On poles of twisted tensor L-functions

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Abstract

It is shown that the only possible pole of the twisted tensor L-functions in $\mathrm{Re}(s) \geq 1$ is located at $s = 1$ for *all* quadratic extensions of global fields.

0. Introduction.

Let E be a quadratic separable field extension of a global field F. Denote by $\mathbf{A}_E$, $\mathbf{A}_F$ the corresponding rings of adeles. Put G_n for GL_n and Z_n for its center. Then $\mathrm{Z}_n(\mathbf{A}_E)$ is the group $\mathbf{A}_E^\times$ of ideles of $\mathbf{A}_E$. Fix a cuspidal representation π of the adele group $\mathrm{G}_n(\mathbf{A}_E)$. Without lost of generality, we may assume that the central character of π is trivial on the split component of $\mathbf{A}_E^\times$. This is the multiplicative group $\mathbf{R}^\times$ of the field of real numbers embedded in $\mathbf{A}_E^\times$ via $x \mapsto (x, ..., x, 1, ...)$ (x in the archimedean, 1 in the finite components). Let S be a finite set of places of F (depending on π), including the places where E/F ramify, and the archimedean places, such that for each place v' of E above a place v outside S the component $\pi_{v'}$ of π is unramified. Following [1], let r be the twisted tensor representation of $\widehat{\mathrm{G}} = [\mathrm{GL}(n, \mathbf{C}) \times \mathrm{GL}(n, \mathbf{C})] \times \mathrm{Gal}(E/F)$ on $\mathbf{C}^n \otimes \mathbf{C}^n$. It acts by $r((a, b))(x \otimes y) = ax \otimes by$ and $r(\sigma)(x \otimes y) = y \otimes x$ ($\sigma \in \mathrm{Gal}(E/F)$, $\sigma \neq 1$). Let q_v be the cardinality of the residue field $R_v/\boldsymbol{\pi}_v R_v$ of the ring R_v of integers in F_v. We define the twisted tensor L-function to be the Euler product

$$L(s,\, r(\pi),\, S) = \prod_{v \notin S} \det\,[1 - q_v^{-s} r(t_v)]^{-1}.$$

The representation π is called *distinguished* if its central character is trivial on $\mathbf{A}_F^\times$ and there is an automorphic form $\phi \in \pi$ in $L^2(\mathrm{G}_n(E)\mathrm{Z}_n(\mathbf{A}_F)\backslash \mathrm{G}_n(\mathbf{A}_E))$, such that $\int \phi(g) dg \neq 0$. The integral is taken over the closed subspace $\mathrm{G}_n(F)\mathrm{Z}_n(\mathbf{A}_F)\backslash \mathrm{G}_n(\mathbf{A}_F)$ of $\mathrm{G}_n(E)\mathrm{Z}_n(\mathbf{A}_F)\backslash \mathrm{G}_n(\mathbf{A}_E)$.

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The following theorem is proven in [1, p. 309] for a quadratic extension E/F of global fields, such that each archimedean place of F splits in E. We prove it for any quadratic extension of global fields, i.e. also for number fields with completions $E_v/F_v = \mathbf{C}/\mathbf{R}$.

Theorem. *The product $L(s,\, r(\pi),\, S)$ converges absolutely, uniformly in compact subsets, in some right half-plane. It has analytic continuation as a meromorphic function to the right half plane $\mathrm{Re}(s) > 1-\epsilon$, for some small $\epsilon > 0$. The only possible pole of $L(s,\, r(\pi),\, S)$ in $\mathrm{Re}(s) > 1-\epsilon$ is simple, located at $s = 1$. The function $L(s,\, r(\pi),\, S)$ has a pole at $s = 1$ if and only if π is distinguished.*

Proof. The proof of this theorem is the same as that of the Theorem of [1, §4], pp. 309-310. On lines 14 and 18 of page 310 of [1], we use the proposition below. It holds in the non-split archimedean case too. Hence the restriction put in [1] on the extension E/F can be removed.

For the functional equation satisfied by $L(s,\, r(\pi),\, S)$, see [1]. For the local L-factors at all non-archimedean places of F, see [2]. The non-vanishing of this L-function on the edge $\mathrm{Re}(s) = 1$ of the critical strip has been shown by Shahidi [6]. Twisted tensor L-functions are used in the study (see Kon-no [5]) of the residual spectrum of unitary groups.

1. Local computations.

From now on, we consider the local case only. Let E/F be a quadratic extension of local fields. Thus in the archimedean case $E/F = \mathbf{C}/\mathbf{R}$. Denote by $x \mapsto \bar{x}$ the non-trivial automorphism of E over F. Let $\iota \neq 0$ be an element of E, such that $\bar{\iota} = -\iota$. Put G_n for GL_n. The groups of F and E-points are denoted by $\mathrm{G}_n(F)$ and $\mathrm{G}_n(E)$. Denote by N_n the unipotent radical of the upper triangular subgroup of G_n, and by A_n the diagonal subgroup. Let ψ_0 be a non trivial additive character of F. For example, if $F = \mathbf{R}$ then $\psi_0(x) = e^{2\pi i x}$. Let ψ be the (non-trivial) character $\psi(z) = \psi_0((z-\bar{z})/\iota)$ of E. It is trivial on F. For $u \in \mathrm{N}_n(E)$, set $\theta(u) = \psi(\sum_{i=1}^{n-1} u_{i,i+1})$.

Fix an irreducible admissible representation π of $\mathrm{G}_n(E)$ on a complex vector space V. The representation π is called *generic* if there exists a non-zero linear form λ on V, such that $\lambda(\pi(u)v) = \theta(u)\lambda(v)$ for all v in V and u in $\mathrm{N}_n(E)$. The dimension of the space of such λ is bounded by one. Let $W(\pi;\, \theta)$ be the space of functions W on $\mathrm{G}_n(E)$ of the form $W(g) = \lambda(\pi(g)v)$, where $v \in V$. We have $W(ug) = \theta(u)W(g)$ ($g \in \mathrm{G}_n(E)$, $u \in \mathrm{N}_n(E)$). Denote by $W_0(\pi;\, \theta)$ those functions in $W(\pi;\, \theta)$ whose corresponding vectors v are in the space of K-finite vectors, where $\mathrm{K}=\mathrm{K}_n(E)$ is the standard maximal compact subgroup of $\mathrm{G}_n(E)$.

For $\Phi \in S(F^n)$, define the integral

$$\Psi(s, \Phi, W) = \int_{N_n(F)\backslash G_n(F)} W(g)\Phi(\epsilon_n g)|\det\, g|^s dg,$$

where $\epsilon_n = (0, 0, ..., 0, 1)$ is a row vector of size n.

Proposition. (i) *There exists some small constant* ϵ*,* $\epsilon > 0$*, such that the integral* $\Psi(s, \Phi, W)$ *converges absolutely, uniformly in compact subsets, for* $\mathrm{Re}(s) > 1 - \epsilon$*;*

(ii) *There exists* W *in* $W_0(\pi; \theta)$ *and* Φ *in* $S(F^n)$*, such that* $\Psi(s, \Phi, W) \neq 0$.

Proof. When E/F is an extension of non-archimedean local fields, (i) and (ii) are treated in the Proposition of [1], §4, p. 308. We prove (i) in general, including the case $(E,\, F) = (\mathbf{C},\, \mathbf{R})$, following Jacquet and Shalika [3], pp. 204-206.

Using the Iwasawa decomposition $\mathrm{G}_n(F) = \mathrm{N}_n(F)\mathrm{A}_n(F)\mathrm{K}_n(F)$, and the associated measure decomposition, we need to show the convergence of the integral

$$\int_{A_n(F)K_n(F)} |W(ak)||\det\, a|^s \delta_{n,F}^{-1}(a) da dk.$$

Here $a = \mathrm{diag}(a_1,\, a_2,\, \dots,\, a_{n-1},\, 1)$. Recall that

$$\delta_{n,F}(a) = \delta_{n-1,F}(a)|\det\, a| = |\det\, a| \prod_{1 \le i < j \le n-1} \frac{|a_i|}{|a_j|},$$

and (see e.g. [1], p. 307) that $\delta_{n,E}(a) = \delta_{n,F}^2(a)$.

By Proposition 3 of Jacquet and Shalika [3, §4] there is a finite set $\mathbf{X}$ of finite functions in $n-1$ variables such that $|W(ak)|$ is bounded by a finite sum of expressions of the form

$$C_\chi \delta_{n-1,E}^{1/2}(a) \Phi\left(\frac{a_1}{a_2},\, \frac{a_2}{a_3},\, ...,\, a_{n-1}\right).$$

Here C_χ is the absolute value of some element of $\mathbf{X}$ and $\Phi \ge 0$ is in $S(F^{n-1})$. Thus, it suffices to show that the integral obtained by replacing W by this estimate is convergent. Using that

$$\delta_{n-1,E}^{1/2}(a)\delta_{n,F}^{-1}(a) = \delta_{n-1,F}(a)\delta_{n,F}^{-1}(a) = |\det\, a|^{-1},$$

we arrive at the finite sum of integrals

$$\int C_\chi \Phi\left(\frac{a_1}{a_2},\, \frac{a_2}{a_3},\, ...,\, a_{n-1}\right) |\det\, a|^{s-1} da.$$

The change of variables $a_1 = t_1 ... t_{n-1}$, $a_2 = t_2 ... t_{n-1}$, ..., $a_{n-1} = t_{n-1}$, has the Jacobian $t_2 t_3^2 ... t_{n-2}^{n-3}$. We obtain a sum of expressions of the form

$$\int C_\chi \Phi(t_1,\, t_2,\, ...,\, t_{n-1}) \prod_{j=2}^{n-2} t_j^{j-1} \prod_{j=1}^{n-1} t_j^{j(s-1)} dt.$$

Again, by Proposition 3 of Jacquet and Shalika [3, §4] the set $\mathbf{X}$ is such that any χ in it is the product of (1) a polynomial in the logarithms of the absolute values of the variables, and (2) a character of the form

$$\chi_1(t_1)\chi_2(t_2)...\chi_{n-1}(t_{n-1}),$$

with $\mathrm{Re}(\chi_i) > 0$, for each i. It follows that the above integral converges uniformly in compact subsets of $\mathrm{Re}(s) > 1 - \epsilon$, for some small $\epsilon > 0$. This completes the proof of (i).

For (ii) we will follow the proof of Proposition 7.3 of Jacquet and Shalika [3]. Assume that $\Psi(s, \Phi, W) = 0$ for all choices of W in $W_0(\pi; \theta)$ and Φ in $S(F^n)$. We will show that $W(e) = 0$ for all W, a contradiction which will imply (ii) of the lemma. Since Φ is arbitrary, it follows that for all W we have

$$\int_{N_{n-1}(F)\backslash G_{n-1}(F)} W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] |\det g|^s dg = 0.$$

Define

$$I_k(W) = \int_{N_k(F)\backslash G_k(F)} W\left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-k} \end{pmatrix}\right] |\det g|^s dg.$$

We claim that $I_k(W)$ is zero for all W and all k with $0 \leq k \leq n-1$. The lemma would then follow, since $W(e) = I_0(W)$. We will show this claim by descending induction on k. We have just seen that $I_{n-1}(W) = 0$. So fix $k \leq n-1$ with $I_k(W) = 0$ for all W. We proceed to show that $I_{k-1}(W) = 0$ for all W.

We apply the fact that $I_k(W) = 0$ to the function W_Φ defined by

$$W_\Phi(g) = \int_{F^k} W\left[g\begin{pmatrix} 1_k & \iota u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix}\right] \Phi(u) du.$$

Here u is a column of size k, $\Phi \in S(F^k)$ and $W \in W_0(\pi; \theta)$. Proposition 2.4 of Jacquet and Shalika [4; II], p. 784, and the remark following it (top of p. 786), assure us that this function is in the space $W_0(\pi; \theta)$.

Note that

$$W_\Phi\left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-k} \end{pmatrix}\right] = W\left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-k} \end{pmatrix}\right] \widehat{\Phi}(\epsilon_k g),$$

where $\widehat{\Phi}(y) = \int_{F^k} \Phi(u)\psi_0(y \cdot u) du$ denotes the Fourier transform of $\Phi \in S(F^k)$. Indeed

$$\widehat{\Phi}(\epsilon_k g) = \int_{F^k} \Phi(u)\psi_0(\epsilon_k g \cdot u) du = \int_{F^k} \Phi(u)\psi_0\left(\sum_{j=1}^{k} g_{kj} u_j\right) du.$$

Further, since

$$\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix} \begin{pmatrix} 1_k & \iota u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix} = \begin{pmatrix} 1_k & \iota g u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix} \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix},$$

we have

$$\begin{aligned} & W\left[\begin{pmatrix} 1_k & \iota g u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix} \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix}\right] \\ = \ & \theta\left(\begin{pmatrix} 1_k & \iota g u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix}\right) W\left[\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix}\right] \\ = \ & \psi\left(\sum_{j=1}^{k} \iota g_{kj} u_j\right) W\left[\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix}\right] = \psi_0\left(\sum_{j=1}^{k} g_{kj} u_j\right) W\left[\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix}\right]. \end{aligned}$$

Now substituting W_Φ for W in $I_k(W) = 0$, we obtain

$$\int_{N_k(F)\backslash G_k(F)} W\left[\begin{pmatrix} g & 0 \\ 0 & 1_{n-k} \end{pmatrix}\right] \widehat{\Phi}(\epsilon_k g) |\det g|^s dg = 0$$

for all $\Phi \in S(F^k)$ and all $W \in W_0(\pi;\theta)$. In this integral $\widehat{\Phi}$ can be replaced by any element of $S(F^k)$. Hence $I_{k-1}(W) = 0$ for all W and we are done.

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