mathfrak{g}^-} = \mathfrak{N}^-$, and

$$(\mathfrak{P}^- + (d\psi_{\tilde{\gamma}})\mathfrak{g}^+)^\perp_{\mathfrak{g}^-} = \mathfrak{N}^- \cap \text{Ad}(\sigma(\gamma)^{-1})\mathfrak{T}$$

is empty. Hence $\mathfrak{P}^- + (d\psi_{\tilde{\gamma}})\mathfrak{g}^+ = \mathfrak{g}^-$, as required. □

Fix a σ-regular $\tilde{\gamma}$ in $\tilde{G}'$, and a parabolic subgroup P of G. Then Proposition 2 implies that the map $G^+ \times \tilde{P} \to \tilde{G}$, $(x, p) \mapsto px\tilde{\gamma}x^{-1}\sigma(p)^{-1}$, is submersive. Theorem 11 of [HCD], p. 49, then asserts that there exists a unique linear map $C_c^\infty(G^+ \times \tilde{P}) \to C_c^\infty(\tilde{G})$, $\alpha \mapsto \phi_{\alpha,\tilde{\gamma}}$, such that for every F in $C_c^\infty(\tilde{G})$ we have

$$\int_{G^+} \int_{\tilde{P}} \alpha(x,p) F(px\tilde{\gamma}x^{-1}\sigma(p)^{-1}) dx d_\ell p = \int_{\tilde{G}} \phi_{\alpha,\tilde{\gamma}}(\tilde{g}) F(\tilde{g}) d\tilde{g}.$$

Here dx is a Haar measure on G^+, $d_\ell p$ is a left invariant Haar measure on $\tilde{P}$, and $d\tilde{g}$ is a Haar measure on $\tilde{G} = G/G^+$.

3. Proposition. *Fix α in $C_c^\infty(G^+\times\tilde{P})$. Then the map $\tilde{G}'\to C_c^\infty(\tilde{G})$, $\tilde{y}\mapsto\phi_{\alpha,\tilde{y}}$, is locally constant.*

Proof. The map $\tilde{G}'\times G^+\times\tilde{P}\to\tilde{G}'\times\tilde{G}$, $(\tilde{y},x,p)\mapsto(\tilde{y},px\tilde{y}x^{-1}\sigma(p)^{-1})$ is submersive. By [HCD], Theorem 11, there is a unique linear map $\beta\mapsto\phi_\beta$ such that

$$\iiint\beta(\tilde{y},x,p)\Phi(\tilde{y},px\tilde{y}x^{-1}\sigma(p)^{-1})d\tilde{y}dxd_\ell p=\iint\phi_\beta(\tilde{y},\tilde{g})\Phi(\tilde{y},\tilde{g})d\tilde{y}d\tilde{g}$$

for all Φ in $C_c^\infty(\tilde{G}'\times\tilde{G})$. We shall use this relation with $\Phi(\tilde{y},\tilde{g})=\lambda(\tilde{y})F(\tilde{g})$, $\lambda\in C_c^\infty(\tilde{G})$. Then

$$\int_{\tilde{G}'}\lambda(\tilde{y})d\tilde{y}\int_{G^+}\int_{\tilde{P}}\beta(\tilde{y},x,p)F(px\tilde{y}x^{-1}\sigma(p)^{-1})dxd_\ell p=\int_{\tilde{G}'}\lambda(\tilde{y})d\tilde{y}\int_{\tilde{G}}\phi_\beta(\tilde{y},\tilde{g})F(\tilde{g})d\tilde{g}$$

holds for all $\lambda\in C_c^\infty(\tilde{G}')$, and so we conclude that for all $\tilde{y}\in\tilde{G}'$ and F in $C_c^\infty(\tilde{G})$, we have

$$\int_{G^+}\int_{\tilde{P}}\beta(\tilde{y},x,p)F(px\tilde{y}x^{-1}\sigma(p)^{-1})dxd_\ell p=\int_{\tilde{G}}\phi_\beta(\tilde{y},\tilde{g})F(\tilde{g})d\tilde{g}.$$

Fix $\tilde{y}_0\in\tilde{G}'$, and consider $\beta(\tilde{y},x,p)$ of the form $\lambda(\tilde{y})\alpha(x,p)$, where $\alpha\in C_c^\infty(G^+\times\tilde{P})$, and $\lambda\in C_c^\infty(\tilde{G}')$ has $\lambda(\tilde{y}_0)=1$. Let $\tilde{G}'_0$ denote a neighborhood of $\tilde{y}_0$ in $\tilde{G}'$ on which $\lambda=1$. Then for any $\tilde{y}$ in $\tilde{G}'_0$ and any F in $C_c^\infty(\tilde{G})$, we have

$$\int_{\tilde{G}}\phi_{\alpha,\tilde{y}}(\tilde{g})F(\tilde{g})d\tilde{g}=\int_{G^+}\int_{\tilde{P}}\alpha(x,p)F(px\tilde{y}x^{-1}\sigma(p)^{-1})dxd_\ell p$$
$$=\int_{G^+}\int_{\tilde{P}}\beta(\tilde{y},x,p)F(px\tilde{y}x^{-1}\sigma(p)^{-1})dxd_\ell p=\int_{\tilde{G}}\phi_\beta(\tilde{y},\tilde{g})F(\tilde{g})d\tilde{g}.$$

Consequently $\phi_{\alpha,\tilde{y}}(\tilde{g})=\phi_\beta(\tilde{y},\tilde{g})$ for all $\tilde{y}$ in $\tilde{G}'_0$ and $\tilde{g}$ in $\tilde{G}$. Since $\phi_\beta\in C_c^\infty(\tilde{G}'\times\tilde{G})$, the map $\tilde{y}\mapsto\phi_{\alpha,\tilde{y}}$ is locally constant, as required. □

Proof of theorem 1. Consider first a σ-regular element $\tilde{\gamma}$ in $\tilde{G}'$ which is elliptic in G; namely it lies in no proper parabolic subgroup P of G, or equivalently, its centralizer $Z_G(\tilde{\gamma})$ in G is compact modulo the center $Z=Z(G)$ of G. Then the centralizer $Z_{G^+}(\tilde{\gamma})$ of $\tilde{\gamma}$ in G^+ is $G^+\cap Z_G(\tilde{\gamma})$, and it is compact modulo $Z^+=G^+\cap Z$; Z^+ is not necessarily the center $Z(G^+)$ of G^+. For such $\tilde{\gamma}\in\tilde{G}'$, we have

$$\mu_{\tilde{\gamma}}(\tilde{f})=\int_{G^+/Z^+}\tilde{f}(\mathrm{Int}(x)\tilde{\gamma})dx=\sum_{m\in M_-/{}^0MZ^+}|K^+mK^+|\int_{K^+}\tilde{f}^{K^+}(mk\tilde{\gamma}k^{-1}m^{-1})dk.$$

Here (P,A) is the σ-invariant minimal parabolic pair in G fixed prior to the statement of Theorem 1, $P=MN$ is the corresponding Levi decomposition, $P^+=P\cap G^+$, $A^+=A\cap G^+$, and $P^+=M^+N^+$ is the Levi decomposition in G^+, $M_-=\{m\in M^+;\beta(m)\le 1$ for each root β of A^+ in $P^+\}$ and ${}^0M=\{m\in M^+;\beta(m)=1$ for each root β of A^+ in $P^+\}$. Also $|K^+mK^+|$ denotes the volume of K^+mK^+ in G^+, and we normalize the measures on the compacta (e.g. K^+ and $Z_{G^+}(\tilde{\gamma})/Z^+$) to assign these groups the volume one. Finally, $\tilde{f}^{K^+}(\tilde{\gamma}) = \int_{K^+} \tilde{f}(\mathrm{Int}(k)\tilde{\gamma})dk$.

The Assumption stated prior to the theorem implies that since M^+/A^+ is compact, so $M_- \subset CA_-$ where C is a compact subset of M^+, we can choose an open compact subgroup P_0 of P with $\mathrm{Int}(m)(P_0 \cap \tilde{P}) \subset K_0$ for all m in M_-. Let α be the characteristic function of $K^+ \times P_0 \cap \tilde{P}$, and $\phi_{\tilde{y}} = \phi_{\alpha,\tilde{y}}$ the corresponding function on $\tilde{G}$. Then the map $\tilde{G}' \to C_c^\infty(\tilde{G})$, $\tilde{y} \mapsto \phi_{\tilde{y}}$, is smooth, and

$$\int_{K^+} \int_{\tilde{P}\cap P_0} F(pk\tilde{\gamma}k^{-1}\sigma(p)^{-1})dkdp = \int_{\tilde{G}} \phi_{\tilde{\gamma}}(\tilde{g})F(\tilde{g})d\tilde{g} \qquad (\tilde{\gamma} \in \tilde{G}', F \in C_c^\infty(\tilde{G})).$$

Applying this identity with $F(\tilde{g}) = \tilde{f}^{K^+}(m\tilde{g}m^{-1})$, we have

$$\int_{K^+} \int_{\tilde{P}\cap P_0} \tilde{f}^{K^+}(mpk\tilde{\gamma}k^{-1}\sigma(p)^{-1}m^{-1})dkdp = \int_{\tilde{G}} \phi_{\tilde{\gamma}}(\tilde{g})\tilde{f}^{K^+}(m\tilde{g}m^{-1}).$$

Our Assumption then implies that $\mathrm{Int}(m)(P_0 \cap \tilde{P}) \subset K_0$ for all $m \in M_-$. Since K_0 can be chosen to be sufficiently small, so that $\mathrm{Int}(K^+)K_0 \subset K_0$, we have $\tilde{f}(kk_0\tilde{g}\sigma(k_0)^{-1}k^{-1}) = \tilde{f}(k\tilde{g}k^{-1})$ for all k in K^+, k_0 in K_0, hence $\tilde{f}^{K^+}(k_0\tilde{g}\sigma(k_0)^{-1}) = \tilde{f}^{K^+}(\tilde{g})$. We conclude that

$$\begin{aligned}\int_{K^+} \tilde{f}^{K^+}(mk\tilde{\gamma}k^{-1}m^{-1})dk &= \int_{\tilde{G}} \phi_{\tilde{\gamma}}(\tilde{g})\tilde{f}^{K^+}(m\tilde{g}m^{-1})d\tilde{g}\\ = \int_{\tilde{G}} \phi_{\tilde{\gamma}_0}(\tilde{g})\tilde{f}^{K^+}(m\tilde{g}m^{-1})d\tilde{g} &= \int_{K^+} \tilde{f}^{K^+}(mk\tilde{\gamma}_0k^{-1}m^{-1})dk\end{aligned}$$

for all $m \in M_-$, and all $\tilde{\gamma}$ in a sufficiently small neighborhood of $\tilde{\gamma}_0$ in $\tilde{G}'$ (since $\tilde{\gamma} \mapsto \phi_{\tilde{\gamma}}$ is locally constant, namely constant in a sufficiently small neighborhood of $\tilde{\gamma}_0$). Hence $\mu_{\tilde{\gamma}}(\tilde{f}) = \mu_{\tilde{\gamma}_0}(\tilde{f})$ for all $\tilde{\gamma}$ in some neighborhood of $\tilde{\gamma}_0$ in $\tilde{G}'$, this neighborhood depending only on K_0.

Suppose that the σ-regular $\tilde{\gamma} \in \tilde{G}'$ is not elliptic in G. Then $T = Z_G(\tilde{\gamma})$ is a torus which is σ-invariant (since $\sigma(\tilde{\gamma}) = \tilde{\gamma}^{-1}$). Let A be the maximal split torus in T. Then $\sigma(A) = A$, and $M = Z_G(A)$ is a Levi component of a parabolic subgroup. Then M is σ-invariant, and we assume that it is the Levi component of a σ-invariant parabolic subgroup $P = MN$. Then $\tilde{\gamma} \in M$, we decompose $x = knm$ in $G^+ = K^+N^+M^+$, and write

$$\int_{G^+/A^+} \tilde{f}(x\tilde{\gamma}x^{-1})dx = \int_{M^+/A^+} \int_{N^+} \tilde{f}^{K^+}(m\tilde{\gamma}m^{-1} \cdot n')dmdn,$$

where $n' = (m\tilde{\gamma}m^{-1})^{-1} \cdot n \cdot m\tilde{\gamma}m^{-1} \cdot n^{-1} \in N$. Since $m\tilde{\gamma}m^{-1}$ is in M, while n' is in N, the function $\tilde{f}^{K^+}$ on $M \times N$ can be considered to be a (linear combination of) product(s) of two functions, $\tilde{f}_1$ on $\tilde{M}$, and f_2 on some subset of N. Extend f_2 to a function in $C_c^\infty(N)$, and let n range over N, rather than only N^+. Then

$$\int_N f_2(n')dn = |\det(\mathrm{Ad}(m\tilde{\gamma}m^{-1})^{-1} - 1)|\mathfrak{N}| \int_N f_2(n)dn = |\det(\mathrm{Ad}(\tilde{\gamma})^{-1} - 1)|\mathfrak{N}| \int_N f_2(n)dn,$$

and $|\det(\mathrm{Ad}(\tilde{\gamma})^{-1} - 1)|\mathfrak{N}|$ is a locally constant function in $\tilde{\gamma} \in \tilde{G}'$. Then it suffices to restrict attention only to $\tilde{\gamma} \in \tilde{G}' \cap M$, for which the centralizer $Z_{M^+}(\tilde{\gamma})$ in M^+ is compact modulo $M^+ \cap Z(M)$. This is the case of the elliptic-in-M^+ elements, considered above. $\square$

12. Spherical characters control orbital integrals. Let F be a local non archimedean field, $\mathbf{G}$ a reductive group over F, $\boldsymbol{\sigma}$ an involution of $\mathbf{G}$ over F, $\mathbf{H} = \mathbf{G}^{\boldsymbol{\sigma}}$ the group of fixed points of $\boldsymbol{\sigma}$ in $\mathbf{G}$, put $G = \mathbf{G}(F)$, $H = \mathbf{H}(F)$, and $Z = \mathbf{Z}(F)$, where $\mathbf{Z}$ is the center of $\mathbf{G}$. These are ℓ-groups in the terminology of [BZ]. Define $\tilde{G}$ to be the image of the map $G/H \to G$, $g \mapsto \tilde{g} = g\sigma(g)^{-1}$, where σ is the involution of G induced by $\boldsymbol{\sigma}$. Given f in $C_c^\infty(G/Z)$, define $\tilde{f}(\tilde{g}) = \int_H f(gh)dh$. Then $\tilde{f}$ lies in $C_c^\infty(\tilde{G}/\tilde{Z})$, $\tilde{Z} = \{z\sigma(z)^{-1}; z \in Z\}$, and we have

$$\mu_\gamma(f) = \int_{H\times H/Z_{H\times H}(\gamma)} f(x\gamma y)dxdy = \int_{H/Z_H(\tilde{\gamma})} \tilde{f}(\mathrm{Int}(x)\tilde{\gamma})dx = \mu_{\tilde{\gamma}}(\tilde{f}).$$

Here $Z_{H\times H}(\gamma) = \{(x,y) \in H \times H; x\gamma y = z\gamma$ for some z in $Z\}$, and $Z_H(\tilde{\gamma}) = \{x \in H; \mathrm{Int}(x)\tilde{\gamma} = \tilde{z}\tilde{\gamma}$ for some z in $Z\}$.

Let π be an admissible irreducible G-module, and $L = L_\pi$ an element of the space $\mathrm{Hom}_H(\pi, \mathbb{C})$ of H-invariant linear forms on π. Let $\tilde{\pi}$ denote the contragredient of π, and $\hat{L} = \hat{L}_{\tilde{\pi}}$ an element of $\mathrm{Hom}_H(\tilde{\pi}, \mathbb{C})$. Then $\hat{L}$ lies in the dual $\tilde{\pi}^*$ of $\tilde{\pi}$. For simplicity we assume that π transforms trivially under Z. For every $f \in C_c^\infty(G/Z)$, the image $\pi(f)\hat{L}$ of $\hat{L}$ under the action of the convolution operator $\pi(f) = \int_{G/Z} f(g)\pi(g)dg$ (a choice of a Haar measure dg is implicit), lies in the smooth part $(\tilde{\pi})^*_{\mathrm{sm}} = \tilde{\tilde{\pi}} = \pi$ of π^*. Hence $f \mapsto < L, \pi(f)\hat{L} >$ defines a distribution on G, which is bi-H-invariant.

Bernstein localization principle ([BZ], (2.37); [Be], (1.4)). *If X, Y are ℓ-spaces, H is an ℓ-group acting on X, and $q : X \to Y$ is a continuous H-invariant ($q(hx) = q(x)$ for all h in H, x in X) map, then $\oplus_{y\in Y} C_c^\infty(X_y)^{*H}$ is dense in $C_c^\infty(X)^{*H}$, where $X_y = q^{-1}(y)$.*

Using this with $X = \tilde{G}/\tilde{Z}$ and Y being the space of semi simple H-conjugacy classes $\tilde{s}$ in $\tilde{G}/\tilde{Z}$, we conclude that $C_c^\infty(\tilde{G}/\tilde{Z})^{*H}$ is generated by $C_c^\infty(\tilde{G}_{\tilde{s}})^{*H}$, where $\tilde{G}_{\tilde{s}}$ denotes the (closed) set of $\tilde{g}$ in $\tilde{G}/\tilde{Z}$ whose semi simple part is H-conjugate to $\tilde{s}$. Here $\tilde{s}$ ranges over the set of semi simple H-conjugacy classes in $\tilde{G}/\tilde{Z}$. Each of the spaces $C_c^\infty(\tilde{G}_{\tilde{s}})^{*H}$ is finite dimensional, the dimension being bounded by the number of H-conjugacy classes in $\tilde{G}_{\tilde{s}}$ which carry an H-invariant measure (i.e., the centralizer of a representative of the H-orbit is unimodular). It is one dimensional, spanned by the orbital integral, when $\tilde{s}$ is σ-regular in $\tilde{G}$.

In the group case, $C_c^\infty(G_s)^{*G}$ is spanned by the orbital integrals on the G-orbits in G_s (which "converge", namely extend to $C_c^\infty(G)^{*G}$, by Rao [R]). In the general symmetric space situation, a basis of $C_c^\infty(\tilde{G}_{\tilde{s}})^{*H}$ will consist only of linear combinations of extensions of orbital integrals. A single orbital integral, namely a generator of $C_c^\infty(\mathrm{Int}(H)\tilde{g})^{*H}$ (where $\tilde{g} \in \tilde{G}/\tilde{Z}$ has semi simple part $\tilde{s}$), may not extend to $C_c^\infty(\tilde{G}_{\tilde{s}})^{*H}$ (and $C_c^\infty(\tilde{G}/\tilde{Z})^{*H}$). Note that since $\tilde{G}_{\tilde{s}}$ is closed in $\tilde{G}/\tilde{Z}$, any element of $C_c^\infty(\tilde{G}_{\tilde{s}})^{*H}$ extends to one in $C_c^\infty(\tilde{G}/\tilde{Z})^{*H}$.

Definition. The elements of $C_c^\infty(\tilde{G}_{\tilde{s}})^{*H}$ will be called *generalized orbital integrals* on $\tilde{G}_{\tilde{s}}$.

0. Corollary. *The distribution $\mathbb{L}_\pi(\tilde{f}) =< L, \pi(f)\hat{L} >$ in $C_c^\infty(\tilde{G}/\tilde{Z})^{*H}$ vanishes at $\tilde{f} \in C_c^\infty(\tilde{G}/\tilde{Z})$ if all generalized orbital integrals of $\tilde{f}$ are zero.*

Following Kazhdan's density theorem [K;Appendix] in the group case, we shall consider now a (partial) converse to this statement.

Definition. The pair (G,H) has the *multiplicity one property* if $\dim_{\mathbb{C}} \operatorname{Hom}_H(\pi,\mathbb{C}) \leq 1$ for every admissible irreducible G-module π.

1. Theorem. *Suppose that (G,H) has the multiplicity one property, and $\tilde{f} \in C_c^\infty(\tilde{G}/\tilde{Z})$ satisfies that $\mathbb{L}_\pi(\tilde{f}) =< L_\pi, \pi(f)\hat{L}_{\tilde{\pi}} >$ is zero for every admissible G-module π (with trivial central character). Then $\mu_{\tilde{\gamma}}(\tilde{f}) = 0$ for every σ-regular σ-elliptic element $\tilde{\gamma}$ in $\tilde{G}$.*

One would expect a stronger conclusion to hold, namely that every generalized orbital integral of $\tilde{f}$ is zero, or that the $\mathbb{L}_\pi$, as π ranges over the set of all admissible irreducible G-modules, span $C_c^\infty(\tilde{G}/\tilde{Z})^{*H}$. Perhaps this holds for any pair (G,H), without assuming the multiplicity one property. However, even to prove the Theorem as stated, we shall assume a natural symmetric space extension of Arthur's fundamental work [A2] on the trace formula. One may then view our work as a motivation to carry out the computation of the spherical – or bi-period – summation formula, as stated below.

Recall that $\tilde{\gamma} \in \tilde{G}$ is σ-regular if $Z_{\mathbf{H}}(\tilde{\gamma})$ is a torus, and it is σ-elliptic if $Z_{\mathbf{H}}(\tilde{\gamma})$ is an elliptic torus of $\mathbf{H}$ ($Z_H(\tilde{\gamma})$ is compact modulo center). The proof of the Theorem – following Kazhdan [K;Appendix] – is global, and uses the notion of stable σ-conjugacy.

Definition. The elements $\tilde{\gamma}, \tilde{\gamma}'$ of $\tilde{G}$ are *stably σ-conjugate* if there exists x in $\overline{H} = \mathbf{H}(\overline{F})$, where $\overline{F}$ is an algebraic closure of F, such that $\tilde{\gamma}' = \operatorname{Int}(x)\tilde{\gamma}$, and $\operatorname{Int}(x) : \mathbf{Z}_{\mathbf{H}}(\tilde{\gamma}) \to \mathbf{Z}_{\mathbf{H}}(\tilde{\gamma}')$ is defined over F. As usual, $\tilde{\gamma}, \tilde{\gamma}'$ of $\tilde{G}$ are called *σ-conjugate* if $\tilde{\gamma}' = \operatorname{Int}(x)\tilde{\gamma}$ for some x in H. Also we say that γ, γ' in G are (stably) σ-conjugate if $\tilde{\gamma}, \tilde{\gamma}'$ are.

The σ-conjugacy classes within the stable σ-conjugacy class of $\tilde{\gamma}$ in $\tilde{G}$ are parametrized by

$$B_\sigma(\tilde{\gamma}/F) = H\backslash A_\sigma(\tilde{\gamma}/F)/Z_{\overline{H}}(\tilde{\gamma}),$$

where $A_\sigma(\tilde{\gamma}/F) = \{x \in \overline{H}; \tilde{\gamma}' = \operatorname{Int}(x)\tilde{\gamma}$ lies in $\tilde{G}$, and $\operatorname{Int}(x) : \mathbf{Z}_{\mathbf{H}}(\tilde{\gamma}) \to \mathbf{Z}_{\mathbf{H}}(\tilde{\gamma}')$ is defined over $F\}$. Note that if $\tilde{\gamma}' = \operatorname{Int}(x)\tilde{\gamma}$, then for any τ in $\operatorname{Gal}(\overline{F}/F)$ we have $x\tilde{\gamma}x^{-1} = \tau(x)\tilde{\gamma}\tau(x)^{-1}$, and $x_\tau = x^{-1}\tau(x)$ lies in $Z_{\overline{H}}(\tilde{\gamma})$. Then via the morphism $x \mapsto \{\tau \mapsto x_\tau = x^{-1}\tau(x), \tau \in \operatorname{Gal}(\overline{F}/F)\}$ of pointed sets, we have

$$B_\sigma(\tilde{\gamma}/F) = \ker[H^1(F, \mathbf{Z}_{\mathbf{H}}(\tilde{\gamma})) \to H^1(F, \mathbf{H})].$$

Thus given $\tilde{\gamma}$ in $\tilde{G}$, a $\tilde{\gamma}'$ which is stably σ-conjugate to $\tilde{\gamma}$ determines (and is determined by) a unique element of $B_\sigma(\tilde{\gamma}/F)$. The element $\tilde{\gamma}'$ is σ-conjugate to $\tilde{\gamma}$ precisely when it determines the trivial element of $H^1(F, \mathbf{Z}_{\mathbf{H}}(\tilde{\gamma}))(= H^1(\operatorname{Gal}(\overline{F}/F), Z_{\overline{H}}(\tilde{\gamma})))$.

Following [K], we need to embed our local situation in a global one. Given a local field F', group G', involution σ' and $H' = G'^{\sigma'}$, an element $\tilde{\gamma}'$ in G' which is σ'-regular, put $T' = Z_{G'}(\tilde{\gamma}')$(this is a σ'-invariant torus since $\sigma'(\tilde{\gamma}') = \tilde{\gamma}'^{-1}$) and $T'_{H'} = Z_{H'}(\tilde{\gamma}') = T' \cap H'$. Fix a galois extension F''/F' of F' over which G', H', T' split. Then by [F1], I, §4, Lemma, there exists a global galois extension E/F such that at least at two places w of F we have that the completion F_w of F at w is F', $E_w = E \otimes_F F_w$ is F'', $\operatorname{Gal}(E_w/F_w) \simeq \operatorname{Gal}(E/F)$. Moreover, there exist a reductive F-group $\mathbf{G}$, an involution $\boldsymbol{\sigma}$ of $\mathbf{G}$ over F (put $\mathbf{H} = \mathbf{G}^{\boldsymbol{\sigma}}$ for the group of fixed points), a σ-invariant torus $\mathbf{T}$ over F (put $\mathbf{T_H} = \mathbf{T} \cap \mathbf{H}$), such that $G_w = \mathbf{G}(F_w)$ is G', $\sigma_w = \boldsymbol{\sigma}|G_w$ is σ', H_w is H', T_w is T' and $T_{H,w} = \mathbf{T_H}(F_w)$ is $T'_{H'}$. The existence of *two* such places w guarantees (by [CF], middle of p. 361) that $G = \mathbf{G}(F), H, T, T_H$ are dense in $G_w, H_w, T_w, T_{H,w}$, and $\tilde{G} = \{\tilde{g}; g \in G\}$ in $\tilde{G}_w = \{\tilde{g}; g \in G_w\}$, $\tilde{T}$ in $\tilde{T}_w$. Moreover

$Z_{\tilde{G}_w}(\tilde{T}_w) = Z_{\tilde{G}_w}(\tilde{T})$ contains $Z_{\tilde{G}}(\tilde{T})$ as a dense subset. Consequently, every neighborhood in $Z_{\tilde{G}_w}(\tilde{T}_w)$ contains an element of $Z_{\tilde{G}}(\tilde{T})$. Given $\tilde{\gamma}_0$ in $\tilde{G}_w^{\sigma\text{-reg}}$, and $\tilde{f}_w \in C_c^\infty(\tilde{G}_w/\tilde{Z}_w)$ with $\mu_{\tilde{\gamma}_0}(\tilde{f}_w) \neq 0$, there exists $\tilde{\gamma}$ in $\tilde{G}^{\sigma\text{-reg}}$ in any given neighborhood of $\tilde{\gamma}_0$ such that $\mu_{\tilde{\gamma}}(\tilde{f}_w) \neq 0$.

We shall use the following observation. Given $\tilde{\gamma}, \tilde{\gamma}'$ in $\tilde{G}$ which are stably σ-conjugate in $\tilde{G}_v$ for some place v of F, they are σ-conjugate over an algebraic closure of F_v, hence over a finite extension of F, namely they are stably σ-conjugate in $\tilde{G}$.

Let $\mathbb{A}$ denote the ring of adeles of F, and $\mathbb{A}^u$ the ring of adeles with no component at the place u of F. Introduce the pointed direct sums $B_\sigma(\tilde{\gamma}/\mathbb{A}) = \oplus_v B_\sigma(\tilde{\gamma}/F_v)$ and $B_\sigma(\tilde{\gamma}/\mathbb{A}^w) = \oplus_{v\neq w} B_\sigma(\tilde{\gamma}/F_v)$, as well as $H^1(\mathbb{A}, \mathbf{T_H}) = \oplus_v H^1(F_v, T_{H,v})$ and $H^1(\mathbb{A}^w, \mathbf{T_H}) = \oplus_{v\neq w} H^1(F_v, T_{H,v})$. If $\tilde{\gamma}, \tilde{\gamma}'$ are σ-conjugate in $\tilde{G}_v$ for all $v \neq w$, then $\tilde{\gamma}'$ defines the identity element in the pointed set $H^1(\mathbb{A}^w, \mathbf{T_H})$. Hence $\tilde{\gamma}, \tilde{\gamma}'$ are σ-conjugate in $\tilde{G}$, by:

2. Theorem (Tate [T]). *$H^1(\mathbb{A}^w, \mathbf{T_H})$ is isomorphic to $H^1(F, \mathbf{T_H})$.*

The proof of Theorem 1 is based on a *bi-period summation formula*, which we proceed to describe. Let $\mathbb{G} = \mathbf{G}(\mathbb{A}), \mathbb{H} = \mathbf{H}(\mathbb{A}), \mathbb{Z} = \mathbf{Z}(\mathbb{A})$ be the groups of adele points of $\mathbf{G}, \mathbf{H}, \mathbf{Z}$. Let $L^2 = L^2(\mathbb{Z}G\backslash\mathbb{G})$ denote the space of complex valued, smooth on the right, functions on $\mathbb{G}$ which transform trivially under $\mathbb{Z}$ and left action of $G = \mathbf{G}(F)$, and are square integrable on $\mathbb{Z}G\backslash\mathbb{G}$. Then $\mathbb{G}$ acts on L^2 by right translation: $(r(g)\phi)(h) = \phi(hg)$ $(\phi \in L^2; g, h \in \mathbb{G})$. Denote by $C_c^\infty(\mathbb{Z}\backslash\mathbb{G})$ the span of the functions $\otimes_v f_v$ on $\mathbb{Z}\backslash\mathbb{G}$, where $f_v \in C_c^\infty(G_v/Z_v)$ for every place v of F, and $f_v = f_v^0$ for almost all v. Here f_v^0 is the constant measure of volume one whose support is $Z_v K_v$, where K_v is a standard maximal compact subgroup of G_v.

The convolution operator $r(f) = \int_{\mathbb{Z}\backslash\mathbb{G}} f(y) r(y) dy$ on L^2 is an integral operator: $r(f)\phi(x) = \int_{\mathbb{Z}G\backslash\mathbb{G}} K_f(y,x) f(y) dy$ with kernel $K_f(y,x) = \sum_{\gamma\in Z\backslash G} f(y^{-1}\gamma x)$. This is the "geometric" expression for the kernel, but there is an alternative expression for the kernel, namely the "spectral" expression. This expression is rather complicated, see Arthur [A1]. We shall use only a part of it here. A full discussion will remain for a future work.

The trace formula is obtained on integrating both the geometric and the spectral expressions over the diagonal $x = y \in \mathbb{Z}G\backslash\mathbb{G}$. The bi-period summation formula, in which no traces appear, is obtained on integrating both expressions over x, y in $\mathbb{Z}_H \cdot H\backslash\mathbb{H}$, where $\mathbb{Z}_H = \mathbb{Z} \cap \mathbb{H}$. Such formulas have been derived by Jacquet-Lai [JL] for $G = GL(2,E), H = GL(2,F)$, E/F a quadratic extension, by [FH] when $G = GL(n,E), H = GL(n,F)$, and by [F2] when $G = GL(2n,F)$, σ is conjugation by $\operatorname{diag}(I_n, -I_n)$, where I_n is the identity in $GL(n,F)$, and $H \simeq GL(n,F) \times GL(n,F)$, diagonally embedded in G.

Definition. The function $f_v \in C_c^\infty(G_v/Z_v)$ is called *σ-discrete* if $f_v(g) \neq 0$ implies that $Z_{H_v}(\tilde{g})/Z_{H_v}$ is a compact torus. Similarly, $\tilde{f}_v \in C_c^\infty(\tilde{G}_v/\tilde{Z}_v)$ is σ-discrete if $\tilde{f}_v(\tilde{g}) \neq 0$ implies that $Z_{H_v}(\tilde{g})/Z_{H_v}$ is a compact torus. The function $f = \otimes f_v \in C_c^\infty(\mathbb{G}/\mathbb{Z})$ is σ-discrete if given $\gamma \in G$, $f(x\gamma y) \neq 0$ for any x, y in $\mathbb{H}$ implies that $Z_{\mathbb{H}}(\tilde{\gamma})$ is an elliptic torus, namely $\tilde{\gamma}$ is σ-regular and $Z_{\mathbb{H}}(\tilde{\gamma})/Z_H(\tilde{\gamma})\mathbb{Z}_H$ is compact. Similarly for $\tilde{f} = \otimes \tilde{f}_v \in C_c^\infty(\tilde{\mathbb{G}}/\tilde{\mathbb{Z}})$.

In particular, a σ-discrete $\tilde{f}_v$ is supported on the σ-regular σ-elliptic set, a σ-discrete $\tilde{f}$ vanishes on the $\mathbb{H}$-orbits of rational elements $\tilde{\gamma}$ in $\tilde{G}$ unless they are σ-regular and σ-elliptic, and $\tilde{f} = \otimes \tilde{f}_v$ is σ-discrete when it has a σ-discrete component.

Let us compute the integral over $(\mathbb{Z}_H H\backslash\mathbb{H})^2$ of the geometric expression for $K_f(x,y)$, when $f = \otimes f_v$ is σ-discrete. It is

$$\iint_{(\mathbb{H}/H\mathbb{Z}_H)^2} \sum_{\gamma \in G/Z} f(x\gamma y^{-1})dxdy = \iint \sum_{\eta \in H/Z_H} \sum_{\gamma \in G/HZ} f(x\gamma\eta y^{-1})dxdy$$

$$= \int_{\mathbb{H}/H\mathbb{Z}_H} dx \int_{\mathbb{H}/\mathbb{Z}_H} dy \sum_{\gamma \in G/HZ} f(x\gamma y) = \int_{\mathbb{H}/H\mathbb{Z}_H} dx \sum_{\tilde{\gamma} \in \tilde{G}/\tilde{Z}} \tilde{f}(\mathrm{Int}(x)\tilde{\gamma})dx,$$

where $\tilde{\gamma} = \gamma\sigma(\gamma)^{-1}$ and $\tilde{f}(\tilde{g}) = \int_{\mathbb{H}/\mathbb{Z}_H} f(gh)dh$. Since $\tilde{f}$ is σ-discrete, this is equal to

$$\int_{\mathbb{H}/H\mathbb{Z}_H} \sum_{\tilde{\gamma} \in (\tilde{G}/\tilde{Z})/H} \sum_{\eta \in H/Z_H(\tilde{\gamma})} \tilde{f}(\mathrm{Int}(x\eta)\tilde{\gamma})dx$$

$$= \sum_{\{\tilde{\gamma}\} \in (\tilde{G}/\tilde{Z})/H} |Z_{\mathbb{H}}(\tilde{\gamma})/Z_H(\tilde{\gamma})\mathbb{Z}_H| \int_{\mathbb{H}/Z_{\mathbb{H}}(\tilde{\gamma})} \tilde{f}(\mathrm{Int}(x)\tilde{\gamma})dx.$$

Here $\{\tilde{\gamma}\}$ ranges over the set $(\tilde{G}/\tilde{Z})/H$ of H-conjugacy classes in $\tilde{G}/\tilde{Z}$, $\mathbf{Z_H}(\tilde{\gamma})$ is a torus, and the volume $|Z_{\mathbb{H}}(\tilde{\gamma})/Z_H(\tilde{\gamma})\mathbb{Z}_H|$ is finite by the assumption that $\tilde{f}$ is σ-regular. This assumption also implies that the sum ranges only over σ-regular σ-elliptic H-conjugacy classes $\tilde{\gamma}$ of rational elements, in $\tilde{G}/\tilde{Z}$. The sum is called the *geometric part of the bi-period summation formula.*

3. Proposition. *For a σ-discrete $\tilde{f}$, the geometric part of the bi-period summation formula is finite.*

Proof. Given a compact set C in $\mathbb{G}/\mathbb{Z}$, the set of G-orbits $\mathrm{Int}(G)\gamma$, $\gamma \in G/Z$, whose $\mathbb{G}$-orbit $\mathrm{Int}(\mathbb{G})\gamma$ intersects C non trivially, is finite (see, e.g., [F1], I, §3, Proposition). We shall apply this observation with $C = \mathrm{supp}\tilde{f}$, $\tilde{\gamma} \in \tilde{G}/\tilde{Z}$. Fix $\tilde{\gamma} \in \tilde{G}/\tilde{Z}$ such that $\mathrm{Int}(\mathbb{G})\tilde{\gamma}$ intersects C. We need to show that the number of H-orbits $\mathrm{Int}(H)\tilde{\gamma}_1$, where $\tilde{\gamma}_1$ lies in $\mathrm{Int}(G)\tilde{\gamma}$, such that $\mathrm{Int}(\mathbb{H})\tilde{\gamma}_1$ intersects C non trivially, is finite. Thus $\tilde{\gamma}_1 = \mathrm{Int}(\eta)\tilde{\gamma}$, $\eta \in G/Z_G(\tilde{\gamma})$, with $\tilde{\gamma}_1 \in \tilde{G}$. Then $1 = \tilde{\gamma}_1\sigma(\tilde{\gamma}_1) = \eta\tilde{\gamma}\eta^{-1}\sigma(\eta)\tilde{\gamma}^{-1}\sigma(\eta)^{-1}$, and $\eta^{-1}\sigma(\eta)$ lies in $Z_G(\tilde{\gamma})$.

A theorem of Harish-Chandra [HCD], p. 52, asserts that given $\tilde{\gamma}$ and the compact C in $\mathbb{G}/\mathbb{Z}$, there exists a compact C_1 in $\mathbb{G}/Z_{\mathbb{G}}(\tilde{\gamma})$, such that if $\mathrm{Int}(g)\tilde{\gamma} \in C$ for g in $\mathbb{G}/Z_{\mathbb{G}}(\tilde{\gamma})$, then $g \in C_1$. Applying this with $\mathrm{Int}(h)\tilde{\gamma}_1 \in C, h \in \mathbb{H}$, we conclude that $h\eta \in C_1$, where $C_1 = C_1(\tilde{\gamma}, C)$ is a compact in $\mathbb{G}/Z_{\mathbb{G}}(\tilde{\gamma})$.

Consider the natural projection $u : \mathbb{G}/Z_{\mathbb{G}}(\tilde{\gamma}) \to \mathbb{H}\backslash\mathbb{G}/Z_{\mathbb{G}}(\tilde{\gamma})$. Then $u(C_1)$ is compact, while the image $u(G/Z_G(\tilde{\gamma}))$ of $G/Z_G(\tilde{\gamma})$ is discrete, in $\mathbb{H}\backslash\mathbb{G}/Z_{\mathbb{G}}(\tilde{\gamma})$. Hence $u(\eta)$ lies in the finite set $u(C_1) \cap u(G)$. In other words, the set $H\backslash\{\eta \in G/Z_G(\tilde{\gamma}); \mathrm{Int}(\mathbb{H}\eta)\tilde{\gamma}$ intersects $C\}$ is finite. It follows that the set $\{\tilde{\gamma} \in \tilde{G}/\tilde{Z}; \mathrm{Int}(\mathbb{H})\tilde{\gamma} \cap C$ non empty $\}$ consists of a finite number of H-orbits. Hence there are only finitely many terms, indexed by $\{\tilde{\gamma}\} \in (\tilde{G}/\tilde{Z})/H$, which contribute to the geometric part of the bi-period summation formula, for any given σ-discrete test function $\tilde{f}$, as asserted. □

Note that each of the orbital integrals $\mu_{\tilde{\gamma}}(\tilde{f}) = \int_{\mathbb{H}/Z_{\mathbb{H}}(\tilde{\gamma})} \tilde{f}(\mathrm{Int}(x)\tilde{\gamma})dx$ is a product of the local orbital integrals $\mu_{\tilde{\gamma}}(\tilde{f}_v) = \int_{\mathbb{H}_v/Z_{\mathbb{H}_v}(\tilde{\gamma})} \tilde{f}_v(\mathrm{Int}(x)\tilde{\gamma})dx$, when $f = \otimes f_v$. For almost all v, the component $\tilde{f}_v$ is the function $\tilde{f}_v^0$ which is supported on $\tilde{K}_v\tilde{Z}_v$, and takes the value

$|\tilde{K}_v\tilde{Z}_v|^{-1}$ there. Then $\tilde{f}_v^0(\mathrm{Int}(x)\tilde{\gamma}) \neq 0$ precisely for $x \in (K_v \cap H_v)Z_{H_v}(\tilde{\gamma})/Z_{H_v}(\tilde{\gamma})$, and the measures on these compact subgroups are normalized so that the product of the $\mu_{\tilde{\gamma}}(\tilde{f}_v)$ over v is convergent.

4. Corollary. *For a σ-discrete $\tilde{f} = \otimes\tilde{f}_v$, the geometric part of the formula is the finite sum*

$$\sum_{\{\tilde{\gamma}\}\in(\tilde{G}/\tilde{Z})/H} |Z_{\mathbb{H}}(\tilde{\gamma})/Z_H(\tilde{\gamma})\mathbb{Z}_H| \prod_v \mu_{\tilde{\gamma}}(\tilde{f}_v).$$

We now summarize what we need on the geometric side of the bi-period summation formula for the proof of Theorem 1. Suppose we are given $\tilde{f}_w$ and a σ-regular σ-elliptic $\tilde{\gamma}_w \in \tilde{G}_w/\tilde{Z}_w$ such that $\mu_{\tilde{\gamma}_w}(\tilde{f}_w) \neq 0$. Then we can embed the local situation $F_w, G_w, \dots$ in a global situation $F, G, \dots$ as above, find a global σ-regular σ-elliptic element $\tilde{\gamma}$ in $\tilde{G}/\tilde{Z}$ such that $\mu_{\tilde{\gamma}}(\tilde{f}_w) \neq 0$, and components $\tilde{f}_v \in C_c^\infty(\tilde{G}_v/\tilde{Z}_v)$ for all $v \neq w$, with $\tilde{f}_v = \tilde{f}_v^0$ for almost all v, such that $\mu_{\tilde{\gamma}}(\tilde{f}_v) \neq 0$ for all $v \neq w$. Put $\tilde{f} = \otimes\tilde{f}_v$. Then $\mu_{\tilde{\gamma}}(\tilde{f}) \neq 0$.

5. Lemma. *There exists a function $\tilde{f}^w = \otimes_{v\neq w}\tilde{f}_v$ such that $\tilde{f} = \tilde{f}_w \otimes \tilde{f}^w$ has the property that $\mu_{\tilde{\delta}}(\tilde{f}) \neq 0$ for $\tilde{\delta} \in (\tilde{G}/\tilde{Z})/H$ precisely when $\tilde{\delta}$ is the class* $\mathrm{Int}(H)\tilde{\gamma}\tilde{Z}$.

Proof. There are only finitely many H-orbits $\tilde{\delta}$ in $\tilde{G}/\tilde{Z}$ such that $\mu_{\tilde{\delta}}(\tilde{f}) \neq 0$. As noted above, $\tilde{\delta}$ and $\tilde{\gamma}$ lie in the same H-orbit if they lie in the same H_v-orbit for all $v \neq w$. Thus for each $\tilde{\delta} \neq \tilde{\gamma}$ in $(\tilde{G}/\tilde{Z})/H$ with $\mu_{\tilde{\delta}}(\tilde{f}) \neq 0$, we can find a place $v \neq w$ such that $\mathrm{Int}(H_v)\tilde{\delta}\tilde{Z}_v \neq \mathrm{Int}(H_v)\tilde{\gamma}\tilde{Z}_v$. These orbits are closed. We can then replace $\tilde{f}_v$ by its product with the characteristic function of an open closed neighborhood of $\mathrm{Int}(H_v)\tilde{\gamma}\tilde{Z}_v$ which does not intersect $\mathrm{Int}(H_v)\tilde{\delta}\tilde{Z}_v$. Repeating this process a finite number of times we obtain the function $\tilde{f}^w$ with the required properties. □

We shall next consider part of the *spectral expression for the kernel* $K_f(x,y)$. This part corresponds to the discrete spectrum L_d^2 of the $\mathbb{G}$-module (r, L^2), which consists of subrepresentations of L^2. Then the irreducible admissible $\mathbb{G}$-modules π which occur in L^2, appear with finite multiplicities. Let $\{\Phi\}_\pi$ denote an orthonormal basis for the isotypical component of π, and put $m(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{G}}(\pi, L_d^2)$ for the multiplicity of π in L^2. Then the restriction of $r(f)$, $f \in C_c^\infty(\mathbb{G}/\mathbb{Z})$, to L_d^2, is represented by the kernel

$$K_{f,d}(x,y) = \sum_{\{\pi\}} \sum_{\{\Phi\}_\pi} \Phi(x)\overline{(\pi(f)\Phi)}(y).$$

Here π ranges over the set of equivalence classes of the irreducible admissible $\mathbb{G}$-modules in L_d^2. Integrating over x, y in $\mathbb{Z}_H H\backslash\mathbb{H}$, we obtain

$$\sum_{\{\pi\}} m(\pi) \sum_{\{\Phi\}_\pi} \int_{\mathbb{Z}_H H\backslash\mathbb{H}} \Phi(x)dx \cdot \int_{\mathbb{Z}_H H\backslash\mathbb{H}} \overline{(\pi(f)\Phi)}(x)dx,$$

where now $\{\Phi\}_\pi$ extends over an orthonormal basis of the $\mathbb{G}$-module $\pi \subset L_d^2$. Indeed, the linear form

$$P(f) = \sum_{\{\Phi\}_\pi} \int_{\mathbb{Z}_H H\backslash\mathbb{H}} \Phi(x)dx \cdot \int_{\mathbb{Z}_H H\backslash\mathbb{H}} \overline{(\pi(f)\Phi)}(x)dx$$

on $C_c^\infty(\mathbb{G}/\mathbb{Z})$ is independent of the choice of the basis $\{\Phi\}_\pi$ of $\pi \subset L_d^2$.

We shall now use the multiplicity one assumption for the pair (G_v, H_v) for every place v of F. Fix an H_v-invariant linear form L_{π_v} on π_v and $L_{\tilde{\pi}_v}$ on $\tilde{\pi}_v$ for each admissible irreducible H_v-spherical G_v-module π_v, such that $L_{\pi_v}(\xi_v^0) = 1$ for almost all v where π_v is unramified and ξ_v^0 is the K_v-fixed vector used in the definition of π as a restricted tensor product $\otimes \pi_v$ of local G_v-modules. The form $\Phi \mapsto \int_{\mathbb{Z}_H H\backslash\mathbb{H}} \Phi(x)dx$ on π is then a multiple of $\otimes L_v$.

A basis for π can also be formed on fixing an orthonormal basis for each component π_v, say $\{\xi_v\}$, and taking the products $\otimes_v \xi_v$, where $\xi_v = \xi_v^0$ for almost all v. Then $P(f)$ is

$$\sum_{\otimes\xi_v} \prod_v < L_{\pi_v}, \xi_v > \cdot \prod_v < L_{\tilde{\pi}_v}, \tilde{\pi}_v(f_v^*)\hat{\xi}_v >= \prod_v < L_{\pi_v}, \pi_v(f_v)L_{\tilde{\pi}_v} >=< L_\pi, \pi(f)L_{\tilde{\pi}} >$$

up to a scalar multiple, where $f_v^*(g) = f_v(g^{-1})$, and $L_\pi = \otimes L_{\pi_v}$, $L_{\tilde{\pi}} = \otimes L_{\tilde{\pi}_v}$.

6. Corollary. *The discrete part of the spectral side of the bi-period summation formula for the pair (G, H) and the function $f \in C_c^\infty(\mathbb{G}/\mathbb{Z})$, $f = \otimes f_v$, is $\sum_{\{\pi\}} c(\pi) < L_\pi, \pi(f)L_{\tilde{\pi}} >$, $c(\pi) \in \mathbb{C}$.* □

When the group $\mathbf{G}$ is anisotropic, namely the quotient $\mathbb{Z}G\backslash\mathbb{G}$ is compact, the entire spectrum $L^2(\mathbb{Z}G\backslash\mathbb{G})$ is discrete, namely it decomposes as a direct sum of irreducible subspaces. However, in general, in addition to the discrete spectrum L_d^2, the space L^2 would contain also a continuous spectrum. This has been studied in depth by Arthur [A] in the group case, where the bi-period summation formula reduces to the trace formula. We shall make the following assumption, which asserts that the natural extension of Arthur's work holds in the symmetric space situation. Of course, carrying out the proof of this extension would require a serious effort. Our assumption is that this extension can be carried out.

Proposition*. *Let $f = \otimes f_v \in C_c^\infty(\mathbb{G}/\mathbb{Z})$ be a test function such that for sufficiently many places v of F, the component f_v is σ-discrete. Then the geometric side of the bi-period summation formula for (G, H) is equal to the discrete part of the spectral side of the bi-period summation formula, where the sum over π may include some other automorphic representations, i.e. constituents of L^2, not necessarily in the discrete spectrum.*

Proof of Theorem 1. We embed the local situation of $f_w, \dots$ in a global situation, where the place w is repeated sufficiently many times, numbered $w_0, w_1, \dots$, with w_0 being the original place w. The σ-elliptic σ-regular element $\tilde{\gamma}_{w_0}$ where $\mu_{\tilde{\gamma}_{w_0}}(f_{w_0}) \neq 0$ can be approximated by a global element $\tilde{\gamma}$ in $\tilde{G}/\tilde{Z}$, where $\mu_{\tilde{\gamma}}(f_{w_0}) \neq 0$, and $\tilde{\gamma}$ is σ-elliptic σ-regular at all of the places w_i. Then the function $\tilde{f}^{w_0} = \otimes_{v \neq w_0} f_v$ can be chosen by Lemma 5 such that $\tilde{f}_{w_i}$ is σ-discrete at each of the places w_i, and such that $\mu_{\tilde{\gamma}}(\tilde{f}) \neq 0$, $\tilde{f} = \tilde{f}^{w_0} \otimes \tilde{f}_{w_0}$, and such that if $\mu_{\tilde{\delta}}(\tilde{f}) \neq 0$ for a rational $\tilde{\delta}$ in $\tilde{G}/\tilde{Z}$, then $\tilde{\delta}$ is in the H-orbit of $\tilde{\gamma}$.

It follows that the geometric part of the bi-period summation formula reduces to a multiple of $\mu_{\tilde{\gamma}}(\tilde{f})$ by a volume factor. Yet this geometric side is equal to the discrete part of the spectral side, for our $\tilde{f}$, which has sufficiently many σ-discrete components so that Proposition* applies. But the discrete part of the spectral side is zero by the assumption on $\tilde{f}_{w_0}$, that $< L_{\pi_w}, \pi_w(f_w)L_{\tilde{\pi}_w} >= 0$ for every admissible irreducible G_w-module π_w. The resulting contradiction implies that $\mu_{\tilde{\gamma}_w}(\tilde{f}_w) = 0$ for every σ-regular σ-elliptic element $\tilde{\gamma}_w$ in $\tilde{G}_w/\tilde{Z}_w$, as required. □

Remark. (1) "Sufficiently many" in Proposition *, is likely to be more than the rank of the symmetric space G/H. This rank is the dimension of a maximal commutative subspace consisting of semi simple elements in the -1 eigenspace $\mathfrak{g}^-$ of the Lie algebra $\mathfrak{g}$ of G under the action of the involution σ on $\mathfrak{g}$.
(2) It will be interesting to extend the conclusion of Theorem 1 to include elements $\tilde{\gamma} \in \tilde{G}/\tilde{Z}$ other than σ-regular σ-elliptic ones.

References

[A1] J. Arthur, A trace formula for reductive groups. I. Terms associated to classes in $G(Q)$, *Duke Math. J.* 45 (1978), 911-952.

[A2] J. Arthur, On a family of distributions obtained from Eisenstein series II: explicit formulas, *Amer. J. Math.* 104 (1982), 1289-1336.

[Be] J. Bernstein, P-invariant distributions on $GL(N)$ and the classification of unitary representations of $GL(N)$ (non-archimedean case), in *Lie Groups Representations II*, SLN 1041 (1984), 50-102.

[BZ] J. Bernstein, A. Zelevinskii, Representations of the group $GL(n,F)$ where F is a nonarchimedean local field, *Uspekhi Mat. Nauk* 31(1976), 5-70.

[Bo] A. Borel, *Linear Algebraic Groups*, GTM 126, Springer-Verlag, New-York, 1991.

[CF] J. Cassels, A. Frohlich, *Algebraic Number Theory*, Academic Press, San Diego, 1968.

[D] G. van Dijk, Invariant Eigendistributions on the tangent space of a rank one semisimple symmetric space, *Math. Ann.* 268 (1984), 405-416.

[DP] G. van Dijk, M. Poel, The irreducible unitary $GL(n-1,\mathbb{R})$-spherical representations of $SL(n,\mathbb{R})$, *Compos. Math.* 73(1990), 1-30.

[F1] Y. Flicker, Rigidity for automorphic forms, *J. Analyse Math.* 49 (1987), 1-68.

[F2] Y. Flicker, Quadratic cycles on $GL(2n)$ cusp forms, *J. Algebra* 174 (1995), 678-697.

[F2$'$] Y. Flicker, Cusp forms on $GL(2n)$ with $GL(n)\times GL(n)$ periods, and simple algebras, *Math. Nachrichten* (1996).

[F3] Y. Flicker, A Fourier summation formula for the symmetric space $GL(n)/GL(n-1)$, *Compos. Math.* 88 (1993), 39-117.

[F4] Bernstein isomorphism and good forms, *Proc. Sympos. Pure Math.* 58 II (1995), 171-196.

[FH] Y. Flicker, J. Hakim, Quaternionic distinguished representations, *Amer. J. Math.* 116 (1994), 683-736.

[H] J. Hakim, Admissible distributions on p-adic symmetric spaces, *J. reine angew. Math.* 455 (1994), 1-19.

[HCD] Harish-Chandra, notes by G. van Dijk, *Harmonic Analysis on Reductive p-adic*

Groups, Lecture Notes in Mathematics 162, Springer-Verlag, New-York, 1970.

[HC1] Harish-Chandra, Admissible invariant distributions on reductive p-adic groups, *Queen's Papers in Pure and Applied Math.* 48 (1978), 281-346.

[HC2] Harish-Chandra, A submersion principle and its applications, in *Collected Papers*, vol. IV, Springer-Verlag, New-York, pp. 439-446, 1984.

[J] H. Jacquet, Principal L-functions of the linear group, in *Proc. Sympos. Pure Math.* 33 II (1979), 63-86.

[JL] H. Jacquet, K. Lai, A relative trace formula, *Compos. Math.* 54 (1985), 243-310.

[K] D. Kazhdan, Cuspidal geometry on p-adic groups, *J. Analyse Math.* 47 (1986), 1-36.

[Ke] T. Kengmana, Characters of the discrete series for pseudo Riemannian symmetric spaces, in *Representation theory of reductive groups*, (P. Trombi, ed), Birkhauser, Boston-Basel, pp. 177-183, 1983.

[KR] B. Kostant, S. Rallis, Orbits and representations associated with symmetric spaces, *Amer. J. Math.* 93 (1971), 753-809.

[P] D. Prasad, Trilinear forms for representations of $GL(2)$ and local ε-factors, *Compos. Math.* 75 (1990), 1-46.

[RR] C. Rader, S. Rallis, Spherical characters on p-adic symmetric spaces, *Amer. J. Math.* 118 (1996), 91-178.

[R] R. Rao, Orbital integrals in reductive groups, *Ann. of Math.* 96 (1972), 505-510.

[Ri] R. Richardson, Orbits, invariants, and representations associated to involutions of reductive groups, *Invent. Math.* 66 (1982), 287-312.

[S1] J. Sekiguchi, Invariant spherical hyperfunctions on the tangent space of a symmetric space, in *Algebraic groups and related topics*, Advanced Studies in Pure Mathematics 6, pp. 83-126, 1985.

[S2] J. Sekiguchi, The nilpotent subvariety of the vector space associated to a symmetric pair, *Publ. RIMS Kyoto Univ.* 20 (1984), 155-212.

[S] J.-P. Serre, *Lie groups and Lie algebras*, Lecture Notes in Mathematics 1500, Springer-Verlag, New-York, 1992.

[Sh] J. Shalika, A theorem on semi-simple p-adic groups, *Ann. of Math.* 95 (1972), 226-242.

[SS] T. Springer, R. Steinberg, Conjugacy classes, in *Seminar on Algebraic Groups and Related Finite Groups*, Lecture Notes in Mathematics 131, Springer-Verlag, New-York, 1970.

[St] R. Steinberg, Endomorphisms of algebraic groups, *Mem. Amer. Math. Soc.* 80 (1968).

[T] J. Tate, The cohomology groups of tori in finite galois extensions of number fields, *Nagoya Math. J.* 27 (1966), 709-719.

[V] M.-F. Vigneras, Caractérisation des intégrales orbitales sur un groupe réductif p-adique, *J. fac. sci. univ. Tokyo* 28 (1982), 945-961.