Examples of Metric Spaces

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We should carry with ourselves many examples of metric spaces. So far, we have subsets of $\mathbb{R}$ and the discrete metric space.

1 One quick example

Proposition 1.1. *Let* $S^1 = [0, 2\pi)$*, with the metric*

$$d_{S^1}(x, y) = \min \{x - y, 2\pi - (x - y)\}$$

Then (S^1, d_{S^1}) *is a metric space.*

Remark 1.1.1. Thinking geometrically, S^1 is just the unit circle; the distance between points x and y is the length of the shortest arc joining x and y.

2 Building new spaces from old

There are a few techniques we will introduce here:

- the product of two metric spaces,
- subsets of metric spaces,
- rescaling the metric
- the union of two metric spaces

Definition 2.0.1. Let (X, d_X) and (Y, d_Y) be metric spaces. The **product** of X and Y (written $X \times Y$) is the set of pairs

$$\{(x, y) : x \in X, y \in Y\}$$

with the metric

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \max \{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

Proposition 2.1. *The space* $(X \times Y, d_{X\times Y})$ *is a metric space.*

Example 2.1.1. We normally write $\mathbb{R} \times \mathbb{R}$ as $\mathbb{R}^2$. Note that $d_{\mathbb{R}\times\mathbb{R}}$ as defined above is *not* the usual Euclidean metric.

Exercise 2.1.1. What "shape" does a ball of radius R have in the metric space $(\mathbb{R}\times\mathbb{R}, d_{\mathbb{R}\times\mathbb{R}})$?

Remark 2.1.1. Recall that any subset of a metric space is still a metric space.

Proposition 2.2. *Let (X, d) be a metric space, and pick $\epsilon > 0$. Then $(X, \epsilon \cdot d)$ is again a metric space (where $\epsilon \cdot d$ means the metric multipled by ϵ).*

Proposition 2.3. *The metric spaces $(\mathbb{R}, d)$ and $(\mathbb{R}, 1000d)$ are isometric.*

Proposition 2.4. *The metric spaces (S^1, d_{S^1}) and $(S^1, 2d_{S^1})$ are not isometric, but they are homeomorphic.*

Definition 2.4.1. Suppose (X_1, d_1) and (X_2, d_2) are metric spaces. The **disjoint union** $X_1 \sqcup X_2$ is the metric space having points $X_1 \sqcup X_2$, and metric

$$\begin{aligned}
d_{X_1 \sqcup X_2}(x_1, y_1) &= d_1(x_1, y_1) \text{ if } x_1, y_1 \in X_1,\\
d_{X_1 \sqcup X_2}(x_2, y_2) &= d_2(x_2, y_2) \text{ if } x_2, y_2 \in X_2,\\
d_{X_1 \sqcup X_2}(x_1, y_2) &= 1 \text{ if } x_1 \in X_1 \text{ and } y_2 \in X_2,\\
d_{X_1 \sqcup X_2}(x_2, y_1) &= 1 \text{ if } x_2 \in X_2 \text{ and } y_1 \in X_1.
\end{aligned}$$

We can take the disjoint union of an indexed family of metric spaces: say (X_i, d_i) are metric spaces for $i \in I$. Then

$$\bigsqcup_{i \in I} (X_i, d_i)$$

can be defined as the above.

Example 2.4.1. Let $(\{\star\}, d)$ be *the* metric space consisting of a single point. Then

$$\bigsqcup_{i \in I} (\{\star\}, d)$$

is the set I with the discrete metric.

Definition 2.4.2. Let (X, d) be a metric space, and suppose $\sim$ is an equivalence relation on the points of X, such that

$$x \sim x' \text{ and } y \sim y' \Rightarrow d(x, y) = d(x', y').$$

In other words, the equivalence relation is compatible with the metric. Then the **quotient** space $(X/\sim, d)$ consists of equivalence classes of points of X, and metric

$$d([x], [y]) = d(x, y),$$

where $[x]$ is the equivalence class containing x.

Definition 2.4.3. Suppose (X_1, d_1) and (X_2, d_2) are metric spaces, and $f_i : (Y, d) \to (X_i, d)$ is an isometry. Then the **union** of X_1 and X_2 along Y, written $X_1 \cup_Y X_2$ is

$$X_1 \sqcup X_2 / \sim$$

where $x_1 \sim x_2$ if there exists $y \in Y$ with $f_i(y) = x_i$.

This definition makes precise the notion of gluing together two metric spaces.

3 The 2-adic numbers

Definition 3.0.4. For $p/q \in \mathbb{Q}$, find integers a, b, n so that

$$\frac{p}{q} = 2^n \frac{a}{b}$$

so that neither a nor b are divisible by two. Then the **2-adic valuation** of p/q, written

$$|p/q|_2$$

is defined to be 2^{-n}. By convention, $|0|_2 = 0$.

We think of the 2-adic valuation as measuring how many twos there are in p/q. Numbers containing a lot of twos (like 16) are small.

Definition 3.0.5. For $x, y \in \mathbb{Q}$, define the 2-adic distance between them to be

$$d_2(x, y) = |x - y|_2$$

Definition 3.0.6. The **2-adics** $(\mathbb{Q}_2, d_2)$ are the completion of $\mathbb{Q}$ with respect to the 2-adic metric d_2.

By construction, $\mathbb{Q}_2$ is a complete metric space.

Exercise 3.0.1. Find an element of $\mathbb{Q}_2$ which is not in $\mathbb{Q}$.

Exercise 3.0.2. Is $\mathbb{Q}_2$ connected?

Remark 3.0.1. There is nothing special about 2—we can similarly define $\mathbb{Q}_p$ for any prime p.