ON ZEROES OF THE TWISTED TENSOR L-FUNCTION

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A. Notations, results, remarks. Let E/F be a separable quadratic extension of global fields, $\mathbb{A} = \mathbb{A}_F$ and $\mathbb{A}_E$ the associated rings of adeles, and $\mathbb{A}^\times, \mathbb{A}_E^\times$ their multiplicative groups of ideles. Signify by $\underline{G}$ the group scheme $GL(n)$ over F, and put $G = \underline{G}(F)$, $G' = \underline{G}(E)$, $\mathbb{G} = \underline{G}(\mathbb{A})$, $\mathbb{G}' = \underline{G}(\mathbb{A}_E)$, and $Z(\simeq F^\times)$, $Z'(\simeq E^\times)$, $\mathbb{Z}(\simeq \mathbb{A}^\times)$, $\mathbb{Z}'(\simeq \mathbb{A}^\times)$, $\mathbb{Z}'(\simeq \mathbb{A}_E^\times)$ for their centers. Fix a unitary character ε of $\mathbb{Z}'/Z'$, and denote by π a cuspidal $\mathbb{G}'$-module whose central character is ε. Such a π is called *distinguished* if there is a form ϕ in π such that $\int_{\mathbb{Z}G\backslash\mathbb{G}} \phi(x)dx \neq 0$; clearly ε is trivial on $\mathbb{Z}$ if π is distinguished.

If $\underline{G}' = Res_{E/F}\underline{G}$ is the group obtained from $\underline{G}$ by restriction of scalars from E to F, then ($G' = \underline{G}'(F), \mathbb{G}' = \underline{G}'(\mathbb{A})$ and) its dual group $\widehat{\underline{G}}'$ is $[\underline{G}(\mathbb{C}) \times \underline{G}(\mathbb{C})] \rtimes Gal(E/F)$, where the non-trivial element σ of the Galois group $Gal(E/F)$ acts by permuting the two copies of $\underline{G}(\mathbb{C})$. As in [F1] the twisted tensor representation r of $\widehat{\underline{G}}'$ is defined on the n^2-dimensional complex space $\mathbb{C}^n \otimes \mathbb{C}^n$ by

$$(r(a,b))(x\otimes y) = ax\otimes by \text{ and } (r(\sigma))(x\otimes y) = y\otimes x \quad (a,b\in \underline{G}(\mathbb{C});\ x,y\in\mathbb{C}^n).$$

The irreducible admissible $\mathbb{G}'$-module π factorizes as a local product $\otimes_v \pi_v$ (v ranges over all F-places) of G'_v-modules π_v. Here F_v is the completion of F at v (we also write R_v for its ring of integers, $\underline{\pi} = \underline{\pi}_v$ for a generator of its maximal ideal, and q_v for the cardinality of $R_v/(\underline{\pi}_v)$, when v is non-archimedean), and $G_v = \underline{G}(F_v)$, $G'_v = \underline{G}(E_v)(= \underline{G}'(F_v))$.

For almost all F-places v the component π_v of π is unramified. If v stays prime in E, such π_v is determined by the semi-simple conjugacy class $t(\pi_v) = (z(\pi_v)\times 1)\times\sigma$ in $\widehat{\underline{G}}'_v = [\underline{G}(\mathbb{C}) \times \underline{G}(\mathbb{C})] \rtimes Gal(E_v/F_v)$, where $z(\pi_v)$ is the diagonal matrix whose eigenvalues $(z_i(\pi_v);\ 1\le i\le n)$ are the Hecke eigenvalues of the unramified G'_v-module π_v.

If v splits into v' and v'' in E, then $E_v = F_v \oplus F_v$, $G'_v = G_v \times G_v$, and $\pi_v = \pi_{v'} \times \pi_{v''}$ is determined by the semi-simple conjugacy class $t(\pi_v) = z(\pi_{v'}) \times z(\pi_{v''})$ in $\widehat{\underline{G}}'_v = \underline{G}(\mathbb{C}) \times \underline{G}(\mathbb{C})$, where $z(\pi_{v'})$ is a diagonal matrix whose diagonal entries $z_i(\pi_{v'})(1\le i\le n)$ are the Hecke eigenvalues of $\pi_{v'}$ (same for $\pi_{v''}$ and $z(\pi_{v''})$).

Correspondingly, as in [F1] we introduce the L-factors

$$L(s, r(\pi_v)\otimes\omega_v) = \det[I - q_v^{-s}\omega_v(\underline{\pi}_v) r(t(\pi_v))]^{-1},$$

where $\omega = \otimes\omega_v$ is a unitary character of $\mathbb{Z}/Z$ unramified at our v.

At a place which stays prime this L-factor is equal to

$$\prod_{1\le i\le n}(1 - q_v^{-s}\omega_v(\underline{\pi}_v) z_i(\pi_v))^{-1} \cdot \prod_{1\le j<k\le n}(1 - q_v^{-2s}\omega_v^2(\underline{\pi}_v) z_j(\underline{\pi}_v) z_k(\underline{\pi}_v))^{-1},$$

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while at v which splits in E it is

$$\prod_{1\leq i,j\leq n}(1-q_v^{-s}\omega_v(\underline{\pi}_v)z_i(\pi_{v'})z_j(\pi_{v''}))^{-1}=L(s,\omega_v\otimes\pi_{v'}\times\pi_{v''}).$$

Denote by V a set of F-places containing the archimedean places and those where E, ω or π ramify. The partial twisted tensor L-function is the infinite product

$$L^V(s,r(\pi)\otimes\omega)=\prod_{v\notin V}L(s,r(\pi_v)\otimes\omega_v),$$

which converges absolutely in some right half plane $Re(s)>>1$.

The local L-factors $L(s,r(\pi_v)\otimes\omega_v)$ can be introduced for non-archimedean $v\in V$ too as nowhere vanishing functions of the form $P(q_v^{-s})^{-1}$, where $P(X)$ is a polynomial in X with $P(0)=1$. The definition in the case of v which splits is given as in [JPS], Theorem 2.7, where $L(s,\pi_{1v}\times\pi_{2v})$ is defined. The definition and properties of these factors for a non-split v are analogously proven in the Appendix to this paper.

At the archimedean places the L-factors are the associated L-factors of the representation of the Weil groups which parametrize π_v (and so $r(\pi_v)\otimes\omega_v$). But the local functional equation has been proven in [JS1], Theorem 5.1, only in the split case, where $E_v=F_v\oplus F_v$. It will be interesting to extend the work of [JS1] to apply in the non-split case too.

We shall then assume that every archimedean place v of F splits in E. Under this assumption the complete L-function $L(s,r(\pi)\otimes\omega)$ is defined to be the product over all places of the local factors. We shall assume throughout this paper that if $\omega\neq 1$ then ω does not factorizes through $z\mapsto\nu(z)=|z|$, as this case can easily be reduced to the case of $\omega=1$. Indeed, if $\nu_E(x)=|x\overline{x}|^{1/2}(x\in\mathbb{A}_E^\times)$, then

$$L(s,r(\pi)\otimes\omega\nu^t)=L(s+t,r(\pi)\otimes\omega)=L(s,r(\pi\otimes\nu_E^{t/2})\otimes\omega).$$

For the same reason we may and will assume that the central character ε of π is trivial on $\mathbb{A}^\times$.

The work of [F1] then extends at once to show that $L(s,r(\pi)\otimes\omega)$ has analytic continuation to the entire complex s-plane with possible poles only at $s=0,1$. This L-function satisfies a functional equation relating s and $1-s$. These poles are at most simple, and occur precisely when $r(\pi)\otimes\omega$ is of the form $r(\pi')$, with a distinguished π'. By $r(\pi)\otimes\omega=r(\pi')$ we mean that $r(\pi_v)\otimes\omega_v=r(\pi'_v)$ for almost all v. See the Remark at the end of this paper concerning such ω and π.

Let $L(T)$ be a separable field extension of F of degree n. Its multiplicative group T is isomorphic over F to the group of F-points of an elliptic torus $\underline{T}$ over F of $\underline{G}$, thus $\underline{T}(F)=T$. The torus T is uniquely determined up to conjugacy in G, and its Lie algebra is isomorphic to $L(T)$, over F. Denote by ω_T the character $x\mapsto\omega(\det x)$ of $\mathbb{A}_{L(T)}^\times=\underline{T}(\mathbb{A})$, and by $L(s,\omega_T)$ the Hecke L-function associated with the character ω_T. Similarly we have $L(s,\omega)$. The function $L(s,\omega)$ has analytic continuation to the entire complex s-plane, with at most simple poles at $s=0,1$. These poles occur precisely when $\omega=1$.

Note that by class field theory ω can be identified with a character of the Weil group $W(\overline{F}/F)$, where $\overline{F}$ is a separable algebraic closure of F containing $L(T)$, and ω_T with the restriction to the subgroup $W(\overline{F}/L(T))$. The Hecke L-functions can be viewed as Artin L-functions associated with these Galois representations. The main result of this paper is the following.

1. Theorem. *Let π be a distinguished cuspidal $\mathbb{G}'$-module with a supercuspidal component, and ω a unitary character of $\mathbb{Z}/Z$. Let s_0 be a complex number such that for every separable field extension $L(T)$ of F of degree n, the L-function $L(s,\omega_T)$ vanishes at $s=s_0$ to the order m. Then $L(s,r(\pi)\otimes\omega)$ vanishes at $s=s_0$ to the order m.*

Note that if $L(s,\omega_T)$ vanishes at $s=s_0$, then $|Re\, s_0-\frac{1}{2}|<\frac{1}{2}$.

For $n=2$ the assumption on the $L(s,\omega_T)$ can be replaced by a single assumption about the vanishing of $L(s,\omega)$ at $s=s_0$, since for an abelian extension $L(T)/F$ one has the factorization $L(s,\omega_T)=\prod_{\zeta} L(s,\zeta\omega)$, where ζ runs through the characters of $\mathbb{A}^\times/F^\times N_{L(T)/F}\mathbb{A}^\times_{L(T)}$, or equivalently, by class field theory, of $Gal(L(T)/F)$. For $n>2$, and $\omega=1$, it is known that $L(s,\omega)$ divides $L(s,\omega_T)$ if $L(T)/F$ is a normal extension (see, e.g., [CF], p. 225), and also when the Galois group of the normal closure of $L(T)$ over F is solvable (see [W] for this and related results).

In general this divisibility follows from Artin's conjecture. Indeed, denote by $Ind_T^F\omega_T$ the representation of $W(\overline{F}/F)$ induced from the character ω_T of $W(\overline{F}/L(T))$. Then

$$L(s,\omega_T)=L(s,Ind_T^F(\omega_T))=L(s,\omega)L(s,\rho),$$

since $Ind_T^F(\omega_T)$ contains the character ω with multiplicity one (by Frobenius reciprocity); ρ is the quotient of $Ind_T^F\omega_T$ by ω. If ω is of finite order, it can be viewed as a character of $Gal(L/F)$ for some Galois field extension L of F containing $L(T)$. Then ω_T can be viewed as a character of the subgroup $Gal(L/L(T))$, and $Ind_T^F\omega_T$ is a representation of the finite group $Gal(L/F)$. Artin's conjecture for the group $Gal(L/F)$ asserts that $\mathrm{L}(s,\rho)$ is entire unless $\omega_T=1$ and $\omega\neq 1$, in which case $L(s,\rho)$ is holomorphic except at $s=0,1$, where it has a simple pole. In particular, when $n=3$ or $n=4$, and $\omega=1$, since Artin's conjecture is known for the symmetric groups S_3 and S_4, the vanishing of the L-function $L(s,\mathbb{1})$ associated with the trivial character $\mathbb{1}$ of $\mathbb{A}^\times/F^\times$ implies the vanishing of $L(s,\mathbb{1}_T)$ (at $s=s_0$, to the order m), for each extension $L(T)$ of F of degree 3 or 4, and the assumption of the Theorem 1 on the $L(s,\mathbb{1}_T)$ can be replaced by a single assumption on $L(s,\mathbb{1})$.

The work in this paper was motivated by an observation of the introduction to [F2]. Let $r(\rho)$ be the (finite dimensional) representation of the Weil group $W(\overline{F}/F)$ obtained on composing with the twisted tensor representation $r:\underline{\widehat{G}}'\to Aut(\mathbb{C}^n\otimes\mathbb{C}^n)$, a representation ρ which parametrizes (conjecturally) a distinguished representation of $\mathbb{G}'$ (ρ factorizes through a base change map b from the dual group $\widehat{U} = \underline{G}(\mathbb{C}) \rtimes W(\overline{F}/F)$ of the unitary group U in n variables associated with E/F). The formal observation in [F2] is that $r(\rho)$ contains a copy of the trivial representation of $W(\overline{F}/F)$; the fixed vector is written out in [F2]. Theorem 1 is an L-function reflection of the underlying representation theoretic fact.

The proof is based on integrating the kernel $K_\varphi(x, y)$ of the usual convolution operator $r(\varphi)$ on the space of cusp forms on $\mathbb{G}'$, against an Eisenstein series in x, over x and y in $\mathbb{Z}G\backslash\mathbb{G}$. The integral is expanded geometrically and spectrally. Theorem 1 is deduced from the resulting equality for a family of test functions. We can work in the context of $GL(n)$ with a general $n \geq 2$ since we use ideas which were previously constructive in developing a simple form of the trace formula (see, e.g., [FK] and [F3]), although we do not use the trace formula in this work.

For related results in the split case $E = F \oplus F$ and the adjoint representation L-function $L(s, \omega \otimes \pi \times \check{\pi})/L(s, \omega)$, see [Z], [JZ], in the context of $GL(2)$, and [F4] in the context of $GL(n)$.

B. Core identity. We shall work with the space $L(G')$ of complex valued functions ϕ on $G'\backslash\mathbb{G}'$ which satisfy (1) $\phi(zg) = \varepsilon(z)\phi(g)$ ($z \in \mathbb{Z}', g \in \mathbb{G}'$), (2) ϕ is absolutely square integrable on $\mathbb{Z}'G'\backslash\mathbb{G}'$. The group $\mathbb{G}'$ acts on $L(G')$ by right translation: $(r(g)\phi)(h) = \phi(hg)$. The action is unitary since ε is.

Definition. The function $\phi \in L(G')$ is called *cuspidal* if for each proper parabolic subgroup $\underline{P}'$ of $GL(n)$ over E with unipotent radical $\underline{N}'$ we have $\int \phi(ng)dg = 0$ ($n \in N'\backslash\mathbb{N}'$) for all $g \in \mathbb{G}'$.

Let r_0 be the restriction of r to the space $L_0(G')$ of cusp forms in $L(G')$. The space $L_0(G')$ decomposes as a direct sum with finite multiplicities of invariant irreducible unitary $\mathbb{G}'$-modules called *cuspidal* $\mathbb{G}'$-modules.

Denote by $C_c^\infty(G'_v, \varepsilon_v^{-1})$ the convolution algebra of complex valued functions φ_v on G'_v with $\varphi_v(g) = \varepsilon_v(z)\varphi_v(zg)$ ($z \in Z'_v, g \in G'_v$) which are compactly supported modulo Z'_v, smooth if v is archimedean and locally constant if not. Implicit is a choice of a Haar measure dg_v on G'_v/Z'_v. It is chosen to have that the product of the volumes $|K'_v/Z'_v \cap K'_v|$ over all F-places v converges. Here K'_v is the standard maximal compact subgroup of G'_v; when v is non-archimedean we have $K'_v = \underline{G}(R'_v)$, where R'_v is the ring of integers in E_v (R'_v is $R_{v'} \times R_{v''}$ if v splits into v', v'' in E, and R_v is the ring of integers in F_v). Denote by $\mathbb{H}_v$ the convolution algebra of K'_v-biinvariant functions in $C_c^\infty(G'_v, \varepsilon_v^{-1})$, and by φ_v^0 its unit element.

Denote by $C_c^\infty(\mathbb{G}', \varepsilon^{-1})$ the linear span of the products $\otimes\varphi_v$, $\varphi_v \in C_c^\infty(G'_v, \varepsilon_v^{-1})$ for all v, and $\varphi_v = \varphi_v^0$ for almost all v. Put $dg = \otimes dg_v$. The convolution operator $r(\varphi) = \int_{\mathbb{G}'/\mathbb{Z}'} \varphi(g)r(g)dg$ is an integral operator on $L(G)$ with the kernel $K_\varphi(x, y) = \sum \varphi(x^{-1}\gamma y)$ ($\gamma \in G'/Z'$).

Definition. (1) Denote by a bar the Galois action of $Gal(E/F)$ on E. For $g = (g_{ij}) \in GL(n, E)$, put $\overline{g} = (\overline{g}_{ij})$.
(2) The element γ of G' is called *r-elliptic* (resp. *r-regular*) if the element $\gamma\overline{\gamma}^{-1}$ of

G' is elliptic (resp. regular). The analogous definition holds in the local case with F_v, E_v, G'_v replacing F, E, G'.
(3) The function $\varphi \in C_c^\infty(\mathbb{G}', \varepsilon^{-1})$ is called *r-discrete* if for every x, y in $\mathbb{G}$ and γ in G' we have $\varphi(x\gamma y) = 0$ unless γ is r-elliptic r-regular.
(4) The elements γ, γ' in G' (resp. G'_v) are *r-conjugate* if there are x, y in G (resp. G_v) with $\gamma' = x\gamma y$.

Here "r-" is an abbreviation for "relatively-". Recall that δ in $G' = GL(n, E)$ (resp. $G'_v = GL(n, E_v)$) is called *regular* if its centralizer in $\underline{G}'$ (resp. G'_v) is an F-torus $\underline{T}'$ (thus $\underline{T}'(F) = T'$ is a torus in G') (resp. E_v-torus T'_v). Such δ is *elliptic* if it lies in a torus G' (resp T'_v) and $\mathbb{T}'/T'\mathbb{Z}'$ has finite volume (resp. T'_v/Z'_v is compact). Thus δ is elliptic regular if and only if it lies in no proper E-parabolic subgroup of G' (resp. E_v-parabolic subgroup of G'_v). The centralizer of an elliptic regular $\gamma \in G'$ is the multiplicative group of a field extension of E of degree n.

Consider the set $S = \{x \in G'; x\overline{x} = 1\}$. By [F2], Proposition 10, we have

2. Lemma. *(1) The map $G'/G \to S$, $x \mapsto x\overline{x}^{-1}$, is a bijection. It bijects the double coset GxG with the orbit $Ad(G)(x\overline{x}^{-1})$ under the adjoint action of G. (2) If $x, y \in S$ are conjugate by an element of G', then they are conjugate by an element of G.*

Note that the centralizer of $\gamma\overline{\gamma}^{-1}$ is defined over F since $x\gamma\overline{\gamma}^{-1}x^{-1} = \gamma\overline{\gamma}^{-1}$ implies $\overline{x}(\gamma\overline{\gamma}^{-1})^{-1}\overline{x}^{-1} = (\gamma\overline{\gamma}^{-1})^{-1}$. We obtain the following description of the r-regular r-conjugacy classes in G'.

2.1 Corollary. *Let $\{\underline{T}\}$ be a set of representatives for the G-conjugacy classes of (maximal) F-tori in $\underline{G}$, $T' = \underline{T}(E)$ the group of E-points on $\underline{T}$, and $T'^{,\text{r-reg}}$ the set of r-regular elements in T'. Denote by $W(T) = N_G(T)/T$ the Weyl group of T in G, and write $t' \sim t''$ for t', t'' in G' if there are $w \in W(T)$ and $t \in T$ with $wt'w^{-1} = tt''$. Then a set of representatives for the set of r-conjugacy classes of the r-regular elements in G' is given by the union over $\{T\}$ of the $T'^{,\text{r-reg}}/\sim$. A set of representatives for the subset of r-conjugacy classes of the r-regular r-elliptic elements in G' is given by the union over the set $\{T\}_{ell}$ of the elliptic tori in $\{T\}$, of the $T'^{,\text{ r-reg}}/\sim$.*

The kernel $K_\varphi(x, y) = \sum \varphi(x^{-1}\gamma y)$ ($\gamma \in G'/Z'$) can now be expressed as

$$\sum_{\{T\}_{ell}} \sum_{\gamma \in T'^{,\text{r-reg}}/Z'} \sum_{\delta_1 \in G/T} \sum_{\delta_2 \in N(T)\backslash G} \varphi(x^{-1}\delta_1\gamma\delta_2 y)$$

$$= \sum_{\{T\}_{ell}} [W(T)]^{-1} \sum_{\gamma \in T'^{,\text{r-reg}}/TZ'} \sum_{\delta_1 \in G/T} \sum_{\delta_2 \in G/Z} \varphi(x^{-1}\delta_1\gamma\delta_2 y), \tag{2.2}$$

for an r-discrete function φ.

Definition. The function $\varphi_v \in C_c^\infty(G'_v, \varepsilon_v^{-1})$ is called *r-discrete* if for every x, y in G_v and $\gamma \in G'_v$ we have $\varphi_v(x\gamma y) = 0$ unless γ is r-elliptic r-regular.

It is clear that $\varphi = \otimes \varphi_v$ is r-discrete if it has an r-discrete component. Indeed, an element $\delta \in G'$ is elliptic (resp. regular) if it is elliptic (resp. regular) in G'_v for some v.

This kernel will be integrated against an Eisenstein series in x. Identify $GL(n-1)$ with a subgroup of $GL(n)$ via $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$, and put $\underline{Q} = GL(n-1)\underline{N}$, where $\underline{N}$ is the unipotent upper triangular group. Let $S(F_v^n)$ be the space of smooth and rapidly decreasing (if v is archimedean), or locally constant compactly supported (otherwise) complex valued functions on F_v^n. Denote by Φ_v^0 the characteristic function of R_v^n in F_v^n if v is non-archimedean. Let $S(\mathbb{A}^n)$ be the linear span of the functions $\Phi = \otimes \Phi_v$, $\Phi_v \in S(F_v^n)$ for all $v, \Phi_v = \Phi_v^0$ or almost all v. Put $\underline{\varepsilon} = (0, \dots, 0, 1) \in \mathbb{A}^n$. The integral in

$$f(g,s) = \omega(\det g)|\det g|^s \int_{\mathbb{A}^\times} \Phi(a\underline{\varepsilon}g)|a|^{ns}\omega^n(a)d^\times a \tag{2.3}$$

converges absolutely, uniformly in compact subsets of $Re(s) \geq \frac{1}{n}$. The absolute value is normalized as usual, and ω is a unitary character of $\mathbb{A}^\times/F^\times$.

It follows from Lemmas (11.5), (11.6) of [GoJ] that the Eisenstein series

$$E(g,\Phi,\omega,s) = \sum f(\gamma g, s) \quad (\gamma \in ZQ\backslash G)$$

converges absolutely in $Re(s) > 1$. In [JS], (4.2), p. 545, and [JS2], (3.5), p. 7 (with a slight modification due to the position of ω here), it is shown that $E(g,\Phi,\omega,s)$ extends to a meromorphic function on $Re(s) > 0$, in fact to the entire complex s-plane with a functional equation $E(g,\Phi,\omega,s) = E({}^t g^{-1}, \widehat{\Phi}, \omega^{-1}, 1-s)$, where ${}^t g$ is the transpose of g, and $\widehat{\Phi}$ is the Fourier transform of the "Schwartz" function Φ (with respect to some additive character $\psi \neq 1$ of $\mathbb{A}/F$). Moreover $E(g,\Phi,\omega,s)$ is slowly increasing (with respect to some Siegel domain) in $g \in G\backslash\mathbb{G}$, and is holomorphic except possibly at $s = 0, 1$, where the pole is at most simple. Note that $f(g)$ and $E(g,s)$ are $\mathbb{Z}$-invariant.

3. Proposition. *For any character ω of $\mathbb{Z}^\times/F^\times$, Schwartz function Φ in $S(\mathbb{A}^n)$, and r-discrete function φ on $\mathbb{G}'$, for each field extension $L(T)$ of degree n of F there is an entire holomorphic function $A(\Phi,\omega,\varphi,L(T),s)$ in s in $\mathbb{C}$ such that*

$$\begin{aligned} &\iint_{(\mathbb{Z}G\backslash\mathbb{G})^2} K_\varphi(x,y)E(x,\Phi,\omega,s)dx\,dy \\ &= \sum_{L(T)} A(\Phi,\omega,\varphi,L(T),s)L(s,\omega_T) \end{aligned} \tag{3.1}$$

on $Re\, s > 1$. The sum over $L(T)$ ranges over a finite set (of field extensions $L(T)$ of degree n of F, up to isomorphism over F) depending on (the support of) φ.

Proof. Since $K_\varphi(x,y)$ is left G-invariant as a function in x (and in y), the first expression (on the left) of (3.1),

$$\iint_{(\mathbb{Z}G\backslash\mathbb{G})^2} K_\varphi(x,y)\cdot\sum_{\gamma\in ZQ\backslash G} f(\gamma x,s)dx\,dy,$$

is equal, in the domain of convergence of the series defining the Eisenstein series, to

$$\int_{\mathbb{Z}Q\backslash\mathbb{G}}(\int_{G\mathbb{Z}\backslash\mathbb{G}} K_\varphi(x,y)dy)f(x,s)dx.$$

Substituting (2.2) this is equal to

$$\int_{\mathbb{Z}Q\backslash\mathbb{T}} dx\cdot\int_{\mathbb{Z}G\backslash\mathbb{G}} dy\cdot\sum_{\{T\}_{ell}}[W(T)]^{-1}\sum_{\gamma\in T^{\prime,\text{r-reg}}/Z'T}\sum_{\delta_1\in G/T}\sum_{\delta_2\in Z\backslash G}\varphi(x^{-1}\delta_1\gamma\delta_2 y)f(x,s)$$

$$=\sum_{\{T\}_{ell}}[W(T)]^{-1}\sum_{\gamma\in T^{\prime,\text{r-reg}}/Z'T}\int_{\mathbb{Z}\backslash\mathbb{G}} dx\int_{\mathbb{Z}\backslash\mathbb{G}}\varphi(x^{-1}\gamma y)f(x,s)dy.$$

The last equality follows from the decomposition $G=QT$, $Q\cap T=\{1\}$, and the left Q-invariance of $f(x,s)$ as a function in x.

To justify the change of summations and integrations, note that given φ the sums over T and γ (in T'/TZ') are finite. Indeed, consider $x^{-1}\gamma\overline{\gamma}^{-1}x$. Its characteristic polynomial has rational coefficients (in F), which lie in a compact depending on the support of φ (the intersection of a discrete and a compact set if finite). Hence the sum over T is finite, as asserted in the proposition. Moreover, the sum over $\gamma\in T^{\prime,\text{r-reg}}/TZ'$ is finite. The T are elliptic since φ is r-discrete.

Now for any elliptic regular $\gamma\overline{\gamma}^{-1}$, if $x^{-1}\gamma\overline{\gamma}^{-1}x$ lies in the compact $supp\,\varphi$ in $\mathbb{G}'/\mathbb{Z}'$, then $x(\in\mathbb{G})$ lies in a compact of $\mathbb{T}\backslash\mathbb{G}$. Moreover, the function $\Phi(\underline{\varepsilon}tx)$ in $t\in\mathbb{T}$, is compactly supported, uniformly in x in a compact of $\mathbb{T}\backslash\mathbb{G}$. Hence x lies in a compact of $\mathbb{Z}\backslash\mathbb{G}$ if $\varphi(x^{-1}\gamma y)f(x,s)\neq 0$. But now $x^{-1}\gamma y$ lies in the compact $supp\,\varphi$, x lies in a compact, and γ in a finite set. Hence y lies in a compact of $\mathbb{G}/\mathbb{Z}$, our integrals are absolutely convergent, and the change of sums and integrals is justified.

Substituting now the expression (2.3) for $f(x,s)$, we obtain a sum over T and γ of the product of $[W(T)]^{-1}$ and

$$\iint_{(\mathbb{Z}\backslash\mathbb{G})^2}\varphi(x^{-1}\gamma y)f(x,s)dx\,dy=\int_{\mathbb{G}} dx\int_{\mathbb{Z}\backslash\mathbb{G}}\varphi(x^{-1}\gamma y)dy\cdot\omega(\det x)|\det x|^s\Phi(\underline{\varepsilon}x)$$

$$=\int_{\mathbb{T}\backslash\mathbb{G}}\int_{\mathbb{G}/\mathbb{Z}}\varphi(x^{-1}\gamma xy)dy\cdot\int_{\mathbb{T}}\Phi(\underline{\varepsilon}tx)\omega(\det tx)|\det tx|^s dt\,dx.$$

The inner integral over $\mathbb{T}$ is a "Tate integral" which defines the L-function $L(s,\omega_T)$. Note that the integral in x is taken over a compact in $\mathbb{T}\backslash\mathbb{G}$, and the integral over y ranges over a compact in $\mathbb{Z}\backslash\mathbb{G}$. The proposition follows.

C. Spectral analysis. There is another expression for the kernel $K_\varphi(x,y)$, which we proceed to describe in the special case where φ is cuspidal.

Definition. The function φ on $\mathbb{G}'$ is called *cuspidal* if for every x,y in $\mathbb{G}'$ and every proper F-parabolic subgroup $\underline{P}'$ of $\underline{G}'$, we have $\int_{\mathbb{N}'}\varphi(xny)dn = 0$, where $\mathbb{N}' = \underline{N}'(\mathbb{A})$ is the unipotent radical of the parabolic subgroup $\mathbb{P}' = \underline{P}'(\mathbb{A})$ of $\mathbb{G}'$.

For a cuspidal φ, the convolution operator $r(\varphi)$ factorizes through the projection on the space $L_0(G')$ of cusp forms. The kernel $K_\varphi(x,y)$ has then the spectral decomposition

$$K_\varphi(x,y) = \sum_\pi K_\varphi^\pi(x,y), \quad \text{where} \quad K_\varphi^\pi(x,y) = \sum_{\phi^\pi}(r(\varphi)\phi^\pi)(x)\overline{\phi}^\pi(y).$$

The π range over all cuspidal $\mathbb{G}'$-modules in $L_0(G')$. The ϕ^π range over an orthonormal basis consisting of $\mathbb{K}' = \prod_v K_v'$-finite vectors in π (K_v' is the standard maximal compact subgroup in G_v'). The ϕ^π are rapidly decreasing functions, and the sum over ϕ^π is finite for each φ (uniformly in x and y) since φ is $\mathbb{K}'$-finite. The sum over π converges in L^2, hence also in the space of rapidly decreasing functions. Hence $K_\varphi(x,y)$ is rapidly decreasing in x and y, and the product of $K_\varphi(x,y)$ with the slowly increasing function $E(x,\Phi,\omega,s)$ is integrable over $(\mathbb{Z}G\backslash\mathbb{G})^2$. Consequently (3.1) can be expressed in the form

$$\sum_\pi\sum_{\phi^\pi}\int_{\mathbb{Z}G\backslash\mathbb{G}}(r(\varphi)\phi^\pi)(x)E(x,\Phi,\omega,s)dx\cdot\int_{\mathbb{Z}G\backslash\mathbb{G}}\overline{\phi}^\pi(y)dy.$$

A cuspidal $\mathbb{G}'$-module which contains a vector ϕ^π whose integral over $\mathbb{Z}G\backslash\mathbb{G}$ is non-zero is called distinguished. Hence the sum over π ranges over the distinguished cuspidal $\mathbb{G}'$-modules only.

To prove Theorem 1 let s_0 be a complex number such that for every separable field extension $L(T)$ of F degree n, the L-function $L(s,\omega_T)$ vanishes at $s=s_0$ to the order $m\geq 1$. It is well-known that then $|Re(s_0)-\frac{1}{2}|<\frac{1}{2}$. It follows that (3.1) vanishes at $s=s_0$ to the order m, and thus for all $j(0\leq j\leq m)$ we have

$$\text{(3.2)} \qquad \sum_\pi\sum_{\phi^\pi}\int_{\mathbb{Z}G\backslash\mathbb{G}}(\pi(\varphi)\phi^\pi)(x)E^{(j)}(x,\Phi,\omega,s_0)dx\cdot\int_{\mathbb{Z}G\backslash\mathbb{G}}\overline{\phi}^\pi(y)dy = 0.$$

Here $E^{(j)}(*,s_0) = \frac{d^j}{ds^j}E(*,s)|_{s=s_0}$.

The test function φ is an arbitrary cuspidal discrete function on $\mathbb{G}'$, and our aim is to show the vanishing of a single summand in the last double sum over π and

ϕ^π. In fact, fix a cuspidal distinguished $\mathbb{G}'$-module π' whose component at some F-place v_2 is supercuspidal, for which Theorem 1 will be proven.

Let V be a finite set of F-primes, containing the archimedean primes and those where π', ω or E/F ramify. Consider $\varphi = \otimes\varphi_v$ (product over all F-places v) with $\varphi_v \in C_c^\infty(G'_v, \varepsilon_v^{-1})$ for all v, and $\varphi_v = \varphi_v^0$ (= the unit element in the Hecke algebra $\mathbb{H}_v$) for almost all v. For all $v \notin V$ the component φ_v is taken to be spherical, namely $\varphi_v \in \mathbb{H}_v$. Each of the operators $\pi_v(\varphi_v)$ ($v \notin V$) factorizes through the projection on the subspace $\pi_v^{K'_v}$ of K'_v-fixed vectors in π_v. This subspace is zero unless π_v is unramified, in which case $\pi_v^{K'_v}$ is one-dimensional. On this K'_v-fixed vector, the operator $\pi_v(\varphi_v)$ acts as the scalar $\varphi_v^\vee(t(\pi_v))$, where $\varphi_v^\vee$ denotes the Satake transform of φ_v. Put $\varphi^\vee(t(\pi^V))$ for the product over $v \notin V$ of $\varphi_v^\vee(t(\pi_v))$, $\pi_V(\varphi_V) = \underset{v\in V}{\otimes} \pi_v(\varphi_v)$, and $\pi^{\mathbb{K}',V}$ for the space of $\prod_{v\notin V} K'_v$-fixed vectors in π. Then (3.2) takes the form

$$\sum_{\{\pi;\pi^{\mathbb{K}',V}\neq 0\}} \varphi^\vee(t(\pi^V))a(\pi,\varphi_V,j,\Phi,\omega,s_0) = 0, \tag{3.3}$$

where

$$a(\pi,\varphi_V,j,\Phi,\omega,s) = \sum_{\phi^\pi} \int_{\mathbb{Z}G\backslash\mathbb{G}} (\pi_V(\varphi_V)\phi^\pi)(x)E^{(j)}(s,\Phi,\omega,s)dx \cdot \int_{\mathbb{Z}G\backslash\mathbb{G}} \overline{\phi}^\pi(y)dy. \tag{3.4}$$

The sum over π ranges over the set of distinguished cuspidal $\mathbb{G}'$-modules $\pi = \otimes\pi_v$ such that π_v is unramified outside V. The sum over ϕ^π ranges over those elements in the orthonormal basis of π which appears in (3.2), which are K_v-invariant and eigenfunctions of $\pi_v(\varphi_v)$ ($\varphi_v \in \mathbb{H}_v$, necessarily with the eigenvalues $t(\pi_v)$) as functions in $x \in G_v$, for any $v \notin V$. In particular, such ϕ^π factorizes as $\phi^\pi(x) = \phi_V^\pi(x_V)\prod_{v\notin V}\phi_v^{\pi_v}(x_v)$; here $\phi_v^{\pi_v}$ is a right K'_v-invariant function on G'_v whose value at 1 is $vol(K'_vZ'_v/Z'_v)^{-1}$ and which transforms under Z'_v via ε_v, which is an eigenfunction of the convolution operators $r(\varphi_v)$ ($\varphi_v \in \mathbb{H}_v$) with the eigenvalue $t(\pi_v)$.

A standard argument (see, e.g., Theorem 2 in [FK] in a more involved situation), based on the absolute convergence of the sum over π in (3.3), standard estimates on the Hecke parameters $t(\pi_v)$ of the unitary unramified $\pi_v (v \notin V)$, and the Stone-Weierstrass theorem, implies the following.

4. Proposition. *Let π be a cuspidal distinguished $\mathbb{G}'$-module which has a supercuspidal component, ω a unitary character of $\mathbb{Z}/Z$, and s_0 a complex number as in Theorem 1. Then for any j, Φ and a function φ_V for which φ is cuspidal and discrete with any choice of $\otimes\varphi_v(v \notin V)$, the sum (3.4) is zero.*

D. Constant term expanded. We shall now proceed to recall from [F1] the relation between the integral over x in (3.4) and the L-function $L(s, r(\pi) \otimes \omega)$.

First we need a lemma, and some notations. Let $\underline{\psi} \neq 1$ be a character of $\mathbb{A}/F$, and $\underline{\psi}'$ the character of $\mathbb{A}_E/(\mathbb{A}+E)$ defined by $\underline{\psi}'(x) = \underline{\psi}((x-\overline{x})/(x_0-\overline{x}_0))$ on $x \in \mathbb{A}_E$. Here x_0 is a fixed element of $E-F$, and – as usual – bar signifies the Galois action of $Gal(E/F)$. Denote by $\underline{\psi}'_v$ the component at an F-place v.

Definition. A G'_v-module π_v is called *generic* if $Hom_{N'_v}(\pi_v, \psi'_v) \neq \{0\}$, where ψ'_v is the character $n = (n_{ij}) \mapsto \underline{\psi}'(\sum_{1\leq i<n} n_{i,i+1})$ of the unipotent upper triangular subgroup N'_v of G'_v.

By [GK], or Corollary 5.17 of [BZ], a generic π_v embeds in the induced G'_v-module $Ind(\psi'_v; G'_v, N'_v)$. Moreover, the dimension of $Hom(\pi_v, Ind(\psi'_v))$ is at most one, equivalently the dimension of $Hom_{N'_v}(\pi_v, \psi'_v)$ is at most one.

Definition. If π_v is generic, denote by $W(\pi_v)$ its realization in $Ind(\psi'_v)$; $W(\pi_v)$ is called the *Whittaker model* of π_v.

Any component of a cuspidal $\mathbb{G}'$-module is generic. Since π_v is admissible, each Whittaker function in $W(\pi_v)$ is smooth (under right action of G'_v). Denote by $W(\pi)$ the linear span of the functions $W(x) = \prod_v W_v(x_v)$, where $W_v \in W(\pi_v)$ for all v, and W_v is the normalized (by $W_v^0(1) = 1$) unramified (right-K'_v-invariant) vector W_v^0 for all v outside V.

Given W in $W(\pi)$, the function $\phi_W(x) = \sum_{p\in N'\backslash Q'} W(px)$ is a cuspidal function in the space of $\pi \subset L_0(G')$, and the space of π is spanned by such ϕ_W. If π is distinguished, namely there is $\phi \in \pi$ with $\int_{\mathbb{Z}G\backslash\mathbb{G}} \phi(x)dx \neq 0$, then $\phi = \sum_i \phi_{W_i}$ and we conclude that π has a distinguished vector of the form ϕ_W.

Given a cusp form ϕ in π, consider the Whittaker function $W_\phi(x) = \int_{N'\backslash\mathbb{N}'} \phi(nx)\overline{\psi}'(n)dn$ in $W(\pi)$. Here $\overline{\psi}'(n) = \psi'(n^{-1})$. It is easy to see that if $\phi = \phi_W$, then W_ϕ is W. The following simple fact is used in [F1]; a proof is included here, since it was not given there.

5. Lemma. *Given $W = \otimes W_v$ in $W(\pi)$ and $\phi(x) = \sum_{p\in N'\backslash Q'} W(px)$, we have*

$$\int_{N\backslash\mathbb{N}} \phi(nx)dn = \sum_{p\in N\backslash Q} W(px).$$

Proof. Let ϕ be a cusp form. We first recall the proof of the expansion

$$\phi(x) = \sum_{p\in N'_n\backslash Q'_n} W_\phi(px).$$

The index (n here) signifies the size of the matrix, and prime means entries in E (rather than F). Embed G'_m in G'_n ($m \leq n$) via $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$, and denote by

V_n' the unipotent radical of the parabolic subgroup of G_n' of type $(n-1,1)$. For $v=(v_{ij})$ in $\mathbb{V}_n'$ put $\psi'(v)=\underline{\psi}'(v_{n-1,n})$, and consider

$$F_\phi(p)=\int_{V_n'\backslash\mathbb{V}_n'}\phi(vp)\psi'(v)dv.$$

Since ϕ is cuspidal, only non-trivial characters of $\mathbb{V}_n'/V_n'$ need be considered here. These make a single orbit under the action of G_{n-1}'. The stabilizer of $v\mapsto\psi'(v)$ is $G_{n-2}'V_{n-1}'$. Hence we have the Fourier expansion

$$\phi(e)=\sum_{p_{n-1}\in G_{n-2}'V_{n-1}'\backslash G_{n-1}'}F_\phi(p_{n-1}).$$

Now F_ϕ is a cusp form on $\mathbb{G}_{n-1}'$. Hence by induction on n we have

$$F_\phi(p)=\sum_{p_{n-2}\in G_{n-3}'V_{n-2}'\backslash G_{n-2}'}\cdots\sum_{p_1\in G_1'}W_{F_\phi}(p_1p_2\cdots p_{n-2}p).$$

But $W_{F_\phi}(x)=W_\phi\begin{pmatrix}x&0\\0&1\end{pmatrix}$. Hence

$$\phi(e)=\sum_{p_{n-1}}\sum_{p_{n-2}}\cdots\sum_{p_1}W_\phi(p_1p_2\cdots p_{n-2}p_{n-1})=\sum_{p\in N_n'\backslash Q_n'}W_\phi(p),$$

as required.

To prove the lemma, using the same notations we consider

$$\phi_{N_n}(x)=\int_{N_n\backslash\mathbb{N}_n}\phi(mx)dm.$$

Since

$$\phi_{N_{n-1}}(v)=\sum_{p_{n-1}'\in G_{n-2}'V_{n-1}'\backslash G_{n-1}'}F_{\phi_{N_{n-1}}}(p_{n-1}'v),$$

the integral

$$\phi_{N_n}(e)=\int\limits_{V_n\backslash\mathbb{V}_n}\phi_{N_{n-1}}(v)dv$$

is equal to

$$\int_{V_n\backslash\mathbb{V}_n}\sum_{p_{n-1}'}F_{\phi_{N_{n-1}}}(p_{n-1}'v)dv=\sum_{p_{n-1}}F_{\phi_{N_{n-1}}}(p_{n-1}).$$

The last sum ranges over $p_{n-1}\in G_{n-2}V_{n-1}\backslash G_{n-1}$ since

$$\int_{V_n\backslash\mathbb{V}_n}\psi'(p_{n-1}'vp_{n-1}'^{\,-1})dv\neq 0$$

implies that $p'_{n-1} \in G'_{n-1}$ must lie in G_{n-1}. Now

$$F_{\phi_{N_{n-1}}}(p_{n-1}) = \int_{V'_n \backslash \mathbb{V}'_n} \phi_{N_{n-1}}(v_n p_{n-1}) \psi'_n(v_n) dv_n,$$

and by induction we have

$$\phi_{N_{n-1}}(x) = \sum_{p_{n-2} \in N_n \cap G_{n-2} \backslash G_{n-2}} W_{\phi, \psi'_{n-1}}(p_{n-2} x).$$

The index of ψ'_n indicates that it is a character on the group $\mathbb{N}'_n$. Substituting we obtain

$$\int_{V'_n \backslash \mathbb{V}'_n} \sum_{p_{n-2}} W_{\phi, \psi'_{n-1}}(p_{n-2} v_n p_{n-1}) \psi'_n(v_n) dv_n.$$

But given p_{n-2} and v_n there is v'_n in $\mathbb{V}'_n$ with $\psi'_n(v_n) = \psi'_n(v'_n)$ and $p_{n-2} v_n = v'_n p_{n-2}$. By definition of W_{ϕ, ψ'_n}, the last displayed expression can be expressed as

$$(F_{\phi_{N_{n-1}}}(p_{n-1}) =) \sum_{p_{n-2}} W_{\phi, \psi'_n}(p_{n-2} p_{n-1}).$$

We conclude that

$$\phi_{N_n}(e) = \sum_{p_{n-1}} \sum_{p_{n-2}} W_{\phi, \psi'_n}(p_{n-2} p_{n-1}) = \sum_{p \in N_n \backslash Q_n} W_{\phi, \psi'_n}(p),$$

as required.

E. L-functions emerge. We can now return to the integral over x in (3.4) and the fundamental identity of [F1] which expresses it as an L-function. Thus we take $W = \otimes W_v$ in $W(\pi)$ with $W_v = W_v^0 (\in W(\pi_v))$ for all $v \notin V$, such that the cuspidal function $\phi(x) = \sum_{P \in N' \backslash Q'} W(px)$ in the space of $\pi \subset L_0(G')$ is distinguished (its integral over the closed subspace $\mathbb{Z}G \backslash \mathbb{G}$ is non-zero). Substituting the series definition of $E(x, \Phi, \omega, s) = \sum_{ZQ \backslash G} f(\gamma x, s)$, we obtain

$$\int_{\mathbb{Z}G \backslash \mathbb{G}} \phi(x) E(x, \Phi, \omega, s) dx = \int_{\mathbb{Z}Q \backslash \mathbb{G}} \phi(x) f(x, s) dx$$

$$= \int_{Q \backslash \mathbb{G}} \phi(x) \Phi(\underline{\varepsilon} x) \omega(\det x) |\det x|^s dx = \int_{Q \mathbb{N} \backslash \mathbb{G}} \Phi(\underline{\varepsilon} x) |\det x|^s \omega(\det x) \int_{N \backslash \mathbb{N}} \phi(nx) dn\, dx$$

$$= \int_{Q \mathbb{N} \backslash \mathbb{G}} \Phi(\underline{\varepsilon} x) \omega(\det x) |\det x|^s [\sum_{N \backslash Q} W_\phi(px)] dx = \int_{\mathbb{N} \backslash \mathbb{G}} \Phi(\underline{\varepsilon} x) |\det x|^s \omega(\det x) W_\phi(x) dx,$$

using Lemma 5. Choosing $\Phi \in S(\mathbb{A}^n)$ to be factorizable, namely $\Phi(x) = \prod \Phi_v(x_v)$ with $\Phi_v \in S(F_v^n)$ for all v (with $\Phi_v = \Phi_v^0$ for $v \notin V$), since $W_\phi(x) = \prod_v W_v(x_v)$ the last integral is a product over v of the local integrals

$$\int_{N_v \backslash G_v} \Phi_v(\underline{\varepsilon} x) |\det x|_v^s \omega_v(\det x) W_v(x) dx. \tag{5.1}$$

When $W_v = W_v^0$ and $\Phi_v = \Phi_v^0$ (and π_v, ω_v and E_v/F_v are unramified), the last local integral is shown in [F1], Proposition, p. 305, on using Schur function computations, to be equal to $L(s, r(\pi_v) \otimes \omega_v)$ (in [F1], ω_v is taken to be 1, but the general case follows on adjusting the computations there). At the "bad" non-archimedean places $v \in V$, where ramification may occur, the following is shown in the Appendix below, in analogy with the split case – where E is replaced by $F \oplus F$ – which is studied in [JPS], Theorem 2.7, pp. 390-393, 395-398.

First, the integral (5.1) is a rational function in q_v^{-s}. Second, there is a polynomial $P(x; \pi_v, \omega_v)$ with constant term 1 over $\mathbb{C}$, such that the $\mathbb{C}$-span of the integrals (5.1), as Φ_v ranges over $S(F_v^n)$ and W_v over $W(\pi_v)$, is precisely the principal fractional ideal $L(s, r(\pi_v) \otimes \omega_v)\mathbb{C}[q_v^s, q_v^{-s}]$ in the fraction field $\mathbb{C}(q_v^s)$ of the ring $\mathbb{C}[q_v^s, q_v^{-s}]$ of polynomials in q_v^s and q_v^{-s}. Here $L(s, r(\pi_v) \otimes \omega_v)$ is $P(q_v^{-s}; \pi_v, \omega_v)^{-1}$, and t is referred to as the "greatest common denominator", or "g.c.d.", of all the integrals (5.1). The quotient of (5.1) by $L(s, r(\pi_v) \otimes \omega_v)$ satisfies a functional equation $s \leftrightarrow 1 - s$.

In the archimedean case, let ρ_v' be the representation of the Weil group which parametrizes π_v. Define $L(s, r(\pi_v) \otimes \omega_v)$ to be the L-factor $L(s, r(\rho_v) \otimes \omega_v)$ associated with the representation $r(\rho_v) \otimes \omega_v$ of the Weil group of $F_v (= \mathbb{R}$ or $\mathbb{C})$. The local integral (5.1) and its quotient by $L(s, r(\rho_v) \otimes \omega_v)$, and the local functional equation, are studied in [JS1], Theorem 5.1, in the case when v splits in E.

We shall assume that each archimedean place v of F splits in E. Then the total L-function $L(s, r(\pi) \otimes \omega)$ is defined as the product over all v of the local factors. The product, as well as the integrals and sums in the fundamental identity leading to (5.1) above, converge absolutely in some right half plane. Since $E(x, \Phi, \omega, s)$ is holomorphic except at $s = 0$ and 1 (for a suitable Φ and ω; see [F1], Lemma, p. 301), the total L-function $L(s, r(\pi) \otimes \omega)$ is entire except possibly for at most simple poles at $s = 0$ and 1 if $r(\pi) \otimes \omega = r(\pi')$, and π' is distinguished.

Proof of Theorem 1. Let π be a cuspidal distinguished $\mathbb{G}'$-module with a supercuspidal component at v_2, ω a unitary character of $\mathbb{Z}/Z$, and s_0 a complex number such that (3.4) is 0 at $s = s_0$ for all j $(0 \leq j \leq m)$, Φ and φ_V (with discrete cuspidal $\varphi = \varphi_V \otimes (\bigotimes_{v \notin V} \varphi_v))$. In (3.4), V is a finite set of F-places containing the archimedean places, and those where π, ω or E/F ramify. Fix a distinguished factorizable automorphic form $\phi' = \otimes \phi_v'$ in the space of $\pi \subset L_0(G)$, which is K_v'-invariant for all $v \notin V$.

The space of vectors ϕ in $\pi \subset L_0(G')$ which are K_v'-invariant for all $v \notin V$ is spanned by the factorizable, thus $\phi(x) = \prod_v \phi_v(x_v)$, such vectors. Given such a

$\phi = \otimes\phi_v$, our aim is (in particular) to choose a function φ_v such that φ be cuspidal and r-discrete, and $\pi(\varphi_V)\phi' = \phi$.

At v_2 consider the matrix coefficient $\varphi'_{v_2}(x) = \langle \pi_{v_2}(x^{-1})\phi'_{v_2}, \phi_{v_2}\rangle$ of the supercuspidal G'_{v_2}-module π_{v_2}. Note that ϕ_{v_2} and ϕ'_{v_2} are functions in $C_c^\infty(G'_v, \varepsilon_v)$, and $\langle\cdot,\cdot\rangle$ denotes the natural inner product. The function φ'_{v_2} lies in $C_c^\infty(G'_v, \varepsilon_v^{-1})$, and it is a supercusp form ($\int \varphi'_{v_2}(xny)dn = 0$, $n \in N'_{v_2}$ = unipotent radical of any proper parabolic subgroup of G'_{v_2}). A function $\varphi = \otimes\varphi_v$ whose component at a place, say v_2, is a supercusp form, is cuspidal. By the Schur orthogonality relations, the convolution operator $\pi_{v_2}(\varphi'_{v_2})$ maps the vector ϕ'_{v_2} to (a multiple of) ϕ_{v_2}, and any vector orthogonal to ϕ'_{v_2} is mapped to 0. Working with $\varphi = \otimes\varphi_v$ whose component at v_2 is φ'_{v_2} we then have that φ is cuspidal, and that the component of the $\pi_V(\varphi_V)\phi^\pi$ which occurs in (3.4) at v_2 is ϕ_{v_2}.

Put $V'' = V - \{v_2\}$, let v_1 be an F-place in V'', say where π and ω are unramified, W_v is W_v^0 and Φ_v is Φ_v^0, and $\phi_{v_1} = \phi'_{v_1} = \phi^0_{v_1}$, and put $V' = V'' - \{v_1\}$. For each $v \in V'$ there is a congruence subgroup K''_v of K'_v such that both ϕ'_v and ϕ_v are right K''_v-invariant. Namely both ϕ'_v and ϕ_v are non-zero vectors in the finite dimensional space $\pi_v^{K''_v}$ of K''_v-fixed vectors in π_v. The Hecke algebra $\mathbb{H}(K''_v)$ of K''_v-biinvariant functions in $C_c^\infty(G'_v, \varepsilon_v^{-1})$ generate the algebra of endomorphisms of the finite dimensional space $\pi_v^{K''_v}$. Consider $\varphi'_v \in \mathbb{H}(K''_v)$ such that $\pi_v(\varphi'_v)$ maps ϕ'_v to ϕ_v, and any vector orthogonal to ϕ'_v (is mapped by $\pi_v(\varphi'_v)$) to 0. Choosing $\varphi = \otimes\varphi_v$, with $\varphi_v = \varphi'_v$ for all $v \in V'$, we conclude that any automorphic form ϕ^π which may contribute a non-zero term to (3.4), has the component ϕ'_v for all $v \neq v_1$. But ϕ^π is automorphic, and $\mathbb{G}' = G' \prod_{v \neq v_1} G'_{v_1}$, hence ϕ^π is uniquely determined to be ϕ'. The vector $(\underset{v\in V'}{\otimes} \pi_v(\varphi'_v))\phi'$ has the component ϕ_v for every $v \neq v_1$. Since it is automorphic, the same argument implies that $\pi_{V'}(\varphi_{V'})\phi' = \phi$.

We still need to choose the component φ_{v_1} of φ in such a way that φ be r-discrete. Note that we choose v_1 to be a place where π, ω and E/F are unramified, and the components ϕ'_{v_1} of ϕ' and ϕ_{v_1} of ϕ are both equal to the (normalized) K'_v-fixed vector $\phi^0_{v_1}$ in π_{v_1}.

Recall that the function $\varphi_{v_1} \in C_c^\infty(G'_{v_1}, \varepsilon_{v_1}^{-1})$ is called r-discrete if it is supported on the r-regular r-elliptic set of G'_{v_1}. Also, a function $\varphi = \otimes\varphi_v$ whose component at v_1 is r-discrete is necessarily r-discrete. It suffices to choose an r-discrete φ_{v_1} whose support is contained in $Z'_vK'_v$, and which is constant on the intersection of its support with K'_v. Suitably normalized we have that $\pi_{v_1}(\varphi_{v_1})\phi'_{v_1} = \phi_{v_1}$ for such φ_{v_1}.

We conclude that the only non-zero summand in (3.4) is the one indexed by ϕ'. Since $\phi = \pi_V(\varphi_V)\phi'$ is arbitrary, and for a suitable such ϕ the integral over x in (3.4) is equal to $L^{(j)}(s, r(\pi)\otimes\omega)$, we conclude that (3.4) is a non-zero multiple of $L^{(j)}(s, r(\pi)\otimes\omega)$ (for some choice of φ_V and Φ). Here $L^{(j)}(s)$ denotes the jth derivatives of $L(s)$. However, Proposition 4 asserts that (3.4) vanishes at $s = s_0$. Hence $L(s, r(\pi)\otimes\omega)$ vanishes at $s = s_0$ to the order m under the conditions of Theorem 1, whose proof is now complete.

Remark. By [F1] and the following Appendix, the L-function $L(s, r(\pi) \otimes \omega)$ is everywhere holomorphic except possibly at $s = 0$, 1, where it has a simple pole if $r(\pi) \otimes \omega = r(\pi')$ and π' is distinguished. But $L(s, \omega)$ has a simple pole at $s = 0, 1$ when $\omega = 1$. Hence Theorem 1 implies (for a distinguished π and a character ω which satisfies its assumptions) that the "twisted adjoint" function $L(s, r(\pi) \otimes \omega)/L(s, \omega)$ is holomorphic everywhere except possibly at $s = 0, 1$. There it has a simple pole precisely when $\omega \neq 1$ and $r(\pi) \otimes \omega = r(\pi')$ with a distinguished cuspidal π'. The last identity is meant in the local sense, for almost all places.

Suppose that ω and π are such that the poles do occur. Let Ω be a character of $\mathbb{A}_E^\times/E^\times$ whose restriction to $\mathbb{A}^\times/F^\times$ is ω. Then $r(\pi) \otimes \omega = r(\pi \otimes \Omega)$. Since $r(\pi') = r(\pi \otimes \Omega)$, by [F1], Corollary on p. 310, $\pi \otimes \Omega$ is also distinguished. Then π is distinguished and $\int_{\mathbb{Z}G\backslash\mathbb{G}} \phi(x)\omega(x)dx \neq 0$ for some $\phi \in \pi$; hence ω has order n. Such π can be studied along lines suggested by the conjecture and techniques of [F2]. In particular, when ω is primitive of order n, it is associated by class field theory to a cyclic extension T of F, and it is likely that the associated π are parametrized by the non-trivial characters of $\mathbb{T}'^\times/T'^\times\mathbb{T}^\times$, where $T' = T \otimes_F E$.

Put $\pi^*(g) = \check{\pi}(\overline{g})$. By [F2], Proposition 12, if π is distinguished then $\pi \simeq \pi^*$. If $\pi \otimes \Omega$ is also distinguished, then $(\pi \otimes \Omega)^* \simeq \pi \otimes \Omega$. Altogether we have $\pi \simeq \pi^* \simeq \pi \otimes (\Omega/\Omega^*) = \pi \otimes (\Omega\overline{\Omega})$, and $\omega|F^\times N_{E/F}\mathbb{A}_E^\times$ is of order dividing n. If $\omega|F^\times N_{E/F}\mathbb{A}_E^\times$ is primitive of order n, namely $T' = T \otimes_F E$ is a field, then by [K] the G'-module π is parametrized by a character θ of $\mathbb{T}'^\times/T'^\times$, and $\pi(\theta)^* = \pi(\overline{\theta}^{-1})$, where the last bar indicates the non-trivial automorphism of T' over T. Hence θ is trivial on $T'^\times N_{T'/T}\mathbb{T}'^\times$, but we suggest above that π is likely to be parametrized by the θ on $\mathbb{T}'^\times/T'^\times$ which are trivial on $\mathbb{T}^\times$.

Appendix: On the local twisted tensor L-function

Let E/F be a quadratic separable extension of global fields, π an irreducible cuspidal representation of $GL(n, \mathbb{A}_E)$, and r the twisted tensor representation $r : [GL(n, \mathbb{C}) \times GL(n, \mathbb{C})] \rtimes Gal(E/F) \to Aut(\mathbb{C}^n \otimes \mathbb{C}^n)$ of [F1]. Let V be a finite set of places of F, containing the archimedean places and those where E/F or π ramify. The partial twisted tensor L-function $L^V(s, r(\pi))$ is defined to be the product over all $v \notin V$ of the local L-factors $L(s, r(\pi_v))$. The product converges absolutely in some half-plane $Re(s) > c$. When each archimedean place of F splits in E, this $L^V(s, r(\pi))$ is shown in [F1] to be holomorphic on $Re(s) \geq 1$, except when π is $GL(n, \mathbb{A})$-distinguished, in which case a simple pole occurs, on $Re(s) = 1$ (at $s = 1$ if the central character ω_π of π is trivial on $\mathbb{A}^\times$).

At a place v which splits in E we have $E_v = F_v \oplus F_v$ and $GL(n, E_v) = GL(n, F_v) \times GL(n, F_v)$, and the component π_v of π is of the form $\pi_{1v} \times \pi_{2v}$. The local L-factor $L(s, r(\pi_v))$ is simply the tensor product L-function $L(s, \pi_{1v} \times \pi_{2v})$. This last factor was introduced by [JPS], Theorem 2.7, in the non-archimedean case and by [JS1], Theorem 5.1, in the archimedean case, for all generic π_{iv}, not necessarily unramified.

Let $\underline{\psi}_v$ be a non-trivial character of F_v (in $\mathbb{C}^\times$), and $\psi_v(u) = \underline{\psi}_v(\sum_{1 \leq i \leq n} u_{i,i+1})$, where $u = (u_{i,j}) \in N_v$, a character of the unipotent upper triangular subgroup N_v of $G_v = GL(n, F_v)$. Denote by $W(\pi_{iv}, \psi_v)$ the Whittaker ψ_v-model of π_{iv}, and for $W_{1v} \in W(\pi_{1v}, \psi_v)$, $W_{2v} \in W(\pi_{2v}, \psi_v^{-1})$ and $\Phi_v \in C_c^\infty(F_v^n)$, and with $\underline{\varepsilon} = (0, \dots, 0, 1) \in F_v^n$, put

$$\Psi(s, W_{1v}, W_{2v}, \Phi_v) = \int_{N_v \backslash G_v} W_{1v}(g) W_{2v}(g) |\det g|^s \Phi_v(\underline{\varepsilon} g) dg.$$

It is shown in [JPS], [JS1] that the quotient $\Psi(s, W_{1v}, W_{2v}, \Phi_v)/L(s, \pi_{1v} \times \pi_{2v})$ satisfies a functional equation where, in particular, s, π_{1v}, π_{2v} are replaced by $1-s$, and the contragredients $\check{\pi}_{1v}, \check{\pi}_{2v}$.

The purpose of this appendix is to introduce the twisted tensor L-factor $L(s, r(\pi_v))$ for any (possibly ramified) quadratic separable extension E_v/F_v of local non-archimedean fields, and any generic representation π_v of $G'_v = GL(n, E_v)$.

Let $\underline{\psi}'_v$ be a non-trivial character of E_v which is trivial on F_v. Note that $E_v/F_v \simeq F_v$. Any such character is of the form $\underline{\psi}'_v(x) = \underline{\psi}_v((x - \overline{x})/(x_0 - \overline{x}_0))$, where $x \in E_v$ and the action of $Gal(E_v/F_v)$ on E_v is denoted by a bar, for a fixed $x_0 \in E_v - F_v$. Then a character ψ'_v of the unipotent upper triangular subgroup N'_v of $G'_v = GL(n, E_v)$, which is trivial on N_v, is defined as before. Denote by $W(\pi_v, \psi'_v)$ the ψ'_v-Whittaker space of π_v, and for $W_v \in W(\pi_v, \psi'_v)$ and $\Phi_v \in C_c^\infty(F_v^n)$ consider the integral

$$\Psi(s, W_v, \Phi_v) = \int_{N_v \backslash G_v} W_v(g) |\det g|^s \Phi_v(\underline{\varepsilon} g) dg.$$

When $E_v/F_v, \psi_v$ and π_v are unramified, W_v is the unit element W_v^0 of $W(\pi_v, \psi'_v)$, and Φ_v is the characteristic function Φ_v^0 of R_v^n, R_v being the ring of integers in F_v, it is shown in [F1] that $\Psi(s, W_v^0, \Phi_v^0) = L(s, r(\pi_v))$. In analogy with [JPS] we shall introduce $L(s, r(\pi_v))$ for a general π_v as a generator of some fractional ideal (generated by the $\Psi(s, W_v, \Phi_v)$), and show that the quotient $\Psi(s, W_v, \Phi_v)/L(s, r(\psi_v))$ satisfies a functional equation, in which s and π_v are replaced by $1-s$ and the contragredient $\check{\pi}_v$.

Having defined the local L-factor for all non-archimedean places (it is defined in [JS1] for the archimedean places which split in E), the complete L-function $L(s, r(\pi))$ can be defined as the product over all places v of F of the L$(s, r(\pi_v))$, for E/F in which each archimedean place of F splits in E. The global functional equation for the global integrals $\Psi(s, W, \Phi)$ of [F], together with the local functional equations of [JPS] and [JS1] in the split cases, and the one of this note in the non-split non-archimedean case, implies the existence of a monomial $\varepsilon(s, r(\pi)) = c(\pi) e^{\varepsilon(\pi) s}$ in s ($\varepsilon(\pi)$ in $\mathbb{C}$, $c(\pi)$ in $\mathbb{C}^\times$), and the functional equation $L(s, r(\pi)) = \varepsilon(s, r(\pi)) L(1-s, r(\check{\pi}))$ for the twisted tensor L-function.

Moreover it is shown in [F1] that $\Psi(s, W, \Phi)$ is holomorphic in $s \in \mathbb{C}$ except possibly for a simple pole at $s = 1$ and $s = 0$ (when π is $GL(n, \mathbb{A})$-distinguished, whose central character is trivial on $\mathbb{A}^\times$). Since the local work shows that $L(s, r(\pi_v))$ is a sum of $\Psi(s, W_v, \Phi_v)$'s, it follows that $L(s, r(\pi))$, which is initially defined in some right half plane, has analytic continuation to the entire complex s-plane with at most two poles, at $s = 0, 1$, which are simple and occur precisely for distinguished π. The work here also replaces the (complicated proof of the) Lemma on p. 306 of [F1]. It is this function $L(s, r(\pi))$ which is studied in the paper preceding this appendix.

From now on we can use local notations, thus let E/F be a quadratic separable extension of local non-archimedean fields, put $G = GL(n, F)$, $G' = GL(n, E)$, let N, N' be the corresponding unipotent upper-triangular subgroups, and ψ, ψ' their characters, π a generic irreducible G'-module with a unitary central character, and $W(\pi, \psi')$ its ψ'-Whittaker model (for any irreducible G'-module there exists at most one (non-zero) ψ'-Whittaker model; π is called generic when it exists). Denote by R the ring of integers in F, and by q the cardinality of its residue field. The purpose of this appendix is to prove the following.

Theorem. *(i) For each $W \in W(\pi, \psi')$ and $\Phi \in C_c^\infty(F^n)$, the integral $\Psi(s, W, \Phi)$ is absolutely convergent for a large $Re(s)$ to a rational function of $X = q^{-s}$.*

(ii) There exists a polynomial $P(X) \in \mathbb{C}[X]$ with $P(0) = 1$ such that the integrals $\Psi(s, W, \Phi)$ span the fractional ideal $L(s, r(\pi))\mathbb{C}[X, X^{-1}]$ of the ring $\mathbb{C}[X, X^{-1}]$, where $L(s, r(\pi)) = P(X)^{-1}$.

(iii) There exists an integer $m(\pi, \psi')$ and a non-zero complex number $c(\pi, \psi')$, such that

$$\Psi(1 - s, \widetilde{W}, \widehat{\Phi})/L(1 - s, r(\check{\pi})) = \omega_\pi(-1)^{n-1}\varepsilon(s, r(\pi), \psi')\Psi(s, W, \Phi)/L(s, r(\pi))$$

for all $W \in W(\pi, \psi)$, $\Phi \in C_c^\infty(F^n)$. Here ω_π is the central character of $\pi, \check{\pi}$ the contragredient of π, and we put

$$\varepsilon(s, r(\pi), \psi') = c(\pi, \psi')X^{m(\pi,\psi')},\ \widetilde{W}(g) = W(J^t g^{-1}),\ \widehat{\Phi}(x) = \int_{F^n} \Phi(y)\psi(tr\, x{\cdot}y)dy.$$

Here $J \in G$ is the matrix whose non-zero entries are 1, located on the anti-diagonal.

Proof. The proof of (i) and (ii) is similar to that of (i), (ii) in [JPS], Theorem 2.7. Since W and Φ are smooth, using the Iwasawa decomposition we obtain a finite sum

$$\Psi(s, W, \Phi) = \sum_i \int_A W_i(a)\Phi_i(a_n)\delta_B^{-1}(a)|\det a|^s d^\times a.$$

Here A is the diagonal subgroup of G and $B = AN$, and $W_i \in W(\pi, \psi')$, $\Phi_i \in C_c^\infty(F)$. We put

$$a = diag(a_1 a_2 \dots a_n, a_2 \dots a_n, \dots, a_{n-1}a_n, a_n).$$

By [JPS1] there exists a finite set $\Xi = \Xi(\pi, \psi')$ of finite functions ξ on A' (continuous functions whose translates span a finite dimensional vector space), such that for every $W \in W(\pi, \psi')$ there are $\phi_\xi \in C_c^\infty(E^{n-1})$ with

$$W(a) = \sum_{\xi\in\Xi} \phi_\xi(a_1, a_2, \dots, a_{n-1})\xi(a) \qquad (a \in A').$$

Hence

$$|W(a)| \leq \sum_{\xi \in \Xi^+} \phi_\xi^+(a_1, \ldots, a_{n-1})\xi(a)$$

where now Ξ^+ is a finite set of finite functions on A' which take non-negative real values, and ϕ_ξ^+ in $C_c^\infty(E^{n-1})$ are ≥ 0. Each function ϕ_ξ^+ is bounded by a finite sum of positive-valued quasi-characters.

Then $\Psi(s, W, \Phi)$ is a finite sum of terms

$$\int_A \phi(a_1, \ldots, a_{n-1}, a_n)\xi(a)\delta_B^{-1}(a)|\det a|^s d^\times a$$

with $\phi \in C_c^\infty(F^n)$ and ξ in a fixed finite set Ξ of finite functions on A. Each product $(\xi\delta_B^{-1})(a)$ is a finite sum of products $\eta_1(a_1)\ldots\eta_n(a_n)$, where each η is a finite function on $F^\times$ in a fixed finite set. We obtain a finite sum of the integrals

$$(*) \qquad \prod_{1\leq i\leq n} \int_{F^\times} \phi_i(a_i)\eta_i(a_i)|a_i|^{is} d^\times a_i \qquad (\phi_i \in C_c^\infty(F)).$$

Replacing the ϕ_i and η_i by their absolute values it follows that $\Psi(s, W, \Phi)$ is absolutely convergent for large $Re\,(s)$. Each factor in $(*)$ is a sum of geometric series in X which converge to $Q_i(X)(1 - \alpha_i X^{k_i})^{-1}$, where $Q_i \in \mathbb{C}[X]$ and α_i, k_i depend only on η_i. Hence $\Psi(s, W, \Phi)$ is a rational function of X as asserted in (i), with a common denominator independent of W, Φ.

The subspace of the field $\mathbb{C}(X)$ generated by these fractions is an ideal for the ring $\mathbb{C}[X, X^{-1}]$. Indeed, if $W_h(g) = W(gh), \Phi_h(x) = \Phi(xh)$, then $\Psi(s, W_h, \Phi_h)$ is the product of $|\det h|^{-s}$ and $\Psi(s, W, \Phi)$.

It is easy to see (as in [F1], Proposition, (ii) on p. 308, which is proven on the middle of p. 309), that $\Psi(s, W, \Phi)$ is identically 1 for a suitable choice of W, Φ. Hence the ideal contains 1 and admits a unique generator of the form $P(X)^{-1}$, with $P \in \mathbb{C}[X]$ and $P(0) = 1$, as asserted in (ii).

For (iii), note first that if $\varepsilon(s, r(\pi), \psi')$ exists, then it is necessarily a monomial. Indeed, applying the asserted functional equation with $(s, \check{\pi}, \psi'^{-1}, \widetilde{W}, \widehat{\Phi})$ replacing $(1 - s, \pi, \psi', W, \Phi)$, and noting that $\omega_{\check{\pi}}\omega_\pi = 1$, we obtain

$$\Psi(s, W, \Phi)/L(s, r(\pi)) = \omega_\pi(-1)^{n-1}\varepsilon(1-s, r(\check{\pi}), \psi'^{-1})\Psi(1-s, \widetilde{W}, \widehat{\Phi})/L(1-s, r(\check{\pi})).$$

Combining this with the equation of (iii), we conclude that the product of $\varepsilon(s, r(\pi), \psi')$ and $\varepsilon(1-s, r(\check{\pi}), \psi'^{-1})$, both in $\mathbb{C}[X, X^{-1}]$, is 1. Hence $\varepsilon(s, r(\pi), \psi') = c(\pi, \psi')X^{m(\pi,\psi')}$, as asserted.

From its integral representation (for large $Re(s)$), we obtain

$$\Psi(s, \pi(g)W, \rho(g)\Phi) = |\det g|^{-s}\Psi(s, W, \Phi),$$

where $\rho(g)\Phi(x) = \Phi(xg)$. The identity

$$(\rho(g)\Phi)\widehat{\ } = |\det g|^{-s}\rho({}^t g^{-1})\widehat{\Phi}$$

implies

$$\Psi(1 - s, (\pi(g)W)\widetilde{\ },\ (\rho(g)\Phi)\widehat{\ }) = |\det g|^{-s}\Psi(s, \widetilde{W}, \widehat{\Phi}).$$

Then (iii) follows at once from the following.

Proposition. *With the exception of finitely many values of $X = q^{-s}$, the space of bilinear forms B on $W(\pi,\psi') \times C_c^\infty(F^n)$ which satisfy*

$$B(\pi(g)W, \rho(g)\Phi) = |\det g|^{-s} B(W,\Phi)$$

is at most one dimensional.

As in [JPS], (iii) of Theorem 2.7, our proof relies heavily on results of [BZ] (and [BZ1]). Denote by δ_G the modular function on an ℓ-group G, thus δ_G^{-1} is the Δ_G of [BZ], Prop. 1.19. Let H be a closed subgroup of G. Denote by $ind(\rho; G, H)$ the unnormalized induction with compact supports of [BZ], and by $i(\rho; G, H) = ind(\rho\delta_H^{1/2}\delta_G^{-1/2}; G, H)$ the normalized induction with compact supports of [BZ1]. Denote by $Ind(\rho; G, H)$ and $I(\rho; G, H)$ the unnormalized and normalized induction with arbitrary supports of [BZ] and [BZ1]. Then $i(\rho)^\vee = I(\rho^\vee)$ by [BZ], Prop. 2.25(c). The space $Bil_G(\pi_1, \pi_2)$ of bilinear forms B on $\pi_1 \times \pi_2$ (π_i are G-modules) which satisfy $B(\pi_1(g)v_1, \pi_2(g)v_2) = B(v_1, v_2)$ ($v_i \in \pi_i, g \in G$) is isomorphic to $Hom_G(\pi_1, \check{\pi}_2)$. Frobenius reciprocity ([BZ], Theorem 2.28) asserts

$$Bil_G(\pi, i(\rho)) = Hom_G(\pi, I(\rho^\vee)) = Hom_H(\pi, \rho^\vee\delta_H^{1/2}\delta_G^{-1/2}) = Bil_H(\pi, \rho\delta_G^{1/2}/\delta_H^{1/2}).$$

Returning to our usual notations ($G = GL(n,F)$, etc.), let P be the group of $g \in G$ with $\underline{\varepsilon} g = \underline{\varepsilon}$ (a prime will always indicate the same group with E instead of F, thus P' is defined using G'). Then $F^n - \{\underline{0}\} = P\backslash G$. The space of $\Phi \in C_c^\infty(F^n)$ with $\Phi(\underline{0}) = 0$ is isomorphic to

$$C_c^\infty(F^n - \{\underline{0}\}) = C_c^\infty(P\backslash G) = i(\delta_P^{-1/2}; G, P), \quad \delta_P\begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix} = |\det g| \quad (g \in G_{n-1});$$

we write G_m for $GL(m,F)$. Define the character $\nu : E^\times \to \mathbb{C}^\times$ by $\nu(x) = |x\overline{x}|^{1/2}$. Then $\Psi(s, W, \Phi)$ ($W \in W(\pi,\psi'), \Phi \in C_c^\infty(F^n), \Phi(\underline{0}) = 0$) defines an element in

$$Bil_G(\pi \otimes \nu^s, i(\delta_P^{-1/2}; G, P)) = Bil_P(\pi \otimes \nu^s, \delta_P^{-1}) = Bil_P(\pi \otimes \nu^{s-1}, \mathbb{1}),$$

where $\mathbb{1}$ denotes the trivial P-module on the space $\mathbb{C}$, namely a P-invariant form on $\pi \otimes \nu^{s-1}$. The main step in the proof of the proposition is to establish the following

Main Lemma. *With the exception of finitely many values of $X = q^{-s}$, the dimension of $Bil_P(\pi \otimes \nu^{s-1}, \mathbb{1})$ is at most one.*

For each $j (0 \le j < n)$, put $H'_j = G'_j N'$, where G'_j embeds in $G' = G'_n$ via $g \mapsto \begin{pmatrix} g & 0 \\ 0 & I \end{pmatrix}$. Then H'_j consists of $\begin{pmatrix} g & x \\ 0 & u \end{pmatrix}$, $g \in G'_j$, $u \in N'_{n-j}$. Given a G'_j-module ρ and the character ψ' of N'_{n-j}, denote by $\rho \otimes \psi'$ the H'_j-module on the space of ρ on which $\begin{pmatrix} g & x \\ 0 & u \end{pmatrix}$ acts by $\rho(g)\psi'(u)$. Corollary 5.13 of [BZ] asserts:

Lemma 1. *For every irreducible admissible G'_j-module ρ, the induced P'-module $ind(\rho\otimes\psi'; P', H'_j)$ is irreducible. Every irreducible admissible P'-module ξ is equivalent to one of the form $ind(\rho\otimes\psi'; P', H'_j)$, where ρ is an admissible irreducible G'_j-module, uniquely determined (so is j) by ξ.*

For any P'-module V, denote by $V^*_{\psi'} = Hom_{N'}(V, \psi'^{-1}) = Bil_{N'}(V, \psi')$ the space of linear forms $\lambda : V \to \mathbb{C}$ with $\lambda(\pi(u)v) = \psi'(u)\lambda(v)$ $(u \in N', v \in V)$. Since π is generic and irreducible (as a G'-module), the uniqueness of the ψ'-Whittaker model $W = W(\pi, \psi')$ implies that $\dim_{\mathbb{C}} W^*_{\psi'} = 1$. Also one has

Lemma 2. *If ξ is an irreducible admissible P'-module, then $\dim_{\mathbb{C}} \xi^*_{\psi'} \leq 1$, with equality precisely when $\xi \simeq \tau$, where $\tau = ind(\psi'; P', N')$.*

Proof. Indeed, $\xi^*_{\psi'} = Bil_{N'}(\xi, \psi') = Bil_{P'}(\xi, ind(\psi'; P', N'))$ by Frobenius reciprocity, and τ is irreducible by Lemma 1.

Corollary 5.22 of [BZ] establishes the following.

Lemma 3. *The restriction $Res_{P'}\pi$ of π to P' has finite length (as a P'-module). Thus there exist a decomposition $W(\pi, \psi') = \bigcup_{0\leq i\leq I} W_i$, $W_{i+1} \supseteq W_i, W_0 = \{0\}$, W_i is stable under P' and $\xi_i = W_{i+1}/W_i$ is an irreducible admissible P'-module.*

Since the functor $V \to V^*_{\psi'}$ is exact, by Lemma 2 there is a unique index i_0 $(0 \leq i_0 \leq I)$ with $\xi_{i_0} \simeq \tau$, and $\xi^*_{i,\psi'} = \{0\}$ for $i \neq i_0$.

Our proof of the Main Lemma is tantamount to showing, for an irreducible P'_{n-1}-module ρ_1, that the dimension of the space of P-invariant forms on $ind(\rho_1 \otimes \psi'; P'_n, P'_{n-1})$, where $P'_n = P'$ and $P'_{n-1} = H'_{n-2}$, is equal to the dimension of the space of P_{n-1}-invariant forms on ρ_1. By induction this dimension is then equal to the dimension of the space of G_j-invariant forms on the irreducible G'_j-module ρ attached to ρ_1 by Lemma 1. In fact we shall work directly with ρ, instead of applying induction, although the reader can safely read our proofs assuming that $j = n-2$. The twist by ν^s is introduced to guarantee that the only constituent ξ_i in $Res_{P'}\pi$ (see Lemma 3) which has a non-zero P-invariant form is the one indexed by $i = i_0$.

Let K, H be two closed subgroups of an ℓ-group G, and (ρ, V) a G-module. Choose a set of representatives g for $H\backslash G/K$, put $H_g = K \cap g^{-1}Hg$ and denote by ${}^g\rho$ the representation ${}^g\rho(g^{-1}hg) = \rho(h)$ of H_g on V. We shall use the following well-known result (we do not include a proof for it, although this is in fact implicit in the proof of the Main Lemma following Lemma 9 below; see also the functorial Theorem 5.2 of [BZ1] in the case of parabolic subgroups, and [S], §7.3, Proposition 22, in the case of finite groups).

Lemma 4. *The restriction to K of $ind(\rho; G, H)$ has a composition series consisting of $ind({}^g\rho; K, H_g)$, where g ranges over a set of representatives for $H\backslash G/K$.*

This Lemma will be applied to each of the irreducible P'-modules $\xi = \xi_i$ of

Lemma 3. By Lemma 1 we have

$$\xi \otimes \nu^{s-1} = \nu^{s-1} \otimes ind(\rho \otimes \psi'; P', H'_j) = ind((\rho \otimes \nu^{s-1}) \otimes \psi'; P', H'_j)$$

for some $j(0 \leq j < n)$ and irreducible G'_j-module ρ uniquely determined by ξ. Applying Lemma 4 with $G = P'$, $H = H'_j$, $K = P$, we conclude:

Lemma 5. *The restriction to P of the induced P'-module $\xi \otimes \nu^{s-1}$ has a composition series consisting of $ind({}^g[(\rho \otimes \nu^{s-1}) \otimes \psi']; P, P \cap g^{-1}H'_j g)$, where g ranges over $H'_j \backslash P'/P$.*

The double coset space $H'_j \backslash P'/P$ is equal to $N'_{n-1} \cdot G'_j \backslash G'_{n-1}/G_{n-1}$. We have

Lemma 6. *The group G' is the disjoint union of the double cosets $B'\eta G$ over all w in the Weyl group $W(A', G')$ (= Normalizer $(A')/A'$ of A' in G') with $w^2 = 1$. Here $\eta = \eta_w \in G'$ satisfies $\eta\overline{\eta}^{-1} = w$, where w is the representative whose entries are 0 and 1. The double coset is independent of the choice of the representative η.*

Proof. As noted in [F2], Proposition 10(1), the map $G'/G \to S = \{g \in G'; g\overline{g} = 1\}$, by $g \mapsto g\overline{g}^{-1}$, is a bijection. Indeed, it is clearly well defined and injective, and the surjectivity follows at once from the triviality of $H^1(\mathrm{Gal}(E/F), GL(n, E))$ (if $g\overline{g} = 1$, $a_\sigma = g$ defines a cocycle, which is then a coboundary, namely there is $x \in G'$ with $g = a_\sigma = x\overline{x}^{-1}$).

If $g \in G'$ maps to $s \in S$, then $bg \mapsto bs\overline{b}^{-1}$. By the Bruhat decomposition $G' = B'WB'$ applied to S, varying g in its double coset $B'gG$ we may assume that $g \mapsto wb \in S$, where $w \in W$ and $b \in B'$. Since wb lies in S, $1 = wbw\overline{b}$. Hence $w^{-1} = bw\overline{b}$, and the uniqueness of the Bruhat decomposition implies that $w^{-1} = w$. Write now $b = an$ with $a \in A'$, $n \in N'$. Since $1 = wbw\overline{b}$, we have $1 = waw\overline{a}$. Define an action σ of $\mathrm{Gal}(E/F)$ on A' by $\sigma(a') = w\overline{a}'w^{-1}$. Since $a\sigma(a) = 1$, $\{\sigma \mapsto a\}$ defines an element of $H^1(\mathrm{Gal}(E/F), A')$. This last group is trivial, hence there exists some $c \in A'$ with $a = w\overline{c}^{-1}wc$. Since $\overline{c}wanc^{-1} = wcnc^{-1}$, replacing g by $\overline{c}g$ we may assume that $g \mapsto wn$. Again $wn \in S$ implies $1 = wnw\overline{n}$, so if we define a Galois action σ on $N' \cap wN'w$ by $\sigma(n') = w\overline{n}'w$, the map $\{\sigma \mapsto n\}$ defines an element of $H^1(\mathrm{Gal}(E/F), N' \cap wN'w)$. Since this last group is trivial, there exists an $m \in N'(\cap wN'w)$ with $n = w\overline{m}^{-1}wm$. Hence $\overline{m}wnm^{-1} = w$, and replacing g by $\overline{m}g$ we may assume that $g \mapsto g\overline{g}^{-1} = w$. Since $G'/G \simeq S$ the existence of g, and the independence of $B'\eta_w G$ of the choice of η_w, are clear. The lemma follows.

A set of representatives for $N'_{n-1} \cdot G'_j \backslash G'_{n-1}/G_{n-1}$ is then given by $g = g(\eta, a) = \eta a$, where $\eta = \eta_w$ satisfies $\eta\overline{\eta}^{-1} = w$, $w^2 = 1$, $w \in W(A', G'_{n-1})/W(A', G'_j)$, thus w is a product over i of disjoint transpositions (k_i, m_i), $1 \leq k_i < m_i < n$ and $m_i > j$, and a ranges over $A'/\{a = w\overline{a}w^{-1} \in A'\}\ G'_j \cap A'$. When $j = n-2$, the only $w \neq 1$ is represented by $w = (k, m) = (n-2, n-1)$.

By Frobenius reciprocity ([BZ], Prop. 2.29), we have

$$\begin{aligned} &Bil_P(ind({}^g[(\rho \otimes \nu^{s-1}) \otimes \psi'];\ P, P \cap g^{-1}H'_j g), \mathbb{1}) \\ &= Bil_{P \cap g^{-1}H'_j g}({}^g[(\rho \otimes \nu^{s-1}) \otimes \psi']\delta_P/\delta_{P \cap g^{-1}H'_j g}, \mathbb{1}). \end{aligned}$$

Lemma 7. *If $g = g(\eta_w, a)$ and $w \neq 1$, then the last space is zero.*

Proof. Denote by (k, m) the transposition in $w \neq 1$ with maximal m. Let u be the unipotent upper triangular matrix in H'_j whose only non-zero entries outside the diagonal are $x (\in E)$ at the place (row, column)$= (m, m+1)$, and y at $(k, m+1)$. We choose y to be $y = \overline{x}\overline{a}_{1+m} a_k / a_{m+1} \overline{a}_m$ if $a = diag(a_1, \ldots, a_{n-1})$. Then $a^{-1}ua$ has the entry $x' = x a_{1+m}/a_m$ at $(m, m+1)$ and $\overline{x}' = y a_{m+1}/a_k = \overline{x}\overline{a}_{1+m}/\overline{a}_m$ at $(k, m+1)$, hence $g^{-1}ug = \eta^{-1}a^{-1}ua\eta$ lies in $g^{-1}H'_j g \cap P$. This $g^{-1}ug$ acts on ${}^g((\rho \otimes \nu^{s-1}) \otimes \psi')$ by multiplication by $\rho(I)\psi'(u) = \underline{\psi}'(x)$, and trivially on $\mathbb{1}$. Since x is arbitrary in E, and $\underline{\psi}' \neq 1$, the lemma follows.

Lemma 8. *With the exception of at most finitely many values of $X = q^{-s}$, the conclusion of Lemma 7 holds when $j \geq 1$.*

Proof. We may assume (by Lemma 7) that $w = 1$, and take $\eta_w = 1$. The element $h = diag(z, \ldots, z, 1, \ldots, 1)$ of H'_j (with $z \in F^\times$ and $\det h = z^j$) commutes with any a in A', and it lies in P. It acts on $(\rho \otimes \nu^{s-1}) \otimes \psi$ by multiplication by $\omega_\rho(z)|z|^{j(s-1)}$, where ω_ρ is the central character of ρ, and trivially on $\mathbb{1}$. Hence if $j \neq 0$, with the exception of at most finitely many values of $X = q^{-s}$, our space is $\{0\}$.

We clearly have

Lemma 9. *In the remaining case of $j = 0$, $w = 1$ (and $\eta_w = 1$), we have $H'_j = N'$, $g = a$ ranges over A'/A, and $P \cap g^{-1}H'_j g = N$. Then $Hom_N({}^a\psi', \mathbb{1})$ is zero if $g = a \notin A$, for then ${}^a\psi'(u) = \psi'(aua^{-1})$ is non-trivial on $u \in N$. If $g = a$ lies in A we may take $a = 1$, and then $Hom_N(\psi', \mathbb{1}) = \mathbb{C}$ since ψ' is trivial on N.*

Proof of Main Lemma. Note that by Lemma 6 the homogeneous space $X' = H'_j\backslash P' = G'_j N'\backslash P'$ is the disjoint union of the cosets $G'_j N'\backslash G'_j N' A' \eta_w G_{n-1}$, where w ranges over the set of w in $W(A', G'_{n-1})$ with $w^2 = 1$, taken modulo $W(A', G'_j)$. Put X'_1 for the union over all such w with $w \neq 1$. It is an open subset of X'. The space of $ind((\rho \otimes \nu^{s-1}) \otimes \psi'; P', H'_j)$ consists of functions on X'. Lemma 7 implies that any P-invariant linear form on $ind((\rho \otimes \nu^{s-1}) \otimes \psi'; P', H'_j)$ - equivalently a form in $Bil_P(ind((\rho \otimes \nu^{s-1}) \otimes \psi'; P', H'_j), \mathbb{1})$, viewed as a function of its first variable - must vanish on the functions which are supported on X'_1. Consequently its value depends only on the restriction of the functions in $ind((\rho \otimes \nu^{s-1}) \otimes \psi'; P', H'_j)$ to the closed subset $X' - X'_1 = NG_j\backslash A'G_{n-1}$ of X'. Lemma 8 shows that any such bilinear form is zero if $j \neq 0$, except for at most finitely many values of $X = q^{-s}$.

When $j = 0$, denote by A'_1 an open subset of A' which does not contain A. Lemma 9 implies that the P-invariant linear form must vanish on the functions which are supported on the open subset $N\backslash A'_1 G_{n-1}$ of $N\backslash A' G_{n-1}$. Hence its value at a function depends only on the restriction of the function to the closed subset $N\backslash (A' - A'_1)G_{n-1}$. In particular we may choose A'_1 to be the complement in A' of the closed subset A of A'.

In conclusion $Bil_P(\xi_i \otimes \nu^{s-1}, \mathbb{1})$ is zero for each ξ_i of Lemma 3, except for $i = i_0$ when $\xi_{i_0} = \tau (= ind(\psi'; P', N'))$, where $Bil_P(\tau \otimes \nu^{s-1}, \mathbb{1}) = Bil_P(\tau, \mathbb{1}) = \mathbb{C}$. The

Main Lemma follows from this by virtue of Lemma 3.

Proof of Proposition. Put $S = C_c^\infty(F^n)$ and $S_0 = C_c^\infty(F^n - \{\underline{0}\})$. We conclude that any

$$H \in Hom_P(\pi \otimes \nu^{s-1}, \mathbb{1}) = Bil_G(\pi \otimes \nu^s, S_0)$$

restricts to zero on W_{i_0} (in the notations of Lemma 3), and it is uniquely determined by its restriction to W_{i_0+1}, and its quotient $\tau = \xi_{i_0} = W_{i_0+1}/W_{i_0}$. In other words, given non-zero H, H' in $Bil_G(\pi \otimes \nu^s, S_0)$ there is a scalar c such that $H_0 = H' - cH$ is zero (in $Hom_P(W_{i_0+1} \otimes \nu^{s-1}, \mathbb{1})$, hence also in $Hom_P(\pi \otimes \nu^{s-1}, \mathbb{1})$). Consequently, given non-zero H, H' in $Bil_G(\pi \otimes \nu^s, S)$, there is a scalar c such that the restriction of $H_0 = H' - cH$ to $W(\pi \otimes \nu^s, \psi') \times S_0$ is zero. Note that $W(\pi \otimes \nu^s, \psi') \simeq W(\pi, \psi')$ via $W \otimes \nu^s \leftrightarrow W$, $(W \otimes \nu^s)(g) = W(g)\nu(\det g)^s$ for $g \in G'$.

The map $\Phi \mapsto \Phi(\underline{0})$ is an isomorphism of S/S_0 with $\mathbb{C}$. Hence the G-invariant bilinear form H_0 on $W(\pi \otimes \nu^s, \psi') \times S$ is of the form $H_0(W, \Phi) = h(W)\Phi(\underline{0})$, where h is a G-invariant linear form on $\pi \otimes \nu^s$. If $h \neq 0$ then $\omega_\pi(z)|z|^{ns} = 1$ for all $z \in F^\times$, where ω_π is the central character of π. Hence $H_0 = H' - cH$ vanishes, except possibly for a finite number of values of $X = q^{-s}$. With the exception of these values of s, we then have that $Bil_G(\pi \otimes \nu^s, S)$ is at most one dimensional, and the proposition follows, as does ((iii) of) the Theorem.

Remark. Suppose that $\pi = I(\pi_1, \ldots, \pi_m)$ is a G'-module normalizedly induced from the following P'-module, where $P' = M'N'$ is the standard parabolic subgroup of type $(n_1, \ldots, n_m)$. This representation is trivial on the unipotent radical N', and is given by the generic irreducible $GL(n_i, E)$-modules π_i on the ith factor of the Levi factor M'. By [Ze], Theorem 9.7(b) in the non-archimedean case, and [V] in the archimedean case, every generic irreducible G'-module is of this form, with square- integrable π_i.

It is likely that one has

$$(*) \qquad L(s, r(\pi)) = \prod_i L(s, r(\pi_i)) \prod_{j<k} L(s, \pi_j \times \pi_k),$$

and that the analogous relation holds for the ε-factors too. These relations are to be expected in analogy with standard properties of L and ε-functions of representations of Weil groups, and they are established in the split case where $E = F \oplus F$ in [JPS] for non-archimedean F, and in [JS] for archimedean F.

In fact the relation (*) is the basis of the proof of the analogue of our Theorem in the split archimedean case in [JS]. For this reason it will be worthwhile (but we do not plan on doing this, at least soon) to establish (*) in our non-split case, especially for $E/F = \mathbb{C}/\mathbb{R}$ (where the n_i are 1 or 2), for then an archimedean analogue of our Theorem is likely to follow, and the global results of [F1] and our paper above would extend to all separable quadratic extensions E/F, not only those where each real place of F splits in E.

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