Inverse Function Theorem

IBL Analysis

May 27, 2009

1 Nodal singularity

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = xy$.

Draw a picture of the level set where $f(x, y) = 0$. For which (x, y) does the implicit function theorem apply? What (not so terribly) terrifying event happens at $(x, y) = (0, 0)$?

2 Cuspidal singularity

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^3 - y^2$.

Draw a picture of the level set where $f(x, y) = 0$. For which (x, y) does the implicit function theorem apply? What truly terrifying event happens at $(x, y) = (0, 0)$?

3 Lines

Consider the function

$$f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$$

given by

$$f(x, y, t) = (xt, y(1 - t))$$

That is, f sends (x, y, t) to the point in $\mathbb{R}^2$ which is t of the way between $(x, 0)$ and $(0, y)$.

Qualitatively describe $f^{-1}(x, y)$ using the implicit function theorem. Are there choices of (x, y) with drastically different behavior?

4 Polar coordinates

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(r, \theta) = (r \cos\theta, r \sin\theta).$$

For which (r, θ) is f locally bijective?

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5 Polynomials

Let P_{n-1} be the (vector) space of degree $n-1$ polynomials in a variable t; note that $\dim P_n = n$. Define a function $f : \mathbb{R}^n \to P_{n-1}$ given by

$$f(x_1, \ldots, x_n) = (t - x_1) \cdots (t - x_n) - t^n$$

So f takes n points in $\mathbb{R}$ to a polynomial having those points as its roots (well, that polynomial minus t^n).

Find points $(x_i) \in \mathbb{R}^n$ for which there exists a neighborhood $U \ni (x_i)$ with $f(U)$ open. Our goal can can be described more colorfully: we are seeking polynomals with n roots, which, when wiggled, still have n roots.

For concreteness, it will help tremendously to explore the special cases $f : \mathbb{R}^2 \to P_1$ and then $f : \mathbb{R}^3 \to P_2$.

6 How to (not) flatten a sphere

Write D^2 for the open unit disk, i.e.,

$$D^2 = \{v \in \mathbb{R}^2 : ||v|| < 1\}.$$

A function from the sphere S^2 to X is given by a *pair* of functions $f : D^2 \to X$ and $g : D^2 \to X$ so that

$$f(v) = g\left(\frac{\frac{3}{2} - ||v||}{||v||} v\right) \quad \text{for } \tfrac{1}{2} < ||v|| < 1$$

You should think of f as defining the function on a bit more than the top hemisphere, and g as defining the function on a bit more than the bottom hemisphere; the functions must agree on the overlap.

Does there exist a function $S^2 \to \mathbb{R}^2$ with everywhere nonvanishing derivative? (Equivalently, is it possible to put coordinates on the Earth so that at every point, four "cardinal directions" are defined?)

7 Square roots

Let M_2 be the four-dimensional space of two-by-two matrices; let $f : M_2 \to M_2$ be the function $f(A) = A \cdot A$.

Compute $Df^{-1}(I)$. Use this to approximate

$$\sqrt{\begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 1.1 \end{bmatrix}}$$

For extra fun, when is Df invertible? This will tell us when perturbations of matrices having square roots also have square roots.

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8 Orthogonal matrices

Recall that a matrix is **orthogonal** if $A \cdot A^{\mathsf{T}} = I$.

Let M_3 be the nine-dimensional vector space of 3-by-3 matrices; let S_3 be the six-dimensional vector space of symmetric 3-by-3 matrices. Define a function $f : M_3 \to S_3$ by $f(A) = A \cdot A^{\mathsf{T}}$. A matrix is orthogonal if it is in $f^{-1}(I)$.

Note that $f(I) = I$; along how many degrees of freedom may we perturb I and still have an orthogonal matrix? (You will have to compute $Df(I)$ to analyze this).

Repeat this problem for the analogously defined function $M_2 \to S_2$ to show that, in two dimensions, there is only one degree of freedom for rotation. For rotations in four dimensions, how many degrees of freedom do you have? (It would indeed be very confusing to be tumbling in four dimensionsal space).

9 Segre variety

Consider $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$f(v_1, v_2, w_1, w_2) = (v_1 w_1, v_1 w_2, v_2 w_1, v_2 w_2).$$

For the linear algebra cognoscenti, this is a special case of a certain (nonlinear!) function $V \oplus W \to V \otimes W$.

Globally, f is not injective—after all,

$$f(0, 0, w_1, w_2) = (0, 0, 0, 0).$$

But are there points for which f is *locally* bijective?

10 Characteristic polynomials

Let M_2 be the four-dimensional vector space of two-by-two matrices; let P_2 be the three-dimensional vector space of degree two polynomials in a variable λ.

The characteristic polynomial of a matrix A is $\det(A - \lambda I)$; let $f : M_2 \to P_2$ be the function sending a matrix to its characteristic polynomial. (If you are very fancy, you might have learned that the roots of the characteristic polynomial are the eigenvalues of the matrix).

Tell me a interesting story involving the implicit function theorem and the function f.

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