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On the Symmetric Square: Orbital Integrals

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0. Introduction

Let F be a local or global field of characteristic 0. Let $\bar{F}$ be an algebraic closure of F. For any field extension K of F in $\bar{F}$ put $G(K)=PGL(3,K)$, $H(K)=SL(2,K)$, $H_1(K)=PGL(2,K)$, $G=G(\bar{F})$, $H=H(\bar{F})$, $H_1=H_1(\bar{F})$, $J=\begin{pmatrix}0&&1\\&-1&\\1&&0\end{pmatrix}$, and $\sigma(g)=J{}^tg^{-1}J$ for g in G. The elements δ, δ' of $G(F)$ are called *(stably) σ-conjugate* if there is h in $G(F)$ (resp. G) with $\delta'=h\delta\sigma(h^{-1})$. Conjugacy and stable conjugacy is defined analogously for $H(F)$ and $H_1(F)$ on omitting σ.

In Sect. 1 we show that the map N described in (1)–(5) below is a well-defined bijection, called the *norm map*, from the set of stable σ-conjugacy classes in $G(F)$, to the set of stable conjugacy classes in $H(F)$. We also define a surjection N_1 from the subset of this set described in (1), (2), (3), (4), to the set of stable conjugacy classes in $H_1(F)$. In the case (4) N_1 actually relates conjugacy classes. Let δ be an element of $G(F)$. The set of eigenvalues of $\delta\sigma(\delta)$ is of the form $\{\lambda, 1, \lambda^{-1}\}$.

(1) If $\delta\sigma(\delta)=1$, put $N\delta=1$ and $N_1\delta=1$.

(2) If $\delta\sigma(\delta)$ is a non-trivial unipotent, let $N\delta$ and $N_1\delta$ be the non-trivial unipotent classes.

(3) If the eigenvalues $\lambda, 1, \lambda^{-1}$ of $\delta\sigma(\delta)$ are distinct, let $N\delta$ be the class in $H(F)$ determined by the eigenvalues λ, λ^{-1}, and $N_1\delta$ the class in $H_1(F)$ with eigenvalues λ, 1.

Put $e=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$. For $h=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, put $h_1=\begin{pmatrix}a&0&b\\0&1&0\\c&0&d\end{pmatrix}$.

(4) If $\delta\sigma(\delta)=h_1$, $h=(-1)\cdot$non-trivial unipotent in $GL(2,F)$, put $N\delta=h$, and $N_1\delta=\begin{pmatrix}0&\alpha\\1&0\end{pmatrix}$ if $\delta=(ae)_1$, with $a=\begin{pmatrix}1&\alpha\\0&1\end{pmatrix}$, $\alpha\neq0$.

(5) If $\delta\sigma(\delta)=h_1$, $h\neq-1$ in $GL(2,F)$, put $N\delta=-1$.

In Sects. 2 and 3, F is a local field, f is a smooth (namely locally constant if F is non-archimedean) compactly supported function on $G(F)$, f_0 is such a function on

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$H_0(F)=H(F)$, f_1 on $H_1(F)$. Put

$$\Phi_f(\delta)=\int_{G_\delta^\sigma(F)\backslash G(F)} f(g^{-1}\delta\sigma(g)), \qquad \Phi_{f_0}(\gamma)=\int_{H_\gamma(F)\backslash H(F)} f_0(g^{-1}\gamma g)$$

and a similar definition for $\Phi_{f_1}(\gamma)$, where $H_\gamma(F)$ is the centralizer of γ in $H(F)$ and $G_\delta^\sigma(F)$ is the σ-centralizer of δ in $G(F)$. If $N\delta \neq 1$ put

$$\Phi_f^{st}(\delta)=\sum_{\delta'}\Phi_f(\delta').$$

The sum ranges over a set of representatives for the σ-conjugacy classes in the stable σ-conjugacy class of δ. If $N\delta=1$ put

$$\Phi_f^{st}(\delta)=\sum_{\delta'}\kappa(\delta')\Phi_f(\delta').$$

The definition of κ is given in Sect. 1; it is too long to recall here. $\Phi_{f_0}^{st}(\gamma)$ is given by $\sum_{\gamma'}\Phi_{f_0}(\gamma')$ for all γ. These orbital integrals depend on a choice of Haar measures. For a suitable choice of measures, studied in Sect. 2, implicit in our notations, we say that f and f_0 have *matching orbital integrals* if they satisfy the relation $\Phi_{f_0}^{st}(\gamma) = \Phi_f^{st}(\delta)$ for all γ, δ with $\gamma=N\delta$. In this case we write $f_0=\lambda^*(f)$. Proposition 3.1.1 asserts that for each f there exists f_0, and for each f_0 there exists f, with $f_0=\lambda^*(f)$.

If $\delta\sigma(\delta)$ has distinct eigenvalues λ, 1, λ^{-1}, put

$$\Phi_f^{lab}(\delta)=\sum_{\delta'}\kappa(\delta')\Phi_f(\delta').$$

We write $f_1=\lambda_1^*(f)$ if

$$\Phi_{f_1}(\gamma)=|(1+\lambda)(1+\lambda^{-1})|^{1/2}\Phi_f^{lab}(\delta).$$

for all $\gamma=N_1\delta$ with distinct eigenvalues, for suitably related measures. Proposition 3.5.1 asserts that for each f there is f_1, and for each f_1 there is f, with $f_1=\lambda_1^*(f)$. The values of the integrals at the singular set are given in 3.6.1 and 3.7.

In Sect. 4, F is a local non-archimedean field with ring R of integers, $K=G(R)$, $K_0=H(R)$; f, f_0 are the characteristic functions of K, K_0 divided by the volumes of K, K_0. The main result, Proposition 4.5, asserts that $f_0=\lambda^*(f)$, more precisely that $\Phi_f^{st}(\delta)=\Phi_{f_0}^{st}(\gamma)$ whenever $\gamma=N\delta$ is elliptic regular (the case of split γ is easy; it is given in [1]). This result is extended to all spherical functions in [2, Sect. 2]. In [4] it is shown that $f_1=\lambda_1^*(f)$ for the above f, if f_1 is the characteristic function of $K_1=H_1(R)$, divided by the volume of K_1. The methods of [4] are completely different from those of the present article. They are global, and rely on the trace formula of [3] and some of the (weaker) results of [2]. The main theorem of the theory is proven in [2] in a special case, and in [5] in general, using our results here. The present paper is the initial part of our symmetric square project. For a general introduction to this project see [1]; for the final statement of the symmetric square theorem, which implies the multiplicity one theorem for all cuspidal representations of $SL(2)$ and the rigidity theorem for packets of $SL(2)$, see [5].

1. Norm Map

1.1. *Conjugacy.* Let F be a local or global field of characteristic 0, fix an algebraic closure $\bar{F}$ of F, G an algebraic group defined over F [so $G=G(\bar{F})$] and $G(F)$ the F-rational points of G, σ an automorphism of G defined over F. The elements δ, δ'

of $G(F)$ are called *σ-conjugate* if there is h in $G(F)$ with $\delta' = h\delta\sigma(h^{-1})$. They are called *stably σ-conjugate* if there is h in G with $\delta' = h\delta\sigma(h^{-1})$. The term (stable) conjugacy (no mention of σ) is employed if σ is the trivial automorphism.

The stable σ-conjugates of δ in $G(F)$ are described by the set $A(\delta)$ of g in G with $g\delta\sigma(g^{-1})$ in $G(F)$. The map $A(\delta) \xrightarrow{\alpha} H^1(F, G_\delta^\sigma)$, where $G_\delta^\sigma = \{g \text{ in } G;\ g\delta\sigma(g^{-1}) = \delta\}$, by $g \mapsto \{\tau \mapsto g_\tau = g^{-1}\tau(g)\}$, factors through

$$1 \longrightarrow D(\delta) \xrightarrow{\alpha} H^1(F, G_\delta^\sigma) \longrightarrow H^1(F, G),$$

where the double coset space $D(\delta) = G(F)\backslash A(\delta)/G_\delta^\sigma$ parametrizes the σ-conjugacy classes within the stable σ-conjugacy class of δ.

The above definitions will be used with $G = PGL(3)$ and the (involution) outer automorphism $\sigma(g) = J\,{}^t g^{-1} J$ $\left(t\text{: transpose; } -1\text{: inverse; } J = \begin{pmatrix} 0 & & 1 \\ & -1 & \\ 1 & & 0 \end{pmatrix}\right)$, and also with $H = H_0 = SL(2)$, $H_1 = PGL(2) = SO(3)$ and the trivial σ. If γ lies in H (or H_1) then H_γ denotes the centralizer of γ in H.

Our purpose is to define maps N and N_1 from the set of stable σ-conjugacy classes in $G(F)$ to the sets of stable conjugacy classes in $H(F)$ and $H_1(F)$, and study their properties. Note that if δ, δ' are (stably) σ-conjugate then $\delta\sigma(\delta)$, $\delta'\sigma(\delta')$ are (stably) conjugate.

1.2. *Identity.* If $\delta\sigma(\delta) = 1$ we write $N\delta = 1$ and $N_1\delta = 1$. Then $\delta J = {}^t(\delta J)$ is symmetric, any two symmetric matrices are equivalent over F, hence for each δ' with $\delta'\sigma(\delta') = 1$ there is S in G with $\delta J = S\delta' J\,{}^t S$, so that $\delta = S\delta'\sigma(S^{-1})$ and the δ with $\delta\sigma(\delta) = 1$ form a single stable σ-conjugacy class.

For such δ the σ-centralizer G_δ^σ is $(PO(3, \delta J) =) SO(3, \delta J)$, the (projective$=$) special orthogonal group with respect to the form δJ. Replacing δ by a σ-conjugate $u\delta\sigma(u^{-1})$ or δJ by $u\delta J\,{}^t u$, implies replacing G_δ^σ by its conjugate $uG_\delta^\sigma u^{-1}$. Hence if F is $\mathbb{R}$ or p-adic then there are two σ-conjugacy classes in the stable σ-conjugacy class of the δ with $N\delta = 1$, corresponding to the split and non-split forms δJ. Put $\kappa(\delta) = 1$ if $G_\delta^\sigma = SO(3, \delta J)$ splits and $\kappa(\delta) = -1$ if it is anisotropic. If we put $\gamma = N\delta (= 1)$ then there is a natural surjection $\varphi: H_\gamma \to G_\delta^\sigma$ with kernel $\{\pm 1\}$. φ is not always defined over F, see (3.3).

1.3 *Unipotent.* If $\delta\sigma(\delta)$ is unipotent but not 1 we check by matrix multiplication that it is a regular unipotent $\left(\text{not conjugate to } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$. Alternatively, $\delta\sigma(\delta)v = v$ if and only if $(\delta J - {}^t(\delta J))w = 0$, where $w = {}^t(\delta J)^{-1}v$. Thus the 1-eigenspace of $\delta\sigma(\delta)$ has the same dimension as the zero-eigenspace of the skew-symmetric matrix $\delta J - {}^t(\delta J)$, namely 1 or 3, and $\delta\sigma(\delta) \neq 1$ is regular unipotent. Up to stable σ-conjugacy we may assume that $\delta\sigma(\delta) = \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, a σ-invariant matrix. Hence δ commutes with $\sigma(\delta)$ and $\delta\sigma(\delta)$, and it is unipotent of the form $\begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$. These make a single σ-conjugacy class. The σ-centralizer G_δ^σ is the additive group $\mathbb{G}_a$, $H^1(F, \mathbb{G}_a)$ is trivial, hence there is a unique σ-conjugacy class of δ with $\delta\sigma(\delta) = \text{unipotent} \neq 1$, and we put $N\delta = \text{unipotent}$ [in $H(F)$]. If $\gamma = N\delta$ then $H_\gamma = \{\pm 1\} \times \mathbb{G}_a$ and there is a natural surjection $\varphi : H_\gamma \to G_\delta^\sigma$ with kernel $\{\pm 1\}$.

1.4 *Negative* I. If δ lies in $GL(3,F)$ then $\delta\sigma(\delta)$ lies in $SL(3,F)$. If $\delta\sigma(\delta)$ has the eigenvalue λ so does ${}^t(\delta\sigma(\delta))$. Hence for some vector v we have $\lambda v = {}^t(\delta\sigma(\delta))v = J\delta^{-1}J^t\delta v$ and $\delta J^t\delta^{-1}J \cdot \delta J v = \lambda^{-1} \cdot \delta J v$, so that $\delta\sigma(\delta)$ has the eigenvalue λ^{-1}. Hence one of the eigenvalues of $\delta\sigma(\delta)$ is 1, and the cases where all three eigenvalues are 1 were dealt with in (1.2) and (1.3). It remains to deal with the cases where two eigenvalues are not 1. Replacing $\delta\sigma(\delta)$ by a conjugate, hence δ by a σ-conjugate, we may assume that $\delta\sigma(\delta)$ is of the form h_1, where for any $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $GL(2)$, we put

$$h_1 = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}.$$

Since δJ takes λ-eigenvectors of ${}^t(\delta\sigma(\delta))$ to λ^{-1}-eigenvectors of $\delta\sigma(\delta)$, the assumption $\delta\sigma(\delta) = h_1$ implies that δJ fixes the subspaces $\begin{pmatrix} 0 \\ * \\ 0 \end{pmatrix}$, $\begin{pmatrix} * \\ 0 \\ * \end{pmatrix}$. So does δ. Hence multiplying by a scalar we have $\delta = a_1$ for some a in $GL(2)$.

1.4.1. Note that if $e = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\delta = (ae)_1$, then $N\delta = h_1$, where $h = aew{}^ta^{-1}ew = \frac{-1}{\det a} a^2$, and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

1.4.2. If $\delta' = (a'e)_1$ and $\delta' = \beta^{-1}\delta\sigma(\beta)$ [hence $\delta'\sigma(\delta') = \beta^{-1}\delta\sigma(\delta)\beta$ and $\beta = b_1$ for some b in $GL(2)$], then $a'e = b^{-1}aew{}^tb^{-1}w$ and $a' = b^{-1}a(ew){}^tb^{-1}(ew)^{-1} = \frac{1}{\det b} b^{-1}ab$. Hence δ, δ' are (stably) σ-conjugate if and only if a, a' are projectively (stably) conjugate.

1.4.3. If $\delta\sigma(\delta) = h_1$ and $h = -I$ in $GL(2)$ then $a^2 = \det a (\delta = (ae)_1)$ and a is a scalar $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$. We put $N\delta = -I$, and note that all δ with $N\delta = -I$ form a single σ-conjugacy class, since

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = \frac{\alpha}{\beta} \begin{pmatrix} \alpha/\beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} \beta/\alpha & 0 \\ 0 & 1 \end{pmatrix}.$$

1.4.4. If $\delta\sigma(\delta) = h_1$ and $h = -\text{unipotent} \neq -I$ in $GL(2)$, then up to conjugacy $h = -\begin{pmatrix} 1 & 2\alpha \\ 0 & 1 \end{pmatrix}$, hence $a = u^{-1}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ with $\alpha \in F^\times$, $u \in F^\times$. But a is equal to $\frac{1}{u}\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \alpha u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$, hence projectively conjugate to $\begin{pmatrix} 1 & \alpha u \\ 0 & 1 \end{pmatrix}$. Now $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ $(\alpha, \beta \in F^\times)$ are (projectively) conjugate only if α/β is a square in

$F^\times$; they are clearly stably conjugate. Hence the σ-conjugacy classes within the single stable σ-conjugacy class of our δ are parametrized by $F^\times/F^{\times 2}$. If $\delta=(ae)_1$, $a=\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}$, $\alpha \neq 0$, we let $N\delta$ be the stable conjugacy class of h in $H(F)$, and define $N_1\delta$ to be the conjugacy class in $H_1(F)$ of elements which generate $F(\sqrt{\alpha})$ over F, and the quotient of whose eigenvalues is -1. Such an element of $GL(2)$ is $\begin{pmatrix}0 & \alpha\\ 1 & 0\end{pmatrix}$.

1.5. *Regulars.* If the eigenvalues λ, 1, λ^{-1} of $\delta\sigma(\delta)$ are distinct then they lie in a quadratic extension of F (or in F) and define a stable conjugacy class $N\delta$ in $H(F)$ (with eigenvalues λ, λ^{-1}) and a (stable) conjugacy class $N_1\delta$ in $H_1(F)$ [with eigenvalues λ, 1, λ^{-1} in $SO(3,F)$ or λ, 1 in $PGL(2,F)$]. Given λ there exist α, β in $F(\lambda)^\times$ with $\alpha/\beta=-\lambda$; here $\beta=\bar{\alpha}$ and we use Hilbert Theorem 90 if $\lambda\notin F$. The pair α, β is determined up to a multiple by a scalar u in $F^\times$. The matrix $\delta\sigma(\delta)$ $(\delta=(ae)_1)$ has eigenvalues $\lambda, 1, \lambda^{-1}$ *iff* a has eigenvalues α, β so that $\frac{-1}{\det a}a^2$ has eigenvalues $-\alpha/\beta$, $-\beta/\alpha$. Hence the norm map is onto the set of regular elements of $H(F)$, and the δ in $G(F)$ with regular $N\delta$ make a single stable σ-conjugacy class, as a and ua (u in $F^\times$) are projectively stably conjugate.

But a and $a'=u^{-1}a$ are projectively conjugate only if $u^{-1}a=\frac{1}{\det b}b^{-1}ab$ for some b in $GL(2,F)$. Then $u^2=\det b^2$, and $u=\pm\det b$. If $u=-\det b$ then $-a=b^{-1}ab$, a has eigenvalues γ, $-\gamma$, and $h=I$ does not have eigenvalues different than 1. Hence $u=\det b$, $a=b^{-1}ab$ and $u=\det b$ lies in $N_{K/F}K^\times$, where $K=F(a)$. It follows that *in the unique stable σ-conjugacy class of δ with regular $\delta\sigma(\delta)$ the σ-conjugacy classes are parametrized by u in $F^\times/NK^\times$, $K=F(\delta\sigma(\delta))$. A set of representatives is given by $\delta=(uae)_1$.*

Corollary. *Let F be a global field, u a place of F, and δ, δ' stably σ-conjugate but non σ-conjugate elements of $G(F)$. Then there is a place $v\neq u$ of F such that δ, δ' are not σ-conjugate in $G(F_v)$.*

1.6. *Kappa.* If $N\delta$ is regular then $N\tilde{\delta}=1$, where $\tilde{\delta}=\frac{1}{2}[\delta J+{}^t(\delta J)]J$. We define $\kappa(\delta)$ to be $\kappa(\tilde{\delta})$ [see (1.2)], namely 1 if $SO(3,\tilde{\delta}J)$ is split and -1 otherwise. Note that if $\delta\sigma(\delta)=1$ then $\delta J={}^t(\delta J)$ and $\tilde{\delta}=\delta$; the present definition then generalizes the one of (1.2).

1.6.1. κ depends only on the σ-conjugacy class of δ. Indeed if δ is replaced by $\beta\delta J^t\beta J$ then $\delta J+{}^t(\delta J)$ is replaced by

$$\beta\delta J^t\beta+\beta J^t\delta^t\beta=\beta[\delta J+{}^t(\delta J)]^t\beta,$$

and the form $\delta J+{}^t(\delta J)$ splits if and only if $\beta[\delta J+{}^t(\delta J)]^t\beta$ does.

1.6.2. If δ, δ' are stably σ-conjugate with regular norm, but they are not conjugate, then the forms $\tilde{\delta}J$ and $\tilde{\delta}'J$ are not equivalent [see (1.5)], and $\kappa(\delta')=-\kappa(\delta)$. Thus if $\delta=(ae)_1$ and $\delta'=(uae)_1$, then $\kappa(\delta')=\chi(u)\kappa(\delta)$, χ being the quadratic character of $F^\times$ trivial on $NK^\times$, $K=F(\delta\sigma(\delta))$.

1.6.3. If $N\delta=\gamma$ is regular in $H(F)$ then $G_\delta^\sigma\cong H_\gamma$. Indeed, if $g^{-1}\delta\sigma(g)=\delta$ then $g^{-1}\delta\sigma(\delta)g=\delta\sigma(\delta)$; if $\delta=(ae)_1$ then $g=b_1$ and $b^{-1}ab=a$, since $\delta\sigma(\delta)=h_1$,

$h=\frac{-1}{\det a}a^2$. Hence $b^{-1}aew^tb^{-1}we=a$, namely $\frac{1}{\det b}b^{-1}ab=1$, so that $\det b=1$. It is clear that $H_\gamma=H_a$.

It is clear that $H_\gamma \cong G^\sigma_\delta$ also in the cases (1.4.3), (1.4.4).

1.7. Lemma. *Suppose that J is a linear algebraic group, defined over a local field F, in the matrix algebra M, δ is in $J(F)$ and ε in the centralizer $J_\delta(F)$ of δ in $J(F)$ is near 1. Then $J_{\varepsilon\delta} \subset J_\delta$.*

Proof. J acts on M by inner automorphisms, and $M=\oplus M(\lambda)$ if we enlarge F; the sum is over the eigenvalues λ of δ and $M(\lambda)$ is the corresponding eigenspace. J_δ is the intersection of J and $M(1)$. Since ε lies in $J_\delta(F)$, $\varepsilon\delta$ leaves each $M(\lambda)$ invariant. If ε is near 1 all fixed vectors of $\varepsilon\delta$ lie in $M(1)$. Indeed, if v lies in $M(\lambda)$, then $v=\varepsilon\delta\cdot v=\lambda\varepsilon\cdot v$ and λ^{-1} is an eigenvalue of ε. This is impossible if $\lambda \neq 1$ and ε is near 1. But then $J_{\varepsilon\delta} \subset J \cap M(1)=J_\delta$, as required.

Applying the lemma with $J=G \rtimes \{1,\sigma\}$ and δ in $G(F)$, we have:

1.7.1. Corollary. *If ε in $G^\sigma_\delta(F)$ is near 1 then $G^\sigma_{\varepsilon\delta} \subset G^\sigma_\delta$.*

1.8. Lemma. *If $N\delta=1$, $\varepsilon \in G^\sigma_\delta(F)$ is near 1 and $N(\varepsilon\delta)$ has distinct eigenvalues, then $\kappa(\varepsilon\delta)=\kappa(\delta)$.*

Proof. $\varepsilon\delta J+{}^t(\varepsilon\delta J)=\varepsilon\delta J+{}^t(\delta J^t\varepsilon^{-1})=\varepsilon\delta J+\varepsilon^{-1}\,{}^t(\delta J)=(\varepsilon+\varepsilon^{-1})\delta J$. $\kappa(\varepsilon\delta)$ is 1 if and only if $G^\sigma_{(\varepsilon+\varepsilon^{-1})\delta/2}$ splits; but this is contained in G^σ_δ by Corollary 1.7.1. Hence the two special orthogonal groups split together.

1.9. Lemma. *If $N\delta=-I$; ε, ε' in $G^\sigma_\delta(F) \cong H(F)$ are stably conjugate but not conjugate, and $N(\varepsilon\delta)$ has distinct eigenvalues, then $\kappa(\varepsilon\delta)=-\kappa(\varepsilon'\delta)$.*

Proof. We may assume that $\delta=e_1$, $e=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$, and then $\varepsilon=a_1$, $\varepsilon'=a'_1$, with a, a' in $SL(2,F)$. $\varepsilon\delta$ and $\varepsilon'\delta$ are σ-conjugate (and define equivalent forms) if and only if a and a' are conjugate (not only projectively conjugate, since $N(\varepsilon\delta)$ has distinct eigenvalues).

2. Differential Forms

2.1. To compare orbital integrals on different groups we need to compare Haar measures, or differential forms which we always take to be invariant of highest degree. To introduce these differential forms we need to recall the construction from [9, Lemma 6.1]. Let $\mathbb{G}_a$ be the additive group and $\zeta: H \to \mathbb{G}_a$ the trace map. If γ has distinct eigenvalues γ_1, $\gamma_2=\gamma_1^{-1}$, then the differential $d\zeta$ *of* ζ *at* γ *is given by*

$$d\zeta=d\gamma_1+d\gamma_2=d\gamma_1-\frac{d\gamma_1}{\gamma_1^2}=\gamma_1\frac{d\gamma_1}{\gamma_1}-\gamma_1^{-1}\frac{d\gamma_1}{\gamma_1}=(\gamma_1-\gamma_2)\frac{d\gamma_1}{\gamma_1},$$

and it is non-zero. At a neighborhood of γ with $\gamma_1=\gamma_2$ we may assume that $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with $a \neq 0$, $d=(1+bc)/a$; then $\zeta(\gamma)=a+d$ has the differential $(1-a^{-2}(1+bc))da+\frac{c}{a}db+\frac{b}{a}dc$. It vanishes only if $a^2=1+bc$, $b=0$, $c=0$, namely at $\gamma=\pm I$. The subset H_r of H where $d\zeta$ is non-zero is called the regular set.

Fix (non-zero invariant) differential forms ω_H and μ (of highest degrees 3 and 1) on H and $\mathbb{G}_a$. μ defines a non-zero invariant form $\omega_\gamma(\mu)$ on H_γ (which is independent of ω_H). If $\mu=dx$ then $\omega_\gamma(\mu)=\frac{d\gamma_1}{\gamma_1}$ or $=dx$ if $\gamma=\pm\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$. If γ is stably conjugate to γ' then $\omega_{\gamma'}(\mu)$ is obtained from $\omega_\gamma(\mu)$ by transport of structure. The fibers of ζ are the stable conjugacy classes in H_r and the quotient of ω_H by μ defines an invariant form on the fibers of ζ in H_r.

In [9, Sect. 6], the corresponding map $\tilde{\zeta}$ is from $GL(2)$ to $X=\mathbb{G}_a^2$, by $\tilde{\zeta}(\gamma)=(\operatorname{tr}\gamma, \det\gamma)=(a+d,\ ad-bc)$. It has 2×4 differential $\begin{pmatrix}1 & 0 & 0 & 1\\ d & -c & -b & a\end{pmatrix}{}^t (da\ db\ dc\ dd)$, which is non-singular if one of $a-d, b, c$ is non-zero. The singular set consists of the scalars. Our $\omega_\gamma(\mu)$ is denoted there by η_γ.

2.2. Similarly, let $\xi: G\to\mathbb{G}_a$ be defined by $\xi(\delta)=\operatorname{tr}N\delta$. To compute its differential note that $\xi(\delta)+1=\operatorname{tr}(\delta J^t\delta^{-1}J)$. Then $d\xi$ is the trace of the differential of the map $\delta\mapsto\delta J^t\delta^{-1}J$, which is

$$d\delta\cdot J^t\delta^{-1}J+\delta J\cdot d({}^t\delta^{-1})\cdot J\,.$$

But

$$0=dI=d(\delta\delta^{-1})=d\delta\cdot\delta^{-1}+\delta\cdot d\delta^{-1}\,,$$

hence

$$d\delta^{-1}=-\delta^{-1}\cdot d\delta\cdot\delta^{-1}\,,$$

and

$$\operatorname{tr}[\delta J\cdot{}^t\delta^{-1}\cdot d({}^t\delta)\cdot{}^t\delta^{-1}\cdot J]=\operatorname{tr}[J\delta^{-1}\cdot d\delta\cdot\delta^{-1}J^t\delta]=\operatorname{tr}[d\delta\cdot\delta^{-1}\cdot J^t\delta J\delta^{-1}]\,.$$

So

$$\delta\xi=\operatorname{tr}d\delta[\sigma(\delta)-\delta^{-1}\sigma(\delta^{-1})\delta^{-1}]\,.$$

Then $d\xi$ is non-zero for all $d\delta$ only if $\delta\sigma(\delta)=(\delta\sigma(\delta))^{-1}$ has square 1, hence has eigenvalues 1 or -1. Since $\delta\sigma(\delta)$ has determinant 1, it is semi-simple and $N\delta$ is $\pm I$. We conclude that the regular set G_r of G of δ where $d\xi\neq 0$ consists of all δ with $N\delta\neq\pm I$.

The fibers of ξ on the regular set G_r are stable σ-conjugacy classes. We fix an invariant differential form ω_G of highest degree on G. As above μ determines an invariant form $\omega_\delta(\mu)$ of maximal degree on G_δ^σ. If δ' is stably σ-conjugate to δ then $G_{\delta'}^\sigma$ is isomorphic to G_δ^σ over $\bar{F}$ and $\omega_\delta(\mu)$ transforms to a form $\omega_{\delta'}(\mu)$ of $G_{\delta'}^\sigma$.

2.3. The map $\varphi: H_\gamma\to G_\delta^\sigma$ of (1.2) and (1.6.3) can be used to pull back the form $\omega_\delta(\mu)$ to a form $\varphi^*(\omega_\delta(\mu))$ on H_γ. The comparison is given by

2.3.1. Lemma. *The form $\varphi^*(\omega_\delta(\mu))$ is equal to $\frac{1}{2}\omega_\gamma(\mu)$.*

2.4. The trace map $\zeta_1: H_1=SO(3)\to\mathbb{G}_a$ is smooth on the regular set H_{1r} of γ_1 with distinct eigenvalues, and $\omega_{\gamma_1}(\mu)$ can be introduced for such γ_1. Note that the centralizer $H_{1\gamma_1}$ of γ_1 in H_1 is isomorphic to G_δ^σ. The pullback of $\omega_\delta(\mu)$ to $H_{1\gamma_1}$ is denoted again by $\omega_\delta(\mu)$.

2.4.1. Lemma. *If $\gamma_1 = N_1\delta$ has distinct eigenvalues $1, \gamma', \gamma'' = \gamma'^{-1}$ (see (1.5)) then $\omega_{\gamma_1}(\mu) = 2(1+\gamma')(1+\gamma'')\omega_\delta(\mu)$.*

The two lemmas are verified below.

2.5. Suppose that $\delta \times \sigma$ is semi-simple in $G \rtimes \langle\sigma\rangle$ (hence $\gamma = N\delta$ and $\gamma_1 = N_1\delta$ are semi-simple in H and H_1). Choose a neighborhood X_δ of the trivial coset G_δ^σ in $G_\delta^\sigma\backslash G$, a section $s: G_\delta^\sigma\backslash G \to G$, and a neighborhood Y_δ of the identity in G_δ^σ so that the morphism $Y_\delta \times X_\delta \to G$, $(\varepsilon, g) \mapsto s(g)^{-1}\varepsilon\delta\sigma(s(g))$ is an immersion (its differential is non-singular at each point). For a local field F the map $Y_\delta(F) \times X_\delta(F) \to G(F)$ is an analytic isomorphism onto an open subset of $G(F)$. X_γ, Y_γ, X_{γ_1}, Y_{γ_1} can be introduced for γ in H, γ_1 in H_1.

2.6 Lemma. *Locally the invariant form ω_G on G can be taken to be $\Theta(\varepsilon)\omega_\delta^1 \wedge \omega^2$, where ω_δ^1 is an invariant form of maximal degree on G_δ^σ, ω^2 a highest degree invariant form on $G_\delta^\sigma\backslash G$, and $\Theta(\varepsilon)$ is the determinant of the transformation* 1-Ad$(\varepsilon\delta \times \sigma)$ *on the* Lie *algebra* Lie$(G_\delta^\sigma\backslash G)$ *of* $G_\delta^\sigma\backslash G$.

Proof. To compute the differential we introduce an extension $F(\eta)$ of F, the quotient of the polynomial ring $F[x]$ by the ideal (x^2). For any algebraic group J over F there is an exact sequence

$$0 \to \mathrm{Lie}\,J(F) \to J(F(\eta)) \to J(F) \to 1,$$

with maps $X \mapsto 1 + \eta X$, $h(1+\eta X) \mapsto h$. To study the map $(\varepsilon, h) \mapsto h^{-1} \cdot \varepsilon\delta \times \sigma \cdot h$ (ε in G_δ^σ, h in $G_\delta^\sigma\backslash G$) we replace h by $(1+\eta Y)h$, where Y is in Lie$(G_\delta^\sigma\backslash G)$, and $\varepsilon\delta \times \sigma$ by $(\varepsilon\delta \times \sigma)(1+\eta X)$. Then $h^{-1} \cdot \varepsilon\delta \times \sigma \cdot h$ becomes

$$\begin{aligned} &h^{-1}(1-\eta Y)(\varepsilon\delta \times \sigma)(1+\eta X)(1+\eta Y)h \\ &= h^{-1} \cdot \varepsilon\delta \times \sigma \cdot (1 - \eta \cdot \mathrm{Ad}(\varepsilon\delta \times \sigma)Y)(1 + \eta(X+Y))h \\ &= h^{-1} \cdot \varepsilon\delta \times \sigma \cdot [1 + \eta(X + [1\text{-}\mathrm{Ad}(\varepsilon\delta \times \sigma)]Y)] \cdot h. \end{aligned}$$

Here we used the relation $(1+\eta Y)^{-1} = 1 - \eta Y$, and $Y\varepsilon = \varepsilon \cdot \mathrm{Ad}\,\varepsilon \cdot Y$. Then

$$\begin{aligned} \omega_G(X+Y) &= \omega^1(X) \wedge \omega^2([1\text{-}\mathrm{Ad}(\varepsilon\delta \times \sigma)]Y) \\ &= \Theta(\varepsilon) \cdot \omega^1(X) \wedge \omega^2(Y), \end{aligned}$$

as required.

2.7. Let $\xi': G_\delta^\sigma \to \mathbb{G}_a$ be $\xi'(\varepsilon) = \xi(\varepsilon\delta) = \mathrm{tr}\,N(\varepsilon\delta)$. Then ξ', μ, and ω_δ^1 can be used as above to define a form $\omega'_\varepsilon(\mu)$ on the centralizer of ε in G_δ^σ, which is equal to $G_{\varepsilon\delta}^\sigma$ by Corollary 1.7.1 if ε is near 1. One has $\omega'_\varepsilon(\mu) = \Theta(\varepsilon)\omega_{\varepsilon\delta}(\mu)$.

Similarly we have $\omega_H = \theta(\eta)\omega_\gamma^1 \wedge \omega^2$, $\omega_{H_1} = \theta_1(\eta_1)\omega_{\gamma_1}^1 \wedge \omega^2$, where $\theta(\eta)$ and $\theta_1(\eta_1)$ are the functions $\det[1\text{-}\mathrm{Ad}(\eta\gamma)]_{\mathrm{Lie}\,H_\gamma\backslash H}$, $\det[1\text{-}\mathrm{Ad}(\eta_1\gamma_1)]_{\mathrm{Lie}\,H_{1\gamma_1}\backslash H_1}$, on H_γ and $H_{1\gamma_1}$. The maps $\zeta'(\eta) = \mathrm{tr}(\eta\gamma)$, $\zeta'_1(\eta_1) = \mathrm{tr}(\eta_1\gamma_1)$ are used to define $\omega'_\eta(\mu)$, $\omega'_{\eta_1}(\mu)$, and we have $\omega'_\eta(\mu) = \theta(\eta)\omega_{\eta\gamma}(\mu)$, $\omega'_{\eta_1}(\mu) = \theta_1(\eta_1)\omega_{\eta_1\gamma_1}(\mu)$.

2.8. If $\gamma = N\delta$, $\gamma_1 = N_1\delta$ and ε is in G_δ^σ, then $\varepsilon\delta\sigma(\varepsilon\delta) = \varepsilon^2\delta\sigma(\delta)$ and ε commutes with $\delta\sigma(\delta)$, so that $N(\varepsilon\delta) = \eta\gamma$ (η in H_γ), $N_1(\varepsilon\delta) = \eta_1\gamma_1$ (η_1 in $H_{1\gamma_1}$). To compute $\Theta(\varepsilon)$, $\theta(\eta)$, $\theta_1(\eta_1)$ we may assume that ε, hence η, η_1, is semi-simple, since these functions depend only on the semi-simple parts of ε, η, η_1 in their Jordan decomposition.

Further, we can work over the algebraic closure $\bar{F}$, and take δ to be the diagonal matrix (a, b, c). Then ε can also be taken to be diagonal; hence $\varepsilon=(d,1,d^{-1})$ since it lies in G_δ^σ. If the eigenvalues of $N(\varepsilon\delta)$ are denoted by β_1, $\beta_2=\beta_1^{-1}$, then it is easily checked that:

2.8.1. If $\gamma=1$ then $\theta(\eta)=1$ and

$$\Theta(\varepsilon)=2(1+d)(1+d^{-1})(1+d^2)(1+d^{-2}).$$

2.8.2. If $\gamma=-1$ then $\theta(\eta)=1$ and

$$\Theta(\varepsilon)=2(1+d^2)(1+d^{-2}).$$

2.8.3. If $\gamma \neq \pm 1$ then $\theta_1(\eta_1)=(1-\beta_1)(1-\beta_2)$,

$$\Theta(\varepsilon)=2(1-\beta_1^2)(1-\beta_2^2), \qquad \theta(\eta)=(1-\beta_1^2)(1-\beta_2^2).$$

2.9. To verify the lemmas it suffices to take the standard form $\mu=dx$ on $\mathbb{G}_a$. If $N\delta$ has distinct eigenvalues then G_δ^σ is abelian, one-dimensional, and isomorphic to H_γ and to $H_{1\gamma_1}$. As in (2.1) we compute

$$(\xi')^*(\mu)=d\xi'=(\beta_1-\beta_2)\frac{d\beta_1}{\beta_1}.$$

But $\omega_\delta^1=e\dfrac{d\beta_1}{\beta_1}$ for some constant e. It is the product of $\omega_\varepsilon'(\mu)$ and the quotient $\omega_\delta^1/(\xi')^*(\mu)=e/(\beta_1-\beta_2)$ of one-forms on G_δ^σ and $\mathbb{G}_a$. The same computation yields the same value for $\omega_\eta'(\mu)$ and $\omega_{\eta_1}'(\mu)$. So it remains to note that $\Theta(\varepsilon)/\theta(\eta)=2$ and $\Theta(\varepsilon)/\theta_1(\eta_1)=2(1+\beta_1)(1+\beta_2)$, and $\beta_i=\gamma_i$ when $\varepsilon=1$, to have Lemmas (2.3.1), (2.4.1) for δ with $N\delta \neq \pm I$.

2.10. If $\gamma=N\delta$ is I or $-I$ then the epimorphism $\varphi: H_\gamma \to G_\delta^\sigma$, $\varphi(\eta_1)=\varepsilon$, satisfies $\eta=N(\varphi(\eta_1))=\eta_1^a$ with $a=4$ if $\gamma=I$ and $a=2$ if $\gamma=-I$. Indeed, if $\gamma=I$ we may take $\delta=I$ and

$$\eta_1=\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\in H_\gamma$$

$$=SL_2 \overset{\varphi}{\mapsto} \begin{pmatrix} a^2 & & 0 \\ & 1 & \\ 0 & & a^{-2}\end{pmatrix} \in G_\delta^\sigma \overset{N}{\mapsto} \begin{pmatrix} a^4 & & 0 \\ & 1 & \\ 0 & & a^{-4}\end{pmatrix} \cong \eta=\begin{pmatrix} a^4 & 0 \\ 0 & a^{-4}\end{pmatrix}.$$

If $\gamma=-I$ we may take $\delta=\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1\end{pmatrix}$ and

$$\eta_1=\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\in H_\gamma$$

$$=SL_2 \overset{\varphi}{\mapsto} \begin{pmatrix} a & & 0 \\ & 1 & \\ 0 & & a^{-1}\end{pmatrix} \in G_\delta^\sigma \overset{N}{\mapsto} \begin{pmatrix} a^2 & & 0 \\ & 1 & \\ 0 & & a^{-2}\end{pmatrix} \cong \eta=\begin{pmatrix} a^2 & 0 \\ 0 & a^{-2}\end{pmatrix}.$$

Given ε near 1 we may choose η_1 near 1: then $H_{\eta_1\gamma}=H_{\eta\gamma}$ and $G^\sigma_{\varepsilon\delta}=\varphi(H_{\eta_1\gamma})$. *All we need to show is that* $\varphi^*(\omega'_\varepsilon(\mu))=a^2\omega'_\eta(\mu)$ at a unipotent ε in G^σ_δ, for then $\Theta(\varepsilon)\varphi^*(\omega_{\varepsilon\delta}(\mu))=a^2\theta(\eta)\omega_{\eta\gamma}(\mu)$, and at $\varepsilon=1$, $2\varphi^*(\omega_\delta(\mu))=\omega_\gamma(\mu)$.

Let O_η, O_{η_1}, O_ε be the conjugacy classes of η, η_1, ε. Since we have a commutative diagram

$$\begin{array}{ccc} H_{\eta_1\gamma}\backslash H_\gamma\cong O_{\eta_1} & \hookrightarrow & H_\gamma \\ \cong\downarrow\varphi & & \downarrow\varphi \\ G^\delta_{\varepsilon\delta}\backslash G^\sigma_\delta\cong O_\varepsilon & \hookrightarrow & G^\sigma_\delta \end{array},$$

the pullback $\varphi^*(\omega'_\varepsilon(\mu))$ of the form $\omega'_\varepsilon(\mu)$ on $G^\sigma_{\varepsilon\delta}$ is a form on $H_{\eta_1\gamma}$ defined by the function $\xi'\circ\varphi: H_\gamma\to\mathbb{G}_a$ and the form $\varphi^*(\omega^1_\delta)$ on H_γ. Note that if $\lambda(\eta_1)=\eta_1^a$, then

$$\xi'(\varphi(\eta_1))=\operatorname{tr}N(\varepsilon\delta)=\operatorname{tr}\eta\gamma=\operatorname{tr}(\eta_1^a\gamma)=\zeta'(\lambda(\eta_1)).$$

There is also a commutative diagram

$$\begin{array}{ccc} H_{\eta_1\gamma}\backslash H_\gamma\cong O_{\eta_1} & \hookrightarrow & H_\gamma \\ \cong\downarrow\lambda & & \downarrow\lambda \\ H_{\eta\gamma}\backslash H_\gamma\cong O_\eta & \hookrightarrow & H_\gamma \end{array},$$

hence $\varphi^*(\omega'_\varepsilon(\mu))=\lambda^*(\omega'_\eta(\mu))$. But

$$\lambda^*(\omega'_\eta(\mu))/\omega'_\eta(\mu)=\lambda^*(\varphi^*(\omega^1_\delta))/\varphi^*(\omega^1_\delta)=\theta(\eta)/\theta(\eta_1)=\frac{(1-\beta_1^{2a})(1-\beta_2^{2a})}{(1-\beta_1^2)(1-\beta_2^2)}$$

is equal to a^2 as $\beta_1\to 1$, as required.

3. Orbital Integrals

3.1. Let F be a local field. The highest degree invariant differential form ω_G determines a Haar measure on $G(F)$. A maximal degree invariant form ω_δ on G^σ_δ determines a measure on $G^\sigma_{\delta'}(F)$ for any δ' in $G(F)$ stably σ-conjugate to δ. The two forms determine a measure on the quotient $G^\sigma_{\delta'}(F)\backslash G(F)$. Let f be a smooth compactly supported function on $G(F)$, and put

$$\Phi_f(\delta)=\Phi_f(\delta;\omega_\delta,\omega_G)=\int_{G^\sigma_\delta(F)\backslash G(F)} f(g^{-1}\delta\sigma(g)).$$

If $N\delta\neq 1$ put

$$\Phi^{st}_f(\delta)=\Phi^{st}_f(\delta;\omega_\delta,\omega_G)=\sum_{\delta'}\Phi_f(\delta').$$

The sum is over a set of representatives for the σ-conjugacy classes in the stable σ-conjugacy class of δ. If $N\delta=1$ put

$$\Phi^{st}_f(\delta)=\sum_{\delta'}\kappa(\delta')\Phi_f(\delta').$$

If f_0 is a smooth compactly supported function on $H(F)$ define

$$\Phi_{f_0}(\gamma)=\Phi_{f_0}(\gamma;\omega_\gamma,\omega_H)=\int_{H_\gamma(F)\backslash H(F)} f_0(g^{-1}\gamma g),$$

and

$$\Phi^{st}_{f_0}(\gamma)=\Phi^{st}_{f_0}(\gamma;\,\omega_\gamma,\omega_H)=\sum_{\gamma'}\Phi_{f_0}(\gamma')\,.$$

If $\gamma=N\delta$ then there is $\varphi:H_\gamma\twoheadrightarrow G^\sigma_\delta$, and we take $\omega_\gamma=[\ker\varphi]^{-1}\varphi^*(\omega_\delta)$. If the functions f and f_0 satisfy the relation

$$\Phi^{st}_{f_0}(\gamma;\,\omega_\gamma,\omega_H)=\Phi^{st}_f(\delta;\,\omega_\delta,\omega_H)$$

for all γ, δ *with* $\gamma=N\delta$, we write $f_0=\lambda^*(f)$.

3.1.1 Proposition. *For each* f *there is* f_0 *with* $f_0=\lambda^*(f)$. *For each* f_0 *there is* f *with* $f_0=\lambda^*(f)$.

3.2. Applying partition of unity and translating, when passing from f to f_0 (resp. f_0 to f) we may assume that f (resp. f_0) is supported in a small neighborhood of a semi-simple element δ_0 (resp. γ_0). The lemma is proved by dealing with the various possible γ_0, δ_0. If δ_0 and γ_0 are such that $\gamma_0=N\delta_0$ is non-scalar then the proof is simple, and it remains to deal with $\gamma_0=-I$ and $\gamma_0=I$.

Suppose that $\gamma_0=-I$. Given f and η_1 in $H_{\gamma_0}(F)=H(F)$, put $\varepsilon=\varphi(\eta_1)$, and define [after choosing a section s of $G^\sigma_{\delta_0}(F)\backslash G(F)$ in $G(F)$]

$$f_0'(\eta_1)=\int_{G^\sigma_{\delta_0}(F)\backslash G(F)} f(g^{-1}\varepsilon\delta_0\sigma(g))\,\frac{\omega_G}{\omega_{\delta_0}}\,, \tag{3.2.1}$$

and

$$f_0(\eta\gamma_0)=f_0'(\eta_1)\qquad(\eta_1=\lambda^{-1}(\eta)) \tag{3.2.2}$$

if η is near 1; note that $\lambda:\eta_1\to\eta=\eta_1^a$ [see (2.10)] has an analytic inverse; put $f_0(\eta\gamma_0)=0$ otherwise. Note that $\varphi(H)=G^\sigma_{\delta_0}$, that $\varphi(H_{\eta_1'})=(G^\sigma_{\delta_0})_\varepsilon=G^\sigma_{\varepsilon'\delta_0}$ if η_1' is near 1 and $\varepsilon'=\varphi(\eta_1')$, and that $\omega_H=\omega_{\gamma_0}=\varphi^*(\omega_{\delta_0})$, $\omega_{\eta_1}=\varphi^*(\omega_{\varepsilon\delta_0})$, $\omega_\eta=\omega_{\eta_1}$, yields

$$\begin{aligned}\Phi^{st}_{f_0}(\eta\gamma_0;\,\omega_\eta,\omega_{\gamma_0})&=\sum_{\eta'}\int_{H_{\eta'}(F)\backslash H(F)} f_0(h^{-1}\eta'\gamma_0 h)\,\frac{\omega_{\gamma_0}}{\omega_\eta}\\
&=\sum_{\eta_1'}\int_{H_{\eta_1'}(F)\backslash H(F)} f_0'(h^{-1}\eta_1'h)\,\frac{\omega_{\gamma_0}}{\omega_{\eta_1}}=\Phi^{st}_{f_0'}(\eta_1;\,\omega_{\eta_1},\omega_{\gamma_0})\\
&=\sum_{\eta_1'}\int_{H_{\eta_1'}(F)\backslash H(F)}\int_{G^\sigma_{\delta_0}(F)\backslash G(F)} f(g^{-1}\varphi(h^{-1}\eta_1'h)\delta_0\sigma(g))\,\frac{\omega_{\gamma_0}}{\omega_{\eta_1}}\,\frac{\omega_G}{\omega_{\delta_0}}\\
&=\sum_{\varepsilon'}\int_{G^\delta_{\varepsilon'\delta_0}(F)\backslash G(F)} f(g^{-1}\varepsilon'\delta_0\sigma(g))\,\frac{\omega_G}{\omega_{\varepsilon\delta_0}}=\Phi^{st}_f(\varepsilon\delta_0;\,\omega_{\varepsilon\delta_0},\omega_G)\,.\end{aligned}$$

Here η is near 1, and η' ranges over a set of representative for the conjugacy classes within the stable conjugacy class of η. η' can be taken to be near 1; the same comment applies to η_1'. Then $\varepsilon'\delta_0$ ranges over a set of representatives for the σ-conjugacy classes within the stable σ-conjugacy class of $\varepsilon\delta_0$. Note that $\eta\gamma_0=N(\varepsilon\delta_0)$, so that f_0 is the desired function.

Conversely, given f_0 with support near γ_0, (3.2.2) defines f_0' for η_1 near 1, and f is defined by

$$f(s(g)^{-1}\varepsilon\delta_0\sigma(s(g)))=f_0'(\eta_1)\beta(g)\,,$$

where β is a smooth compactly supported function on $G^\sigma_{\delta_0}(F)\backslash G(F)$ with

$$\int_{G^\sigma_{\delta_0}(F)\backslash G(F)} \beta(g)dg=1\,.$$

3.3. Suppose that $\gamma=1$, and replace H by an inner form H' if necessary, so that $\varphi:H'\to G^\sigma_\delta$ is defined over F. Then $\varphi:H'(F)\to G^\sigma_\delta(F)$ is a local isomorphism and (3.2.1) defines a function f'_0 on $H'(F)$. If $\eta_1 \neq \pm I$ then φ restricted to $H_{\eta_1}=H'_{\eta_1}$ is not $\varphi_{\eta_1}:H_{\eta_1}\to(G^\sigma_\delta)_\varepsilon=G^\sigma_{\varepsilon\delta}$, but its square. Here we take η_1 near $\pm I$. Hence $\omega_{\eta_1}=\frac{1}{2}\varphi^*(\omega_{\varepsilon\delta_0})$; we have taken $\omega_{\gamma_0}=\frac{1}{2}\varphi^*(\omega_{\delta_0})$. As in (3.2) we have

$$\Phi^{st}_{f'_0}(\eta_1;\omega_{\eta_1},\omega_{\gamma_0})=\Phi^{st}_f(\varepsilon\delta_0;\omega_{\varepsilon\delta_0},\omega_G)\,.$$

Both sides are 0 when η_1 is not close to $\pm I$. Since $\lambda:\eta_1\mapsto\eta'=\eta_1^a$ $(a=4)$ has an analytic inverse on $H'(F)$ in a neighborhood of I, we may define a function f''_0 on $H'(F)$ by $f''_0(\eta')=f'_0(\eta_1)$. As is well-known, the orbital integrals of f''_0 can be transferred to $H(F)$. This is clear if H' is isomorphic to H over F. Otherwise there exists f_0 on $H(F)$ with

$$\Phi^{st}_{f_0}(\eta;\omega_\eta,\omega_H)=\Phi^{st}_{f''_0}(\eta';\omega_{\eta'},\omega_{H'})$$

when η in $H(F)$ is regular and corresponds to η' in $H'(F)$, and with

$$\Phi^{st}_{f_0}(\eta;\omega_\eta,\omega_H)=0$$

if η has distinct eigenvalues in $F^\times$ or it is a scalar multiple of a non-trivial unipotent. In this case $f_0(\pm I)=-f''_0(\pm I)$. This is the required f_0. The passage back from f_0 to f is done as in (3.2), but we have to choose δ_0 with $N\delta_0=I$ so that $\kappa(\delta_0)=1$.

3.4 Corollary. *If f, f_0 are compactly supported smooth functions on $G(F)$, $H(F)$ with*

$$\Phi^{st}_{f_0}(\gamma;\omega_\gamma,\omega_H)=\Phi^{st}_f(\delta;\omega_\delta,\omega_G)$$

for all $\gamma=N\delta$ with distinct eigenvalues, then $\lambda^*(f)=f_0$.

Proof. Choose f'_0 with $f'_0=\lambda^*(f)$. Then the stable orbital integrals of $f_0-f'_0$ are 0 on the regular semi-simple set, hence identically 0, since the germs of Φ^{st}_f at $u=\pm I$ are scalar multiples of $f_0(u)$ and $\Phi^{st}_{f_0}\left(u\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)$.

3.5. Analogous discussion has to be carried out for the transfer of functions from $G(F)$ to $H_1(F)=SO(3,F)$. If $\gamma=N_1\delta$ has eigenvalues $1,\gamma',\gamma''$ with $\gamma'\neq\gamma''$ put

$$\Phi^{\text{lab}}_f(\delta)=\Phi^{\text{lab}}_f(\delta,\omega_\delta,\omega_G)=\sum_{\delta'}\kappa(\delta')\Phi_f(\delta';\omega_\delta,\omega_G)\,.$$

If f_1 is a smooth compactly supported function on $H_1(F)$ then

$$\Phi_{f_1}(\gamma)=\Phi_{f_1}(\gamma;\omega_\gamma,\omega_{H_1})=\int_{H_{1\gamma}(F)\backslash H_1(F)} f_1(h^{-1}\gamma h)\,,$$

for all regular semi-simple γ. We say that $f_1=\lambda_1^*(f)$ if

$$\Phi_{f_1}(\gamma)=|(1+\gamma')(1+\gamma'')|^{1/2}\Phi^{\text{lab}}_f(\delta)$$

for all $\gamma=N_1\delta$ with distinct eigenvalues, where $\omega_\gamma=\varphi^*(\omega_\delta)$ and $\varphi:H_{1\gamma}\xrightarrow{\sim}G^\sigma_\delta$.

3.5.1 Proposition. *For each f there is f_1, and for each f_1 there is f, with $f_1=\lambda_1^*(f)$.*

This is easily verified for a function f with support near δ_0 and a function f_1 with support near γ_0, if $\gamma_0=N_1\delta_0$ has distinct eigenvalues, due to Lemma 2.4.1. The difficulty is when $N\delta_0$ is $-I$, for then there are several conjugacy classes in $H_1(F)$ of elements γ_0 with eigenvalues $1, -1, -1$. For each quadratic extension of F there is such γ_0 in $H_1(F)$ $\left(\text{with representative } \begin{pmatrix}0 & \theta\\ 1 & 0\end{pmatrix} \text{ in } GL(2,F), \theta \text{ in } F \text{ but not in } F^2\right)$. The lemma defines $\Phi_{f_1}(\gamma;\omega_\gamma,\omega_{H_1})$ at any γ in $H_1(F)$ with distinct eigenvalues; it is 0 unless the eigenvalues of γ are close to those of γ_0. It has to be shown that the function $\Phi_{f_1}(\gamma)$ is smooth at γ_0 to use the classification theorem of orbital integrals on $H_1(F)$ to deduce the existence of f_1. Namely, we have to establish the smoothness at γ_0 of the sum

$$\sum_{\varepsilon'}\kappa(\varepsilon'\delta)\Phi_f(\varepsilon'\delta_0;\omega_\delta,\omega_G)=\sum_{\eta_1'}\kappa(\varepsilon'\delta_0)\Phi_{f_0'}(\eta_1';\omega_{\eta_1},\omega_{\gamma_0})$$

of (3.2), multiplied by

$$|(1+\gamma')(1+\gamma'')|^{1/2}=|\gamma''|^{1/2}\,|1+\gamma'|\,.$$

Here $\varphi(\eta_1')=\varepsilon'$, $\varphi: H\to G_\delta^\sigma$, and the product is smooth by, e.g., Proposition 2 on p. 231 of [6], as required. Note that the eigenvalues γ', γ'^{-1} of $\gamma=N(\varepsilon\delta_0)$ are near -1.

3.6. It was noted above that there is a natural bijection between the conjugacy classes of γ in $H_1(F)$ with eigenvalues $1, -1, -1$ and the quotient $F^\times/F^{\times 2}$. The σ-conjugacy classes of δ in $G(F)$ with $N\delta$ equals the product of -1 and a non-trivial unipotent are also parametrized by $F^\times/F^{\times 2}$. The Hilbert symbol defines a pairing, which we denote by $\langle\gamma,\delta\rangle$.

3.6.1 Proposition. *If γ in $H_1(F)$ has eigenvalues $1, -1, -1$, and $f_1=\lambda_1^*(f)$, then*

$$\lim_{\gamma_1\to\gamma}|(1+\gamma_1')(1+\gamma_1'')|^{1/2}\Phi_{f_1}(\gamma_1;\omega_{\gamma_1}(\mu),\omega_{H_1})=\sum_\delta\langle\gamma,\delta\rangle\Phi_f(\delta;\omega_\delta(\mu),\omega_G)\,.$$

The sum is over σ-conjugacy classes of δ in $G(F)$ with $N\delta=-1$ times a non-trivial unipotent. The eigenvalues of γ_1 are $1, \gamma_1', \gamma_1''$.

Proof. As in (3.5) the expression on the left is

$$|(1+\gamma_1')(1+\gamma_1'')|^{1/2}\Phi_{f_1}(\gamma_1;\omega_{\gamma'}(\mu),\omega_{H_1})=\Phi_f^{\mathrm{lab}}(\delta_1;\omega_{\delta'}(\mu),\omega_G)$$

where $\delta_1=\varepsilon\delta_0$ and $N\delta_1=\gamma_1$. If $\varphi(\eta_1')=\varepsilon'$, $\varphi: H\tilde{\to}G_{\delta_0}^\sigma$, by Lemma 2.3.1 this is equal to (the sum is over the conjugacy classes η_1' in the stable class)

$$\sum_{\eta_1'}\kappa(\varphi(\eta_1')\delta_0)\Phi_{f_0'}(\eta_1';\omega_{\eta_1'}(\mu),\omega_H)\,.$$

η_1' is a regular element of $H(F)$, and lies in some torus $T(F)$.

The right side

$$\sum_{\{\delta;\, N\delta=-\mathrm{unip}\neq -I\}}\langle\gamma,\delta\rangle\Phi_f(\delta;\omega_\delta(\mu),\omega_G)$$

is equal to

$$\sum_{\eta_1}\langle\gamma,\varphi(\eta_1)\delta_0\rangle\Phi_{f_0'}(\eta_1;\omega_{\eta_1}(\mu),\omega_H)\,,$$

where $\delta=\varphi(\eta_1)\delta_0$, and the sum ranges over the non-trivial unipotent classes η_1 in $H(F)$. It suffices to show the equality of the two sums only for f supported on a small neighborhood of $\delta'=\varphi(\eta_1')\delta_0$, where δ' is close to $\delta=\varphi(\eta_1)\delta_0$, where η_1 is a non-trivial unipotent in $H(F)$.

So we may assume that

$$\delta_0=\begin{pmatrix}-1&&0\\&1&\\0&&1\end{pmatrix},\quad \delta=\begin{pmatrix}1&&x\\&1&\\0&&1\end{pmatrix}\delta_0,\quad \delta_1=\begin{pmatrix}\alpha&&\alpha x\\&1&\\\alpha\varepsilon&&\alpha\end{pmatrix}\delta_0,$$

where $x\in F^\times$, $\eta_1=\begin{pmatrix}1&x\\0&1\end{pmatrix}$, ε is near 0, $\eta_1'=\begin{pmatrix}\alpha&\alpha x\\\alpha\varepsilon&\alpha\end{pmatrix}$ where $\alpha^2(1-\varepsilon x)=1$ since $1-\varepsilon x\in F^{\times 2}$ as ε is small; we may assume that α is also a square, since it is close to 1. It has to be shown that: when $N\delta_1=\gamma_1\to\gamma$, and δ_1 is near δ, namely η_1' lies in the centralizer $H_\gamma(F)$ of γ in $H(F)$ $\left(\text{as } N\delta_1=\frac{-1}{\det\eta_1'}\eta_1'^2\right)$, and it is near η_1, then $\kappa(\delta')=\langle\gamma,\delta\rangle$. But

$$\tfrac{1}{2}[\delta' J+{}^t(\delta' J)]=\begin{pmatrix}x\alpha&&0\\&-1&\\0&&-\varepsilon\alpha\end{pmatrix},$$

hence $\kappa(\delta')=(x,-\varepsilon)$. The centralizer H_γ of γ splits over $F(\lambda)$ with $\lambda^2-c=0$ for some c in $F^\times$, hence $\langle\gamma,\delta\rangle=(c,x)$. But η_1' lies in H_γ only if $(\lambda-1)^2-\varepsilon x=0$ splits in $F(\lambda)$, namely if $\varepsilon x/c$ is a square in $F^\times$. Hence

$$\langle\gamma,\delta\rangle=(x,c)=(x,\varepsilon x)=(x,-\varepsilon)=\kappa(\delta_1),$$

as required.

3.7 Proposition. *If $\lambda_1^*(f)=f_1$ then $f_1(1)=|2|\sum\Phi_f(\delta)$, where the sum is over the σ-conjugacy classes of δ with $N\delta=1$. If $\gamma=N\delta$ is a non-trivial unipotent then*

$$(3.7.1)\qquad \Phi_{f_1}(\gamma;\omega_\gamma(\mu),\omega_{H_1})=|2|\Phi_f(\delta;\omega_\delta(\mu),\omega_G).$$

Proof. If $N\delta=1$ and f_0' is defined by (3.2.1) then

$$\Phi_f^{\mathrm{lab}}(\varepsilon\delta;\omega_{\varepsilon\delta},\omega_G)=\kappa(\delta)\Phi_{f_0'}(\eta_1;\omega_{\eta_1},\omega_H)$$

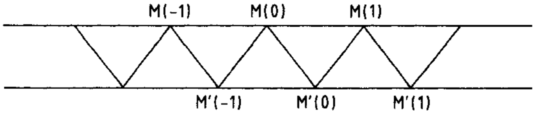
where $\varphi: H\to G^\sigma_{\delta_0}$, η_1 is near 1 with $\varphi(\eta_1)=\varepsilon$, hence $\kappa(\varepsilon\delta)=\kappa(\delta)$ by Lemma 1.8. The factor $|(1+\gamma')(1+\gamma'')|^{1/2}$ is smooth for γ' near 1, the asymptotic behavior permits the application of [10, Lemma 6.1], hence f_1 satisfies $f_1(1)=\kappa(\delta)|2|f_0'(1)$. When $\kappa(\delta)=1$ the right side of (3.7.1) is the limit of $(\Delta\Phi_{f_0'}(\eta_1)$ as $\eta_1\to 1$, and the left side is the corresponding limit of $\Delta\Phi_{f_1}$ as $N(\varepsilon\delta)=\varepsilon^2 N\delta=\varepsilon^2=\eta_1^4\to 1$; η_1 can be taken in the split set.

4. Unit of Hecke Algebra

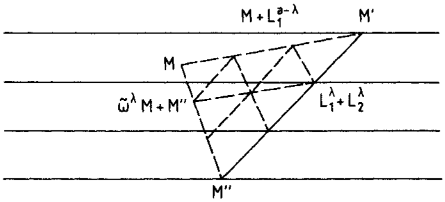
4.1. Let $\mathfrak{X}=\mathfrak{X}_F(G)$ denote the Bruhat-Tits building (see [11]) of $G=PGL(3)$ over the local non-archimedean field F of characteristic zero and odd residual characteristic. It is a simplical complex of dimension two. To describe its vertices let $\mathscr{R}$ be the ring of integers in F, and X the space of column 3-vectors. The set of vertices of $\mathfrak{X}$ is the quotient of the set of $\mathscr{R}$-lattices in X by the equivalence relation $M_1 \sim M_2$ if $M_1 = \lambda M_2$, λ in $F^\times$. Two vertices are joined by an edge if there are representatives M_1, M_2 with $\varpi M_1 \subsetneqq M_2 \subsetneqq M_1$. Here ϖ denotes a local uniformizer of the maximal ideal of $\mathscr{R}$. Three vertices form a two-simplex if there are representatives M_1, M_2, M_3 with

$$\varpi M_1 \subsetneqq M_3 \subsetneqq M_2 \subsetneqq M_1 .$$

Write X as a direct sum $X_1 \oplus X_2$ with $\dim_F X_i = i$. Let $\mathfrak{Y}$ denote the Bruhat-Tits building associated with X_2. It is a simplical complex of dimension 1, or the tree of $H = SL(2)$. A vertex of $\mathfrak{Y}$ is the equivalence class of an $\mathscr{R}$-lattice L_2 in X_2, two vertices are joined by an edge if there are representatives L_2, L_2' with $\varpi L_2 \subsetneqq L_2' \subsetneqq L_2$. If L_1^0 is any $\mathscr{R}$-lattice in X_1 then any lattice in X_1 has the form $\varpi^{-\lambda} L_1^0$. The lattices $M(\lambda) = \varpi^{-\lambda} L_1^0 + L_2$ (λ in $\mathbb{Z}$) define a line in $\mathfrak{X}$, identifying $\mathfrak{Z} = \mathfrak{Y} \times \mathbb{Z}$, or $\mathfrak{Y} \times$ line, with the set of vertices in $\mathfrak{X}$ with a representative M in X which satisfies $M = M \cap X_1 + M \cap X_2$. The convention when drawing diagrams will be to increase λ to the right. If $\varpi L_2 \subsetneqq L_2' \subsetneqq L_2$ put $M'(\lambda) = \varpi^{-\lambda} L_1^0 + L_2'$. The strip associated to the edge joining L_2' and L_2 is described by



Any vertex in $\mathfrak{X}$, represented by a lattice M, determines a unique pair in $\mathfrak{Z}$, represented by M' and M'', and a diagram



The equilateral triangle with vertices represented by M, M', M'' will be called the *characteristic triangle* of M; it lies in an apartment and its intersection with $\mathfrak{Z}$ is the segment, called the *characteristic segment* of M, from M' to M''. If Pr_1, Pr_2 are the projections of X on X_1, X_2 then $M' = \mathrm{Pr}_1 M + \mathrm{Pr}_2 M$ and $M'' = M \cap X_1 + M \cap X_2$. Put $L_1^0 = \mathrm{Pr}_1 M$; then $M \cap X_1 = \varpi^a L_1^0$ for some $a \geqq 0$. Since

$$\mathrm{Pr}_i M / M \cap X_i \cong M/(M \cap X_1 + M \cap X_2),$$

there is an isomorphism

$$\mathrm{Pr}_1 M / M \cap X_1 \cong \mathrm{Pr}_2 M / M \cap X_2,$$

and we denote by $L_2^\lambda (M \cap X_2 \subset L_2^\lambda \subset \mathrm{Pr}_2 M)$ the image of $L_1^\lambda = \varpi^\lambda L_1^0 (0 \leqq \lambda \leqq a)$. The base of the characteristic triangle consists of the vertices $L_1^\lambda + L_2^\lambda$ $(0 \leqq \lambda \leqq a)$. The vertices of the edge from M to M' are $M + L_1^{a-\lambda}$ $(0 \leqq \lambda \leqq a)$, and those from M to M'' are $\varpi^\lambda M + M''$ $(0 \leqq \lambda \leqq a)$. In fact elementary divisor theory yields the existence of three rank one $\mathscr{R}$-lattices N_1, N_2, N_3 with $M' = N_1 + N_2 + N_3$, $M = \varpi^a N_1 + N_2 + N_3$, $M'' = \varpi^a N_1 + \varpi^a N_2 + N_3$. The vertices in the characteristic triangle are $\varpi^{\lambda_1} N_1 + \varpi^{\lambda_2} N_2 + N_3$ $(0 \leqq \lambda_2 \leqq \lambda_1 \leqq a)$.

4.2. The group $G(F)$ acts on $\mathfrak{X}$, δ maps the lattice M to δM. $\mathfrak{Z}=\mathfrak{Z}(X_1,X_2)$ is mapped to $\mathfrak{Z}(\delta X_1,\delta X_2)$ and characteristic segments and triangles with respect to X_1,X_2 are mapped to such objects with respect to $\delta X_1,\delta X_2$. Note that an edge in $\mathfrak{X}$, from the vertex represented by M_1 to the vertex of M_2, has positive direction if the representatives are such that $\varpi M_2 \subsetneqq M_1 \subsetneqq M_2$ and M_2/M_1 is a module of rank one over the residue field. The positive direction on the lines in $\mathfrak{Z}$ is from left to right, and the action of δ preserves the direction.

Let σ denote the outer automorphism of $G(F)$ defined by $\sigma(g)=J{}^tg^{-1}J$, where $J=\begin{pmatrix}0&&1\\&-1&\\1&&0\end{pmatrix}$. To define its action on $\mathfrak{X}$ fix the bilinear pairing of X with itself given by $\langle x,y\rangle={}^tyJx$. Given M, the dual lattice is

$$M^\vee=\{y \text{ in } X;\ {}^tyJx \text{ in } \mathscr{R} \text{ for all } x \text{ in } M\}.$$

Then σ takes the vertex represented by M to the vertex of $M^\vee$. It takes $\mathfrak{Z}(X_1,X_2)$ to $\mathfrak{Z}(X_2^\perp,X_1^\perp)$, where $X_i^\perp$ is the orthogonal complement of X_i, but it reverses the directions on lines. The extremes vertices M', M'' in the characteristic segment of M are mapped to the extreme vertices $(\sigma M)'=(M'')^\vee$, $(\sigma M)''=(M')^\vee$ in the characteristic segment of $\sigma M=M^\vee$ with respect to $X_2^\perp$, $X_1^\perp$. The actions of $G(F)$ and σ are compatible and extend to an action of $G(F)\rtimes\langle\sigma\rangle=G'(F)$ on $\mathfrak{X}$.

Replacing δ in $G(F)$ by a σ-conjugate $g\delta\sigma(g^{-1})$ [g in $G(F)$], if necessary, we may assume that δ is of the form $(\alpha e)_1$, where

$$\alpha=\begin{pmatrix}a&b\\c&d\end{pmatrix},\quad e=\begin{pmatrix}-1&0\\0&1\end{pmatrix},\quad \alpha_1=\begin{pmatrix}a&0&b\\0&1&0\\c&0&d\end{pmatrix}.$$

Then $N\delta=\delta\sigma(\delta)=\left(\frac{-1}{\det\alpha}\alpha^2\right)_1$, and we consider here only δ with regular $\gamma=\frac{-1}{\det\alpha}\alpha^2$; thus γ is an element of $H(F)=SL(2,F)$ with distinct eigenvalues. We choose X_1 to be the space of column vectors $\{{}^t(0,x,0);\,x \text{ in } F\}$, and $X_2=\{{}^t(x,0,y);\, x,y \text{ in } F\}$. Then $\delta X_i=X_i$ and as $X_i^\perp=X_j\,(i\neq j)$ we have that $\mathfrak{Z}=\mathfrak{Z}(X_1,X_2)$ is stable under $\delta\times\sigma$. Also γ acts on the building $\mathfrak{Y}$ associated with X_2. As $\delta\times\sigma$ acts on $\mathfrak{Z}$ by transforming one line into another, reversing its direction, it acts on $\mathfrak{Z}$ too. The square of $\delta\times\sigma$ is $\delta\sigma(\delta)\times 1$, fixing each vector in X_1 and inducing on $\mathfrak{Y}$ the transformation γ.

4.3. Suppose that γ is regular elliptic, namely $E=F(\gamma)$ is a quadratic extension of F. In this case the σ-centralizer G_δ^σ is isomorphic to the centralizer H_γ of γ in H over F, and $G_\delta^\sigma(F)$ is compact. If f is the characteristic function of $K=G(\mathscr{R})$ divided by the volume $|K|$ of K, then its orbital integral is

$$\begin{aligned}\Phi_f(\delta)&=\int_{G_\delta^\sigma(F)\backslash G(F)} f(g^{-1}\delta\sigma(g))\frac{\omega_G}{\omega_\delta}\\&=|G_\delta^\sigma(F)|^{-1}\int_{G(F)} f(g^{-1}\delta\sigma(g))\omega_G\\&=|G_\delta^\sigma(F)|^{-1}\sum_{(\delta\times\sigma)P=P}1,\end{aligned}$$

where the sum is taken over the vertices P represented by $M=gM_0$ with $(\delta\times\sigma)P=P$. M_0 is a fixed vertex, say the lattice of vectors in X with integral entries. It is clear that $\Phi_f(\delta)$ depends only on the σ-conjugacy class of δ. There are two σ-conjugacy classes in the stable σ-conjugacy class of the above δ; if $\delta=(\alpha e)_1$ and $\delta_u=(u\alpha e)_1$ with a scalar u in $F^\times$, then δ, δ' are σ-conjugate if and only if u lies in $N_{E/F}E^\times$, where $E=F(\gamma)$. Note that all $\delta_u\times\sigma$ define the same action on $\mathfrak{Y}$, and $(\delta_u\times\sigma)^2=\gamma_1\times 1$ is independent of u. The stable σ-orbital integral of f at δ is the product of $|G_\delta^\sigma(F)|^{-1}$ and

$$\frac{1}{[U_0:U]}\sum_{u\in U_0/U}\sum_{i=0,1}\sum_{(\delta_{ui}\times\sigma)P=P}1. \tag{4.3.1}$$

Here we put $\delta_{ui}=(u\varpi^i\alpha e)_1$, and observe that each element of $F^\times$ can be expressed in the form $u\varpi^i$ for a unique unit u in $U_0=\mathscr{R}^\times$ and integer i, if ϖ is fixed; of course $\varpi^2=N_{E/F}\varpi$ as ϖ is in F. U is a compact open subgroup of $N_{E/F}E^\times$, u ranges over a set of representatives in U_0 for U_0/U, and each σ-conjugacy class is obtained $[U_0:U]$ times as the first two sums range over $2[U_0:U]$ values of u and i.

Remark. In [4] we take δ such that $G_{\delta'}^\sigma$ is split over F, where $\delta'=\frac{1}{2}(\delta+J^t\delta J)$, and study by completely different means the unstable σ-orbital integral

$$\frac{1}{[U_0/U]}\sum_{u\in U_0/U}\sum_{i=0,1}\sum_{(\delta_{ui}\times\sigma)P=P}\kappa(u),$$

where κ is the non-trivial character of $F^\times/N_{E/F}E^\times$.

4.4. If the lattice M represents a vertex P of $\mathfrak{Z}$ then it specifies a vertex $p=p(P)$ of $\mathfrak{Y}$. Let d be the maximum distance between two fixed points of γ, and take U to be in

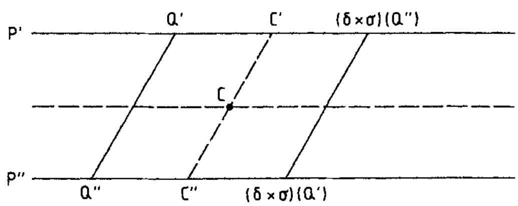
$$U_d=\{x \text{ in } \mathscr{R}^\times;\ x\equiv 1(\operatorname{mod}\varpi^d)\}.$$

Lemma. *Suppose p', p'' lie in $\mathfrak{Y}$ and $(\delta\times\sigma)p'=p''$, $(\delta\times\sigma)p''=p'$. Then* (a) *For each (u,i) there is at most one pair P', P'' in $\mathfrak{Z}$ with $p'=p(P')$, $p''=p(P'')$, $(\delta_{ui}\times\sigma)P'=P''$, $(\delta_{ui}\times\sigma)P''=P'$, such that P', P'' form the extreme vertices of a characteristic segment.*

(b) *The set of (u,i); u in U_0/U; $i=0,1$; for which P', P'' exists, consists of $[U_0/U]$ elements; the pair P', P'' is the same for all such u.*

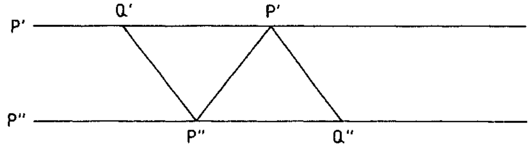
(c) *The number of pairs $((u,i);P)$ with (u,i) such that P', P'' exists, and $(\delta_{ui}\times\sigma)P=P$, and P has characteristic segment (P',P''), is $[U_0/U]$.*

Proof. (a) and (b) are verified on considering the diagram



The broken lines are reflected about the center point C by $\delta\times\sigma$, and P', P'' exist for $\delta\times\sigma$ if the points C', C'' are vertices. If C', C'' are not vertices they lie at midpoints of edges. Replacing $\delta\times\sigma$ by $\delta_u\times\sigma$ with $|u|=|\varpi|^\lambda$ shifts C a distance $\frac{1}{2}\lambda$ to the right, and (a), (b) follow.

To prove (c) we may assume by the proof of (a), (b) that P', P'' exists for $i=0$ and $\delta_{ui} \times \sigma = \delta_u \times \sigma$, u in U_0. We may choose three rank one $\mathscr{R}$-modules N_1, N_2, N_3 so that the vertex P' is represented by $N_1+N_2+N_3$ and P'' by $\varpi^k N_1+\varpi^k N_2+N_3$, and the vertices Q', Q'' are represented by $\varpi^k N_1+N_2+N_3$ and $N_1+\varpi^k N_2+N_3$.



A vertex P with characteristic segment (P', P'') is represented by a lattice $L+N_3$, where L is an $\mathscr{R}$-lattice in the space Z generated by $N=N_1+N_2$. L satisfies

(i) $$\varpi^k N \subsetneqq L \subseteq N,$$

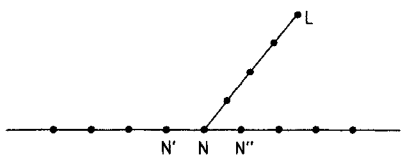
(ii) $$\varpi^{k-1} N \subsetneqq L,$$

(iii) $$L \subsetneqq \varpi N,$$

(iv) $$L \subsetneqq \varpi N_1 + N_2 = N',$$

(v) $$L \subsetneqq N_1 + \varpi N_2 = N''.$$

Namely L is at distance at most k from N, at least k from N, not in the direction of P'', or Q', or Q'', respectively. In the Bruhat-Tits tree of Z we have



The transformation $\delta \times \sigma$ acts on the set of P represented by $L+N_3$ with the above L, and so do the transformations $u_1 = \mathrm{diag}(1, u, 1)$ in $PGL(3, F)$, with u in U_0. The induced action on the set of lattices L is defined by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$ on the tree of Z. Thus U_0 acts transitively on the set of L, the stabilizer of any point being U_d. Hence for each P there exists u in U_0 with

$$P = u_1((\delta \times \sigma)P) = (\delta_u \times \sigma)P.$$

This u is uniquely determined modulo U_d, and (c) follows.

4.5. It is clear from Lemma 4.4 that the sum (4.3.1) is equal to the number of vertices p' in $\mathfrak{Y}$ which are fixed by γ, as a fixed point p' of γ is equivalent to an ordered pair p', p'' *in* $\mathfrak{Y}$ with $p''=(\delta \times \sigma)p'$, $p'=(\delta \times \sigma)p''$. If f_0 is the characteristic function of $K_0 = SL(2, \mathscr{R})$ divided by the volume $|K_0|$ of K_0, and $H_\gamma(F)$ is the (compact) centralizer of γ in $H(F)$, then the stable orbital integral of f_0 at γ is computed as usual to be the product of $|H_\gamma(F)|^{-1}$ and the integral over h in $PGL(2, F)$ of $f_0(h^{-1}\gamma h)$. This integral is the number of vertices p in $\mathfrak{Y}$ fixed by γ, and as we choose the measures with $|H_\gamma(F)| = |G_\delta^\sigma(F)|$ we deduce that:

Proposition. *We have*

$$\Phi_f^{st}(\delta) = \Phi_{f_0}^{st}(\gamma)$$

whenever $\gamma = N\delta$ *is regular elliptic and* f, f_0 *are the characteristic functions of* $K = G(R)$, $K_0 = H(R)$ *divided by their volumes.*

Concluding remarks. (1) Less elementary but conceptually clear, representation theoretic proofs of the existence assertions of Propositions 3.1.1 and 3.5.1, can now be given (see [0] for analogous cases), on using the elegant results of [7].

(2) The delicate germ computations in Sect. 3 here are not indispensible for the work of [1–5].

(3) Proposition 4.5 is crucial for [1–5]; its proof was suggested to me by R. Langlands. It would be interesting to find a conceptual, representation theoretic proof of this result, perhaps along the lines of [4].

(4) The proofs of this paper apply to any local field F of characteristic $\neq 2$. Alternatively, by virtue of Theorem A of [8] (see also the lines prior to Proposition 1 in [4]), our results can be transferred from the case of F with characteristic zero and residual characteristic $p > 0$, to the case of a local F with char $F = p$.

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